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Quasi-stationarity distributions and diffusion models in population dynamics

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QUASI-STATIONARY DISTRIBUTIONS AND DIFFUSION MODELS IN POPULATION DYNAMICS.

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Abstract. In this paper, we study quasi-stationarity for a large class of Kolmogorov diffusions, that is, existence of a quasi-stationary distribution, conditional convergence to such a distribution, construction of a Q-process (process conditioned to be never extinct). The main novelty here is that we allow the drift to go to $-\infty$ at the origin, and the diffusion to have an entrance boundary at $+\infty$.

These diffusions arise as images, by a deterministic map, of generalized Feller diffusions, which themselves are obtained as limits of rescaled birth–death processes. Generalized Feller diffusions take non-negative values and are absorbed at zero in finite time with probability 1. A toy example is the logistic Feller diffusion.

We give sufficient conditions on the drift near 0 and near $+\infty$ for the existence of quasi-stationary distributions, as well as rate of convergence, and existence of the Q-process.

We also show that under these conditions, there is exactly one conditional limiting distribution (which implies uniqueness of the quasi-stationary distribution) if and only if the process comes down from infinity.

Proofs are based on spectral theory. Here the reference measure is the natural symmetric measure for the killed process, and we use in an essential way the Girsanov transform.

Key words. quasi-stationary distribution, birth-death process, population dynamics, density-dependence, logistic growth, diffusion approximation, generalized Feller diffusion, Dirichlet form, spectral theory, Yaglom limit, convergence rate, Q-process, return from infinity.

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1. **Introduction.**

The main motivation of this work is the existence, domain of attraction, and uniqueness, of a quasi-stationary distribution for some diffusion models arising from population dynamics. After a change of function, the problem is stated in the framework of Kolmogorov diffusion processes with a drift behaving like $-1/2x$ near the origin. Here, we study quasi-stationarity for a slightly larger class of one-dimensional Kolmogorov diffusions, with drift possibly exploding near the origin.

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1.1. **Population Dynamics and Quasi-Stationary Distributions.** Our aim is to study the asymptotic behavior of the size $(Z_t; t \geq 0)$ of some isolated biological population. Since competition for limited resources impedes natural populations with no immigration to grow indefinitely, they are all doomed to become extinct after some finite time T_0. However, T_0 can be large compared to human timescale and it is common that population sizes fluctuate for large amounts of time before extinction actually occurs. This behavior is captured by the mathematical notion of quasi-stationarity. (See [23] for a regularly updated extensive bibliography, [24, 27] for a description of the biological meaning, [10, 13, 26] for the Markov chain case).

Specifically, a *quasi-stationary distribution* (in short QSD) for Z is a probability measure ν satisfying

$$\mathbb{P}_\nu(Z_t \in A \mid T_0 > t) = \nu(A), \quad \forall \text{ Borel set } A \subseteq (0, \infty). \tag{1.1}$$

A specific quasi-stationary distribution is defined, if it exists, as the limiting law, as $t \to \infty$, of Z_t conditioned on $T_0 > t$, when starting from a fixed population. That is, if the limit

$$\mu(A) = \lim_{t\to\infty} \mathbb{P}_x(Z_t \in A \mid T_0 > t)$$

exists and defines a probability distribution, then it is a QSD called quasi-limiting distribution, or (as we will do here) *Yaglom limit*.

We will also study the existence of the so-called Q-process which is obtained as the the law of the process Z conditioned to be never extinct, and it is defined as follows. For any $\mathcal{B}$ which is $\mathcal{F}_s$-measurable, consider

$$\mathbb{Q}_x(\mathcal{B}) = \lim_{t\to\infty} \mathbb{P}_x(Z \in \mathcal{B} \mid T_0 > t).$$

When it exists, this limit procedure defines a diffusion that never reaches 0.

Here, we want to study quasi-stationarity for diffusions that arise as scaling limits of general birth–death processes. More precisely, let $(Z_t^N)_N$ be a sequence of birth–death processes renormalized by the weight N^{-1}, hence taking values in $N^{-1}\mathbb{N}$. Assume that their birth and death rates from state x are respectively equal to $b_N(x)$ and $d_N(x)$, and $b_N(0) = d_N(0) = 0$, ensuring that the state 0 is absorbing. We also assume that for each N and for some constant B_N, $b_N(x) \leq (x+1)B_N,\ x \geq 0$ and that there exist a non-negative constant γ and a function $h \in C^1([0, +\infty))$, $h(0) = 0$, such that

$$\forall x \in (0, +\infty): \quad \lim_{N\to\infty} \frac{1}{N}(b_N(x) - d_N(x)) = h(x) \quad ; \quad \lim_{N\to\infty} \frac{1}{2N^2}(b_N(x) + d_N(x)) = \gamma. \tag{1.2}$$

Assuming further that $(Z_0^N)_N$ converges as $N \to \infty$ (we thus model the evolution of a population whose size is of order N), we may prove, following Lipow [18] or using the techniques of Joffe-Métivier [15], that the sequence $(Z_t^N, t \geq 0)$ converges weakly to a continuous limit $(Z_t, t \geq 0)$. The parameter γ can be interpreted as a demographic parameter describing the ecological timescale. There is a main qualitative difference depending on whether $\gamma = 0$ or not.

If $\gamma = 0$, then the limit Z is a deterministic solution to the dynamical system $\dot{Z}_t = h(Z_t)$. Since $h(0) = 0$, the state 0 is always an equilibrium, but it can be unstable, and in many usual cases, one can prove the existence of a non-trivial asymptotically stable equilibrium.

If $\gamma > 0$, the sequence $(Z^N)_N$ converges in law to the process Z, solution to the following stochastic differential equation

$$dZ_t = \sqrt{\gamma Z_t} dB_t + h(Z_t)dt. \tag{1.3}$$

The acceleration of the ecological process has generated the noise. The function h thus models the growth function of the population. Note that $h'(0^+)$ is the mean *per capita* growth rate for *small* populations. The fact that it is finite is mathematically convenient, and biologically reasonable. Since $h(0) = 0$, the population undergoes no immigration, so that 0 is an absorbing state. One can easily check that when time goes to infinity, either Z goes to ∞ or is absorbed at 0.

When $h \equiv 0$, we get the classical Feller diffusion, so we call *generalized Feller diffusions* the diffusions driven by (1.3). Notice that when h is linear, we get the general continuous-state branching process with continuous paths. When h is quadratic, we get the logistic Feller diffusion [9, 16].

The latter comes from one of the most simple and familiar biological examples. Indeed, suppose that $b_N(x) = (\gamma N + \lambda)Nx$ and $d_N(x) = (\gamma N + \mu)Nx + \frac{c}{N}Nx(Nx-1)$. The quadratic term in the death rate describes the interaction between individuals. Remark that since the number of individuals is of order N, the biomass of each individual is of order N^{-1}, which explains the value c/N of the interaction coefficient. In this case, $(Z_N)_N$ converges when $\gamma = 0$ to a solution of the famous logistic equation

$$\dot{z} = (\lambda - \mu)z - cz^2.$$

The parameter $r = \lambda - \mu$ describes the intrinsic growth rate of the population. It is easily checked that when $r > 0$, this equation has two equilibria, 0 which is unstable, and r/c (called carrying capacity) which is asymptotically stable. When $\gamma \neq 0$, the sequence $(Z_N)_N$ converges to the logistic Feller equation

$$dZ_t = \sqrt{\gamma Z_t} dB_t + (rZ_t - cZ_t^2)dt.$$

It is easy to see (e.g. by stochastic domination) that the process Z becomes extinct in finite time, and we will show in Section 7 that the absorption time from infinity has exponential moments.

Other famous ecological examples concerned by our results are (i) the linear Malthusian case, where the individual growth rate r is negative and $c = 0$ (subcritical branching process); (ii) dynamics governed by an individual growth rate of the form $r(\frac{z}{K_0} - 1)(1 - \frac{cz}{r})$, where the population size has a threshold K_0 to growth, below which it cannot take over. Observe that in this last case, the individual growth rate is no longer a monotonically decreasing function of the population size k and instead shows an Allee effect, i.e. a positive density-dependence for certain ranges of density, corresponding to cooperation in natural populations.

1.2. **The growth function.** As we said, referring to the previous construction of the generalized Feller diffusion (1.3), $h(z)$ can be viewed as the expected growth rate of a population of size z and $h(z)/z$ as the mean *per capita* growth rate. Indeed, $h(z)$ informs of the resulting action of density upon the growth of the population, and $h(z)/z$ indicates the resulting action of density upon each individual. In the range of densities z where $h(z)/z$ increases with z, the most important interactions are of the *cooperative* type, one speaks of *positive* *density-dependence*. On the contrary, when $h(z)/z$ decreases with z, the interactions are of the *competitive* type, and density-dependence is said to be *negative*. In many cases, such as the logistic one, the limitation of resources forces harsh competition in large populations, so that, as $z \to \infty$, $h(z)/z$ is negative decreasing, and in particular $h(z)$ goes to $-\infty$. The shape of h at infinity determines the long time behavior of the diffusion Z.

Let us examine the case of the continuous-state branching process, where $h(z) = rz$. In the subcritical case $r < 0$ it is shown in [17] that there are infinitely many QSD's, but no QSD when $r = 0$ (critical case). When $r > 0$ (supercritical case), the Yaglom limit is meaningless, since the conditioning to be non extinct at time t forces the process to go to infinity as $t \to \infty$. However, the process Z conditioned on extinction is exactly the subcritical branching process with $h(z) = -rz$, which allows to study quasi-stationarity of the process, provided it is first conditioned on its *eventual* extinction.

Now, let h be a general function. If we assume that there exists $h_\infty := \lim\limits_{z\to\infty} h(z)$ with $h_\infty \in [-\infty, +\infty]$, we can distinguish three cases, exactly as in the case of the branching process. If $h_\infty = -\infty$, then the process is almost surely absorbed at 0 at a finite time, and we will be able to prove the existence of the Yaglom limit. We will also show that when $1/h$ is integrable at $+\infty$, then the diffusion Z comes down from infinity, and the Yaglom limit is the *unique* QSD. If $h_\infty \in (-\infty, +\infty)$ the case is critical, and nothing seems to be known about the existence of any QSD. When $h_\infty = +\infty$, then the absorption at 0 is not certain, but we can go back to the previous case by conditioning the process on its eventual extinction. Indeed, the following statement ensures that, similarly to the branching case, conditioning on extinction roughly amounts to *replacing h with $-h$*.

Proposition 1.1. *Assume that Z is given by (1.3), where $h \in C^1([0, +\infty))$, $h(0) = 0$, $\lim_{x\to\infty} h(x) = +\infty$ and h satisfies the technical assumption $\lim_{x\to\infty} xh'(x)h(x)^{-2} = 0$. Define $u(x) := \mathbb{P}_x(\lim\limits_{t\to\infty} Z_t = 0)$ and let Y be the diffusion Z conditioned on eventual extinction. Then Y is given by*

$$dY_t = \sqrt{\gamma Y_t} dB_t + \left(h(Y_t) + \gamma Y_t \frac{u'(Y_t)}{u(Y_t)} \right) dt.$$

In addition

$$h(y) + \gamma y \frac{u'(y)}{u(y)} \sim_{y\to\infty} -h(y).$$

The proof of this result is postponed to the appendix.

From now on, we make the following assumptions on h.

Definition 1.2. *(HH) We say that h verifies the condition (HH) if*

$$(i) \lim_{x\to\infty} \frac{h(x)}{\sqrt{x}} = -\infty, \qquad (ii) \lim_{x\to\infty} \frac{xh'(x)}{h(x)^2} = 0.$$

In particular (HH) holds for any subcritical branching diffusion, and any logistic Feller diffusion. Concerning Assumption (i), the fact that h goes to $-\infty$ indicates strong competition in large populations resulting in negative growth rates (as in the logistic case). On the other hand, in the spirit of the previous discussion, (i) can be turned into $\lim_{x\to\infty} \frac{h(x)}{\sqrt{x}} = \pm\infty$, provided that the population is *conditioned to eventually become extinct*. Assumption (ii) is fulfilled for most classical biological models, and it appears as a mere technical condition.

We may state one of the main results of this work.

Theorem 1.3. *Let Z be the solution of (1.3). If h satisfies Assumption (HH), then for all initial laws with bounded support, the law of Z_t conditioned on $\{Z_t \neq 0\}$ converges exponentially fast to a probability measure ν, called the Yaglom limit.*
The law $\mathbb{Q}_x$ of the process Z starting from x and conditioned to be never extinct exists and defines the so-called Q-process. This process converges, as $t \to \infty$, in distribution, to its unique invariant probability measure. This probability measure is absolutely continuous w.r.t. ν with a nondecreasing Radon-Nikodym derivative.
If in addition, the following integrability condition is satisfied

$$\int_1^\infty \frac{dx}{-h(x)} < \infty,$$

then Z comes down from infinity and the convergence of the conditional one-dimensional distributions holds for all initial laws, so that the Yaglom limit ν is the unique quasi-stationary distribution.

