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# On Hausdorff Measures of Curves in Sub-Riemannian Geometry

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**Abstract**

In sub-Riemannian geometry, the length of a non-horizontal path is not defined (or is equal to $+\infty$). However several other notions allow to measure a path, such as the Hausdorff measures, the class of $k$-dimensional lengths introduced in [2], or notions based on approximations by discrete sets, like the nonholonomic interpolation complexity and the entropy (see [8, 10]). The purpose of this paper is to compare these different notions and to extend our knowledge of the $k$-lengths, the entropy and the complexity to the Hausdorff measures. We show in particular that $k$-lengths coincide with the Hausdorff measures, thus providing an integral representation of these measures and the value of the Hausdorff dimension. We also give asymptotics of the entropy and of the nonholonomic interpolation complexity in function of the Hausdorff measures, which in turn allow to compute these measures in many cases.

## 1 Introduction

Let $(M,D,g)$ be a $C^\infty$ sub-Riemannian manifold: $M$ is a $C^\infty$ manifold, $D \subset TM$ a $C^\infty$ distribution on $M$ and $g$ a $C^\infty$ Riemannian metric on $D$ (such manifolds are also called Carnot-Carathéodory spaces).

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We assume that Chow's Condition is satisfied: let $D^s$ denote the $\mathbb{R}$-linear span of brackets of degree $\leq s$ of vector fields tangent to $D^1 = D$; then, at every $p \in M$, there exists an integer $r = r(p)$ such that $D^r(p) = T_pM$.

Horizontal paths are those absolutely continuous paths which are almost everywhere tangent to $D$. The length of horizontal paths is then obtained as in Riemannian geometry integrating the norm of their tangent vectors. Chow's condition implies that one can join any two points of the manifold by a horizontal path and therefore a distance $d$ can be defined.

For a non-horizontal path, the length is not defined (or is equal to $+\infty$, depending on the convention). However several notions allow to measure any path. We consider here three kinds of such notions: first, the Hausdorff measures, the usual ones $\mathcal{H}^k$ and the spherical ones $\mathcal{S}^k$; secondly, the class of $k$-dimensional lengths introduced in [2]; finally, the notions based on approximations by discrete sets, such as the nonholonomic interpolation complexity $\sigma_{\text{int}}$ and the entropy $e$ (see [8, 10]).

Some results already exist for these quantities. We have given in [11] weak equivalents of the entropy and the complexity $\sigma_{\text{int}}$ in an analytic setting. Gauthier and Zakalyukin have computed strong equivalents of $\sigma_{\text{int}}$ for particular classes of sub-Riemannian manifolds (when the distribution is of corank one or when $D^2 = TM$ for instance). We have also computed in [2] the $k$-dimensional lengths in sub-Riemannian contact manifolds. But almost no result exist at the moment for the Hausdorff measures.

The purpose of this paper is to compare these different notions in the general case and to extend our knowledge of the $k$-lengths, the entropy and the complexity to the Hausdorff measures. Our first main result is that $\mathcal{H}^k$, $2^{k-1}\mathcal{S}^k$ and the $k$-length coincide when the path is almost everywhere regular. This gives in particular the value of the Hausdorff dimension of such a path $\gamma$: it is the smallest integer $k$ such that $\gamma$ is almost everywhere tangent to $D^k$. Note also that, for $k > 1$, the $k$-dimensional spherical and non spherical Hausdorff measures do not coincide (except when they both vanish), even for smooth paths. As a last consequence, the $k$-lengths provide integral representations of the Hausdorff measures.

The second result states that, for a $C^1$ and equiregular curve $C$, the limits of $2\varepsilon^k e(C,\varepsilon)$ and of $\varepsilon^k \sigma_{\text{int}}(C,\varepsilon)$ are equal to $\mathcal{H}^k(C)$. This, together with the results of [8], give the values of the Hausdorff measures as well as strong equivalents for the entropy, in many different situations.

This paper is organized as follows. Section 2 is devoted to the statement of the main results, firstly for Hausdorff measures and $k$-dimensional lengths, and secondly for entropy and nonholonomic interpolation complexity. Proofs are provided in Section 3.

# 2 Main results

## 2.1 Hausdorff measures and classes of lengths

We consider the metric space $(M,d)$ and we denote by $\operatorname{diam} S$ the diameter of a set $S \subset M$.

