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A Competitive Dynamic Equilibrium with Different Asymmetric Information

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Abstract

In this paper, we consider a complete continuous-time financial market with discontinuous prices. We consider a small investors model, where the agents are price takers and the prices are exogenous. Our definition of equilibrium is stated in term of a portfolio market clearing condition. The investors who trade in this market have different types of side-information. They are "strong"-informed (initially or progressively), meaning that they know a functional ω-wise (immediately and only at time $t = 0$ or the information is getting clearer to them as time evolves). Or they are "weak"-informed, meaning that they only know the law of a functional. Our purpose is to see if an equilibrium can be achieved in such an asymmetrical financial market. We show that the more informed an agent is, the less weight he must invest. We simulate an equilibrium and the maximal weight an insider can invest in the market.

Keywords : Strong and weak information, Risk neutral probability measure and minimal probability measure, Competitive dynamic equilibrium, Portfolio market clearing condition.

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1 Introduction

This paper deals with the existence of a competitive dynamic equilibrium between agents who do not have the same information about the market structure. The agents can invest in financial assets, whose prices small variations can be modeled by a continuous process driven by a Brownian motion. Merton (1976) [30] and Jarrow and Rudd (1983) [20] pointed out that the analysis of prices evolution reveals sudden and rare breaks, that it is natural to model by the means of a point process, whose jumps occur at rare and unpredictable intervals. Thus, we fix a finite horizon time $T > 0$ and a filtered probability space $(\Omega, \mathrm{F}_T, \mathbb{P})$, with a multidimensional Brownian motion W and a multidimensional point process N. A market model is built, with one bond and d discontinuous risky assets.

A large litterature about equilibrium under asymmetric information is written in the setting of Kyle (1985) [27] and Back (1992) [4]. In their model, there are three types of agents : one insider, noise traders who are not rational and trade for liquidity reasons, and a market maker who sets the prices using the total order. In his PhD Thesis [32] (1999), Wu studies particular cases of noisy information, of delayed information, and a model with two degrees of information. In each case, Wu requires that the insider's strategy is unconspicious, in the sense that the total supply is a Brownian motion. Therefore, the insider's trades can not be seen in the market. In the same framework, Campi et al. [9] (2006) study an equilibrium model for the pricing of a defaultable zero coupon bond. The insider is assumed to know the default time. Here again, the insider's trades are inconspicuous at equilibrium. Ly Vath (2006) [29] considers a competitive equilibrium with noise traders and two rational traders: an "ordinary" one, and an informed one whose side-information consists of the total supply of the risky asset. The existence of an equilibrium is proved in the particular case where the total supply dynamics is a Brownian motion, that is in a "inconspicious" setting.
The equilibrium model of this article is more related to a competitive dynamic stochastic equilibrium, with only rational traders and no market makers. It is studied in Duffie and Huang (1985) [12], Karatzas, Lehoczky and Shreve (1990) [26] or Dana and Pontier (1992) [11] (to name a few), in the classical setting of a purely diffusive financial market where all agents share the same information flow, which is conveyed by the prices. Karatzas et al. consider two related problems : the moneyed model (prices are measured in dollars) and the moneyless model (prices are measured in units of commodity). In the moneyed model, the spot price of the commodity is determined at equilibrium, whereas in the moneyless model, the interest rate and the mean rates of return of the stocks are determined (that is equivalent to the fact that the interest rate and the risk neutral probability measure are determined).
Our purpose in this paper is not to study the impact of asymmetric information on stock prices. Therefore, we will consider a quite unusual competitive dynamic equilibrium in a moneyless model, in the sense that the prices are assumed exogenous and neither the agents strategies nor the total supply affect them. We can think of a "local" economy

where agents try to set transactions of assets (whose prices are given, for example determined by an other economy), such that those assets are in zero net supply in their local economy. Thus, the prices are exogenous and our aim is to see if the transactions can occur, and under which conditions about the private information. This naïve approach allows us to consider only rationals traders : there are no noise traders, therefore the insiders information is not hidden by the non rational noise traders supply. Likewise, the insiders strategies are not assumed unconspicious. Furthermore, we will give a framework without specifying the side information. We will consider three general types of side-information an insider may have.

The first type is called "initial strong" information: from the beginning the insider has an extra information available about the outcome of some variable L of the prices. The cornerstone of this modelization is the theory of initial enlargement of filtration by a random variable, which was developed by Jeulin (1980) [22], in the series of papers in the "Séminaire de Calcul Stochastique (1982/83)" of the University Paris VI, by Jacod (1985) [19], Jeulin and Yor (1985) [23], Föllmer and Imkeller (1993) [14] and further by Amendinger et al. (1998) [2] and Amendinger (2000) [1]. The second type is called "progressive strong" information: the insider's information is perturbed by an independent noise changing throughout time. This case deals with the theory of progressive enlargement of filtration and is studied by Corcuera et al. (2004) [10] in a purely diffusive model, using Malliavin's calculus. The third type is called "weak" information or anticipation: the insider anticipates the law of a random variable L that will be realized at a future date. In this latter case, enlargement of filtration techniques are irrelevant. This notion of weak information is defined by Baudoin (2002) in [6] and [7].

In the moneyeless model of Karatzas et al. (1990), the risk neutral probability measure is determined at equilibrium. In our setting of asymmetric information, each agent has is own probability measure and the main result of this paper states a necessary and sufficient condition for existence of an equilibrium, that gives a relation linking the densities of the probability measures of each agent. That is why we will first recall the construction of a risk neutral probability measure for the strong-informed agents, and a "minimal probability measure" for the weak-informed agents. These probability measures summarize the information of each insider, whichever the type of side information they have. Then, with the assets prices being given, we would like to see if an equilibrium can be achieved, and under which constraints.

This article is organized as follows.
In section 2, we define the market and introduce the general framework and notation that are valid throughout the paper.
In section 3, we solve the optimization problem of the consumption of an agent having a side-information. We point out the similarities and the differences between an initial strong information on the one hand and the other types of side information on the other

hand. The key point of this section is that whatever the type of the side information, the change of the probability measure (risk neutral probability measure for a strong insider, minimal probability measure for a weak insider) summarizes his information.
In section 4, we study the formation of an equilibrium. The assets prices being exogenous, our definition of equilibrium is that the financial assets are in zero net supply. We consider a logarithmic utility and different optimization problems (maximization of the consumption and/or the terminal wealth). Although our framework of exogenous prices seems naïve, we believe it is worth to study it because it leads to a meaningful necessary and sufficient condition for existence of an equilibrium, that gives a relation linking the densities of the probability measures of each agents. The meaning of this relation is that the transactions between agents can take place if and only if their information and their endowments are well-balanced. Thus, in order to reach an equilibrium in these three cases of side information, we show that the more informed an insider is, the less weight he must have in the financial market. In a market model with two agents, we simulate the maximal weight an insider can invest in the market. Figures are given in annex.

2 The market

Let W be a real m-dimensional Brownian motion on its canonical probability space $(\Omega^W, \mathrm{F}_T^W := (\mathcal{F}_t^W)_{t\in[0,T]}, \mathbb{P}^W)$. Let N be a n-dimensional point process on its canonical probability space $(\Omega^N, \mathrm{F}_T^N := (\mathcal{F}_t^N)_{t\in[0,T]}, \mathbb{P}^N)$, with a positive, F_T^N-predictable intensity κ satisfying $E_{\mathbb{P}^N}[\int_0^T \kappa(t)dt] < +\infty$. The process M defined by $M(t) := N(t) - \int_0^t \kappa(s)ds$ is a $(\mathrm{F}_T^N, \mathbb{P}^N)$-martingale, called the compensated martingale of the point process N.
Let $(\Omega, \mathrm{F}_T, \mathbb{P}) := (\Omega^W \times \Omega^N, \mathrm{F}_T^W \otimes \mathrm{F}_T^N, \mathbb{P}^W \otimes \mathbb{P}^N)$ be the product space. W and N are independent processes. Let $d = m + n$. Let $A < T$.

We consider a financial market with K agents who invest in one bond and d stocks (financial assets), the prices of which are expressed in units of a single perishable commodity and driven by the following stochastic differential equations :

$$P_0(t) = P_0(0)\exp(\int_0^t r(s)ds) \tag{2.1}$$

$$dP_i(t) = P_i(t^-)[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)d(W^*, N^*)_j^*(t)] \qquad i = 1, \cdots, d \tag{2.2}$$

where X^* denote the transposed process of process X.
σ is a given strongly non-degenerate $d \times d$-matrix-valued process. The processes r, b and σ are assumed to be uniformly bounded on $[0, T] \times \Omega$ and F_T-predictable. We assume that $\sigma_{ij} > -1$ for all $m + 1 \leq j \leq d$ and $1 \leq i \leq d$. Thus (2.2) has a unique strong

solution and the market is complete (cf. Bardhan and Chao [5]).

If the k-th agent has a strong information (initial or progressive), he receives an individual flow of "side-information", represented by the filtration $\mathrm{H}_T^k := (\mathcal{H}_t^k)_{t\in[0,T]}$. Finally, for all $k = 1, \cdots, K$, we introduce the individual agent's filtration $\mathrm{G}_T^k := (\mathcal{G}_t^k)_{t\in[0,T]}$ of available information, with

$$\mathcal{G}_t^k := \mathcal{F}_t \vee \mathcal{H}_t^k, \qquad 0 \le t \le T.$$

In other words, this agent possesses all informations about the market up to the present time t, plus his own "side-information". Besides, if agent k has a weak information, he only has the filtration $\mathrm{G}_T^k := \mathrm{F}_T$ of the market available.