1.3. **Outline.** Starting from (1.3), the change of variable $x = 2\sqrt{z/\gamma}$ yields a new diffusion process X of Kolmogorov type, that is a drifted Brownian motion. Of course the study of QSD for the initial Z reduces to the study of QSD for X up to a change of variable in the corresponding QSD's.
For such Kolmogorov diffusions, the study of QSD is a long standing problem starting with Mandl's paper [20] in 1961, and developed by many authors (see in particular [5, 21, 28]). All these works assume Mandl's conditions. Mandl's conditions are not satisfied in the situation described above, since in particular the drift of X behaves like $-1/2x$ near 0. For instance in the logistic case the drift is given by $-q(x)$ where

$$q(x) = \frac{1}{2x} - \frac{rx}{2} + \frac{c\gamma x^3}{8}\,.$$

It is worth noticing that the behavior of q at infinity also violates Mandl's conditions, since $1/q$ is integrable at $+\infty$.
This unusual behavior is due to the square root in the diffusive term in (1.3), and prevents us from using earlier results on QSD's of solutions of Kolmogorov equation. Hence we are led to develop new techniques allowing to cope with this situation.
In Section 2 we start with the study of a general Kolmogorov diffusion process on the half line and introduce the hypothesis (H1) which is equivalent to reaching 0 in finite time with probability 1. Then we introduce its symmetric measure μ, describe the Girsanov transform and show how to use it in order to obtain $\mathbb{L}^2(d\mu)$ estimates for the heat kernel. In the present paper we work in $\mathbb{L}^p(d\mu)$ spaces rather than $\mathbb{L}^p(dx)$, because it simplifies the presentation of the spectral theory.
This spectral theory is done in Section 3, where we introduce the hypothesis (H2) ensuring the discreteness of the spectrum.
Section 4 gives some sharper properties on the eigenfunctions defined in the previous section, using in particular properties of the Dirichlet heat kernel. Either of hypotheses (H3) or (H4) ensures that the eigenfunctions belong to $\mathbb{L}^1(d\mu)$.
Section 5 contains the proofs of the existence of the Yaglom limit as well as the exponential decay to equilibrium, under hypotheses (H1) and (H2), along with (H3) or (H4).

In Section 7 we introduce condition (H5) which is equivalent to the existence of an entrance law at $+\infty$. Condition (H5) is satisfied for the biological model as soon as $\int^{+\infty} -1/h < +\infty$. Under (H5), the repelling force at infinity imposes to the process starting from infinity to reach any finite interval in finite time. The process is then said to 'come down from infinity'. We show that the process comes down from infinity if and only if the Yaglom limit is the conditional limit distribution starting from any initial law. It is then the *unique* QSD. We do not know if this relationship between uniqueness of QSD's and return from $+\infty$ has been noticed in any previous study.

Appendix A is devoted to the proofs of Proposition 1.1 and Theorem 1.3. Appendix B contains the proof of a intermediate technical lemma.

2. **One dimensional diffusion processes on the positive half line.**

We consider a one dimensional drifted Brownian motion on $(0, +\infty)$

$$dX_t \,=\, dB_t \,-\, q(X_t)\, dt \quad , \quad X_0 = x > 0 \tag{2.1}$$

where q is defined and C^1 on $(0, +\infty)$. In particular q is allowed to explode at the origin. A pathwise unique solution of (2.1) thus exists up to the explosion time $\tau = T_0 \wedge T_{+\infty}$ where T_y is the first time the process hits y. The law of the process starting from x will be denoted by $\mathbb{P}_x$. In the sequel, we shall often make an abuse of notation, writing $X_.$ instead of $\omega_.$ for the canonical path.

Define

$$Q(x) = \int_1^x 2q(u)du\,, \tag{2.2}$$

$$\Lambda(x) = \int_1^x e^{Q(y)}\, dy \qquad \text{and} \qquad \kappa(x) = \int_1^x e^{Q(y)} \left(\int_1^y e^{-Q(z)}\, dz\right) dy\,. \tag{2.3}$$

We shall from now on assume

Hypothesis (H1):

$$\text{for all } x > 0\ , \quad \mathbb{P}_x(\tau = T_0 < +\infty) = 1\,. \tag{2.4}$$

It is well known (see e.g. [14] Theorem 3.2 p.450) that (2.4) holds if and only if

$$\Lambda(+\infty) = +\infty \ \text{ and } \ \kappa(0^+) < +\infty\,. \tag{2.5}$$

Example 2.1. The main cases that we are interested in are the following ones.

(1) Consider the generalized Feller diffusion defined in (1.3)

$$dZ_t = \sqrt{\gamma Z_t}\, dB_t + h(Z_t)dt \ \ , \ Z_0 = z > 0\,.$$

with $h(0) \geq 0$. If we define $X_t = 2\sqrt{Z_t/\gamma}$ then

$$dX_t = dB_t + \frac{1}{X_t}\left(\frac{2}{\gamma}\, h\left(\frac{\gamma X_t^2}{4}\right) - \frac{1}{2}\right) dt \ \ , \ X_0 = x = 2\sqrt{z/\gamma} > 0\,,$$

so that $X_.$ is a drifted Brownian motion as before. Of particular interest is the case when h vanishes at the origin. In this case, $Q(x)$ behaves like $\log(x)$ near 0 hence $\kappa(0^+) < +\infty$. Notice that if $z = 0$, $Z_t = 0$ for all t is then the unique solution of (1.3) (see [14]), so that as soon as the diffusion reaches 0 it stays at 0.

The logistic Feller diffusion corresponds to $h(z) = rz - cz^2$ for some constants c and r, hence to

$$q(x) = \frac{1}{2x} - \frac{rx}{2} + \frac{c\gamma x^3}{8}\,.$$

It is easily seen that (2.5) is satisfied in this case provided $c > 0$ or $c = 0$ and $r < 0$.

Conversely, starting with (2.1), define $Z_t = X_t^2$. It holds

$$dZ_t = 2\sqrt{Z_t}\, dB_t + h(Z_t)\, dt$$

where

$$h(z) = 1 - 2\sqrt{z}q(\sqrt{z})\,.$$

Hence $h(0) = 0$ if and only if $\lim_{z\to 0} zq(z) = 1/2$. $\diamondsuit$

(2) $\lim_{x\to 0^+} q(x)$ exists and is finite, hence $\kappa(0^+) < +\infty$.

We shall now discuss some properties of the law of $X_.$ up to T_0. The first result is a Girsanov type result.

Proposition 2.2. *Assume (H1). For any Borel bounded function F defined on $\Omega = C^0([0,t],(0,+\infty))$ it holds*

$$\mathbb{E}_x\left[F(\omega)\,\mathbb{I}_{t<T_0(\omega)}\right] = \mathbb{E}^{\mathbb{W}_x}\left[F(\omega)\,\mathbb{I}_{t<T_0(\omega)}\,\exp\left(\frac{1}{2}\,Q(x) - \frac{1}{2}\,Q(\omega_t) - \frac{1}{2}\int_0^t (q^2 - q')(\omega_s)ds\right)\right]$$

where $\mathbb{E}^{\mathbb{W}_x}$ denotes the expectation w.r.t. the Wiener measure starting from x.

Proof. It is enough to show the result for F non-negative and bounded. Let $\varepsilon \in (0,1)$ and $\tau_\varepsilon = T_\varepsilon \wedge T_{1/\varepsilon}$. Choose some ψ_ε which is a non-negative C^∞ function with compact support included in $]\varepsilon/2, 2/\varepsilon[$ such that $\psi_\varepsilon(u) = 1$ if $\varepsilon \le u \le 1/\varepsilon$. For all x such that $\varepsilon \le x \le 1/\varepsilon$ the law of the diffusion (2.1) coincides up to τ_ε with the law of a similar diffusion process X^ε obtained by replacing q with the cutoff $q_\varepsilon = q\psi_\varepsilon$. For the latter we may apply Novikov criterion ensuring that the law of X^ε is given via Girsanov formula. Hence

$$\begin{aligned}
\mathbb{E}_x\left[F(\omega)\,\mathbb{I}_{t<\tau_\varepsilon(\omega)}\right] &= \mathbb{E}^{W_x}\left[F(\omega)\,\mathbb{I}_{t<\tau_\varepsilon(\omega)}\,\exp\left(\int_0^t -q_\varepsilon(\omega_s)d\omega_s - \frac{1}{2}\int_0^t (q_\varepsilon)^2(\omega_s)ds\right)\right]\\
&= \mathbb{E}^{W_x}\left[F(\omega)\,\mathbb{I}_{t<\tau_\varepsilon(\omega)}\,\exp\left(\int_0^t -q(\omega_s)d\omega_s - \frac{1}{2}\int_0^t q^2(\omega_s)ds\right)\right]\\
&= \mathbb{E}^{W_x}\left[F(\omega)\,\mathbb{I}_{t<\tau_\varepsilon(\omega)}\,\exp\left(\frac{1}{2}\,Q(x) - \frac{1}{2}\,Q(\omega_t) - \frac{1}{2}\int_0^t (q^2 - q')(\omega_s)ds\right)\right]
\end{aligned}$$

integrating by parts the stochastic integral. But $\mathbb{I}_{t<\tau_\varepsilon}$ is non-decreasing in ε and converges almost surely to $\mathbb{I}_{t<T_0}$ both for $\mathbb{P}_x$ (thanks to (H1)) and $\mathbb{W}_x$. Indeed, almost surely,

$$\lim_{\varepsilon\to 0} X_{\tau_\varepsilon} = \lim_{\varepsilon\to 0} X_{T_\varepsilon} = \lim_{\varepsilon\to 0} \varepsilon = 0$$

so that $\lim_{\varepsilon\to 0} T_\varepsilon \ge T_0$. But $T_\varepsilon \le T_0$ yielding the equality. It remains to use Lebesgue monotone convergence theorem to finish the proof. $\square$

The next theorem is inspired by the calculation in Theorem 3.2.7 of [25]. It will be useful to introduce the following measure defined on $(0,+\infty)$

$$\mu(dy) = e^{-Q(y)}\, dy\,. \tag{2.6}$$

It is a non-negative but non necessarily bounded measure.

Theorem 2.3. *Assume (H1). For all $x > 0$ and all $t > 0$ there exists some density $r(t,x,.)$ that verifies*

$$\mathbb{E}_x[f(X_t)\, \mathbb{1}_{t<T_0}] = \int_0^{+\infty} f(y)\, r(t,x,y)\, \mu(dy)$$

for all bounded Borel f.
If in addition there exists some $C > 0$ such that $q^2(y) - q'(y) \geq -C$ for all $y > 0$, then for all $t > 0$ and all $x > 0$,

$$\int_0^{+\infty} r^2(t,x,y)\, \mu(dy) \;\leq (1/2\pi t)^{\frac{1}{2}}\, e^{Ct}\, e^{Q(x)}\,.$$

Proof. Define

$$G(\omega) = \mathbb{1}_{t<T_0(\omega)}\, \exp\left(\frac{1}{2}\, Q(\omega_0) - \frac{1}{2}\, Q(\omega_t) - \frac{1}{2}\int_0^t (q^2 - q')(\omega_s) ds\right)\,.$$

Denote by

$$e^{-v(t,x,y)} = (2\pi t)^{-\frac{1}{2}}\, \exp\left(-\frac{(x-y)^2}{2t}\right)$$

the density at time t of the Brownian motion starting from x. According to Proposition 2.2 we have

$$\begin{aligned}
\mathbb{E}_x[f(X_t)\, \mathbb{1}_{t<T_0}] &= \mathbb{E}^{\mathbb{W}_x}[f(\omega_t)\mathbb{E}^{\mathbb{W}_x}[G|\omega_t]] \\
&= \int f(y)\, \mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]\, e^{-v(t,x,y)}\, dy \\
&= \int_0^{+\infty} f(y)\, \mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]\, e^{-v(t,x,y)+Q(y)}\, \mu(dy)\,,
\end{aligned}$$

because $\mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y] = 0$ if $y \leq 0$. In other words, the law of X_t restricted to non extinction has a density with respect to μ given by

$$r(t,x,y) = \mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]\, e^{-v(t,x,y)+Q(y)}\,.$$

Hence

$$\begin{aligned}
\int_0^{+\infty} r^2(t,x,y)\, \mu(dy) &= \int \left(\mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]\, e^{-v(t,x,y)+Q(y)}\right)^2\, e^{-Q(y)+v(t,x,y)}\, e^{-v(t,x,y)}\, dy \\
&= \mathbb{E}^{\mathbb{W}_x}\left[e^{-v(t,x,\omega_t)+Q(\omega_t)}\, \left(\mathbb{E}^{\mathbb{W}_x}[G|\omega_t]\right)^2\right] \\
&\leq \mathbb{E}^{\mathbb{W}_x}\left[e^{-v(t,x,\omega_t)+Q(\omega_t)}\, \mathbb{E}^{\mathbb{W}_x}[G^2|\omega_t]\right] \\
&\leq e^{Q(x)}\, \mathbb{E}^{\mathbb{W}_x}\left[\mathbb{1}_{t<T_0(\omega)}\, e^{-v(t,x,\omega_t)}\, e^{-\int_0^t (q^2-q')(\omega_s) ds}\right]\,,
\end{aligned}$$

where we have used Cauchy-Schwarz's inequality. Since $e^{-v(t,x,.)} \leq (1/2\pi t)^{\frac{1}{2}}$ the proof is completed. $\square$

Remark 2.4. It is interesting to discuss a little bit the conditions we have introduced.