**Definition 1.** Let $k \geq 0$ be a real number. For every set $A \subset M$, we define the *k-dimensional Hausdorff measure* $\mathcal{H}^k$ of $A$ as $\mathcal{H}^k(A) = \lim_{\varepsilon \to 0^+} \mathcal{H}^k_\varepsilon(A)$, where

$$\mathcal{H}^k_\varepsilon(A) = \inf\{\sum_{i=1}^{\infty} (\operatorname{diam} S_i)^k : A \subset \bigcup_{i=1}^{\infty} S_i,\ \operatorname{diam} S_i \leq \varepsilon,\ S_i \text{ closed set}\},$$

and the *k-dimensional spherical Hausdorff measure* $\mathcal{S}^k$ of $A$ as $\mathcal{S}^k(A) = \lim_{\varepsilon \to 0^+} \mathcal{S}^k_\varepsilon(A)$, where

$$\mathcal{S}^k_\varepsilon(A) = \inf\{\sum_{i=1}^{\infty} (\operatorname{diam} S_i)^k : A \subset \bigcup_{i=1}^{\infty} S_i,\ S_i \text{ is a ball},\ \operatorname{diam} S_i \leq \varepsilon\}.$$

*Remark* 2. In the Euclidean space $\mathbb{R}^n$, the $k$-dimensional Hausdorff measures are often defined as $2^{-k}\alpha(k)\mathcal{H}^k$ and $2^{-k}\alpha(k)\mathcal{S}^k$, where $\alpha(k)$ is defined from the usual gamma function as $\alpha(k) = \Gamma(\frac{1}{2})^k/\Gamma(\frac{k}{2}+1)$. This normalization factor is necessary for the $n$-dimensional Hausdorff measures and the Lebesgue measure to coincide on $\mathbb{R}^n$.

For a given set $A \subset M$, $\mathcal{H}^k(A)$ is a decreasing function of $k$, infinite when $k$ is smaller than a certain value, and zero when $k$ is greater than this value. We call *Hausdorff dimension* of $A$ the real number

$$\dim_{\mathcal{H}} A = \sup\{k : \mathcal{H}^k(A) = \infty\} = \inf\{k : \mathcal{H}^k(A) = 0\}.$$

Note that $\mathcal{H}^k \leq \mathcal{S}^k \leq 2^k\mathcal{H}^k$, so the Hausdorff dimension can be defined equally from Hausdorff or spherical Hausdorff measures.

When the set $A$ is a curve, another kind of dimensioned measures exist, namely the class of lengths defined in [2].

**Definition 3.** Let $\gamma : [a,b] \to M$ be an absolutely continuous path and $C = \gamma([a,b])$. For $k \geq 1$, we define the *k-dimensional length* of $C$ as

$$\mathrm{Length}_k(C) = \int_a^b \mathrm{meas}_t^k(\gamma)\, dt, \tag{1}$$

where $\mathrm{meas}_t^k(\gamma) = \left[\limsup_{s\to 0} \frac{1}{s} d\left(\gamma(t), \gamma(t+s^k)\right)\right]^k$.

Given a curve $C$, $\mathrm{Length}_k(C)$ is a decreasing function of $k$. More precisely, let $k_C$ be the smallest integer $k$ such that $\dot\gamma(t)$ belongs to $D^k(\gamma(t))$ for a.e. $t \in [a,b]$; then, from [2, Cor. 17],

$$\mathrm{Length}_k(C) = \begin{cases} 0 & \text{for } k > k_C, \\ +\infty & \text{for } k < k_C. \end{cases}$$

To compare Hausdorff measures and $k$-lengths, we need additional characterizations of curves.

**Definition 4.** Let $A \subset M$. A point $p \in A$ is said *A-regular* if the sequence of dimensions $\dim D^1(q) \leq \dim D^2(q) \leq \cdots$ remains constant for $q \in A$ near $p$, and *A-singular* otherwise. The set $A$ is said *equiregular* if every point of $A$ is $A$-regular. A path $\gamma : [a,b] \to M$ is said *almost everywhere regular* (a.e. regular) if the set $\{t \in [a,b] \ : \ \gamma(t)\ C\text{-singular}\}$, where $C = \gamma([a,b])$, is of zero Lebesgue measure.

**Theorem 5.** *Let $\gamma : [a,b] \to M$ be an absolutely continuous and a.e. regular path, and $C = \gamma([a,b])$. Then, for any $k \geq 1$,*

$$\begin{aligned} \mathcal{H}^k(C) &= \mathrm{Length}_k(C), \\ \mathcal{S}^k(C) &= 2^{k-1}\mathrm{Length}_k(C). \end{aligned}$$

When $k = 1$, since the 1-dimensional length equals the usual length [2, Lem. 17], we recover the classical relation $\mathcal{H}^1(C) = \mathcal{S}^1(C) = \mathrm{length}(C)$ (see [3, 2.10.13]). However, for $k > 1$, we obtain that the $k$-dimensional Hausdorff and spherical Hausdorff measures of the curves do not coincide. In [14], this difference is already noticed for the $Q$-dimensional Hausdorff measures in a Carnot group of homogeneous dimension $Q$ and is related to the nonexistence of an isodiametric inequality.