The agents strategies are produced on $[0, A]$, with $A < T$. The k-th agent has his own nontrivial endowment process ϵ_k, expressed in units of the commodity and assumed to be a nonnegative uniformly bounded G_A^k-adapted process. We introduce the cumulative endowment rate $\varepsilon := \sum_{k=1}^K \epsilon_k$. The k-th agent is free to choose a nonnegative consumption rate process c_k G_A^k-adapted, and a $\mathbb{R}^d$-valued portfolio process π_k G_A^k-predictable. He chooses both these processes to satisfy the integrability requirement $\int_0^A (c_k(t) + ||\sigma^*(t)\pi_k(t)||^2)dt < \infty$ $\mathbb{P}$ almost surely. $\pi_{ki}(t)$ represents the amount invested by the k-th agent at time t in the i-th stock $(i = 1, \cdots, d)$. As usually, we assume that the strategy is self-financing, so the k-th agent's discounted wealth is given by

$$\begin{aligned} \beta(t)X_k(t) = \int_0^t \beta(s)(\epsilon_k - c_k)(s)ds \quad &+ \quad \int_0^t \beta(s)\pi_k^*(s)(b - rI_d)(s)ds \\ &+ \quad \int_0^t \beta(s)\pi_k^*(s^-)\sigma(s)d(W^*, N^*)^*(s) \end{aligned} \tag{2.3}$$

where $\beta(t) := (P_0(t))^{-1}$ is the deflator process and $I_d = (1, \cdots, 1)^* \in \mathbb{R}^d$. But on an enlarged filtration, the process (W^*, N^*) could no more be a semi-martingale. We will add in section 3 sufficient conditions to obtain a meaningful wealth equation for a strong insider.

Definition 2.1 *A G_A^k-admissible strategy for the k-th agent is a pair (π_k, c_k) (portfolio, consumption) such that c_k is G_A^k-adapted, $\frac{\pi_{ki}}{P_i}$, $i = 1, \cdots, d$, are G_A^k-predictable and satisfying the integrability requirement $\int_0^A (c_k(t) + ||\sigma^*(t)\pi_k(t)||^2)dt < \infty$ $\mathbb{P}$ almost surely and so that the corresponding wealth process X_k is bounded from below $dt \otimes d\mathbb{P}$ a.s. and satisfies $X_k(A) \ge 0$ $\mathbb{P}$ almost surely.*

Agent k chooses his strategy so as to optimize his consumption c_k, or more precisely the expectation of his utility from consumption. An utility function $U_k : (t, c) \to U_k(t, c)$ is continuous, strictly increasing and strictly concave in its second variable and $\forall t \in [0, A]$ $I_k(t, \cdot)$ is the inverse of the strictly decreasing mapping $\frac{\partial}{\partial c}U_k(t, \cdot)$ and satisfies :
$I_k(t, +\infty) = \lim_{c\to+\infty} I_k(t, c) = 0, \qquad I_k(t, 0) = \lim_{c\searrow 0} I_k(t, c) = +\infty.$

Let us introduce some notations:

- $I_n = (1, \cdots, 1)^* \in \mathbb{R}^n$.
- If v_1 et v_2 are two vectors of same dimension d, we note $v_1.v_2$ the vector with components $(v_1.v_2)_i = v_{1,i}.v_{2,i}, \ i = 1, \cdots, d$.
- $\mathcal{E}$ denote the Doléans exponential.

We define $\Theta(t) :=$ m first lines of $(\sigma(t))^{-1}(b(t) - r(t)I_d)$ and the n dimensional process q such that $q(t).\kappa(t) :=$ n last lines of $-(\sigma(t))^{-1}(b(t) - r(t)I_d)$.
We assume that q is a process with positive components (otherwise arbitrage opportunities can occur, see [21]).

$$\widehat{W}(t) := W(t) + \int_0^t \Theta(s)ds, \qquad t \in [0,T]. \tag{2.4}$$

$$\widehat{M}(t) := N(t) - \int_0^t q(s)\kappa(s)ds, \qquad t \in [0,T]. \tag{2.5}$$

We denote $\widehat{S} := (\widehat{W}^*, \widehat{M}^*)^*$.

3 Change of probability measure and optimization of consumption

In Hillairet (2005) [17], we have studied the case of an agent who knows a functional L_k ω-wise from the beginning. We say that he is an initial strong insider. The side information for a initial strong insider is the following

Assumption 3.1 *$\forall t \in [0,T], \mathcal{H}_t^k = \sigma(L_k)$ where L_k is an $\mathcal{F}_T$-measurable random variable with values in a Polish space $(E_k, \mathcal{E}_k)$ (the k-th agent receives his additional information immediately and only at time $t = 0$) and moreover, L_k satisfies the assumption : $\mathbb{P}(L_k \in \cdot|\mathcal{F}_t)(\omega) \sim \mathbb{P}(L_k \in \cdot)$ for $\mathbb{P}$ almost all $\omega \in \Omega$, for all $t \in [0, A]$.*

Remark: Assumption 3.1 is equivalent to : There exists a probability measure equivalent to $\mathbb{P}$ on $\mathcal{F}_A \vee \sigma(L_k)$ and under which $\forall t \in [0, A]$, $\mathcal{F}_t$ and $\sigma(L_k)$ are independent (Lemma 3.1 and 3.4 in [16]). We consider the only one that is identical to $\mathbb{P}$ on $\mathcal{F}_A$ and on $\sigma(L_k)$ (cf. Lemma 3.1 in [16] for the construction of this measure). We denote it $\mathbb{Q}^{L_k}$. We introduce the density process $Z_k(t) := E_{\mathbb{Q}^{L_k}}[\frac{d\mathbb{P}}{d\mathbb{Q}^{L_k}}|\mathcal{G}_t^k]$. There exist G_A^k-predictable processes ρ_1^k and ρ_2^k such that $dZ_k(t) = Z_k(t^-)[\rho_1^{k*}(t)dW(t) + (\rho_2^k(t) - I_n)^* dM(t)]$.
$\widetilde{W}^k(\cdot) := W(\cdot) - \int_0^\cdot \rho_1^k(t)dt$ is a $(\mathrm{G}_A^k, \mathbb{P})$-Brownian motion and $\widetilde{M}^k(\cdot) := N(\cdot) - \int_0^\cdot \kappa.\rho_2^k(t)dt$ is the compensated process of a $(\mathrm{G}_A^k, \mathbb{P})$-point process with intensity $(\kappa.\rho_2^k)$. Therefore the wealth equation (2.3) is meaningful under assumption 3.1.

Definition 3.2

$$Y_k := \mathcal{E}\left(\int_0^{\cdot}\left(-(\Theta+\rho_1^k(s))^* d\widetilde{W}^k(s) + \left(\frac{q}{\rho_2^k(s)} - I_n\right)^* d\widetilde{M}^k(s)\right)\right).$$

Y_k is a positive $(\mathrm{G}_A^k, \mathbb{P})$-local martingale. A straightforward calculus yields :
$d(Y_k^{-1})(t) = Y_k^{-1}(t^-) l_k^*(t) d\widehat{S}(t) \qquad \text{with} \quad l_k^* := \left((\Theta+\rho_1^k)^*, (\frac{\rho_2^k}{q} - I_n)^*\right)$.

Assumption 3.3 *We assume that Y_k defined in (3.2) is a $(\mathrm{G}_A^k, \mathbb{P})$-martingale.*

Under this assumption, $\widehat{\mathbb{P}^k} := Y_k(A)\mathbb{P}$ is a risk neutral probability measure for the initial strong insider and [17] gives his optimal strategy. We extend here these results with two other types of side information and we stress on the similarities and the differences between these three settings of side information. We will point out the common fact that all the relevant information for the insider is contained in the density probability of an equivalent change of probability measure (a risk neutral probability measure for a strong insider, respectively a minimal probability measure for a weak insider).

3.1 Progressive strong information

In this subsection, we consider an agent whom additional information changes through time. His knowledge is disturbed by an independent noise, and is getting to him clearer as time evolves.

Assumption 3.4
$\forall t \in [0,T], \mathcal{H}_t^k = \sigma(L_k(s), s \le t)$ where $L_k(s) = h_k(L_k, B_k(s))$ with
- *$h_k : \mathbb{R}^2 \to \mathbb{R}$ is a given measurable function.*
- *$B_k = \{B_k(t), 0 \le t \le T\}$ is independent of $\mathcal{F}_T$.*
- *L_k is an $\mathcal{F}_T$-measurable random variable such that :*

$\mathbb{P}(L_k \in \cdot|\mathcal{F}_t)(\omega) \ll \mathbb{P}(L_k \in \cdot)$ for all $t \in [0,T[$ for $\mathbb{P}$ almost all $\omega \in \Omega$.

G_T^k and Y_T^k denote the usual enlarged filtrations

$$\mathrm{G}_t^k = \cap_{u>t}\left(\mathcal{F}_u \vee \sigma(L_k(s), s \le u)\right) \text{ and } \mathcal{Y}_t^k = \cap_{u>t}\left(\mathcal{F}_u \vee \sigma(L_k)\right), \quad t \in [0,T[.$$

L_k contains the additional information available to the insider, and B_k represents an additional noise that perturbs this “side-information”. Therefore one expects in general that $B_k(T) = 0$ and that the variance of the noise decreases to zero as revelation time T approaches. We denote by $P_t^k(\omega, dx)$ a regular version of the conditional law of L_k given $\mathcal{F}_t$ and by P_{L_k} the law of L_k.