(1) Since q is assumed to be regular, the condition $q^2 - q'$ bounded from below has to be checked near $+\infty$ or near 0.

(2) Consider the behavior near $+\infty$. Let us show that if $\liminf_{y\to+\infty}(q^2(y) - q'(y)) = -\infty$ then $\limsup_{y\to+\infty}(q^2(y) - q'(y)) > -\infty$ i.e. the drift q is strongly oscillating. Indeed, assume that $q^2(y) - q'(y) \to -\infty$ as $y \to +\infty$. It follows that $q'(y) \to +\infty$, hence $q(y) \to +\infty$. For y large enough we may thus write $q(y) = e^{u(y)}$ for some u going to infinity at infinity. So $e^{2u(y)}(1 - u'(y)e^{-u(y)}) \to -\infty$ implying that $u'e^{-u} \geq 1$ near infinity. Thus if $g = e^{-u}$ we have $g' \leq -1$ i.e. $g(y) \to -\infty$ as $y \to +\infty$ which is impossible since g is non-negative.

(3) If X comes from a generalized Feller diffusion, we have

$$q(y) = \frac{1}{y}\left(\frac{1}{2} - \frac{2}{\gamma}h\left(\frac{\gamma y^2}{4}\right)\right).$$

Hence, since h is of class C^1, and under the absorption assumption $h(0) = 0$, $q^2(y) - q'(y)$ behaves near 0 like $3/4y^2$ so that $q^2 - q'$ is bounded from below near 0 (see Appendix for further conditions fulfilled by h to get the same result near ∞). $\diamondsuit$

3. $\mathbb{L}^2$ and spectral theory of the diffusion process.

Theorem 2.3 shows that for a large family of initial laws, the law of X_t before extinction has a density belonging to $\mathbb{L}^2(\mu)$. This measure μ is natural since the kernel of the killed process is symmetric in $\mathbb{L}^2(\mu)$, which allow us to use spectral theory.

Let $C_0^\infty((0,+\infty))$ be the vector space of infinitely differentiable functions on $(0,+\infty)$ with compact support. We denote

$$\langle f, g\rangle_\mu = \int_0^{+\infty} f(u)g(u)\mu(du)\,.$$

Consider the symmetric form

$$\mathcal{E}(f,g) = \langle f', g'\rangle_\mu \quad , \quad D(\mathcal{E}) = C_0^\infty((0,+\infty)). \tag{3.1}$$

This form is Markovian and closable. The proof of the latter assertion is similar to the one of Theorem 2.1.4 in [11] just replacing the real line by the positive half line. Its smallest closed extension, again denoted by $\mathcal{E}$, is thus a Dirichlet form which is actually regular and local. According to the theory of Dirichlet forms (see [11] or [12]) we thus know that

- there exists a non-positive self adjoint operator L on $\mathbb{L}^2(\mu)$ with domain $D(L)$ such that for all f and g in $C_0^\infty((0,+\infty))$ the following holds (see [11] Theorem 1.3.1)

$$\mathcal{E}(f,g) = -2\int_0^{+\infty} f(u)\,Lg(u)\,\mu(du) \;=\; -2\,\langle f, Lg\rangle_\mu\,. \tag{3.2}$$

We point out that for $g \in C_0^\infty((0,+\infty))$,

$$Lg = \frac{1}{2}\,g'' - qg'.$$

- L is the generator of a strongly continuous symmetric semi-group of contractions on $\mathbb{L}^2(\mu)$ denoted by $(P_t)_{t\geq 0}$. This semi-group is (sub)-Markovian, i.e. $0 \leq P_t f \leq 1$ μ a.e. if $0 \leq f \leq 1$. (see [11] Theorem 1.4.1)
- There exists a unique μ-symmetric Hunt process with continuous sample paths (i.e. a diffusion process) up to its explosion time τ whose Dirichlet form is $\mathcal{E}$. (see [11] Theorem 6.2.2)

The last assertion in particular implies that for μ quasi-all $x > 0$ one can find a probability measure $\mathbb{Q}_x$ on $C^0(\mathbb{R}^+, (0, +\infty))$ such that for all $f \in C_0^\infty((0, +\infty))$,

$$f(\omega_{t\wedge\tau}) - f(x) - \int_0^{t\wedge\tau} Lf(\omega_s)ds$$

is a local martingale with quadratic variation $\int_0^{t\wedge\tau} |f'|^2(\omega_s)ds$. Due to our hypotheses we know that this martingale problem admits a unique solution given by $\mathbb{P}_x$.

In other words, the semi-group $P_t f(x) = \mathbb{E}[f(X_t^x)\, \mathbb{1}_{t<T_0}]$ defined for all smooth and compactly supported f, X^x being the solution of (2.1) starting from x, extends to a symmetric sub-Markovian semi-group of contractions on $\mathbb{L}^2(\mu)$.

Let $(E_\lambda : \lambda \geq 0)$ be the spectral family of $-L$. We can restrict to $\lambda \geq 0$ because $-L$ is non-negative. Then $\forall\, t \geq 0$, f, $g \in \mathbb{L}^2(\mu)$,

$$\int P_t f\, g\, d\mu = \int_0^{+\infty} e^{-\lambda t}\, d\langle E_\lambda f, g\rangle_\mu\,, \tag{3.3}$$

see e.g. [11] p.16 for the definitions.
Note that for $f \in \mathbb{L}^2(\mu)$ and all closed interval $K \in (0, +\infty)$,

$$\begin{aligned}
\int (P_t f)^2 d\mu &= \int (P_t(f\mathbb{1}_K + f\mathbb{1}_{K^c}))^2 d\mu \\
&\leq 2\int (P_t(f\mathbb{1}_K))^2 d\mu + 2\int (P_t(f\mathbb{1}_{K^c}))^2 d\mu \\
&\leq 2\int (P_t(f\mathbb{1}_K))^2 d\mu + 2\int (f\mathbb{1}_{K^c})^2 d\mu\,.
\end{aligned}$$

We may choose K large enough in order that the second term in the latter sum is bounded by ε. Similarly we may approximate $f\mathbb{1}_K$ in $\mathbb{L}^2(\mu)$ by $\tilde{f}\mathbb{1}_K$ for some continuous and bounded $\tilde{f}$, up to ε (uniformly in t). Now, thanks to (H1), that is $\mathbb{P}_x(\lim_{t\to\infty} X_t = 0) = 1$, we know that $P_t(\tilde{f}\mathbb{1}_K)(x)$ goes to 0 as t goes to infinity for any x. Since

$$\int (P_t(\tilde{f}\mathbb{1}_K))^2 d\mu = \int_K \tilde{f}\, P_{2t}(\tilde{f}\mathbb{1}_K)\, d\mu\,,$$

we may apply Lebesgue bounded convergence theorem and conclude that $\int (P_t(\tilde{f}\mathbb{1}_K))^2 d\mu \to 0$ as $t \to +\infty$. Hence, we have shown that,

$$\text{for any } f \in \mathbb{L}^2(\mu), \qquad \int (P_t f)^2 d\mu \to 0 \text{ as } t \to +\infty\,. \tag{3.4}$$

Now we shall introduce the main assumption for the spectral aspect of the study.

Definition 3.1. ***(H2)*** *We say that hypothesis (H2) holds if* $\inf_{y>0} q^2(y) - q'(y) = -C$ $0 < C < +\infty$ *and if in addition*

$$\lim_{y\to+\infty} q^2(y) - q'(y) = +\infty\,. \tag{3.5}$$

Proposition 3.2. *If (H2) holds then* $|q(x)|$ *tends to infinity as* x *grows to infinity, and* $q^-(x) = (-q(x)) \vee 0$ *or* $q^+(x) = q(x) \vee 0$ *tend to 0 as* $x \downarrow 0$. *Moreover if (H1) holds then* $q(x) \to +\infty$, *as* $x \to +\infty$.

Proof. Since $q^2 - q'$ tends to $+\infty$, as $x \to +\infty$, q does not change sign for large x. Indeed, if q is bounded near infinity, we arrive at a contradiction because q' tends to $-\infty$ and therefore q tends to $-\infty$ as well. So we may assume that q is unbounded and has constant sign for large x. If $\liminf_{x\to\infty} |q(x)| = a < \infty$ then we can construct a sequence $x_n \to \infty$ of local maxima, or local minima of q whose value $|q(x_n)| < a + 1$, but then $q^2(x_n) - q'(x_n)$ stays bounded, which is a contradiction.
Now we prove that $q^-(x)$ or $q^+(x)$ tend to 0 as $x \downarrow 0$. In fact, assume there exist $\epsilon > 0$ and a sequence $x_n \downarrow 0$ such that $q(x_{2n}) = -\epsilon, q(x_{2n+1}) = \epsilon$. Then we can construct another sequence $z_n \downarrow 0$ such that $|q(z_n)| \le \epsilon$ and $q'(z_n) \to \infty$, contradicting (H2).
Finally, assume (H1) and assume $q(x) \le -1$ for all $x > x_0$. Then for all t

$$\mathbb{P}_{x_0+1}(T_0 > t) \ge \mathbb{P}_{x_0+1}(T_{x_0} > t) \ge \mathbb{P}_{x_0+1}(T_{x_0} = \infty) \ge 1 - e^{-2},$$

where the last quantity comes from the Brownian motion with constant drift 1. This contradicts (H1), and therefore under (H2) we have $q(x) \to +\infty$ as $x \to +\infty$. □

We may now state

Theorem 3.3. *If (H2) is satisfied,* $-L$ *has a purely discrete spectrum* $0 \le \lambda_1 < \lambda_2 < \ldots$. *Furthermore each* λ_i *is associated to a unique (up to a multiplicative constant) eigenfunction* η_i *of class* $C^2((0,\infty))$ *and they satisfy the ODE*

$$\frac{1}{2}\eta_i'' - q\eta_i' = -\lambda_i\eta_i. \tag{3.6}$$

$(\eta_k)_{k\ge1}$ *is an orthonormal basis of* $\mathbb{L}^2(\mu)$, η_1 *can be chosen nonnegative and if so this eigenfunction is strictly positive in* $(0,\infty)$.
For $g \in \mathbb{L}^2(d\mu)$,

$$P_t g = \sum_k e^{-\lambda_k t}\langle \eta_k, g\rangle_\mu \eta_k, \quad \text{in} \quad \mathbb{L}^2(d\mu),$$

and then for $f, g \in \mathbb{L}^2(d\mu)$,

$$\lim_{t\to+\infty} e^{\lambda_1 t}\langle g, P_t f\rangle_\mu = \langle \eta_1, f\rangle_\mu \, \langle \eta_1, g\rangle_\mu\,.$$

If in addition (H1) holds then $\lambda_1 > 0$.

Proof. For $f \in \mathbb{L}^2(dx)$ define $\tilde{P}_t(f) = e^{-Q/2}P_t(f\, e^{Q/2})$ which exists since $f\, e^{Q/2} \in \mathbb{L}^2(d\mu)$. $(\tilde{P}_t)_{t\ge0}$ is then a strongly continuous semi-group in $\mathbb{L}^2(dx)$, whose generator $\tilde{L}$ coincides on $C_0^\infty((0,+\infty))$ with $\frac{1}{2}\, d^2/dx^2 - \frac{1}{2}\,(q^2 - q')$. The spectral theory of such a Schrödinger operator on the line (or the half line) is well known, but here the potential $v = (q^2 - q')/2$ does not necessarily belong to $\mathbb{L}^\infty_{loc}$ near 0 as it is generally assumed. We shall use [3] chapter 2.