Another important consequence is that the Hausdorff dimension of $C$ is the integer $k_C$.

**Corollary 6.** *If $C = \gamma([a,b])$, where $\gamma : [a,b] \to M$ is absolutely continuous and a.e. regular, then $\dim_{\mathcal{H}} C$ is the smallest integer $k$ such that $T_qC$ belongs to $D^k(q)$ for $\mathcal{H}^1$-a.e. $q \in C$.*

*Remark* 7. When the sub-Riemannian manifold is equiregular, it is already known [9, p. 104] that the Hausdorff dimension of a one-dimensional submanifold $C$ is the smallest integer $k$ such that $T_qC \in D^k(q)$ for every $q \in C$. The corollary above is a generalization of this formula.

*Remark* 8. Under the hypothesis of Corollary 6, the Hausdorff dimension of $C$ is necessary an integer. In particular, in an analytic sub-Riemannian manifold (i.e. when $M, D, g$ are analytic), every analytic curve is a.e. regular and so its Hausdorff dimension is an integer[1].

Theorem 5 gives an integral representation of the Hausdorff measures. This representation can be explicitly computed when $(M,D,g)$ is a sub-Riemannian contact manifold. For in this case, we have expressed the 2-dimensional length as a function of the canonical form and of a geometric invariant $\lambda_{\max}$ (see [2, Sect. 4.1] for the details).

**Corollary 9.** *Let $(M,D,g)$ be a sub-Riemannian contact manifold of dimension $2n+1$ and $\theta$ the canonical form. If $\gamma : [a,b] \to M$ is absolutely continuous, then, for $C = \gamma([a,b])$*

$$\mathcal{H}^2(C) = 4\pi \int_a^b \frac{|\theta(\gamma(t), \dot{\gamma}(t))|}{\lambda_{\max}(\gamma(t))}\, dt.$$

If the contact manifold coincides with the 3-dimensional Heisenberg group, we obtain

$$\mathcal{S}^2(C) = 8\pi \int_a^b |\theta(\gamma(t), \dot{\gamma}(t))|\, dt.$$

This formula, originally due to Pansu [13], is instrumental in the proof of coarea formula in the Heisenberg group (see [12], where the link between 2-lengths and 2-dimensional spherical Hausdorff measures is already pointed up).

## 2.2 Approximations by discrete sets

Another way to measure a curve is to study its approximations by discrete sets (see [10] and [9, p. 278]). We consider two kind of such sets, namely $\varepsilon$-nets and $\varepsilon$-chains.

[1] In [11, p. 495, (d)], we wrongly assert that the Hausdorff dimension of an analytic curve may be not an integer. The mistake arises from the statement of an incorrect result, namely Prop. 2.4-(ii).

Let $C = \gamma([a,b])$ be the image of an absolutely continuous path $\gamma : [a,b] \to M$. An $\varepsilon$-*net* of $C$ is a set of points $q_1, q_2, \ldots$ in $M$ such that the closed balls $B(q_i, \varepsilon)$ cover $C$. An $\varepsilon$-*chain* of $C$ is a set of points $q_1 = \gamma(a), \ldots, q_N = \gamma(b)$ in $C$ such that $d(q_i, q_{i+1}) \leq \varepsilon$.

**Definition 10.** The *metric entropy* $e(C, \varepsilon)$ is the minimal number of points in an $\varepsilon$-net of $C$. The *nonholonomic interpolation complexity* $\sigma_{\mathrm{int}}(C, \varepsilon)$ is the minimal number of points in an $\varepsilon$-chain of $C$.

*Remark* 11. The terminology "nonholonomic interpolation" has been introduced by Gauthier and Zakalyukin [8]: they define an $\varepsilon$-nonholonomic interpolating path of $C$ as an horizontal path with the same extremities as $C$ and formed by a finite number of pieces of length $\leq \varepsilon$ that connect points of $C$. The minimal number of pieces in an $\varepsilon$-nonholonomic interpolating path is $\sigma_{\mathrm{int}}(C, \varepsilon)$.

*Remark* 12. The metric entropy of a set $A$ is related to its spherical Hausdorff measures: for every $k \geq 0$ and $\varepsilon > 0$, $2^{-k} \mathcal{S}^k(A) \leq \varepsilon^k e(A, \varepsilon)$. Hence $\dim_{\mathcal{H}} A$ is not greater than the infimum of $k \geq 0$ such that $e(A, \varepsilon) < \varepsilon^{-k}$. However the latter infimum, called the *entropy dimension*, can be different from the Hausdorff dimension (see [15, p. 26]).