Lemma 3.5 *(Proposition 12 Grorud (2000) [15])*
There exists a measurable version of the conditional density $p^k(t,x)(\omega) = \frac{dP_t^k}{dP_{L_k}}(\omega,x)$ *which is a* $(\mathrm{F}_A, \mathbb{P})$*-martingale and can be written, for all* $x \in \mathbb{R}$*, as*

$$p^k(t,x) = p^k(0,x) + \int_0^t \alpha^k(s,x)dW(s) + \int_0^t \beta^k(s,x)dM(s)$$

where for all x*,* $s \to \alpha^k(s,x)$ *and* $s \to \beta^k(s,x)$ *are* F_A*-predictable processes. Moreover, for all* $s \leq A$*,* $p^k(s,L_k) > 0$ $\mathbb{P}$ *almost surely.* $W(\cdot) - \int_0^\cdot \frac{\alpha^k(s,L_k)}{p^k(s,L_k)}ds$ *is a* $(\mathrm{Y}_A^k, \mathbb{P})$*-Brownian motion and if* $(I_n + \frac{\beta^k(s,L_k)}{p^k(s^-,L_k)})$ *has positive components,* $M(\cdot) - \int_0^\cdot \frac{\left(\kappa(s).\beta^k(s,L_k)\right)}{p^k(s^-,L_k)}ds$ *is the compensated process of a* $(\mathrm{Y}_A^k, \mathbb{P})$*-point process with intensity* $\kappa(s).\left(I_n + \frac{\beta^k(s,L_k)}{p^k(s^-,L_k)}\right)$.

Proof: cf. proof of Proposition 12 in Grorud (2000) [15], using Jacod's results [19] (Theorem 2.1 and Theorem 2.5). □

Theorem 3.6 *We assume that* $E_{\mathbb{P}}\left(||\frac{\alpha^k(t,L_k)}{p^k(t,L_k)}|| + ||I_n + \frac{\beta^k(t,L_k)}{p^k(t^-,L_k)}||\right) < +\infty$ $\forall t \in [0,A]$ *and that* $\left(I_n + \frac{\beta^k(t,L_k)}{p^k(t^-,L_k)}\right)$ *has positive components. Setting* $\rho_1^k(t) = E_{\mathbb{P}}\left(\frac{\alpha^k(t,L_k)}{p^k(t,L_k)} \mid \mathcal{G}_t^k\right)$ *and* $\rho_2^k(t) = E_{\mathbb{P}}\left(\left(I_n + \frac{\beta^k(t,L_k)}{p^k(t^-,L_k)}\right) \mid \mathcal{G}_t^k\right)$*, we assume that* $\int_0^A \left(||\rho_1^k(t)|| + ||(\kappa.\rho_2^k)(t)||\right)dt < +\infty$ $\mathbb{P}$ *almost surely. Then* $\widetilde{W}^k(\cdot) := W(\cdot) - \int_0^\cdot \rho_1^k(s)ds$ *is a* $(\mathrm{G}_A^k, \mathbb{P})$*-Brownian motion and* $\widetilde{M}^k(\cdot) := N(\cdot) - \int_0^\cdot (\kappa.\rho_2^k)(s)ds$ *is the compensated process of a* $(\mathrm{G}_A^k, \mathbb{P})$*-point process with intensity* $(\kappa.\rho_2^k)$.

Proof: cf. Annex 5.1. □

Example 3.7
(Example in Corcuera et al. (2004) [10]).
L_k is the following m-dimensional process

$$(L_k)_i(t) = \delta_i W_i(T) + B_i(g_i(T-t)) \quad i = 1, \cdots, m \quad t \in [0,T]$$

where $\delta_i \neq 0$, $B = (B_i, i = 1, \cdots, m)$ is a m-dimensional Brownian motion independent of $\mathcal{F}_T$, and $g_i : [0,T[\to [0,+\infty[$ is a strictly increasing bounded function with $g_i(0) = 0$. Then, $\rho_2^k \equiv \mathbb{I}_n$ and using Malliavin's Calculus, the authors get

$$\rho_1^k(t) = \left(\frac{\delta_i\left((L_k)_i(t) - \delta_i W_i(t)\right)}{\delta_i^2(T-t) + g_i(T-t)}\right)_{i=1,\cdots,m} \qquad t \in [0,A].$$

□

Theorem 3.6 implies that the wealth equation (2.3) is meaningful and we are now able to construct a risk neutral probability measure for this progressive strong informed agent.

Definition 3.8

$$\begin{cases} l_k^* & := \left((\Theta + \rho_1^k)^*, \left(\frac{\rho_2^k}{q} - I_n \right)^* \right). \\ Y_k & := \mathcal{E} \left(\int_0^{\cdot} \left(-(\Theta + \rho_1^k)^*(s) d\widetilde{W}^k(s) + \left(\frac{q(s)}{\rho_2^k(s)} - I_n \right)^* d\widetilde{M}^k(s) \right) \right). \end{cases}$$

Y_k is a $(\mathrm{G}_A^k, \mathbb{P})$-local martingale. We assume that :

Assumption 3.9 *Y_k is a $(\mathrm{G}_A^k, \mathbb{P})$-martingale.*

Therefore $\widehat{\mathbb{P}^k} := Y_k(A)\mathbb{P}$ is a risk neutral probability measure for agent k.
$\widehat{W}$ is a $(\mathrm{G}_A^k, \widehat{\mathbb{P}^k})$-Brownian motion and $\widehat{M}$ is the compensated process of a $(\mathrm{G}_A^k, \widehat{\mathbb{P}^k})$-point process with intensity $(q.\kappa)$. Furthermore,

$$d(Y_k^{-1})(t) = Y_k^{-1}(t^-) l_k^*(t) d\widehat{S}(t).$$

Besides, if (π_k, c_k) is a G_A^k-admissible strategy for the k-th agent, then it satisfies the budget constraint

$$E_{\mathbb{P}} \left[\int_0^A \beta(s) Y_k(s) \left(c_k(s) - \epsilon_k(s) \right) ds | \mathcal{G}_0^k \right] \leq 0. \tag{3.1}$$

But in contrary to an initial strong information, this necessary condition (3.1) for admissibility is not sufficient. Indeed, the main difference here is that the financial market is not complete for the insider because as a general rule, we do not always have the existence of a probability measure equivalent to $\mathbb{P}$ and under which $\mathcal{F}_t$ and $\mathcal{H}_t^k$ are independent for all $t \in [0, A]$. Therefore we do not have a martingale representation theorem for the $(\mathrm{G}_A^k, \widehat{\mathbb{P}^k})$-local martingales. Thus a consumption satisfying the budget constraint (3.1) is not always attainable.
A progressive strong insider wants to maximize the mapping

$$(\pi_k, c_k) \to V(\pi_k, c_k) := E_{\mathbb{P}}[\int_0^A U_k(t, c_k(t)) dt | \mathcal{G}_0^k] \tag{3.2}$$

over all G_A^k-admissible investment/consumption strategies.
Because of the non completeness of the market for a progressive strong insider, we can not give the optimal consumption and wealth for any utility function. Nevertheless, in the usual case of a logarithmic utility and a deterministic discounted endowment, we can solve this optimization problem explicitly by producing an optimal portfolio: we obtain the same solution as for an initial strong insider (Example 4.7 in [17]), with the corresponding Y_k.

Example 3.10
We assume 3.4, assumptions of Theorem 3.6 and 3.9.
$U_k(t, c) = \ln(c), \forall (t, c) \in [0, A] \times]0, +\infty[$.

We assume that $E_{\mathbb{P}}[\int_0^A Y_k(t)\beta(t)\epsilon_k(t)dt|\mathcal{G}_0^k] > 0$ $\mathbb{P}$ almost surely and that $\beta\epsilon_k$ is deterministic. Then

$$\lambda_k = \frac{A}{E_{\mathbb{P}}[\int_0^A Y_k(t)\beta(t)\epsilon_k(t)dt|\mathcal{G}_0^k]},$$

$$\widehat{c_k}(t) = \frac{1}{\lambda_k\beta(t)Y_k(t)} \qquad \text{and} \qquad \beta(t)Y_k(t)\widehat{X_k}(t) = \frac{A-t}{\lambda_k} - E_{\mathbb{P}}[\int_t^A Y_k(s)\beta(s)\epsilon_k(s)ds|\mathcal{G}_t^k].$$

$$\widehat{\pi}_k(t) = (\sigma^*(t))^{-1}\left(\frac{A-t}{A\beta(t)}\int_0^A \beta(s)\epsilon_k(s)ds\right)\frac{1}{Y_k(t^-)}l_k(t).$$

□

3.2 Weak information

Here we consider that the true model of the stock prices is partially observed. More precisely, the effective probability measure $\mathbb{P}$ of the market is unknown, but the agents know the risk neutral probability measure $\widehat{\mathbb{P}}$ of a non-insider, that is the unique probability measure equivalent to $\mathbb{P}$ such that the discounted prices are $(\mathrm{F}_T, \widehat{\mathbb{P}})$-local martingales. The insider knows there will be a release of information about the outcome of some variable L_k of the prices, but in contrary to a strong information, he does not observe it, therefore he anticipates its law. We say that this agent is weakly informed on this $\mathcal{F}_T$-measurable random variable L_k, meaning that he only has the filtration F_T available (thus his strategy is F_T-admissible), but he anticipates the law of L_k under $\mathbb{P}$.
Let $L_k : \Omega \longrightarrow \mathbb{R}^j$ be a $\mathcal{F}_T$-measurable random variable. We denote by $\widehat{\mathbb{P}}_{L_k}$ the law of L_k under $\widehat{\mathbb{P}}$. With L_k we associate a probability measure ν_k on $\mathbb{R}^j$. We assume that

Assumption 3.11 *ν_k admits a positive bounded density ξ_k with respect to $\widehat{\mathbb{P}}_{L_k}$.*

The insider knows that the law of L_k under the effective probability measure $\mathbb{P}$ is ν_k. ν_k is called a weak information on the functional L_k.