First we follow the proof of Theorem 3.1 in [3]. Since we have assumed that v is bounded from below by $-C$, we may consider $H = \tilde{L} - (C+1)$, i.e. replace v by $v + C + 1 = w \geq 1$, hence translate the spectrum. But since

$$-(Hf, f) = -\int_0^{+\infty} Hf(u)\, f(u)\, du = \int_0^{+\infty} \left(|f'(u)|^2/2 + w(u)f^2(u)\right) du \geq (f, f)\,, \tag{3.7}$$

H has a bounded inverse operator. Hence the spectrum of H (hence of $\tilde{L}$) will be discrete as soon as H^{-1} is a compact operator, i.e. as soon as $M = \{f \in D(H)\,;\, -(Hf, f) \leq 1\}$ is relatively compact. But this is shown in [3] when w is locally bounded, in particular bounded near 0. If w goes to infinity at 0, the situation is even better since our set M is included into the corresponding one with $w \approx 1$ near the origin, which is relatively compact thanks to the asymptotic behavior of v. The conclusion of Theorem 3.1 in [3] is thus still true in our situation, i.e. the spectrum is discrete.

But the discussion in Section 2.3 of [3] pp.59-69 is only concerned with the asymptotic behavior (near infinity) of the solutions of $f'' - 2wf = 0$, hence all the discussion applies to our case. All eigenvalues are thus simple (Proposition 3.3 in [3]), and of course the corresponding set of eigenfunctions $(\psi_k)_{k\geq 1}$ is an orthonormal basis of $\mathbb{L}^2(dx)$.

The system $(e^{Q/2}\,\psi_k)_{k\geq 1}$ is thus an orthonormal basis of $\mathbb{L}^2(d\mu)$, each $\eta_k = e^{Q/2}\,\psi_k$ being an eigenfunction of L. We can choose them to be $C^2((0,\infty))$ and they satisfy (3.6).

For every $t > 0$, and for every $g, f \in L^2(d\mu)$ we have

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \langle \eta_k, g\rangle_\mu \langle \eta_k, f\rangle_\mu = \langle g, P_t f\rangle_\mu = \int\int g(x)f(y)r(t,x,y)e^{-Q(x)-Q(y)}dxdy.$$

In addition if g and f are nonnegative we get

$$0 \leq \lim_{t\to+\infty} e^{\lambda_1 t}\langle g, P_t f\rangle_\mu = \langle \eta_1, f\rangle_\mu \,\langle \eta_1, g\rangle_\mu\,.$$

It follows that $\langle \eta_1, f\rangle_\mu$ and $\langle \eta_1, g\rangle_\mu$ have the same sign. Changing η_1 into $-\eta_1$ if necessary, we may assume that $\langle \eta_1, f\rangle_\mu \geq 0$ for any non-negative f, hence $\eta_1 \geq 0$. Since $P_t\eta_1(x) = e^{-\lambda_1 t}\eta_1(x)$ and η_1 is continuous and not trivial, we deduce that $\eta_1(x) > 0$ for all $x > 0$.

Since L is non-positive, $\lambda_1 \geq 0$. Now assume that (H1) holds. If $g \in \mathbb{L}^2(d\mu)$ then $g = \sum_k \langle g, \eta_k\rangle_\mu\, \eta_k$. Hence

$$\lim_{t\to+\infty} \langle P_t g, P_t g\rangle_\mu = \lim_{t\to+\infty} e^{-2\lambda_1 t}\langle g, \eta_1\rangle_\mu^2 = 0$$

thanks to (3.4) showing that $\lambda_1 > 0$. $\square$

We are moreover able to obtain a pointwise representation of the density r.

Proposition 3.4. *Uniformly on compact sets of* $(0,\infty)\times(0,\infty)\times(0,\infty)$*, we have*

$$r(t,x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\eta_k(x)\eta_k(y). \tag{3.8}$$

Therefore on compact sets of $(0,\infty)\times(0,\infty)$ *we have*

$$\lim_{t\to\infty} e^{\lambda_1 t} r(t,x,y) = \eta_1(x)\eta_1(y). \tag{3.9}$$

Proof. For every smooth function g compactly supported on $(0,\infty)$ we have

$$\sum_{k=1}^{n} e^{-\lambda_k t}\langle \eta_k, g\rangle_\mu^2 \leq \sum_{k=1}^{\infty} e^{-\lambda_k t}\langle \eta_k, g\rangle_\mu^2 = \int\int g(x)g(y)r(t,x,y)e^{-Q(x)-Q(y)}dxdy.$$

Then using the regularity of η_k and r we obtain

$$\sum_{k=1}^{n} e^{-\lambda_k t}\eta_k(x)^2 \leq r(t,x,x).$$

We also have that the series $\sum\limits_{k=1}^{\infty} e^{-\lambda_k t}\eta_k(x)^2$ converges pointwise, which by Cauchy Schwarz implies the pointwise absolute convergence of $\zeta(t,x,y) := \sum\limits_{k=1}^{\infty} e^{-\lambda_k t}\eta_k(x)\eta_k(y)$ and the bound for all n

$$\sum_{k=1}^{n} e^{-\lambda_k t}|\eta_k(x)\eta_k(y)| \leq \sqrt{r(t,x,x)}\sqrt{r(t,y,y)}.$$

Using Harnack inequality, compactness and the dominated convergence Theorem we obtain that for all Borel functions g, f with compact support in $(0,\infty)$

$$\int\int g(x)f(y)\zeta(t,x,y)e^{-Q(x)-Q(y)}dxdy = \int\int g(x)f(y)r(t,x,y)e^{-Q(x)-Q(y)}dxdy.$$

Therefore $\zeta(t,x,y) = r(t,x,y)$ $dxdy$-a.s., which proves the almost sure version of (3.8).

On the other hand, since η_k are smooth eigenfunctions we get the pointwise equality

$$\begin{aligned} e^{-\lambda_k t}\eta_k(x)^2 &= \int\int r(t/3,x,y)r(t/3,x,z)e^{-\lambda_k t/3}\eta_k(y)\eta_k(z)e^{-Q(z)-Q(y)}dydz \\ &= e^{-\lambda_k t/3}\langle r(t/3,x,\bullet),\eta_k\rangle_\mu \langle r(t/3,x,\bullet),\eta_k\rangle_\mu \end{aligned}$$

which together with the fact $r(t/3,x,\bullet) \in L^2(d\mu)$ allow us to deduce

$$\begin{aligned} \textstyle\sum_{k=1}^{\infty} e^{-\lambda_k t}\eta_k(x)^2 &= \int\int r(t/3,x,y)r(t/3,x,z)\textstyle\sum_{k=1}^{\infty} e^{-\lambda_k t/3}\eta_k(y)\eta_k(z)e^{-Q(z)-Q(y)}dydz \\ &= \int\int r(t/3,x,y)r(t/3,x,z)r(t/3,y,z)e^{-Q(z)-Q(y)}dydz = r(t,x,x). \end{aligned}$$

Dini's theorem then proves the uniform convergence in compacts of $(0,\infty)$ for the series

$$\sum_{k=1}^{\infty} e^{-\lambda_k t}\eta_k(x)^2 = r(t,x,x),$$

and (3.8) follows, which together with the dominated convergence Theorem yields (3.9).

□

In the previous theorem, notice that $\sum_k e^{-\lambda_k t} = \int r(t,x,x)e^{-Q(x)}dx$ is the $\mathbb{L}^1(d\mu)$ norm of $r(t,x,x)$. This is finite if and only if P_t is Hilbert-Schmidt.

4. **Properties of the eigenfunctions**

In this section, we study some properties of the eigenfunctions η_i, including their integrability with respect to μ.

Proposition 4.1. *Assume that (H1) and (H2) are satisfied. Then* $\int_1^\infty \eta_1 e^{-Q}dx < \infty$, $F(x) = \eta_1'(x)e^{-Q(x)}$ *is a nonnegative decreasing function and the following limits exist*

$$F(0^+) = \lim_{x\downarrow 0} \eta_1'(x)e^{-Q(x)} \in (0,\infty], \quad F(\infty) = \lim_{x\to\infty} \eta_1'(x)e^{-Q(x)} \in [0,\infty).$$

Moreover $\int\limits_0^\infty \eta_1(x)e^{-Q(x)}dx = \frac{F(0^+)-F(\infty)}{2\lambda_1}$. *In particular*

$$\eta_1 \in \mathbb{L}^1(d\mu) \text{ if and only if } F(0^+) < \infty.$$

Note that $g = \eta_1 e^{-Q}$ satisfies the adjoint equation $\frac{1}{2}g'' + (qg)' = -\lambda_1 g$, and then $F(x) = g'(x) + 2q(x)g(x)$ represents the flux at x. Then $\eta_1 \in \mathbb{L}^1(d\mu)$ or equivalently $g \in \mathbb{L}^1(dx)$ if and only if the flux at 0 is finite.

Proof. Since η_1 satisfies $\eta_1''(x) - 2q\eta_1'(x) = -2\lambda_1\eta_1(x),\ x \in (0,\infty)$, we obtain

$$\eta_1'(x)e^{-Q(x)} = \eta_1'(x_0)e^{-Q(x_0)} - 2\lambda_1 \int_{x_0}^x \eta_1(y)e^{-Q(y)}dy, \tag{4.1}$$

and $F = \eta_1' e^{-Q}$ is decreasing. Integrating further gives

$$\eta_1(x) = \eta_1(x_0) + \int_{x_0}^x \left(\eta_1'(x_0)e^{-Q(x_0)} - 2\lambda_1 \int_{x_0}^z \eta_1(y)e^{-Q(y)}dy\right) e^{Q(z)}dz.$$

If for some $z_0 > x_0$ it holds that $\eta_1'(x_0)e^{-Q(x_0)} - 2\lambda_1 \int_{x_0}^{z_0} \eta_1(y)e^{-Q(y)}dy < 0$, then this inequality holds for all $z > z_0$ which implies that for large x the function η_1 is negative, because $e^{Q(z)}$ tends to ∞ as $z \to \infty$. This is a contradiction and we deduce that for all $x > 0$

$$2\lambda_1 \int_x^\infty \eta_1(y)e^{-Q(y)}dy \le \eta_1'(x)e^{-Q(x)}.$$

This implies that η_1 is increasing and, being nonnegative, it is bounded near 0. Moreover $\eta_1(0^+)$ exists. We can take the limit as $x \to \infty$ in (4.1) to get

$$F(\infty) = \lim_{x\to\infty} \eta_1'(x)e^{-Q(x)} \in [0,\infty),$$

and $\eta_1'(x_0)e^{-Q(x_0)} = F(\infty) + 2\lambda_1 \int_{x_0}^\infty \eta_1(y)e^{-Q(y)}dy$. From this the result follows.

□

We give some sufficient conditions, in terms of q, for the integrability of the eigenfunctions in the next results.

Proposition 4.2. *Assume that (H1) and (H2) are satisfied. Let c be defined as* $-C = \inf_{x>0}(q^2(x) - q'(x))$. *Assume in addition that the following hypothesis is satisfied*

Hypothesis (H3):

$$\int_1^{+\infty} e^{-Q(y)}dy < +\infty \ \text{ and } \ \int_0^1 \frac{1}{q^2(y) - q'(y) + C + 1}\,\mu(dy) < +\infty\,.$$

Then η_i *belongs to* $\mathbb{L}^1(d\mu)$ *for all* i.

Proof. Recall that $\psi_i = e^{-Q/2}\eta_i$ is an eigenfunction of the Schrödinger operator H introduced in the proof of Theorem 3.3. Replacing f by ψ_i in (3.7) thus yields

$$(C+1+\lambda_i)\int_0^{+\infty}\psi_i^2(y)dy = \int_0^{+\infty}(|\psi_i'|^2(y)/2 + w(y)\psi_i^2(y))dy\,.$$

Since the left hand side is finite, the right hand side is finite, in particular

$$\int_0^{+\infty} w(y)\eta_i^2(y)\mu(dy) \;=\; \int_0^{+\infty} w(y)\psi_i^2(y)dy < +\infty\,.$$

As a consequence, using Cauchy-Schwarz inequality we get on one hand

$$\int_0^1 |\eta_i(y)|\mu(dy) \;\;\le\;\; \left(\int_0^1 w(y)\,\eta_i^2(y)\mu(dy)\right)^{\frac12}\left(\int_0^1 \frac{1}{w(y)}\,\mu(dy)\right)^{\frac12} < +\infty$$

thanks to (H3). On the other hand

$$\int_1^{+\infty} |\eta_i(y)|\mu(dy) \;\;\le\;\; \left(\int_1^{+\infty} \eta_i^2(y)\mu(dy)\right)^{\frac12}\left(\int_1^{+\infty}\mu(dy)\right)^{\frac12} < +\infty$$

according to (H3). We have thus proved that $\eta_i \in \mathbb{L}^1(d\mu)$. □

We now obtain sharper estimates using properties of the Dirichlet heat kernel.