**Theorem 13.** *Let $\gamma : [a,b] \to M$ be a $C^1$ path without double points. If $C = \gamma([a,b])$ is equiregular, then $\mathcal{H}^k(C) = \mathrm{Length}_k(C)$ (due to Theorem 5) and*

$$
\begin{aligned}
\mathcal{H}^k(C) &= \lim_{\varepsilon \to 0} \varepsilon^k \sigma_{\mathrm{int}}(C, \varepsilon), \\
\frac{1}{2} \mathcal{H}^k(C) &= \lim_{\varepsilon \to 0} \varepsilon^k e(C, \varepsilon).
\end{aligned}
$$

In [8] (see also [4, 5, 6, 7]), Gauthier and Zakalyukin have computed equivalents of the nonholonomic interpolation complexity $\sigma_{\mathrm{int}}$ in a large class of sub-Riemannian manifolds. It follows from Theorem 13 that these equivalents give directly the corresponding Hausdorff measures. Conversely, the results of [8] in the contact case are consequences of Corollary 9 and Theorem 13.

*Remark* 14. The equiregularity hypothesis in the previous theorem is essential. Indeed, it results from [11] that, for a non equiregular curve, entropy and complexity may be non equivalent and that entropy and Hausdorff dimension can be non equal (see Remark 8).

# 3 Proofs of the result

## 3.1 Distance along a path

In this section we will use some basic tools of sub-Riemannian geometry. For a general introduction with references see [1].

Let $\gamma : [a,b] \to M$ be $C^1$ and equiregular. Note in particular that the dimensions $n_i := \dim D^i(\gamma(t))$, $i \geq 1$, do not depend on $t \in [a,b]$.

We begin by choosing $n$ vector fields $Y_1, \ldots, Y_n$ whose values at each $\gamma(t)$ form a basis of $T_{\gamma(t)}M$ adapted to the filtration

$$\{0\} \subset D^1(\gamma(t)) \subset D^2(\gamma(t)) \subset \cdots \subset D^r(\gamma(t)) = T_\gamma(t)M,$$

in the sense that, for every integer $i \geq 1$, $Y_1(\gamma(t)), \ldots, Y_{n_i}(\gamma(t))$ is a basis of $D^i(\gamma(t))$. The local diffeomorphism

$$z \in \mathbb{R}^n \mapsto \exp(z_n Y_n) \circ \cdots \circ \exp(z_1 Y_1)(\gamma(t))$$

defines a system of local coordinates $\Phi^t : q \in M \mapsto (z_1, \ldots, z_n)$ near $\gamma(t)$ which are said to be *privileged* at $\gamma(t)$.

For $i = 1, \ldots, n$, we set $w_i = k$ if $n_{k-1} < i \leq n_k$ and we define a dilation on $\mathbb{R}^n$ by $\delta_s z = (s^{w_1} z_1, \ldots, s^{w_n} z_n)$. Then it follows from [1, Sect. 5.3] that there exists a sub-Riemannian distance $\widehat{d_t}$ on $\mathbb{R}^n$ such that:

- $\widehat{d_t}$ is homogeneous under the dilation, that is $\widehat{d_t}(\delta_s z, \delta_s z') = s\widehat{d_t}(z, z')$;
- when defined, the mapping $t \mapsto \widehat{d_t}(\Phi^t(q), \Phi^t(q'))$ is continuous.

**Lemma 15.** *Let $k \leq r$ be an integer such that $\dot{\gamma}(t) \in D^k(\gamma(t))$ for all $t \in [a,b]$. There exists a continuous mapping $x : [a,b] \to \mathbb{R}^n$ such that, for any $t$ and $t+s$ in $[a,b]$,*

$$d\big(\gamma(t), \gamma(t+s)\big) = s^{1/k} \widehat{d_t}\big(0, x(t)\big) + s^{1/k} \varepsilon_t(s),$$

*where $\varepsilon_t(s)$ tends to zero as $s \to 0$ uniformly with respect to $t$.*

*Moreover, $\dot{\gamma}(t) \in D^{k-1}(\gamma(t))$ if and only if $x(t) = 0$.*

Note that this lemma provides another expression of the $k$-dimensional length of $C = \gamma([a,b])$, as follows:

$$\mathrm{Length}_k(C) = \int_a^b \widehat{d_t}\big(0, x(t)\big)^k \, dt.$$

*Proof.* Since the distance $\widehat{d_t}(0,\Phi^t(q))$ depends continuously on $t$, it results from [1, Th. 7.32] that, for every $t \in [a,b]$,

$$d\big(\gamma(t),q\big) = \widehat{d_t}\big(0,\Phi^t(q)\big)\big(1+\varepsilon_t(\widehat{d_t}(0,\Phi^t(q)))\big),$$

where $\varepsilon_t(s)$ tends to zero as $s \to 0$ uniformly with respect to $t$.