Proposition 3.12 *(Proposition 3 Baudoin (2002) [6])*
On $\mathcal{F}_T$, there exists a unique probability measure $\mathbb{P}^{\nu_k}$ such that
(i) For all $\mathcal{F}_T$-measurable bounded random variable X, $E_{\mathbb{P}^{\nu_k}}(X|L_k) = E_{\widehat{\mathbb{P}}}(X|L_k)$.
(ii) The law of L_k under $\mathbb{P}^{\nu_k}$ is ν_k.
$\mathbb{P}^{\nu_k}$ is called the minimal probability associated with the conditioning (T, L_k, ν_k).

Proof: We easily check that $\mathbb{P}^{\nu_k}$ given by : $\forall B \in \mathcal{F}_T$, $\mathbb{P}^{\nu_k}(B) = \int_{\mathbb{R}^j}\widehat{\mathbb{P}}(B|L_k = y)\nu_k(dy)$ is the unique probability measure on $\mathcal{F}_T$ satisfying (i) and (ii). □

The link between the notion of initial strong information given by an initial enlargement of filtration and the notion of weak information is given in Baudoin (2001) [7] Theorem 2.2 and Proposition 2.3. We need a technical lemma to solve the optimization problem of a weak-insider.

Lemma 3.13 *Under assumption 3.11*

$$(1) \qquad \frac{d\mathbb{P}^{\nu_k}}{d\widehat{\mathbb{P}}} = \frac{d\nu_k}{d\widehat{\mathbb{P}}_{L_k}}(L_k) = \xi_k(L_k).$$

$$(2) \qquad \left(\frac{d\mathbb{P}^{\nu_k}}{d\widehat{\mathbb{P}}}\right)_{|\mathcal{F}_t} = E_{\widehat{\mathbb{P}}}(\xi_k(L_k)|\mathcal{F}_t).$$

Proof: cf. Baudoin (2002) [6]. □

We denote

Definition 3.14

$$\begin{cases} \zeta_k(t) & := E_{\widehat{\mathbb{P}}}(\xi_k(L_k)|\mathcal{F}_t). \\ Y_k & := \frac{1}{\zeta_k}. \end{cases}$$

Since $\zeta_k = \frac{1}{Y_k}$ is a strictly positive $(\mathrm{F}_T, \widehat{\mathbb{P}})$-local martingale, there exists a F_T-predictable process l_k such that $d\frac{1}{Y_k}(t) = \frac{1}{Y_k}(t^-)l_k^*(t)d\widehat{S}(t)$.

Example 3.15
We take $L_k = \widehat{W}_i(T)$ and we study the case where ν_k is the Gaussian measure

$$\nu_k(dy) = \frac{1}{\sqrt{2\pi}s}\exp\left(\frac{-(y-m)^2}{2s^2}\right)dy, \qquad m \in \mathbb{R}, \qquad s^2 \leq T.$$

Then $\quad \xi(y) = \frac{\sqrt{T}}{s}\exp\left(\frac{y^2}{2T} - \frac{(y-m)^2}{2s^2}\right)$ and

$$\frac{1}{Y_k}(t) = \int \frac{1}{\sqrt{2\pi}s}\frac{\sqrt{T}}{\sqrt{T-t}}\exp\left(\frac{y^2}{2T} - \frac{(y-\widehat{W}_i(t))^2}{2(T-t)} - \frac{(y-m)^2}{2s^2}\right)dy.$$

Thus in this example $d\frac{1}{Y_k}(t) = \frac{1}{Y_k}(t^-)l_k^*(t)d\widehat{S}(t)$ with

$$\begin{aligned} (l_k)_i(t) &= \frac{\int \frac{y-\widehat{W}_i(t)}{T-t}\exp\left(\frac{y^2}{2T} - \frac{(y-\widehat{W}_i(t))^2}{2(T-t)} - \frac{(y-m)^2}{2s^2}\right)dy}{\int \exp\left(\frac{y^2}{2T} - \frac{(y-\widehat{W}_i(t))^2}{2(T-t)} - \frac{(y-m)^2}{2s^2}\right)dy}, \\ (l_k)_j &\equiv 0 \quad \text{if} \quad i \neq j. \end{aligned}$$

□

A weak-informed agent wants to maximize the mapping

$$(3.3) \qquad (\pi_k, c_k) \to V(\pi_k, c_k) := E_{\mathbb{P}^{\nu_k}}\left[\int_0^A U_k(t, c_k(t))dt\right]$$

over all F_A-admissible strategies, that are characterized by the budget constraint $E_{\widehat{\mathbb{P}}}[\int_0^A \beta(s)(c_k - \epsilon_k)(s)ds] \leq 0$ (indeed, there is no enlargement of filtration for a weak insider, therefore the market is both viable and complete for his point of view). Then, using Lemma 3.13, we solve his optimization problem by means of the Lagrange multipliers.

Theorem 3.16 *We suppose assumption 3.11 satisfied. If there exists a positive constant λ_k satisfying $E_{\widehat{\mathbb{P}}}[\int_0^A \beta(t)(I_k(t,\lambda_k\beta(t)Y_k(t)) - \epsilon_k(t))dt] = 0$, then there exists a unique solution to the k-th weak informed agent's optimization problem.*
The optimal consumption rate is given by $\widehat{c_k}(t) = I_k(t,\lambda_k\beta(t)Y_k(t))$.
The optimal wealth is given by $X_k^{(\widehat{\pi_k},\widehat{c_k})}(t) = \frac{1}{\beta(t)}E_{\widehat{\mathbb{P}}}[\int_t^A \beta(s)(\widehat{c_k} - \epsilon_k)(s)ds|\mathcal{F}_t]$.

Example 3.17
We assume 3.11. $U_k(t,c) = \ln(c)$, $\forall(t,c) \in [0,A]\times]0,+\infty[$.
We assume that $E_{\widehat{\mathbb{P}}}[\int_0^A \beta(t)\epsilon_k(t)dt] > 0$ $\widehat{\mathbb{P}}$ almost surely. Then since $\frac{1}{Y_k}$ is a density process of a probability measure,

$$\lambda_k = \left(\frac{E_{\widehat{\mathbb{P}}}(\int_0^A \beta(t)\epsilon_k(t)dt)}{E_{\widehat{\mathbb{P}}}(\int_0^A \frac{1}{Y_k}(t)dt)}\right)^{-1} = \left(\frac{E_{\widehat{\mathbb{P}}}(\int_0^A \beta(t)\epsilon_k(t)dt)}{A}\right)^{-1},$$

$$\widehat{c_k}(t) = \frac{1}{\lambda_k\beta(t)Y_k(t)} \qquad \text{and} \qquad \beta(t)\widehat{X_k}(t) = \frac{(A-t)}{\lambda_k Y_k(t)} - E_{\widehat{\mathbb{P}}}[\int_t^A \beta(s)\epsilon_k(s)ds|\mathcal{F}_t].$$

If $\beta\epsilon_k$ is deterministic, the optimal portfolio is

$$\widehat{\pi_k}(t) = (\sigma^*(t))^{-1}\left(\frac{A-t}{A\beta(t)}\int_0^A \beta(s)\epsilon_k(s)ds\right)\frac{1}{Y_k(t^-)}l_k(t).$$

□

Remark 3.18 *In this three types of side information, the optimal strategy of the k-th agent is determined by the density process Y_k of his probability measure. This means that all the relevant information of each agent is contained in his own probability measure. Therefore it is not surprising that a necessary and sufficient condition of equilibrium establishes a link between these densities probabilities. That is what we will see in the next section.*

4 Equilibrium

In Hillairet (2005) [17], the formation of an equilibrium of the consumptions is studied, in the case where all agents are initial strong insiders. The consumption market clearing condition was :

$$\sum_{k=1}^K \widehat{c_k}(t) = \sum_{k=1}^K \epsilon_k(t), \qquad t \in [0,A]. \tag{4.1}$$

A necessary and sufficient condition for existence of an equilibrium of the consumptions is given. Under an assumption about conditional independence of the enlarged filtrations, the insiders have the same risk neutral probability measure. In a moneyless model, this risk neutral probability measure and the interest rate are uniquely determined at equilibrium. Moreover, if an equilibrium of the consumptions exists , then $\sum_{k=1}^{K} \widehat{\pi}_k{}^*(t) = 0_{\mathbb{R}^d}, \quad t \in [0, A]$. Here we will focus our attention on this following portfolio market clearing condition, for the three types of side information :

$$\sum_{k=1}^{K} \widehat{\pi}_k{}^*(t) = 0_{\mathbb{R}^d}, \qquad t \in [0, A]. \tag{4.2}$$

Our setting is not usual for an equilibrium model in the sense that we assume that the price processes are exogenous. The agents we consider are price takers. We can think for example at a small closed structure of agents trying to set transactions such that the stock market clears in their "local" structure. Their transactions do not affect the price processes, that are fixed by an external market. It is a competitive dynamic equilibrium, with no market maker, and the price are not determined by this equilibrium. Although this framework seems naïve, it leads to a particularly interesting and meaningful relation linking the densities of the probabilities measures of each agents. This relation (4.3) means that the transactions can occur if the endowments and the information of the agents are well-balanced. It can be interpreted as following : the more informed an agent is, the less weight he must invest.

4.1 Equilibrium with logarithmic utility and deterministic discounted endowments

In this subsection, we consider the optimization problem of the consumption with a logarithmic utility and deterministic discounted endowments. The price processes being exogenous (given by (2.2)), the question is the following : can the transactions occur, and under which constraints?