Proposition 4.3. *Assume (H2) holds and that the function Q satisfies*
Hypothesis (H4):

$$\int_1^{\infty} e^{-Q(x)}dx < \infty \quad \text{and} \quad \int_0^1 x\, e^{-Q(x)/2}dx < \infty.$$

Then all eigenfunctions η_k belong to $\mathbb{L}^1(d\mu)$, and satisfy a bound

$$|\eta_k| \le K_1\, e^{\lambda_k}\, e^{\frac{Q}{2}}$$

for some constant K_1 independent of k. Moreover η_1 is strictly positive on $\mathbb{R}^+$, and there is a constant $K_2 > 0$ such that for any $x \in (0,1]$ and any k

$$|\eta_k(x)| \le K_2\, x\, e^{2\lambda_k}\, e^{\frac{Q(x)}{2}}.$$

Proof. In Section 3, we introduced the semigroup $\tilde{P}_t$ associated with the Schrödinger equation, and showed that $\eta_k = e^{\frac{Q}{2}}\psi_k$, where ψ_k is the unique eigenfunction related to the eigenvalue λ_k for $\tilde{P}_t$. Using estimates on this semigroup, we will get some properties of ψ_k, and will deduce the theorem.
The semigroup $\tilde{P}_t$ is given for $f \in \mathbb{L}^2(\mathbb{R}^+, dx)$ by

$$\tilde{P}_t f(x) = \mathbb{E}^{\mathbb{W}_x}\left[f(\omega(t))\,\mathbb{I}_{t<T_0}\,\exp\left(-\frac12\int_0^t (q^2-q')(\omega_s)ds\right)\right]\,.$$

We first establish a basic estimate on its kernel $\tilde{p}_t(x,y)$.

Lemma 4.4. *Assume condition (H2) holds. There exists a constant $K > 0$ and a continuous function B defined on $\mathbb{R}^+$, bounded below and satisfying* $\lim_{z\to\infty} B(z) = \infty$ *such that for any* $x > 0, y > 0$ *we have*

$$0 < \tilde{p}_1(x,y) \le e^{-(x-y)^2/4}e^{-B(\max\{x,y\})}\,. \tag{4.2}$$

and

$$\tilde{p}_1(x,y) \le K\, p_1^D(x,y)\ , \tag{4.3}$$

where p_t^D is the Dirichlet heat semigroup in $\mathbb{R}^+$ given for $x, y \in \mathbb{R}^+$ by

$$p_t^D(x,y) = \frac{1}{\sqrt{2\pi t}}\left(e^{\frac{-(x-y)^2}{2t}} - e^{\frac{-(x+y)^2}{2t}}\right).$$

The proof of this lemma is postponed to the Appendix.

It follows immediately from Lemma 4.4 that the kernel $\tilde{p}_1(x,y)$ defines a bounded operator from $\mathbb{L}^2\big(\mathbb{R}^+, dx\big)$ to $\mathbb{L}^\infty\big(\mathbb{R}^+, dx\big)$. As a byproduct, we get that all eigenfunctions ψ_k of $\tilde{P}$ are bounded, and more precisely

$$|\psi_k| \le K_1\, e^{\lambda_k}\ .$$

One also deduces from the previous lemma that the kernel defined for $M > 0$ by

$$\tilde{p}_1^M(x,y) = \mathbb{1}_{x<M}\mathbb{1}_{y<M}\ \tilde{p}_1(x,y)$$

is Hilbert-Schmidt. In addition, it follows at once again from Lemma 4.4 that if $\tilde{P}_1^M$ denotes the operator with kernel $\tilde{p}_1^M$, we have

$$\big\|\tilde{P}_1^M - \tilde{P}_1\big\| \le Ce^{-B(M)}$$

where C is a positive constant independent of M. Since $\lim_{M\to\infty} B(M) = \infty$, the operator $\tilde{P}_1$ is a limit in norm of compact operators and hence compact. Since $\tilde{p}_1(x,y) > 0$, the operator $\tilde{P}_1$ is positivity improving and it follows that the eigenvector ψ_1 is positive (see [7]). We now show that $|\psi_k(x)| \le K_2\ x\ e^{\lambda_k}$ for $0 < x \le 1$. We have from Lemma 4.4 and the explicit expression for $p_1^D(x,y)$ that

$$\left|e^{-\lambda_k}\psi_k(x)\right| \le C\int_0^\infty p_1^D(x,y)\,|\psi_k(y)|\ dy \le C\|\psi_k\|_\infty\sqrt{\frac{2}{\pi}}e^{-x^2/2}\int_0^\infty e^{-y^2/2}\sinh(xy)\ dy\ .$$

We now estimate the integral in the right hand side as follows.

$$\begin{aligned}\int_0^\infty e^{-y^2/2}\sinh(xy)\ dy &\le \int_0^{1/x} e^{-y^2/2}\sinh(xy)\ dy + \int_{1/x}^\infty e^{-y^2/2}\sinh(xy)\ dy\\ &\le x\ \cosh(1)\int_0^{1/x} y\,e^{-y^2/2}\ dy + e^{-1/(4x^2)}\int_{1/x}^\infty e^{-y^2/4}\sinh(xy)\ dy\\ &\le x\ \cosh(1) + x^{-1}\ e^{-1/(4x^2)}\int_1^\infty e^{-z^2/(4x^2)}\sinh(z)\ dz\end{aligned}$$

which is obviously $\mathcal{O}(x)$ for $0 < x < 1$.

Since

$$\eta_k(x) = \psi_k(x)\ e^{Q(x)/2},$$

the results stated in Theorem 4.3 follow immediately from the previous bounds and Assumptions (H4), using Cauchy-Schwarz's inequality. □

Remark 4.5. (1) If q extends continuously up to 0, hypotheses (H2), (H3) and (H4) reduce to their counterpart at infinity.

(2) Consider $q(x) = \frac{a}{x} + g(x)$ with g a C^1 function up to 0 and $a > -\frac{1}{2}$, for (2.5) at the origin to hold. Recall that $a = \frac{1}{2}$ if X comes from a generalized Feller diffusion. Then $\mu(dx) = Kx^{-2a}dx$ near the origin, while $q^2(x) - q'(x) \approx (a + a^2)/x^2$. Hence for (H3) to hold, we need $a \geq 0$. Then

$$\int_0^{\varepsilon} 1/(q^2(x) - q'(x) + c + 1)\, \mu(dx) \approx \int_0^{\varepsilon} Kx^{2(1-a)}dx\,,$$

and

$$\int_0^{\varepsilon} xe^{\frac{-Q(x)}{2}}dx \approx \int_0^{\varepsilon} Kx^{(1-a)}dx\,,$$

so that at the origin, (H3) holds for $a < \frac{3}{2}$ and (H4) holds for $a < 2$. $\diamondsuit$

(3) If $q(x) \geq 0$ for x large, hypothesis (H2) implies the first part of hypotheses (H3) and (H4). Indeed, let $y = e^{-Q/2}$, this function satisfies $y' = -qy$ and $y'' = (q^2 - q')y$. Let $a > 0$ be such that for any $x \geq a$ we have $q(x) > 0$ and $q^2(x) - q'(x) > 1$ (from (H2)). For $b > a$ we get after integration by parts

$$0 = \int_a^b \left((q^2 - q')y^2 - yy''\right)\, dx = \int_a^b \left((q^2 - q')y^2 + y'^2\right)\, dx - y(b)y'(b) + y(a)y'(a)\,.$$

Using $y' = -qy$ we obtain

$$\int_a^b y^2dx \leq \int_a^b \left((q^2 - q')y^2 + y'^2\right)\, dx = q(a)y(a)^2 - q(b)y(b)^2 \leq q(a)y(a)^2 < +\infty$$

and the result follows by letting b tend to infinity.

5. **Quasi-stationary distribution and Yaglom limit.**

Existence of the Yaglom limit and of QSD for killed one-dimensional diffusion processes have already been proved by various authors, following the pioneering work by Mandl [20] (see e.g. [5, 21, 28] and references therein). One of the main assumptions in these works is $\kappa(+\infty) = +\infty$ and

$$\int_1^{\infty} e^{-Q(y)} \left(\int_1^y e^{Q(z)}\, dz\right)\, dy \;=\; +\infty$$

which is not necessarily satisfied in our case. Indeed, under mild conditions, Laplace method yields that $\int_1^y e^{Q(z)}\, dz$ behaves like $e^{Q(y)}/2q(y)$ at infinity, so the above equality will not hold if q grows too fast to infinity at infinity. Actually, we will be particularly interested in these cases (our forthcoming assumption (H5)), since they are exactly those when the diffusion "comes down from infinity", which ensures uniqueness of the QSD. The second assumption therein is that q is C^1 up to the origin which is not true in our case of interest.

We first introduce (H)

Definition 5.1. ***Hypothesis (H):***
We say that condition (H) is verified if (H1) and (H2) hold, and moreover $\eta_1 \in \mathbb{L}^1(d\mu)$ (which is the case for example under (H3) or (H4)).

We now study the existence of QSD and Yaglom limit in our framework. When $\eta_1 \in \mathbb{L}^1(d\mu)$, a natural candidate for being a QSD is the normalized measure $\eta_1\mu/\langle\eta_1, 1\rangle_\mu$, which turns to be the conditional limit distribution.

Theorem 5.2. *Assume that hypothesis (H) holds. Then* $d\nu_1 = \eta_1 d\mu / \int_0^{+\infty} \eta_1(y)\mu(dy)$ *is a quasi-stationary distribution, namely for every* $t \geq 0$ *and any Borel subset* A *of* $(0, +\infty)$,

$$\mathbb{P}_{\nu_1}(X_t \in A \,|\, T_0 > t) = \nu_1(A)\,.$$

Also for any $x > 0$ *and any Borel subset* A *of* $(0, +\infty)$,

$$\lim_{t\to+\infty} e^{\lambda_1 t}\, \mathbb{P}_x(T_0 > t) \;=\; \eta_1(x)\,, \tag{5.1}$$

$$\lim_{t\to+\infty} e^{\lambda_1 t}\, \mathbb{P}_x(X_t \in A\,,\, T_0 > t) \;=\; \nu_1(A)\, \eta_1(x)\,.$$

This implies immediately

$$\lim_{t\to+\infty} \mathbb{P}_x(X_t \in A \,|\, T_0 > t) \;=\; \nu_1(A)\,,$$

and the probability measure ν_1 *is the Yaglom limit distribution. Moreover, for any probability measure* ρ *with compact support in* $(0, \infty)$ *we have*

$$\lim_{t\to+\infty} e^{\lambda_1 t}\, \mathbb{P}_\rho(T_0 > t) \;=\; \int \eta_1(x)\rho(dx); \tag{5.2}$$

$$\lim_{t\to+\infty} e^{\lambda_1 t}\, \mathbb{P}_\rho(X_t \in A\,,\, T_0 > t) \;=\; \nu_1(A) \int \eta_1(x)\, \rho(dx); \tag{5.3}$$

$$\lim_{t\to+\infty} \mathbb{P}_\rho(X_t \in A \mid T_0 > t) \;=\; \nu_1(A). \tag{5.4}$$

Proof. Thanks to the symmetry of the semi-group, we have for all f in $\mathbb{L}^2(\mu)$,

$$\int P_t f \eta_1 d\mu = \int f P_t \eta_1 d\mu = e^{-\lambda_1 t} \int f \eta_1 d\mu\,.$$

Since $\eta_1 \in \mathbb{L}^1(d\mu)$ this equality extends to all bounded f. In particular we may use it for $f = \mathbb{1}_A$ or $f = \mathbb{1}_{(0,+\infty)}$. Noticing that

$$\int P_t(\mathbb{1}_{(0,+\infty)})\, \eta_1 d\mu = \mathbb{P}_{\nu_1}(T_0 > t) \langle \eta_1, 1\rangle_\mu$$

and $\int P_t f \eta_1 d\mu = \mathbb{P}_{\nu_1}(X_t \in A\,,\, T_0 > t)\langle \eta_1, 1\rangle_\mu$, we have shown that ν_1 is a QSD.