The hypothesis on $\dot\gamma$ ensures that there exist continuous functions $c_i : [a,b] \to M$, $i = 1,\dots,n_k$, such that

$$\dot\gamma(t) = \sum_i c_i(t) Y_i(\gamma(t)) \qquad \text{for all } t \in [a,b].$$

We define the continuous function $x : [a,b] \to \mathbb{R}^n$ by $x_i(t) = c_i(t)$ if $w_i = k$ and $x_i(t) = 0$ otherwise. A straightforward adaptation of the proof of [2, Lem. 12] shows that $\delta_{s^{-1}}\Phi^t(\gamma(t+s^k))$ tends to $x(t)$ as $s \to 0$ uniformly with respect to $t$. This completes the proof. □

*Remark* 16. If $\gamma$ is $C^\infty$ and if $\dot\gamma(t) \in D^k/D^{k-1}(\gamma(t))$ for every $t \in [a,b]$, one can choose the vector fields $Y_1,\dots,Y_n$ such that one of them satisfies $Y_j(\gamma(t)) = \dot\gamma(t)$ on $[a,b]$, with $w_j = k$. In this case, $x(t) = \Phi^t_*(\dot\gamma(t))$ is the tangent vector expressed in coordinates $\Phi^t$.

## 3.2 The basic case

In this whole section, we assume the path $\gamma : [a,b] \to M$ equiregular, $C^1$, and such that $\dot\gamma(t) \in D^k/D^{k-1}(\gamma(t))$ for all $t \in [a,b]$. Lemma 15 applies to $\gamma$ and provides a never vanishing function $x(t)$.

Note that the curve $C = \gamma([a,b])$ can be reparameterized by the $k$-length, as follows: let $\alpha$ be the $C^1$ homeomorphism defined by

$$\alpha^{-1}(t) = \text{Length}_k(\gamma([a,t])) = \int_a^t \widehat{d_{t'}}\big(0,x(t')\big)^k dt', \qquad \text{for all } t \in [a,b],$$

and set $\gamma' = \gamma\circ\alpha$. Then $\gamma'([0,\text{Length}_k(C)]) = C$ and, for $t \in [0,\text{Length}_k(C)]$,

$$d\big(\gamma'(t),\gamma'(t+s)\big) = s^{1/k} + s^{1/k}\varepsilon_t(s),$$

where $\varepsilon_t(s)$ tends to zero as $s \to 0$ uniformly with respect to $t$. Note also that the $k$-dimensional length does not depend on the parameterization [2, Lem. 16], so definition (1) of $\text{Length}_k(C)$ holds with both $\gamma$ and $\gamma'$.

**Lemma 17.**

$$\mathcal{H}^k(C) = \mathrm{Length}_k(C) = \lim_{\varepsilon\to 0} \varepsilon^k \sigma_{\mathrm{int}}(C,\varepsilon),$$
$$\lim_{\varepsilon\to 0} \varepsilon^k e(C,\varepsilon) \le \tfrac{1}{2}\mathrm{Length}_k(C).$$

*Proof.* Fix $\delta > 0$. Up to replacing $\gamma$ by $\gamma'$, we assume $C = \gamma([0,T])$, where $T = \mathrm{Length}_k(C)$, and $\widehat{d_t}\big(0,x(t)\big)^k \equiv 1$ on $[0,T]$. From Lemma 15, there exists $\eta > 0$ such that, if $t, t+s \in [0,T]$ and $0 < s < \eta$, then

$$d\big(\gamma(t),\gamma(t+s)\big) = s^{1/k} + s^{1/k}\varepsilon_t(s), \quad \text{with } |\varepsilon_t(s)| \le \delta. \tag{2}$$

Let $\varepsilon > 0$ be smaller than $(1+\delta)\eta^{1/k}$. We denote by $N$ the smallest integer such that $T \le N(\frac{\varepsilon}{1+\delta})^k$ and define $t_0,\dots,t_N$ by:

$$t_i = i\Big(\frac{\varepsilon}{1+\delta}\Big)^k \quad \text{for } i = 0,\dots,N-1, \qquad t_N = T.$$

Set $S_i = \gamma([t_{i-1},t_i])$, $i = 1,\dots,N$. For $t,t'$ in $S_i$, one has $|t-t'| \le \varepsilon^k/(1+\delta)^k$; it follows from (2) that

$$d(\gamma(t),\gamma(t')) \le (1+\delta)|t-t'|^{1/k} \le \varepsilon,$$

which in turn implies $\mathrm{diam}\, S_i \le \varepsilon$. Thus $\mathcal{H}^k_\varepsilon(C) \le \sum_i (\mathrm{diam}\, S_i)^k \le N\varepsilon^k$. Using $(N-1)\varepsilon^k \le T(1+\delta)^k$ and $\mathrm{Length}_k(C) = T$, we obtain

$$\mathcal{H}^k_\varepsilon(C) \le (1+\delta)^k \mathrm{Length}_k(C) + \varepsilon^k.$$