We can express the optimal portfolio for each of the three types of "side-information", see example 4.5 in [17], examples 3.10 and 3.17. For a strong information (initial or progressive) or a weak information, the optimal portfolio is given by

$$\beta(t)\widehat{\pi}_k{}^*(t)\sigma(t) = \frac{A-t}{A}\int_0^A \beta(s)\epsilon_k(s)ds\frac{1}{Y_k(t^-)}l_k^*(t)$$

with the corresponding Y_k defined respectively in Definitions 3.2, 3.8 and 3.14. It is important to notice here that, for these three Y_k,

$$d\frac{1}{Y_k}(t) = \frac{1}{Y_k}(t^-)l_k^*(t)d\widehat{S}(t)$$

with the corresponding l_k and the **same** $\widehat{S}(t)$. Therefore

$$\begin{aligned}
\sum_{k=1}^{K} \widehat{\pi_k}^* \equiv 0_{\mathbb{R}^d} \quad &\Longleftrightarrow \quad \sum_{k=1}^{K} \frac{\int_0^A \beta(s)\epsilon_k(s)ds}{Y_k(t^-)} l_k^*(t) = 0 \qquad t \in [0, A] \\
&\Longleftrightarrow \quad \sum_{k=1}^{K} \frac{\int_0^A \beta(s)\epsilon_k(s)ds}{Y_k(t^-)} l_k^*(t) d\widehat{S}(t) = 0 \qquad t \in [0, A] \\
&\Longleftrightarrow \quad \sum_{k=1}^{K} \frac{\int_0^A \beta(s)\epsilon_k(s)ds}{Y_k(t)} = \sum_{k=1}^{K} \int_0^A \beta(s)\epsilon_k(s)ds \qquad t \in [0, A].
\end{aligned}$$

The last equivalence is obtained by integrating the expression between 0 and t and by using the fact that $Y_k(0) = 1$. We proved the following theorem :

Theorem 4.1 *For an initial-strong insider, we assume 3.1 and 3.3. For a progressive-strong insider, we assume 3.4, assumptions of Theorem 3.6 and 3.9. For a weak-insider, we assume 3.11. In those three types of side information, if* $a_k := \frac{\int_0^A \beta(s)\epsilon_k(s)ds}{\sum_{k=1}^K \int_0^A \beta(s)\epsilon_k(s)ds}$

$$\sum_{k=1}^{K} \widehat{\pi_k}^* \equiv 0_{\mathbb{R}^d} \Longleftrightarrow \sum_{k=1}^{K} a_k \frac{1}{Y_k(t)} = 1 \qquad t \in [0, A] \tag{4.3}$$

with the corresponding Y_k *defined respectively in 3.2, 3.8 and 3.14.*

In the case of an equilibrium of the consumptions with only initial strong insiders, under an assumption about conditional independence of the enlarged filtrations (assumption 5.2 [17]), the insiders have the same risk neutral probability measure, that is uniquely determined at equilibrium. In our setting of an equilibrium of the portfolio with different types of insiders, the densities Y_k of their own probability measures (risk neutral probability measure for a strong insider, minimal probability measure for a weak insider) are linked by relation (4.3). The weights a_k (note that $\sum_{k=1}^K a_k = 1$) represent the proportion of endowments of each agent, and can be seen as the weight of each agent in the market. The process Y_k summarizes the information of insider k. It determines his behavior (cf. the expressions of the optimal consumption and portfolio). The process Y_k is also closely related with the insider's gain. Indeed, in our case of a logarithmic utility, the discounted consumption is proportional to $\frac{1}{Y_k}$. Furthermore, we can represent the insider's additional expected logarithmic utility as the relative entropy of his own probability measure with respect to the risk neutral probability measure of a non-insider. That is why relation (4.3), giving a linear constraint between the weight of the agents and their probability measures, is particularly interesting and meaningful. From a financial point of view, the relation (4.3) means that an equilibrium (the transactions) can occur if the endowments and the information of the agents are well-balanced.

The two following subsections are an illustration and an interpretation of this theorem in two particular cases.

4.1.1 Particular case where agents share the same information

We suppose here that the process Y_k is the same for all agents, thus all agents share the same information. Then equivalence (4.3) implies that at equilibrium, $Y_k \equiv 1$. Therefore $\mathbb{P}$ is the risk neutral probability measure for the strong insiders and $\widehat{\mathbb{P}} = \mathbb{P}^{\nu_k}$ for the weak insiders, that is to say the agents have no side-information, or their side-information is irrelevant. In this case of no side-information, the agents do not move, and consume exactly their endowments distributed over $[0, A]$: for all k,

$$\widehat{\pi_k}^* \equiv 0_{\mathbb{R}^d} \qquad \text{and} \qquad \beta \widehat{c_k} \equiv \frac{\int_0^A \beta(s)\epsilon_k(s)ds}{A}.$$

Remark: link with the consumption market clearing condition (4.1).
In the case of the portfolio market clearing condition with no side information,

$$\sum_{k=1}^K \widehat{c_k}(t) = \frac{\int_0^A \beta(s)(\sum_{k=1}^K \epsilon_k(s))ds}{\beta(t)A} \qquad t \in [0, A].$$

Therefore we do not have generally the relation (4.1), but

$$\int_0^A \beta(s)(\sum_{k=1}^K \widehat{c_k}(s))ds = \int_0^A \beta(s)(\sum_{k=1}^K \epsilon_k(s))ds.$$

This example shows that the market clearing (4.1) is a bit too restrictive, therefore it is justified and more natural to consider the portfolio market clearing condition.

Let us now focus on the more interesting case where the agents do not have the same information.

4.1.2 Case where agents have asymmetric information

Lemma 4.2 *Under assumptions of Theorem 4.1, if the discounted endowment rates are constant, then*

$$\sum_{k=1}^K \widehat{c_k} \equiv \sum_{k=1}^K \epsilon_k \Longleftrightarrow \sum_{k=1}^K \widehat{\pi_k}^* \equiv 0_{\mathbb{R}^d}.$$

Proof: We assume that the discounted endowment rates are constant ($\beta\epsilon_k \equiv a_k$ where a_k is a strictly positive constant), then $\beta(t)\widehat{c_k}(t) = \frac{a_k}{Y_k(t)}$.
Thus

$$\sum_{k=1}^K \widehat{c_k} \equiv \sum_{k=1}^K \epsilon_k \Longleftrightarrow \sum_{k=1}^K \frac{a_k}{Y_k} \equiv \sum_{k=1}^K a_k \Longleftrightarrow \sum_{k=1}^K \widehat{\pi_k}^* \equiv 0_{\mathbb{R}^d}.$$

$\square$

First, let us consider only two agents on the financial market. Y_1 of the first agent being given, what must satisfy Y_2 so that an equilibrium could be achieved? Equivalence (4.3) yields

$$\widehat{\pi}_1{}^* + \widehat{\pi}_2{}^* \equiv 0_{\mathbb{R}^d} \Longleftrightarrow \frac{1}{Y_2} \equiv 1 + \frac{\int_0^A \beta(s)\epsilon_1(s)ds}{\int_0^A \beta(s)\epsilon_2(s)ds}(1 - \frac{1}{Y_1}). \tag{4.4}$$

On the one hand, Y_2 must be a positive process. On the other hand, the more informed is the first agent, the bigger is $\frac{1}{Y_1}$ and therefore the more negative is the process $(1 - \frac{1}{Y_1})$. Thus we have to "counterbalanced" $(1 - \frac{1}{Y_1})$ with a small $\frac{\int_0^A \beta(s)\epsilon_1(s)ds}{\int_0^A \beta(s)\epsilon_2(s)ds}$ so that Y_2 remains strictly positive. Intuitively, the bigger $\int_0^A \beta(s)\epsilon_k(s)ds$ is, the more influential agent k is on the financial market. This is coherent with the theory of the representative agent that was introduced by Karatzas et al. [26]. A representative agent is a fictitious agent who receives the endowment process ε and attempts to maximize his total expected utility $E_{\mathbb{P}}[\int_0^A U(t, c(t), \Lambda)dt|\mathcal{G}_0]$ under the constraint $E_{\mathbb{P}}[\int_0^A \beta(t)Y(t)(c-\varepsilon)(t)dt|\mathcal{G}_0] \leq 0$ (U is the representative agent's utility function and $\mathcal{G}_0$ is the common information shared by all agents at time 0). For all $k = 1, \cdots, K$, the representative agent assigns to the k-th agent the weight Λ_k. Actually, $\Lambda_k = \lambda_k^{-1}$ (up to a multiplicative constant, cf [17]), where λ_k is the Lagrange multiplier of agent k. In the case of logarithmic utility

$$\Lambda_k = (\lambda_k)^{-1} = \frac{E_{\mathbb{P}}\left(\int_0^A \beta(s)\epsilon_k(s)ds\right)}{A}$$

Therefore, the more informed an agent is, the less weight he must have in the financial market (comparatively to the others agents) so that an equilibrium could be achieved (cf. section 4.3 for explicit calculus and simulations).
Besides, the more endowment an insider invests on a financial market, the more abnormal deviations he will induce on this market, and therefore the more he risks to be detected. Furthermore, the abnormal strategies or deviations induced by an insider are as big as his side-information is important. Thus a well-informed agent would be well advised to keep himself unobtrusive and invest a small part of his endowment on the market (see section 4.3).