The rest of the proof is divided into two cases. First assume that μ is a bounded measure. Thanks to Theorem 2.3, we know that for any $x > 0$, any set $A \subset (0, +\infty)$ such that $\mathbb{1}_A \in \mathbb{L}^2(\mu)$ and for any $t > 1$

$$\begin{aligned}
\mathbb{P}_x(X_t \in A\,,\, T_0 > t) \;&=\; \int \mathbb{P}_y(X_{t-1} \in A\,,\, T_0 > t-1)\, r(1,x,y)\, \mu(dy) \\
&=\; \int P_{t-1}(\mathbb{1}_A)(y)\, r(1,x,y)\, \mu(dy) \\
&=\; \int \mathbb{1}_A(y)\, (P_{t-1} r(1,x,.))(y)\, \mu(dy)\,.
\end{aligned}$$

Since both $\mathbb{1}_A$ and $r(1, x, .)$ are in $\mathbb{L}^2(\mu)$ and since (H2) is satisfied, we obtain

$$\lim_{t\to+\infty} e^{\lambda_1 (t-1)}\, \mathbb{P}_x(X_t \in A\,,\, T_0 > t) = \langle \mathbb{1}_A, \eta_1\rangle_\mu\, \langle r(1,x,.), \eta_1\rangle_\mu\,. \tag{5.5}$$

This is enough to get the Yaglom limit starting from x.

If μ is not bounded (i.e. $\mathbb{1}_{(0,+\infty)} \notin L^2(\mu)$) we need an additional result to obtain the Yaglom limit.

Lemma 5.3. *Assume $\eta_1 \in \mathbb{L}^1(d\mu)$ then for all $x > 0$, there exists $C(x)$ such that for all $y > 0$ and all $t > 1$,*

$$r(t, x, y) \leq C(x)\, e^{-\lambda_1 t}\, \eta_1(y)\,. \tag{5.6}$$

We postpone the proof of the lemma and indicate how it is used to conclude the proof of the theorem.
If (5.6) holds, for $t > 1$, $e^{\lambda_1 t}\, r(t, x, .) \in \mathbb{L}^1(d\mu)$ and is dominated by $C(x)\, \eta_1$. Writing again $r(t, x, .) = P_{t-1} r(1, x, .)$ μ a.s., we know that $\lim_{t\to+\infty} e^{\lambda_1 t}\, r(t, x, .)$ exists in $\mathbb{L}^2(\mu)$ and is equal to

$$e^{\lambda_1} \langle r(1, x, .), \eta_1 \rangle_\mu\, \eta_1(.) = \eta_1(x)\, \eta_1(.)\,,$$

since

$$\int r(1, x, y)\eta_1(y)\mu(dy) = (P_1\eta_1)(x) = e^{-\lambda_1}\eta_1(x)\,.$$

Recall that convergence in $\mathbb{L}^2$ implies almost sure convergence along subsequences. Therefore, for any sequence $t_n \to +\infty$ there exists a subsequence t'_n such that

$$\lim_{n\to+\infty} e^{\lambda_1 t'_n}\, r(t'_n, x, y) \;=\; \eta_1(x)\, \eta_1(y) \;\text{ for } \mu\text{-almost all } y > 0\,.$$

Since

$$\mathbb{P}_x(T_0 > t'_n) = \int_0^{+\infty} r(t'_n, x, y)\mu(dy)\,,$$

Lebesgue bounded convergence theorem yields

$$\lim_{n\to+\infty} e^{\lambda_1 t'_n}\, \mathbb{P}_x(T_0 > t'_n) = \eta_1(x) \int_0^{+\infty} \eta_1(y)\mu(dy)\,,$$

that is (5.5) with $A = (0, +\infty)$ for the sequence t'_n. Since the limit does not depend on the subsequence, $\lim_{t\to+\infty} e^{\lambda_1 t}\, \mathbb{P}_x(T_0 > t)$ exists and is equal to the previous limit, hence (5.5) is still true.
For the last part of the theorem we just use Harnack's inequality ($C(.)$ is bounded on compact sets), which gives a uniform bound on compacts included in $(0, \infty)$ and then, the result follows from the dominated convergence theorem.

It remains to prove Lemma 5.3.
Proof of the Lemma. According to the parabolic Harnack inequality, for all $x > 0$, one can find $C(x) > 0$ such that for all $t > 1$, $y > 0$ and z with $|z - x| \leq \rho(x) = \frac{1}{2} \wedge \frac{x}{4}$

$$r(t, x, y) \leq C(x)\, r(t + 1, z, y)\,.$$

It follows that

$$
\begin{aligned}
r(t,x,y) &= \frac{\left(\int_{|z-x|\le\rho(x)} r(t,x,y)\eta_1(z)\mu(dz)\right)}{\left(\int_{|z-x|\le\rho(x)} \eta_1(z)\mu(dz)\right)}\\
&\le C(x)\,\frac{\left(\int_{|z-x|\le\rho(x)} r(t+1,z,y)\eta_1(z)\mu(dz)\right)}{\left(\int_{|z-x|\le\rho(x)} \eta_1(z)\mu(dz)\right)}\\
&\le C(x)\,\frac{\left(\int r(t+1,z,y)\eta_1(z)\mu(dz)\right)}{\left(\int_{|z-x|\le\rho(x)} \eta_1(z)\mu(dz)\right)}\\
&\le C(x)\,\frac{e^{-\lambda_1(t+1)}\,\eta_1(y)}{\left(\int_{|z-x|\le\rho(x)} \eta_1(z)\mu(dz)\right)}\,.
\end{aligned}
$$

But $\int_{|z-x|\le\rho(x)} \eta_1(z)\mu(dz) = c(x) > 0$, otherwise η_1, which is a solution of the linear o.d.e. $\frac{1}{2}g'' - qg' + \lambda_1 g = 0$ on $(0,+\infty)$, would vanish on the whole interval $|z-x|\le\rho(x)$, hence on $(0,+\infty)$ according to the uniqueness theorem for linear o.d.e's. The proof of the lemma is thus completed. □

λ_1 is the natural killing rate of the process. Indeed, the limit (5.1) obtained in Theorem 5.2 shows for any $x>0$ and any $t>0$,

$$
\lim_{s\to+\infty} \frac{\mathbb{P}_x(T_0>t+s)}{\mathbb{P}_x(T_0>s)} = e^{-\lambda_1 t}\,.
$$

Let us also remark that

$$
\mathbb{P}_{\nu_1}(T_0>t) = e^{-\lambda_1 t}.
$$

In order to control the speed of convergence to the Yaglom limit, we first establish the following lemma.

Lemma 5.4. *Under conditions (H2) and (H4), the operator P_1 is bounded from $\mathbb{L}^\infty(d\mu)$ to $\mathbb{L}^2(d\mu)$. Moreover, for any compact subset K of $(0,+\infty)$, there is a constant C_K such that for any function $f\in\mathbb{L}^1(d\mu)$ with support in K we have*

$$
\|P_1 f\|_{\mathbb{L}^2(d\mu)} \le C_K\,\|f\|_{\mathbb{L}^1(d\mu)}
$$

Proof. Let $g\in\mathbb{L}^\infty(d\mu)$, since

$$
\left|P_1 g\right| \le P_1\left|g\right| \le \|g\|_{\mathbb{L}^\infty(d\mu)}\,,
$$

we get from (H4)

$$
\int_1^\infty \left|P_1 g\right|^2 d\mu \le \|g\|^2_{\mathbb{L}^\infty(d\mu)} \int_1^\infty e^{-Q(x)}dx\,.
$$

We now recall that (see section 3)

$$
P_1 g(x) = e^{Q(x)/2}\tilde{P}_1\left(e^{-Q/2}g\right)(x)\,.
$$

It follows from Lemma 4.4 that uniformly in $x \in (0,1]$ we have using hypothesis (H4)

$$\left|\tilde{P}_1\left(e^{-Q/2}g\right)(x)\right| \le \mathcal{O}(1)\ \|g\|_{\mathbb{L}^\infty(d\mu)} \int_0^\infty e^{-Q(y)/2}e^{-y^2/4}y\,dy \le \mathcal{O}(1)\ \|g\|_{\mathbb{L}^\infty(d\mu)}\ .$$

This implies

$$\int_0^1 \left|P_1 g\right|^2 d\mu = \int_0^1 \left|\tilde{P}_1\left(e^{-Q/2}g\right)(x)\right|^2 dx \le \mathcal{O}(1)\|g\|^2_{\mathbb{L}^\infty(d\mu)}\ ,$$

and the first part of the lemma follows. For the second part, we have from the Gaussian bound of Lemma 4.4 that for any $x>0$ and for any f integrable and with support in K

$$\left|\tilde{P}_1\left(e^{-Q/2}f\right)(x)\right| \le \mathcal{O}(1)\int_K e^{-Q(y)/2}e^{-(x-y)^2/2}|f(y)|\,dy$$

$$\le \mathcal{O}(1)\sup_{z\in K} e^{Q(z)/2}\sup_{z\in K} e^{-(x-z)^2/2}\int_K e^{-Q(y)}|f(y)|\,dy \le \mathcal{O}(1)\ e^{-x^2/4}\int_K e^{-Q(y)}|f(y)|\,dy$$

since K is compact. This implies

$$\int_0^\infty \left|P_1 f\right|^2 d\mu = \int_0^\infty \left|\tilde{P}_1\left(e^{-Q/2}f\right)(x)\right|^2 dx \le \mathcal{O}(1)\ \|f\|^2_{\mathbb{L}^1(d\mu)}\ .$$

□

We can now use the spectral decomposition of $r(1,x,.)$ to obtain the following convergence result.

Proposition 5.5. *Under conditions (H2) and (H4), for all $x>0$ and all measurable subset B of $(0,\infty)$, we have*

$$\lim_{t\to+\infty} e^{(\lambda_2-\lambda_1)t}\left(\mathbb{P}_x(X_t\in B \mid T_0>t)-\nu_1(B)\right) = \frac{\eta_2(x)}{\eta_1(x)}\left(\frac{\langle \mathbb{1}_{\mathbb{R}^+},\eta_1\rangle_\mu\langle \mathbb{1}_B,\eta_2\rangle_\mu - \langle \mathbb{1}_{\mathbb{R}^+},\eta_2\rangle_\mu\langle \mathbb{1}_B,\eta_1\rangle_\mu}{(\langle \mathbb{1}_{\mathbb{R}^+},\eta_1\rangle_\mu)^2}\right)\ . \tag{5.7}$$

Proof. Let h be a non negative bounded function, with compact support in $(0,\infty)$. We have, using the semi-group property, Lemma 5.4 and the spectral decomposition

$$\mathbb{P}_h(X_t\in B\,,\,T_0>t) = \langle he^Q\,,\,P_t\mathbb{1}_B\rangle_\mu = \langle P_1\big(he^Q\big)\,,\,P_{(t-2)}P_1\mathbb{1}_B\rangle_\mu$$

$$= \langle P_1\big(he^Q\big)\,,\,\eta_1\rangle_\mu\,\langle\eta_1\,,\,P_1\mathbb{1}_B\rangle_\mu e^{-\lambda_1(t-2)} + \langle P_1\big(he^Q\big)\,,\,\eta_2\rangle_\mu\,\langle\eta_2\,,\,P_1\mathbb{1}_B\rangle_\mu e^{-\lambda_2(t-2)} + R(h,B,t)$$

with

$$|R(h,B,t)| \le e^{-\lambda_3(t-2)}\|P_1\big(he^Q\big)\|_{\mathbb{L}^2(d\mu)}\|P_1\mathbb{1}_B\|_{\mathbb{L}^2(d\mu)}\ .$$

Note that since P_1 is symmetric with respect to the scalar product, we have $\langle P_1\big(he^Q\big)\,,\,\eta_1\rangle_\mu = e^{-\lambda_1}\langle he^Q,\eta_1\rangle_\mu$ and similarly for η_2. We also have $\langle\eta_1\,,\,P_1\mathbb{1}_B\rangle_\mu = e^{-\lambda_1}\langle\eta_1\,,\,\mathbb{1}_B\rangle_\mu$ and similarly for η_2. It follows immediately form Lemma 5.4 that for any fixed compact subset K of $(0,\infty)$, we have for any B and any h satisfying the hypothesis of the proposition, the latter with support in K,

$$|R(h,B,t)| \le \mathcal{O}(1)e^{-\lambda_3(t-2)}\|h\|_{\mathbb{L}^1(d\mu)}\ .$$

Therefore, letting h tend to a Dirac mass, we obtain that for any compact subset K of $(0,\infty)$, there is a constant D_K such that for any $x \in K$, for any measurable subset B of $(0,\infty)$, and for any $t > 2$, we have

$$\left|\mathbb{P}_x(X_t \in B\,,\, T_0 > t) - e^{Q(x)}\eta_1(x)\,\langle \eta_1\,,\, \mathbb{1}_B\rangle_\mu e^{-\lambda_1 t} - e^{Q(x)}\eta_2(x)\,\langle \eta_2\,,\, \mathbb{1}_B\rangle_\mu e^{-\lambda_2 t}\right| \le D_K\, e^{-\lambda_3 t}\,.$$

The proposition follows at once from

$$\mathbb{P}_x(X_t \in B \,|\, T_0 > t) = \frac{\mathbb{P}_x(X_t \in B\,,\, T_0 > t)}{\mathbb{P}_x(X_t \in (0,\infty)\,,\, T_0 > t)}\,.$$

□

6. The Q-process

As in [5] (Theorem B), we can also describe the law of the process conditioned to be never extinct, usually called the Q-process (also see [17]).