In the same way, one can show that $\gamma(t_0)$, ..., $\gamma(t_N)$ is an $\varepsilon$-chain of $C$ and that $\gamma(t_1),\gamma(t_3),\dots,\gamma(t_N)$ is an $\varepsilon$-net of $C$, which imply

$$\begin{aligned} \varepsilon^k\sigma_{\mathrm{int}}(C,\varepsilon) &\le (1+\delta)^k \mathrm{Length}_k(C) + \varepsilon^k, \\ \varepsilon^k e(C,\varepsilon) &\le (1+\delta)^k \frac{1}{2}\mathrm{Length}_k(C) + \varepsilon^k. \end{aligned}$$

Taking the limit as $\varepsilon \to 0$, then the limit as $\delta \to 0$ in the preceding three inequalities, we find

$$\mathcal{H}^k(C) \le \mathrm{Length}_k(C), \quad \lim_{\varepsilon\to 0} \varepsilon^k\sigma_{\mathrm{int}}(C,\varepsilon) \le \mathrm{Length}_k(C)$$
$$\text{and } \lim_{\varepsilon\to 0} \varepsilon^k e(C,\varepsilon) \le \tfrac{1}{2}\mathrm{Length}_k(C).$$

The next step is to prove converse inequalities for $\mathcal{H}^k$ and $\sigma_{\text{int}}$. As above, we fix $\delta > 0$ and we choose $\varepsilon > 0$ small enough.

Consider a countable family $S_1, S_2, \dots$ of closed subsets of $M$ such that $C \subset \bigcup_i S_i$ and $\operatorname{diam} S_i \leq \varepsilon$. For every $i \in \mathbb{N}$, we set $I_i = \gamma^{-1}(S_i \cap C)$. With the help of (2), there holds, for any $t, t'$ in $I_i$,

$$\operatorname{diam} S_i \geq d(\gamma(t), \gamma(t')) \geq (1-\delta)|t-t'|^{1/k},$$

which implies $\mathcal{L}^1(I_i) \leq (\operatorname{diam} S_i)^k/(1-\delta)^k$. Note that $T \leq \sum_i \mathcal{L}^1(I_i)$ since the sets $I_i$ cover $[0,T]$. It follows that $\mathcal{H}^k_\varepsilon(C) \geq T(1-\delta)^k$, that is,

$$\mathcal{H}^k_\varepsilon(C) \geq (1-\delta)^k \text{Length}_k(C). \tag{3}$$

In the same way, an $\varepsilon$-chain $\gamma(t_0) = \gamma(a), \dots, \gamma(t_N) = \gamma(b)$ of $C$ satisfies $N\varepsilon^k \geq T(1-\delta)^k$ since

$$\varepsilon \geq d(\gamma(t_{i-1}), \gamma(t_i)) \geq (1-\delta)|t_i - t_{i-1}|^{1/k}.$$

It follows that $\varepsilon^k \sigma_{\text{int}}(C,\varepsilon) \geq (1-\delta)^k \text{Length}_k(C)$. Taking the limit as $\varepsilon \to 0$, then the limit as $\delta \to 0$ in this inequality and in (3), we find $\mathcal{H}^k_\varepsilon(C) \geq \text{Length}_k(C)$ and $\lim \varepsilon^k \sigma_{\text{int}}(C,\varepsilon) \geq \text{Length}_k(C)$, whic completes the proof. $\square$

**Lemma 18.**

$$\begin{aligned} \mathcal{S}^k(C) &= 2^{k-1}\mathcal{H}^k(C), \\ \lim_{\varepsilon \to 0} \varepsilon^k e(C,\varepsilon) &= \frac{1}{2}\mathcal{H}^k(C). \end{aligned}$$

*Proof.* We begin by recalling a standard result of geometric measure theory (see for instance [3, Th. 2.10.17 and 2.10.18]): *if $\mu$ is a regular measure such that*

$$\lim_{r \to 0} \frac{\mu(C \cap B(q,r))}{(2r)^k} = 1 \quad \forall q \in C,$$

*then* $\mu(C) = \mathcal{S}^k(C)$.

We will show that this result applies to $\mu = 2^{k-1}\mathcal{H}^k$. As in the proof of the preceding lemma, we assume that $C = \gamma([0,T])$ is parameterized by the $k$-length.