Proposition 4.3 *Under assumptions of Theorem 4.1, if Y_k, $k = 1, \cdots, K-1$ of the $(K-1)$ first agents are given, there is an equilibrium (i.e. $\sum_{k=1}^K \widehat{\pi}_k{}^* \equiv 0_{\mathbb{R}^d}$) if and only if*

$$\begin{cases} \sum_{k=1}^{K-1} a_k(1 - \frac{1}{Y_k}) > -1. \\ \frac{1}{Y_K} = 1 + \sum_{k=1}^{K-1} a_k(1 - \frac{1}{Y_k}). \end{cases}$$

Proof: It is a straightforward application of equivalence (4.3). $\square$

4.2 Equilibrium with other optimization problems

A similar work can be done with other optimization problems. We find it interesting to write a short paragraph on them because, although the theory remains the same, the simulation part is sometimes more technical, as we will see.
First, we can consider the problem consisting in optimizing both consumption and terminal wealth (or respectively terminal wealth only), with an initial wealth $X_k(0)$ and no endowment. In the case of a logarithmic utility, by using similar methods as for the consumption optimization problem, we show that the optimal portfolio is given by

$$\beta(t)\widehat{\pi}_k{}^*(t)\sigma(t) = \frac{\beta(0)X_k(0)(1+A-t)}{A+1}\frac{1}{Y_k(t^-)}l_k^*(t)$$

$$\text{(respectively} \qquad \beta(t)\widehat{\pi}_k{}^*(t)\sigma(t) = \beta(0)X_k(0)\frac{1}{Y_k(t^-)}l_k^*(t). \quad)$$

Thus, instead of equivalence (4.3), we have

$$\sum_{k=1}^{K}\widehat{\pi}_k{}^* \equiv 0_{\mathbb{R}^d} \Longleftrightarrow \sum_{k=1}^{K}\frac{X_k(0)}{Y_k(t)} = \sum_{k=1}^{K}X_k(0) \qquad t\in[0,A]. \tag{4.5}$$

Here the weight of agent k in the market is $\Lambda_k = \frac{\beta(0)X_k(0)}{A+1}$ (respectively $\beta(0)X_k(0)$), thus his weight is proportional to his initial wealth. In this setting, similar results can be proven, with $X_k(0)$ instead of $\int_0^A \beta(s)\epsilon_k(s)ds$.

We can also consider the portfolio market clearing condition with different optimization problems according to each agent : they all have a logarithmic utility but the first k_1 maximize their consumption (with no initial wealth and a deterministic discounted endowment), the next k_2 maximize their terminal wealth (with an initial wealth and no endowment) and the others maximize both consumption and terminal wealth (with an initial wealth and no endowment). Thus, instead of equivalence (4.3), we have

$$\sum_{k=1}^{K}\widehat{\pi}_k{}^* \equiv 0_{\mathbb{R}^d} \quad \text{if and only if}$$

$$\begin{aligned} &\textstyle\sum_{k=1}^{k_1} \frac{(\int_0^A \beta(s)\epsilon_k(s)ds)}{A}\left[(A-t)\frac{1}{Y_k}(t) + \int_0^t \frac{1}{Y_k}(s)ds\right] + \sum_{k=k_1+1}^{k_2}\beta(0)X_k(0)\frac{1}{Y_k}(t) \\ &+ \sum_{k=k_2+1}^{K}\frac{\beta(0)X_k(0)}{(A+1)}\left[(1+A-t)\frac{1}{Y_k}(t) + \int_0^t \frac{1}{Y_k}(s)ds\right] \\ &= \sum_{k=1}^{k_1}(\int_0^A \beta(s)\epsilon_k(s)ds) + \sum_{k=k_1+1}^{K}\beta(0)X_k(0), \quad \forall t\in[0,A]. \end{aligned} \tag{4.6}$$

Simulations can be done to exhibit Y_k (that summarize the side-information) such that an equilibrium exists. The optimization problem and the side information of $K-1$ agents being given, we simulate for each one of them the process $\frac{1}{Y_k}$. Using relation (4.6), we simulate (or we deduce straight) the process $\frac{1}{Y_{k_0}}$ of the last agent such that an equilibrium exists. More precisely, if this agent does not maximize his consumption (i.e. $k_1 < k_0 \le k_2$), we can deduce straight from (4.6) his process $\frac{1}{Y_{k_0}}$ because $\int_0^{\cdot} \frac{1}{Y_{k_0}}(s)ds$ does not appear in (4.6). On the other hand, if this agent maximizes his consumption (i.e. $k_0 \le k_1$ or $k_0 > k_2$), $\int_0^{\cdot} \frac{1}{Y_{k_0}}(s)ds$ appears in (4.6) and we simulate then $\frac{1}{Y_{k_0}}$ by iteration, setting $\frac{1}{Y_{k_0}}(0) = 1$ and simulating $\int_0^{\frac{iA}{N}} \frac{1}{Y_k}(s)ds$ with $\frac{A}{N}\sum_{j=1}^{i} \frac{1}{Y_k}(\frac{jA}{N})$, $i = 1, \cdots, N$ (where $\frac{A}{N}$ is the iteration's step).
In our simulation, there are 3 agents. The first one is a non-insider and maximizes his terminal wealth. The second one knows $\mathbb{I}_{[a,b]}(P_1(T))$ (initial strong insider) and maximizes both his consumption and his terminal wealth. We simulate the process $\frac{1}{Y_3}$ of the third agent who maximizes his consumption only.

Results
cf. section 5.2 for the figures. We have simulated the portfolio of each agent (cf. figures 3, 5.2 and 4), so that we can see our equilibrium condition about the portfolio . We have also simulated the discounted consumption and wealth of the third agent (cf. figure 2).

4.3 Maximal weight of an insider

We will estimate the maximal weight that an insider can invest in the market such that an equilibrium can be achieved. More precisely, we consider a market with two agents. The first one has a side information (an initial or progressive strong information, or a weak information). We notice that the weight in the market of an agent i is proportional to $\int_0^A \beta(s)\epsilon_i(s)ds$ if he maximizes his consumption only and has the endowment ϵ_i, or to $\beta(0)X_i(0)$ otherwise (where $X_i(0)$ is his initial wealth), denoted as a_i in both cases. The necessary and sufficient condition (4.6) gives the maximal ratio $\frac{a_1}{a_2}$ so that the process $\frac{1}{Y_2}$ is positive, i.e. so that an equilibrium can be achieved. We simulate this ratio for different information and different optimization problems, but for the same realization of ω.
If both agents have the same optimization problem, it follows from relations (4.4) and (4.5) that the maximal ratio $\frac{a_1}{a_2}$ is equal to $-\left(1 - \max_{t\in[0,A]}(\frac{1}{Y_1(t)})\right)^{-1}$ if $\max_{t\in[0,A]}(\frac{1}{Y_1(t)}) > 1$ (otherwise the maximal ratio is infinite, there is no constraint on the insider's weight).
If the first agent (the insider) maximizes his consumption (respectively his consumption and his terminal wealth) and the second agent maximizes his terminal wealth, it follows from relation (4.6) that the maximal ratio $\frac{a_1}{a_2}$ is :

$$-\left(1 - \frac{1}{A}\max_{t\in[0,A]}(\frac{A-t}{Y_1(t)} + \int_0^t \frac{1}{Y_1(s)}ds)\right)^{-1} \quad \text{if} \quad \frac{1}{A}\max_{t\in[0,A]}(\frac{A-t}{Y_1(t)} + \int_0^t \frac{1}{Y_1(s)}ds) > 1$$

(respectively $\quad -\left(1 - \frac{1}{A+1}\max_{t\in[0,A]}\left(\frac{1+A-t}{Y_1(t)} + \int_0^t \frac{1}{Y_1(s)}ds\right)\right)^{-1}$

$$\text{if} \qquad \frac{1}{A+1}\max_{t\in[0,A]}\left(\frac{1+A-t}{Y_1(t)} + \int_0^t \frac{1}{Y_1(s)}ds > 1\right),$$

otherwise there is no constraint on the insider's weight.

The insider may have the side information i, $i = 1, \cdots, 4$.

$$\begin{cases} \text{information 1} & : \text{ initial strong information } \mathbb{I}_{[a,b]}(P_1(T)). \\ \text{information 2} & : \text{ initial strong information } \ln(P_1(T)) - \ln(P_2(T)). \\ \text{information 3} & : \text{ progressive strong information } 2W_1(T) + a + B_{(T-t)} \text{ (cf. example 3.7)} \\ \text{information 4} & : \text{ weak information of example 3.15.} \end{cases}$$

Both agents can maximize their consumption (optimization number 1), their terminal wealth (optimization number 2) or both their consumption and their terminal wealth (optimization number 3), until a date $A < T$. Here are the data we have used for our simulations. $A = 0.95$ and $T = 1$. Both Brownian motion and Poisson process are 2-dimensional: $m = n = 2$. The intensity of the Poisson process is $\kappa = (3, 2)$. We choose constant market coefficients, but the simulations could be easily extended with time-varying market coefficients. The annual interest rate is 0.02 : $r(t) = 0.02$ for all t.

$$\text{The drift } b(t) = \begin{pmatrix} 0.15 \\ 0.1 \\ 0.084 \\ 0.1 \end{pmatrix} \qquad \forall t \in [0, T].$$

$$\text{The volatility } \sigma(t) = \begin{pmatrix} -0.4 & -0.1 & -0.15 & 0.17 \\ -0.09 & -0.4 & -0.03 & 0.035 \\ 0.048 & -0.12 & 0.1 & -0.12 \\ 0.075 & 0.26 & 0.31 & -0.28 \end{pmatrix} \qquad \forall t \in [0, T],$$

Results

cf. section 5.3 page 24 for the table of the results and the figures. The first simulation is in a diffusive-jump market (first table and figures 5 and 6). The second simulation is in a purely diffusive market (second table and figures 7 and 8) driven by a 4-dimensional Brownian motion. Optimization jk corresponds to the optimization number j for the insider and number k for the second agent. We also have simulated the process $\frac{1}{Y}$ for each side information.

We notice that the larger the process $\frac{1}{Y_1}$ is, the smaller the maximal ratio $\frac{a_1}{a_2}$ is. In particular, the process $\frac{1}{Y}$ for the second information in the second simulation is very high and such an insider must invest a derisory sum (the maximal ratio is in the region of 1e-4) so that an equilibrium could exist between the two agents. In both our simulations, the

process $\frac{1}{Y}$ is larger (and then the maximal ratio smaller) in the case of the progressive strong information (information 3) than in the case of the initial strong information 1. This can be explained by the fact that for the information 1, the insider knows an event that occurs with a positive probability, whereas for the information 3, the event the insider knows (in a disturbed way) occurs with a probability equal to zero. Furthermore, the processes $\frac{1}{Y}$ are larger in the purely diffusive market than in the diffusive-jump market, especially for the informations 2 and 3 where the insiders know an event that occurs with a probability equal to zero. Indeed, in both those cases, the side-information loses some of its relevance if a jump occurs between A and T.
We notice that the maximal ratio depends on the optimization problems of both agents. It is smaller if the insider maximizes only his terminal wealth, and larger if he maximizes only his consumption. Inversely, it is larger if the second agent maximizes only his terminal wealth, and smaller if he maximizes only his consumption.