Corollary 6.1. *Assume (H). Let $\mathcal{B}_s$ be $\mathcal{F}_s$ measurable, where $\mathcal{F}_s$ is the natural filtration of the process. Then for all $x > 0$,*

$$\lim_{t\to+\infty} \mathbb{P}_x(X \in \mathcal{B}_s \mid T_0 > t) = \mathbb{Q}_x(\mathcal{B}_s)\,,$$

where $\mathbb{Q}_x$ is the law of a diffusion process on $(0,+\infty)$, with transition probability densities (w.r.t. Lebesgue measure) given by

$$q(s,x,y) = e^{\lambda_1 s}\,\frac{\eta_1(y)}{\eta_1(x)}\, r(s,x,y)\, e^{-Q(y)}\,,$$

that is, $\mathbb{Q}_x$ is locally absolutely continuous w.r.t. $\mathbb{P}_x$ and

$$\mathbb{Q}_x(\mathcal{B}_s) = \mathbb{E}_x\left(\mathbb{1}_{\mathcal{B}_s(\omega)}\, e^{\lambda_1 s}\,\frac{\eta_1(\omega_s)}{\eta_1(x)}\,, T_0 > s\right)\,.$$

Proof. First check thanks to Fubini's theorem and $\kappa(0^+) < \infty$ in Hypothesis (H1), that $\Lambda(0^+) > -\infty$. We can thus slightly change the notation (for this proof only) and define Λ as $\Lambda(x) = \int_0^x e^{Q(y)}\,dy$. From standard diffusion theory, $(\Lambda(X_t); t \ge 0)$ is a local martingale, from which it is easy to derive that for any $y \ge x \ge 0$, $\mathbb{P}_y(T_x < T_0) = \Lambda(y)/\Lambda(x)$.

Now define $v(t,x) = \frac{\mathbb{P}_x(T_0 > t)}{\mathbb{P}_1(T_0 > t)}$. As in [5, proof of Theorem B], one can prove for any $x \ge 1$, using the strong Markov property at T_x of the diffusion X starting from 1, that $v(t,x) \le \Lambda(x)/\Lambda(1)$. On the other hand, for $x \le 1$, $v(t,x) \le 1$, so that for any $x \ge 0$, $v(t,x) \le 1 + \Lambda(x)/\Lambda(1)$.

Now thanks to Theorem 5.2, for all x, $e^{\lambda_1 t}\mathbb{P}_x(T_0 > t) \to \eta_1(x)$ as $t \to \infty$, and

$$\lim_{t\to+\infty} v(t,x) = \frac{\eta_1(x)}{\eta_1(1)}\,.$$

Using the Markov property, it is easily seen that for t large,

$$\mathbb{P}_x(X \in \mathcal{B}_s \mid T_0 > t) = \mathbb{E}_x\left[\mathbb{1}_{\mathcal{B}_s}(X)\, v(t-s, X_s), T_0 > s\right]\, \frac{\mathbb{P}_1(T_0 > t-s)}{\mathbb{P}_x(T_0 > t)}\,.$$

The random variable in the expectation is (positive and) bounded from above by $1 + \Lambda(X_s)/\Lambda(1)$, which is integrable (see below), so we obtain the desired result using Lebesgue bounded convergence theorem.

To see that $\mathbb{E}_x\left(\Lambda(X_s)\,\mathbb{I}_{s<T_0}\right)$ is finite, it is enough to use Itô's formula with the harmonic function Λ up to time $T_0 \wedge T_M$. Since Λ is non-negative it easily yields $\mathbb{E}_x\left(\Lambda(X_s)\,\mathbb{I}_{s<T_0\wedge T_M}\right) \leq \Lambda(x)$ for all $M > 0$. Letting M go to infinity the indicator goes almost surely to 1 thanks to Hypothesis (H1), so that the monotone convergence theorem yields $\mathbb{E}_x\left(\Lambda(X_s)\,\mathbb{I}_{s<T_0}\right) \leq \Lambda(x)$. $\square$

Corollary 6.2. *Assume (H). Then for any Borel subset B and any x,*

$$\lim_{s\to+\infty} \mathbb{Q}_x(\omega_s \in B) = \int_B \eta_1^2(y)\mu(dy)\,.$$

Proof. We know that $e^{\lambda_1 s} r(s, x, .)$ converges to $\eta_1(x)\,\eta_1(.)$ in $\mathbb{L}^2(d\mu)$. Hence, since $\mathbb{I}_B\eta_1 \in \mathbb{L}^2(\mu)$,

$$\eta_1(x)\mathbb{Q}_x(\omega_s \in B) = \int \mathbb{I}_B(y)\eta_1(y)\,e^{\lambda_1 s} r(s, x, y)\mu(dy) \to \eta_1(x) \int_B \eta_1^2(y)\mu(dy)$$

as $s \to +\infty$. $\square$

Remark 6.3. The previous statement gives the stationary measure of the Q-process as $\eta_1^2(y)\mu(dy)$. Notice that it can also be given in terms of the Yaglom limit ν_1 as $c\eta_1(y)\nu_1(dy)$, where $c = \left(\int \eta_1(dz)\mu(dz)\right)^{-1}$. Thus, the stationary measure of the Q-process is absolutely continuous w.r.t. ν_1, with Radon-Nikodym derivative $c\eta_1$, which, thanks to Proposition 4.1, is *nondecreasing*. In particular, the ergodic measure of the Q-process dominates stochastically the Yaglom limit. We refer to [21, 17] for further discussion of the relationship between QSD and ergodic measure of the Q-process.

7. Domain of attraction, return from infinity and uniqueness of QSD.

7.1. **Main statement.** Once we have proved that ν_1 is a QSD and is the limit of the law of the diffusion process conditioned on non-extinction (or non-killing) starting from any point, it is natural to ask about its uniqueness. Here again, our assumptions on the behavior of q at infinity will allow us to characterize the domain of attraction of the QSD ν_1 associated to η_1. This turns out to be entirely different from the cases studied in [5] for instance.

We say that the diffusion process X *comes down from infinity* or *returns from infinity*, if $+\infty$ is an entrance boundary for X, that is, there is a nonnegative real number y and a time t such that

$$\lim_{x\uparrow\infty} \downarrow \mathbb{P}_x(T_y < t) \;>\; 0.$$

Recall from Theorem 5.2 that under Hypothesis (H), the measure $d\nu_1 = \eta_1 d\mu / \int_0^{+\infty} \eta_1(y)\mu(dy)$ is a quasi-stationary distribution, which in addition is the limiting conditional distribution starting from any initial distribution *with compact support.*

Let us introduce the following condition.

Definition 7.1. ***Hypothesis (H5):***
We say that condition (H5) is verified if

$$\int_1^\infty e^{Q(y)} \left(\int_y^\infty e^{-Q(z)}\, dz \right) dy \,<\, \infty.$$

Theorem 7.2. *Assume (H) holds. Then the following are equivalent:*

(i) *X comes down from infinity*
(ii) *(H5)*
(iii) *ν_1 is the unique limiting conditional distribution, namely*

$$\lim_{t\to\infty} \mathbb{P}_\nu(X_t \in A \mid T_0 > t) = \nu_1(A),$$

for any Borel set A and any initial distribution ν.

Theorem 7.2 follows immediately from the next three lemmas, the first two of which are general results that can be useful in other contexts.

Lemma 7.3. *Assume (H1) holds. If there is a unique limiting conditional distribution π, then X comes down from infinity.*

Lemma 7.4. *The following are equivalent*

(i) *X comes down from infinity*
(ii) *(H5)*
(iii) *for any $A > 0$ there exists $y_A > 0$ such that $\sup_{x>y_A} \mathbb{E}_x[e^{AT_{y_A}}] < \infty$.*

The previous two lemmas are proved in Subsection 7.2, and the next one in Subsection 7.3.

Lemma 7.5. *Assume (H). If there is x_0 such that $\sup_{x\geq x_0} \mathbb{E}_x(e^{\lambda_1 T_{x_0}}) < \infty$, then ν_1 is the one and only limiting conditional distribution.*

Remark 7.6. It is not obvious when Condition (H5) holds. Actually, the following conditions are sufficient for (H5) to hold:

- $q(x) \geq a > 0$ for all $x \geq x_0$
- $\liminf_{x\to\infty} q'(x)/2q^2(x) > -1$
- $\int^\infty \frac{1}{q(x)}\, dx < \infty$.

Indeed, check first that these conditions imply that $q(x)$ goes to infinity as $x \to \infty$. Then set $s(y) := \int_y^\infty e^{-Q(z)}\, dz$, show (thanks to the first condition) that se^Q is bounded and integrate it by parts as $Q'e^Q\; s/2q$. Since $se^Q/2q$ vanishes at infinity, the integration by parts and the third condition imply that $se^Q(1 + q'/2q^2)$ is integrable. Conclude thanks to the second condition.
On the other hand, if (H5) holds and $q'(x) \geq 0$ for $x \geq x_0$, then $q(x)$ goes to infinity as $x \to \infty$ and $\int^\infty \frac{1}{q(x)}\, dx < \infty$.
We can retain that under the assumption that $q'(x) \geq 0$ for $x \geq x_0$ and $q(x)$ goes to infinity as $x \to \infty$, then

$$\text{(H5)} \iff \int^\infty \frac{1}{q(x)}\, dx < \infty\,.$$

7.2. **Proofs of Lemmas 7.3 and 7.4.**

Proof of Lemma 7.3. Since π is a conditional limiting distribution, it is a QSD and it is easy to prove that $\mathbb{P}_\pi(T_0 > t)$ is exponential in t, and we let α its decay parameter, that is $\mathbb{P}_\pi(T_0 > t) = e^{-\alpha t}$. Since absorption is certain, (H1), then $\alpha > 0$. For the rest of the proof let ν be any initial distribution.

We first prove that for any $\lambda < \alpha$, $\mathbb{E}_\nu(e^{\lambda T_0}) < \infty$. The assumption in the lemma can be stated as

$$\lim_{t\to\infty} \int_0^\infty \mathbb{P}_\nu(X_t \in dx \mid T_0 > t) f(x) = \int_0^\infty f(x)\pi(dx)$$

for any bounded measurable f. Now take $f(x) = \mathbb{P}_x(T_0 > s)$ so that the expression in the limit equals $\mathbb{P}_\nu(T_0 > t + s \mid T_0 > t)$ and the r.h.s. equals $\mathbb{P}_\pi(T_0 > s)$, which entails that for any s

$$\lim_{t\to\infty} \frac{\mathbb{P}_\nu(T_0 > t + s)}{\mathbb{P}_\nu(T_0 > t)} = e^{-\alpha s}\,.$$

Now pick $\lambda \in (0, \alpha)$ and ε such that $(1 + \varepsilon)e^{\lambda-\alpha} < 1$. An elementary induction shows that there is t_0 such that for any $t > t_0$, and any integer n

$$\frac{\mathbb{P}_\nu(T_0 > t + n)}{\mathbb{P}_\nu(T_0 > t)} \le (1 + \varepsilon)e^{-\alpha n}\,.$$

Breaking down the integral $\int_{t_0}^\infty \mathbb{P}_\nu(T_0 > s)\, e^{\lambda s}\, ds$ over intervals of the form $(n, n + 1]$ and using the previous inequality, it is easily seen that this integral converges. This proves that $\mathbb{E}_\nu(e^{\lambda T_0}) < \infty$ for any initial distribution ν.

Now fix $\lambda = \alpha/2$ and for any $x \ge 0$, let $g(x) = \mathbb{E}_x(e^{\lambda T_0}) < \infty$. We want to show that g is bounded, which trivially entails that X comes down from infinity. Thanks to the previous step, for any nonnegative random variable Y with law ν

$$\mathbb{E}(g(Y)) = \mathbb{E}_\nu(e^{\lambda T_0}) < \infty.$$

Since Y can be any random variable, this implies that g is bounded. Indeed, observe that g is increasing and $g(0) = 1$, so that $a := 1/g(\infty)$ is well defined in $[0, 1)$. Then check that

$$\nu(dx) = \frac{g'(x)}{(1 - a)g(x)^2}\, dx$$

is a probability density on $(0, \infty)$. Conclude computing $\int g\, d\nu$. □

Proof of Lemma 7.4. Observe that $(iii) \Rightarrow (i)$ is immediate. We now prove $(i) \Rightarrow (ii)$. Because X comes down from infinity, there are y, t, h such that $\mathbb{P}_x(T_y < t) \ge h > 0$ for all $x \ge y$. For any $x > 0$, set

$$f(x) = \mathbb{E}_x(\exp(-T_y)).$$

By a standard coupling argument, f is non-increasing so that $f(x)$ has a nonnegative limit, say a, as $x \to \infty$. But

$$f(x) = \int_0^\infty \mathbb{P}_x(T_y < s)\, e^{-s}\, ds \ge \int_t^\infty h\, e^{-s}\, ds = he^{-t},$$

which entails that a is nonzero.