Fix $\delta > 0$. With the help of (2), we get, for $t \in [0,T]$ and $r > 0$ small enough,

$$\gamma([t - \tfrac{r^k}{(1+\delta)^k}, t + \tfrac{r^k}{(1+\delta)^k}]) \subset C \cap B(\gamma(t), r) \subset \gamma([t - \tfrac{r^k}{(1-\delta)^k}, t + \tfrac{r^k}{(1-\delta)^k}]).$$

For every interval $[t,t'] \subset [0,T]$, it follws from Lemma 17 that $\mathcal{H}^k(\gamma([t,t'])) = |t-t'|$. One has therefore

$$\frac{1}{(1+\delta)^k} \leq \frac{\mathcal{H}^k(C \cap B(\gamma(t),r))}{2r^k} \leq \frac{1}{(1-\delta)^k}.$$

Letting $r \to 0$ and $\delta \to 0$, we obtain the required property for $\mu = 2^{k-1}\mathcal{H}^k$, which in turn imply that $2^{k-1}\mathcal{H}^k(C) = \mathcal{S}^k(C)$.

Finally, recall that $\varepsilon^k e(C,\varepsilon) \geq 2^{-k}\mathcal{S}^k(C)$. Since $2^{-k}\mathcal{S}^k(C) = \mathcal{H}^k(C)/2$, the conclusion follows from Lemma 17. $\square$

## 3.3 Proof of Theorems 5 and 13

**Lemma 19.** *Let $\gamma : [a,b] \to M$ be a $C^1$ equiregular path without double points such that $\dot\gamma(t) \in D^k(\gamma(t))$ for all $t \in [a,b]$. Then*

$$\mathcal{H}^k(C) = \frac{\mathcal{S}^k(C)}{2^{k-1}} = \lim_{\varepsilon\to 0} \varepsilon^k \sigma_{\text{int}}(C,\varepsilon) = 2\lim_{\varepsilon\to 0} \varepsilon^k e(C,\varepsilon) = \text{Length}_k(C).$$

Observe that Theorem 13 is a consequence of this lemma.

*Proof.* Consider the (possibly empty) open subset of $[a,b]$

$$I = \{t \in [a,b] \ : \ \dot\gamma(t) \in D^k/D^{k-1}(\gamma(t))\},$$

and its complementary $I^c = [a,b] \setminus I$. The set $I$ is the union of a disjointed countable family of open subintervals $I_i$ of $[a,b]$. Note that, since $\text{meas}_t^k(\gamma) = 0$ for all $t \in I^c$ (see [2]), one has

$$\text{Length}_k(C) = \int_I \text{meas}_t^k(\gamma)dt = \sum_i \int_{I_i} \text{meas}_t^k(\gamma)dt.$$

Now, for every subinterval $I_i$, we choose an increasing sequence $I_i^1 \subset I_i^2 \subset \cdots$ of closed subintervals of $I_i$ with $\bigcup_j I_i^j = I_i$. Applying Lemma 17 to each path $\gamma(I_i^j)$, we get $\mathcal{H}^k(\gamma(I_i^j)) = \text{Length}_k(\gamma(I_i^j))$. Taking the limit as $j \to \infty$, it follows that

$$\mathcal{H}^k(\gamma(I_i)) = \int_{I_i} \text{meas}_t^k(\gamma)dt, \quad \forall i. \tag{4}$$

Since $\mathcal{H}^k(C) \geq \sum_i \mathcal{H}^k(\gamma(I_i))$, we obtain $\mathcal{H}^k(C) \geq \text{Length}_k(C)$.

The next step is to prove the converse inequality. Let $\delta > 0$ and let $x : [a,b] \to \mathbb{R}^n$ be the continuous mapping resulting from the application of Lemma 15 to $\gamma$. Since the function $t \mapsto \widehat{d}_t(0,x(t))$ is uniformly continuous on $[a,b]$, there exists $\eta > 0$ such that, if $t,t' \in [a,b]$ and $|t-t'| < \eta$, then $|\widehat{d}_t(0,x(t)) - \widehat{d}_{t'}(0,x(t'))| < \delta$.

In the covering $I = \bigcup_i I_i$, only a finite number $N_\delta$ of subintervals $I_i$ may have a Lebesgue measure greater than $\eta$. Up to reordering, we assume $\mathcal{L}^1(I_i) < \eta$ if $i > N_\delta$. Set $J = I^c \cup \bigcup_{i>N_\delta} I_i$. Since the restriction $x_{|I^c}$ of $x$ to $I^c$ is identically zero, there holds $\widehat{d}_t(0,x(t)) < \delta$ for every $t \in J$.

The $k$-dimensional Hausdorff measure of $C$ satisfies

$$\begin{aligned} \mathcal{H}^k(C) &\leq \sum_{i \leq N_\delta} \mathcal{H}^k(\gamma(I_i)) + \mathcal{H}^k(\gamma(J)) \\ &\leq \sum_{i \leq N_\delta} \int_{I_i} \mathrm{meas}_t^k(\gamma) dt + \mathcal{H}^k(\gamma(J)), \end{aligned} \tag{5}$$

in view of (4).