5 Annex

5.1 Proof of Theorem 3.6

We use similar arguments as those of Proposition 1 of Corcuera et al. (2004) [10] to generalize their result in the case of our mixed diffusive-jumps model. In particular, we need to explicit the proof because of the jumps and to explicit the formula for the compensator for the point process.
Let $q \in \mathbb{N}^*$ and $s_1 \leq \cdots \leq s_q \leq s < t$. Let $C \in \mathcal{F}_s$ and h a bounded measurable function on $\mathbb{R}^q$. Set $H = h(L_k(s_1), \cdots, L_k(s_q))$. Using Lemma 3.5 for the second equality, we get

$$\begin{aligned}
&E_{\mathbb{P}}\left[(W_t - W_s)\,\mathbb{I}_C H | B_k(s_1) = b_1, \cdots, B_k(s_q) = b_q\right] \\
&\quad = E_{\mathbb{P}}\left[(W_t - W_s)\,\mathbb{I}_C h\left(h_k\left(L_k, b_1\right), \cdots, h_k\left(L_k, b_q\right)\right)\right] \\
&\quad = E_{\mathbb{P}}\left[\mathbb{I}_C h\left(h_k\left(L_k, b_1\right), \cdots, h_k\left(L_k, b_q\right)\right) \int_s^t \frac{\alpha^k(u, L_k)}{p^k(u, L_k)} du\right] \\
&\quad = E_{\mathbb{P}}\left[\mathbb{I}_C H \int_s^t \frac{\alpha^k(u, L_k)}{p^k(u, L_k)} du | B_k(s_1) = b_1, \cdots, B_k(s_q) = b_q\right].
\end{aligned}$$

Taking expectation with respect to $(B_k(s_1), \cdots, B_k(s_q))$ yields

$$E_{\mathbb{P}}\left[(W_t - W_s)\,\mathbb{I}_C H\right] = E_{\mathbb{P}}\left[\mathbb{I}_C H \int_s^t \frac{\alpha^k(u, L_k)}{p^k(u, L_k)} du\right].$$

Therefore $W(\cdot) - \int_0^{\cdot} \rho_1^k(s) ds$ is a continuous $(\mathbb{G}_A^k, \mathbb{P})$-local martingale. Lévy's characterization theorem implies that $W(\cdot) - \int_0^{\cdot} \rho_1^k(s) ds$ is a $(\mathbb{G}_A^k, \mathbb{P})$-Brownian motion. The same proceeding with N instead of W yield

$$E_{\mathbb{P}}\left[(N_t - N_s)\,\mathbb{I}_C H\right] = E_{\mathbb{P}}\left[\mathbb{I}_C H \int_s^t \kappa(u).\left(I_n + \frac{\beta^k(u, L_k)}{p^k(u^-, L_k)}\right) du\right].$$

Thus $N(\cdot) - \int_0^{\cdot}(\kappa.\rho_2^k)(s)ds$ is a $(\mathrm{G}_A^k, \mathbb{P})$-local martingale and Theorem 5 p. 25 in [8] implies that $N(\cdot) - \int_0^{\cdot}(\kappa.\rho_2^k)(s)ds$ is the compensated process of a $(\mathrm{G}_A^k, \mathbb{P})$-point process with intensity $(\kappa.\rho_2^k)$. $\square$

5.2 Equilibrium

In this simulation, $a = 0.6$, $b = 1.2$, $P_1(T) = 0.3666$. The initial wealth of the two first agents is equal to 1. The third agent has a constant discounted endowment equal to 1. cf. figures 1, 2, 3, and 4.

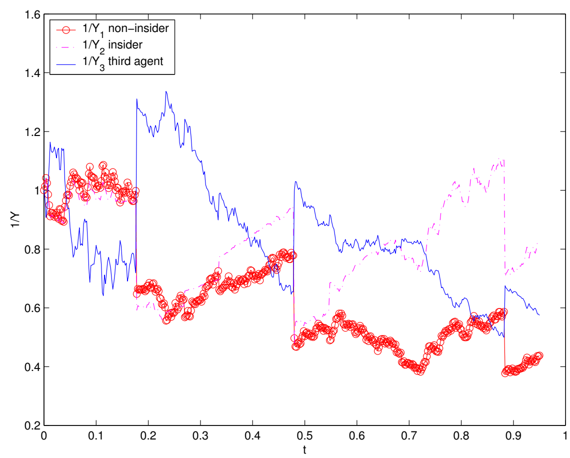
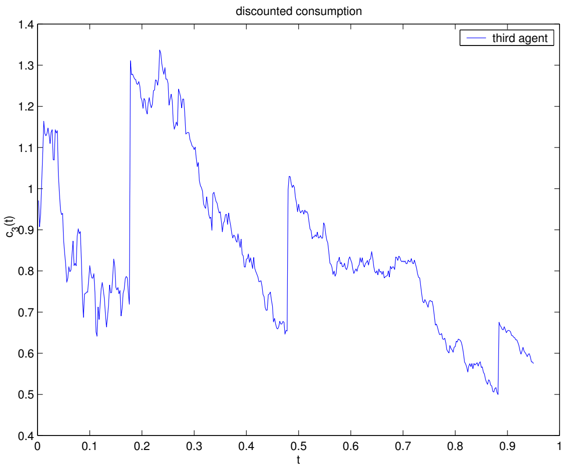
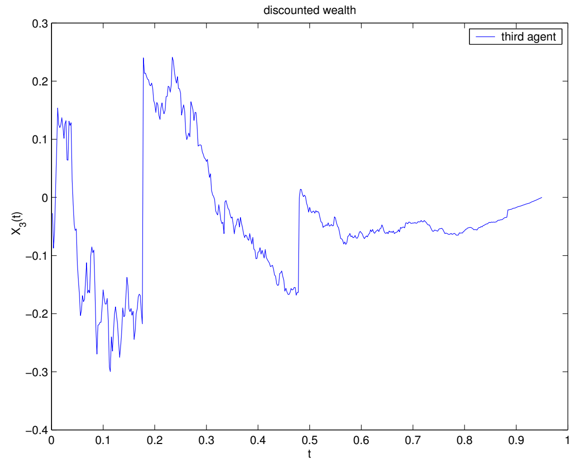


Figure 1: $\frac{1}{Y}$ of the 3 agents at equilibrium



discounted consumption
of the third agent at equilibrium



discounted wealth
of the third agent at equilibrium

Figure 2:

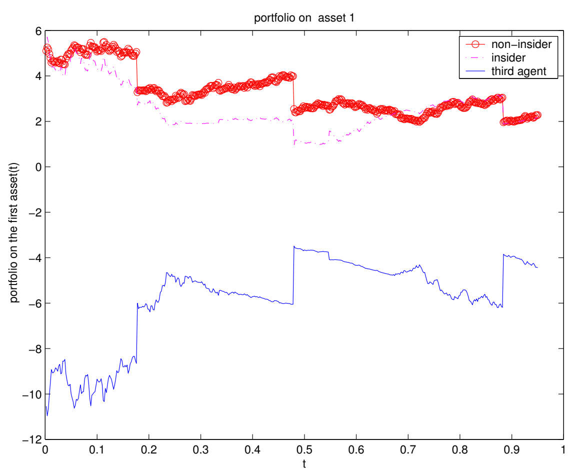
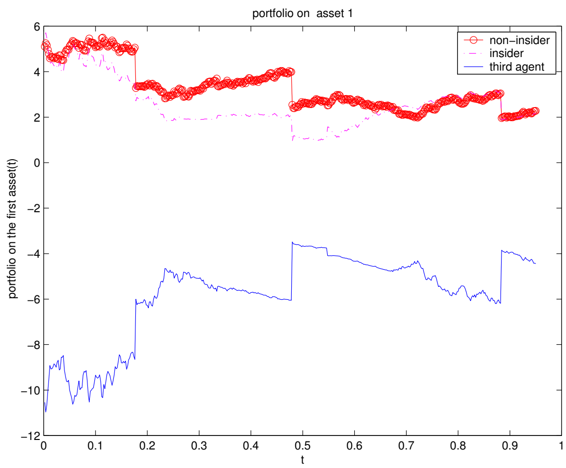
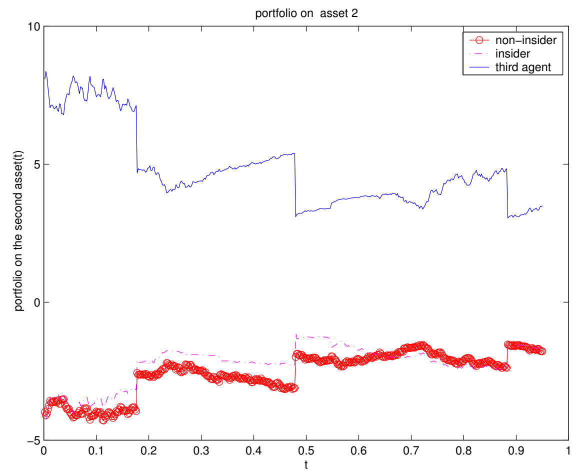


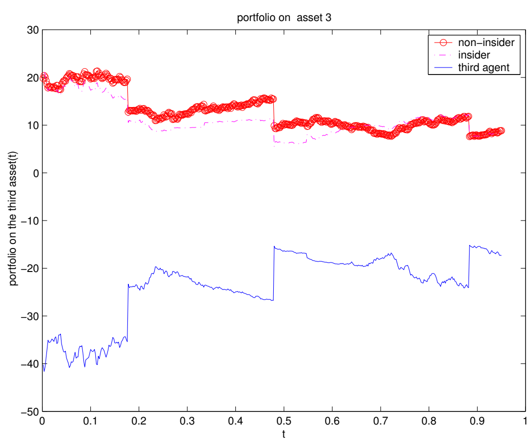
Figure 3: portfolio on the bond of the 3 agents at equilibrium



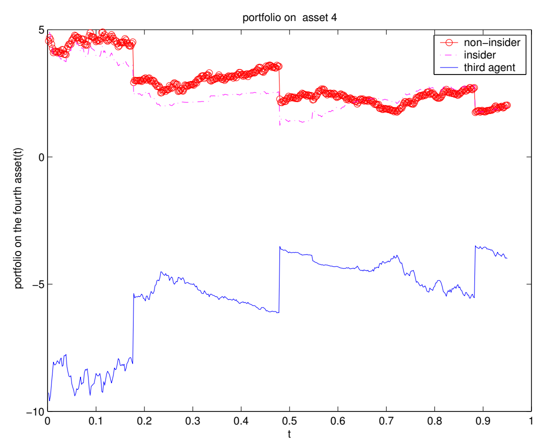
portfolio on asset 1



portfolio on asset 2



portfolio on asset 3



portfolio on asset 4

Figure 4: portfolio of the 3 agents at equilibrium

5.3 Maximal weight

See below the table of the results. Optimization jk corresponds to the optimization number j for the insider and number k for the second agent. We also have simulated the process $\frac{1}{Y}$ for each side information.

5.3.1 First simulation (diffusive-jump market):

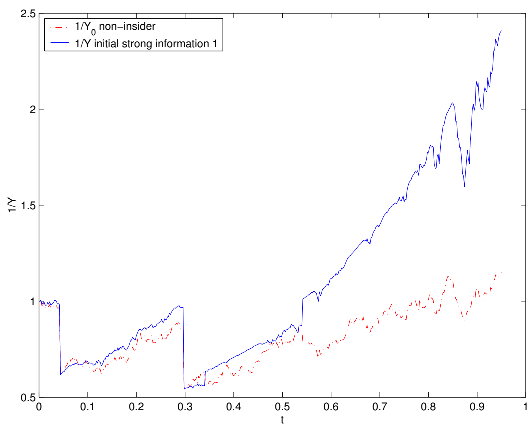
$P_1(T) = 0.8083,\ \ln(P_1(T)) - \ln(P_2(T)) = -0.2986.$ cf. figures 5 and 6.

information	optimization						
	jj	12	32	21	31	13	23
info1	0.7095	5.6045	1.2595	0.0965	0.1566	2.5656	0.4117
info2	0.2118	1.5066	0.3714	0.0212	0.0378	0.8661	0.1207
info3	0.3343	2.7773	0.5967	0.0394	0.0680	1.4481	0.1889
info4	21.4833	82.6345	69.9539	1.3163	1.2123	70.3262	10.7096

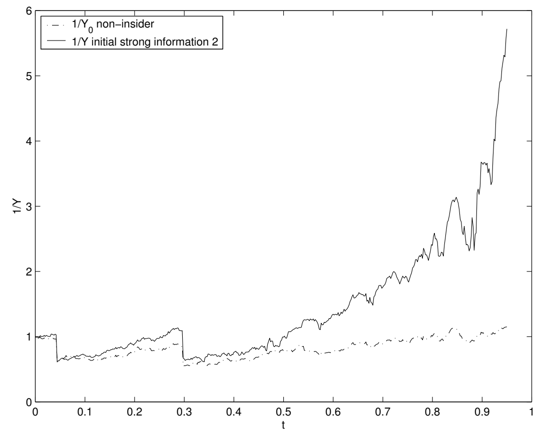
5.3.2 Second simulation (purely diffusive market) :

$P_1(T) = 2.2397,\ \ln(P_1(T)) - \ln(P_2(T)) = 0.7127.$ cf. figures 7 and 8.

information	optimization						
	jj	12	32	21	31	13	23
info1	0.5037	0.8791	0.6404	0.1996	0.2544	0.7019	0.38
info2	0.0092	0.1008	0.0168	6.4122e-4	0.0012	0.0595	0.005
info3	0.0405	0.1080	0.0597	0.0069	0.0117	0.0806	0.0266
info4	0.6637	1.235	0.8728	0.1791	0.2474	0.9954	0.4797

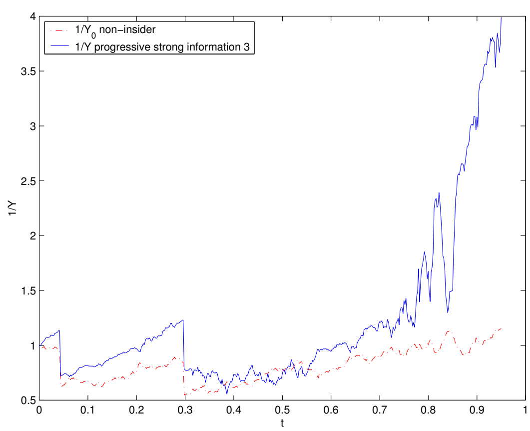


Process $\frac{1}{Y}$ for information 1

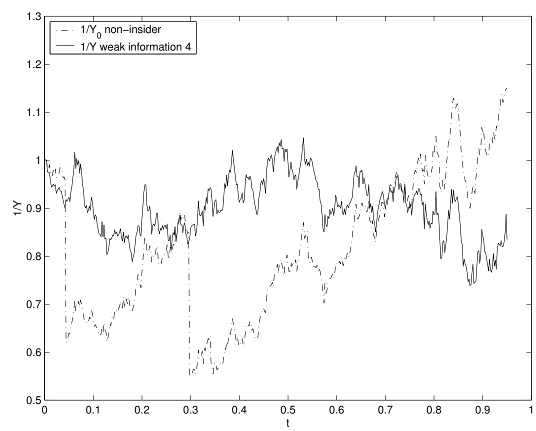


Process $\frac{1}{Y}$ for information 2

Figure 5: diffusive-jump market

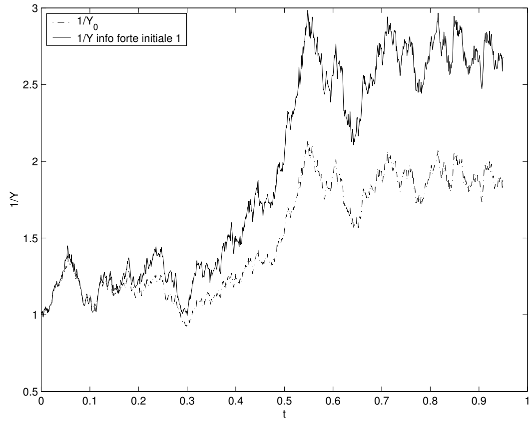


Process $\frac{1}{Y}$ for information 3

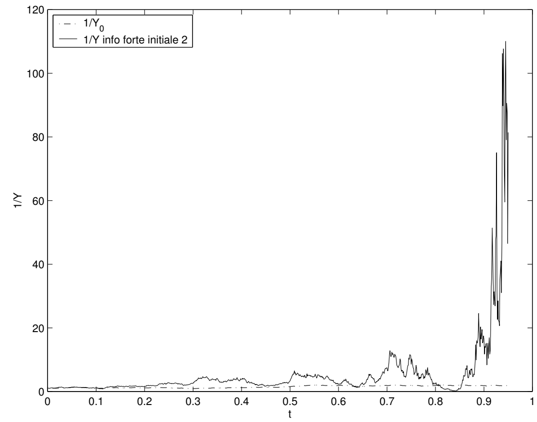


Process $\frac{1}{Y}$ for information 4

Figure 6: diffusive-jump market

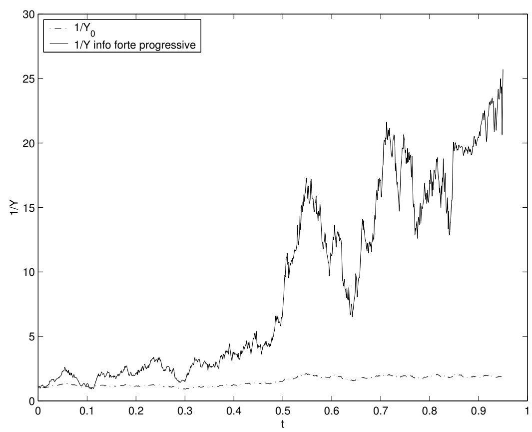


Process $\frac{1}{Y}$ for information 1

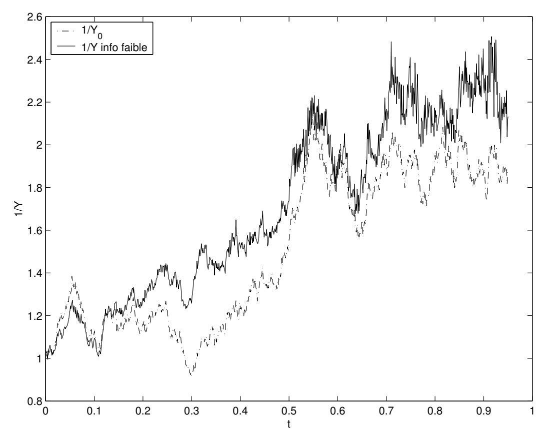


Process $\frac{1}{Y}$ for information 2

Figure 7: purely diffusive market



Process $\frac{1}{Y}$ for information 3



Process $\frac{1}{Y}$ for information 4

Figure 8: purely diffusive market

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