It remains to compute $\mathcal{H}^k(\gamma(J))$. Being the complementary of $\bigcup_{i \leq N_\delta} I_i$ in $[a,b]$, $J$ is the disjointed union of $N_\delta + 1$ closed subintervals $J_i = [a_i,b_i]$ of $[a,b]$. For each one of these interval we will proceed as in the proof of Lemma 17.

Let $\varepsilon > 0$ and $i \in \{1,\dots,N_\delta+1\}$. We denote by $N'$ the smallest integer such that $b_i - a_i \leq N'(\frac{\varepsilon}{2\delta})^k$ and define $t_0,\dots,t_{N'}$ by

$$t_j = a_i + j\Big(\frac{\varepsilon}{2\delta}\Big)^k \quad \text{for } j = 0,\dots,N'-1, \qquad t_{N'} = b_i.$$

We then set $S_j = \gamma([t_j,t_{j-1}])$. Applying Lemma 15, we get, for any $t,t' \in [t_j,t_{j-1}]$,

$$d\big(\gamma(t),\gamma(t')\big) = |t-t'|^{1/k}(\widehat{d}_t\big(0,x(t)\big) + \varepsilon_t(|t-t'|)).$$

Note that $\widehat{d}_t(0,x(t)) < \delta$ since $t \in J$. Note also that, if $\varepsilon$ is small enough, then $\varepsilon_t(|t-t'|)$ is smaller than $\delta$. Therefore $d\big(\gamma(t),\gamma(t')\big) < 2\delta|t-t'|^{1/k} \leq \varepsilon$ and $\mathrm{diam}\, S_j \leq \varepsilon$. As a consequence

$$\mathcal{H}_\varepsilon^k(\gamma(J_i)) \leq N'\varepsilon^k \leq (2\delta)^k(b_i - a_i) + \varepsilon^k,$$

and $\mathcal{H}^k(\gamma(J_i)) \leq (2\delta)^k(b_i-a_i)$. It follows that

$$\mathcal{H}^k(\gamma(J)) \leq \sum_{i \leq N_\delta+1} (2\delta)^k(b_i-a_i) \leq (2\delta)^k(b-a).$$

Finally, Formula (5) yields

$$\mathcal{H}^k(C) \le \sum_{i \le N_\delta} \int_{I_i} \mathrm{meas}_t^k(\gamma) dt + (2\delta)^k (b-a).$$

Letting $\delta \to 0$, we get $\mathcal{H}^k(C) \le \int_I \mathrm{meas}_t^k(\gamma) dt = \mathrm{Length}_k(C)$, and thus $\mathcal{H}^k(C) = \mathrm{Length}_k(C)$.

The same arguments apply to show that $2^{-k+1}\mathcal{S}^k(C)$ and the limits of $\varepsilon^k \sigma_{\mathrm{int}}(C,\varepsilon)$ and $2\varepsilon^k e(C,\varepsilon)$ are equal to $\mathrm{Length}_k(C)$. □

It remains to prove Theorem 5.

*Proof of Theorem 5.* It clearly suffices to prove that, if $\gamma : [a,b] \to M$ is absolutely continuous, a.e. regular, and such that $\dot\gamma(t) \in D^k(\gamma(t))$ a.e. on $[a,b]$, then $\mathcal{H}^k(C) = 2^{-k+1}\mathcal{S}^k(C) = \mathrm{Length}_k(C)$ for $C = \gamma([a,b])$.

The preceding hypothesis on $\gamma$ ensure that $[a,b]$ is a disjointed countable union $[a,b] = I_0 \cup \bigcup_i [a_i,b_i]$, such that $I_0$ is of zero Lebesgue measure, every restriction $\gamma_{|[a_i,b_i]}$ is $C^1$ and equiregular, and $\dot\gamma(t) \in D^k(\gamma(t))$ for $t \in \bigcup_i [a_i,b_i]$. Note that $\mathcal{H}^k(\gamma(I_0)) = 0$ since the path $\gamma$ is absolutely continuous. It follows from the additivity of the Hausdorff measure and from Lemma 19 that

$$\mathcal{H}^k(C) = \sum_i \mathcal{H}^k(\gamma([a_i,b_i])) = \sum_i \mathrm{Length}_k(\gamma([a_i,b_i])).$$

On the other hand, the $k$-dimensional length satisfies

$$\mathrm{Length}_k(C) = \int_{[a,b]} \mathrm{meas}_t^k(\gamma) dt = \sum_i \int_{[a_i,b_i]} \mathrm{meas}_t^k(\gamma) dt,$$

and the proof is complete. □

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