Then check that $f(X_{t\wedge T_y})e^{-(t\wedge T_y)}$ is a martingale, so that

$$\frac{1}{t}\left(\mathbb{E}_x(f(X_t)) - f(x)\right) = \mathbb{E}_x\left(f(X_t)\left(\frac{1-e^{-t}}{t}\right)\mathbb{1}_{t\leq T_y}\right) + \frac{1}{t}\mathbb{E}_x\left(\left(f(X_t)-f(y)\right)e^{-T_y}\mathbb{1}_{t>T_y}\right).$$

Assume that $x \neq y$. As $t \to 0$, the l.h.s. tends to $Lf(x)$. By dominated convergence, the first term in the r.h.s. converges to $f(x)$. The absolute value of the second term is less than $\mathbb{P}_x(T_y < t)/t$ which converges to 0. The result is $Lf = f$, that is an ordinary differential equation,

$$f'' - 2qf' = 2f.$$

This last equation ensures that $f'e^{-Q}$ has nonnegative derivative equal to $2fe^{-Q}$. Then $f'e^{-Q}$ is nondecreasing and nonpositive, so it converges to a nonpositive limit D. We now show that $D < 0$ leads to a contradiction. Indeed, if D is negative, then f' is equivalent to De^{Q}. Since f' is integrable (f converges), we first get $\int_1^\infty e^Q < \infty$. In addition, since $2fe^{-Q}$ is also integrable (it is the derivative of $f'e^{-Q}$, which converges), and because f converges to $a > 0$, we get $\int_1^\infty e^{-Q} < \infty$. But applying the Cauchy-Schwarz inequality to the functions $e^{Q/2}$ and $e^{-Q/2}$ shows that both integrals of e^Q and e^{-Q} cannot converge at the same time. We conclude that $D = 0$ and use this fact to integrate the ordinary differential equation as

$$f'(x) = -2e^{Q(x)}\int_x^\infty f(z)\,e^{-Q(z)}\,dz.$$

Again because f converges, the r.h.s. is integrable, that is,

$$\int_1^\infty e^{Q(x)}\int_x^\infty f(z)\,e^{-Q(z)}\,dz\,dx < \infty.$$

But since f decreases to a positive real number a, we get (H5).

We continue the proof with $(ii) \Rightarrow (iii)$. Let $A > 0$, and pick x_A large enough so that

$$\int_{x_A}^\infty e^{Q(x)}\int_x^\infty e^{-Q(z)}\,dz\,dx \leq \frac{1}{2A}.$$

Let H be the positive increasing function defined on $[x_A, \infty)$ by

$$H(x) = \int_{x_A}^x e^{Q(y)}\int_y^\infty e^{-Q(z)}\,dz\,dy.$$

Then check that $H'' = 2qH' - 1$, so that $LH = -1/2$. Finally, set $y_A = 1 + x_A$, and apply Itô's formula for $x \geq y_A$ and $t > 0$, to get

$$\mathbb{E}_x(e^{A(t\wedge T_{y_A})}\,H(X_{t\wedge T_{y_A}})) = H(x) + \mathbb{E}_x\left(\int_0^{t\wedge T_{y_A}} e^{As}\left(AH(X_s) + LH(X_s)\right)ds\right).$$

But $LH = -1/2$, and $H(X_s) < H(\infty) \leq 1/(2A)$ for any $s \leq T_{y_A}$, so that

$$\mathbb{E}_x[e^{A(t\wedge T_{y_A})}\,H(X_{t\wedge T_{y_A}})] \leq H(x).$$

But H is increasing, hence for $x \geq y_A$, $1/(2A) > H(x) \geq H(y_A) > 0$. It follows that $\mathbb{E}_x(e^{A(t\wedge T_{y_A})}) \leq 1/(2AH(y_A))$ and finally $\mathbb{E}_x(e^{AT_{y_A}}) \leq 1/(2AH(y_A))$ using the Monotone Convergence Theorem. $\square$

7.3. **Proof of Lemma 7.5.** In everything that follows, we assume (H), and that there is x_0 such that $\sup_{x\geq x_0}\mathbb{E}_x(e^{\lambda_1 T_{x_0}})<\infty$.

Lemma 7.7. *Assume $h\in\mathbb{L}^1(d\mu)$ is strictly positive in $(0,\infty)$. Then*

$$\lim_{\epsilon\downarrow 0}\limsup_{t\to\infty}\frac{\int_0^\epsilon h(x)\mathbb{P}_x(T_0>t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0>t)\mu(dx)}=0 \tag{7.1}$$

$$\lim_{M\uparrow\infty}\limsup_{t\to\infty}\frac{\int_M^\infty h(x)\mathbb{P}_x(T_0>t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0>t)\mu(dx)}=0 \tag{7.2}$$

Proof. We start with (7.1). Using Harnack's inequality, we have for $\epsilon<1$ and large t

$$\frac{\int_0^\epsilon h(x)\mathbb{P}_x(T_0>t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0>t)\mu(dx)}\leq\frac{\mathbb{P}_1(T_0>t)\int_0^\epsilon h(z)\mu(dz)}{Cr(t-1,1,1)\int_1^2 h(x)\mu(dx)\int_1^2\mu(dy)},$$

then

$$\limsup_{t\to\infty}\frac{\int_0^\epsilon h(x)\mathbb{P}_x(T_0>t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0>t)\mu(dx)}\leq\limsup_{t\to\infty}\frac{\mathbb{P}_1(T_0>t)\int_0^\epsilon h(z)\mu(dz)}{Cr(t-1,1,1)\int_1^2 h(x)\mu(dx)\int_1^2\mu(dy)}=\frac{e^{-\lambda_1}\int_0^\epsilon h(z)\mu(dz)}{\eta_1(1)\int_1^2 h(x)\mu(dx)\int_1^2\mu(dy)},$$

and the first assertion of the statement follows.
For the second limit, we set $A(x_0)=\sup_{x\geq x_0}\mathbb{E}_x(e^{\lambda_1 T_{x_0}})<\infty$. Then for large $M>x_0$, we have

$$\mathbb{P}_x(T_0>t)=\int_0^t\mathbb{P}_{x_0}(T_0>u)\mathbb{P}_x(T_{x_0}\in d(t-u))+\mathbb{P}_x(T_{x_0}>t).$$

Using that $\lim_{u\to\infty}e^{\lambda_1 u}\mathbb{P}_{x_0}(T_0>u)=\eta_1(x_0)$ we obtain that $B(x_0)=\sup_{u\geq 0}e^{\lambda_1 u}\mathbb{P}_{x_0}(T_0>u)<\infty$. Then

$$\begin{aligned}\mathbb{P}_x(T_0>t)&\leq B(x_0)\int_0^t e^{-\lambda_1 u}\mathbb{P}_x(T_{x_0}\in d(t-u))+\mathbb{P}_x(T_{x_0}>t)\\&\leq B(x_0)e^{-\lambda_1 t}\mathbb{E}_x(e^{\lambda_1 T_{x_0}})+e^{-\lambda_1 t}\mathbb{E}_x(e^{\lambda_1 T_{x_0}})\leq e^{-\lambda_1 t}A(x_0)(B(x_0)+1),\end{aligned}$$

and (7.2) follows immediately. □

Lemma 7.8. *Let ν a probability measure whose support is contained in $(0,\infty)$ then*

$$h(y)=\int r(1,x,y)\nu(dx)$$

belongs to $\mathbb{L}^1(d\mu)$ and it is strictly positive in $(0,\infty)$.

Proof. First notice that $h(y)<\infty$. In fact from Tonelli's theorem we have

$$\int\int r(t,x,y)\nu(dx)\,\mu(dy)=\int\int r(t,x,y)\,\mu(dy)\,\nu(dx)=\int\mathbb{P}_x(T_0>t)\nu(dx)\leq 1,$$

which implies that $\int r(t,x,y)\nu(dx)$ is finite $dy-$a.s.. The rest of the conclusion follows from Harnack's inequality. □

Proof of Lemma 7.5. Thanks to the previous lemma, we can assume that ν has a density $h \in \mathbb{L}^1(d\mu)$, with respect to $\mu(dx)$, which is strictly positive in $(0,\infty)$. Consider $M > \epsilon > 0$ and A any Borel set included in $(0,\infty)$. Then

$$\left|\frac{\int h(x)\mathbb{P}_x(X_t \in A,\, T_0 > t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0 > t)\mu(dx)} - \frac{\int_\epsilon^M h(x)\mathbb{P}_x(X_t \in A,\, T_0 > t)\mu(dx)}{\int_\epsilon^M h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}\right|$$

is bounded by the sum of the following two terms

$$I1 = \left|\frac{\int h(x)\mathbb{P}_x(X_t \in A,\, T_0 > t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0 > t)\mu(dx)} - \frac{\int_\epsilon^M h(x)\mathbb{P}_x(X_t \in A,\, T_0 > t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}\right|$$

$$I2 = \left|\frac{\int_\epsilon^M h(x)\mathbb{P}_x(X_t \in A,\, T_0 > t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0 > t)\mu(dx)} - \frac{\int_\epsilon^M h(x)\mathbb{P}_x(X_t \in A,\, T_0 > t)\mu(dx)}{\int_\epsilon^M h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}\right|.$$

We have the bound

$$I1 \vee I2 \le \frac{\int_0^\epsilon h(x)\mathbb{P}_x(T_0 > t)\mu(dx) + \int_M^\infty h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}.$$

Thus, from Lemma 7.7 we get

$$\lim_{\epsilon\downarrow 0,\, M\uparrow\infty} \limsup_{t\to\infty} \left|\frac{\int h(x)\mathbb{P}_x(X_t \in A,\, T_0 > t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0 > t)\mu(dx)} - \frac{\int_\epsilon^M h(x)\mathbb{P}_x(X_t \in A,\, T_0 > t)\mu(dx)}{\int_\epsilon^M h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}\right| = 0.$$

On the other hand we have

$$\lim_{t\to\infty} \frac{\int_\epsilon^M h(x)\mathbb{P}_x(X_t \in A,\, T_0 > t)\mu(dx)}{\int_\epsilon^M h(x)\mathbb{P}_x(T_0 > t)\mu(dx)} = \frac{\int_A \eta_1(z)\mu(dz)}{\int_{\mathbb{R}^+} \eta_1(z)\mu(dz)},$$

independently of $M > \epsilon > 0$, and the result follows. □

7.4. **Further results and remarks.** The following corollary of Lemma 7.4 describes how fast the process comes down from infinity.

Corollary 7.9. *Assume (H) and (H5). Then for all $\lambda < \lambda_1$, $\sup_{x>0} \mathbb{E}_x[e^{\lambda T_0}] < +\infty$.*

Proof. We have seen in Section 5 that for all $x > 0$, $\lim_{t\to+\infty} e^{\lambda_1 t}\mathbb{P}_x[T_0 > t] = \eta_1(x) < \infty$ i.e. $\mathbb{E}_x[e^{\lambda T_0}] < \infty$ for all $\lambda < \lambda_1$. Applying Lemma 7.4 with $A = \lambda$ and the strong Markov property it follows that $\sup_{x>y_\lambda} \mathbb{E}_x[e^{\lambda T_0}] < +\infty$. Furthermore, thanks to the uniqueness of the solution of (2.1), $X_t^x < X_t^{y_\lambda}$ a.s. for all $t > 0$ and all $x < y_\lambda$, hence $\mathbb{E}_x[e^{\lambda T_0}] \le \mathbb{E}_{y_\lambda}[e^{\lambda T_0}]$ for those x, completing the proof. □

Remark 7.10. The previous corollary can be rephrased as follows : the explosion (absorption, killing) time for the process starting from infinity has exponential moments up to order λ_1. In [16] an explicit calculation of the law of T_0 is done in the case of the logistic Feller diffusion Z (hence the corresponding X) and also for other related models. In particular it is shown in Corollary 3.10 therein, that the absorption time for the process starting from infinity has a finite expectation. As we remarked in studying examples, a very general family of diffusion processes (including the logistic one) satisfy all assumptions in Corollary 7.9, which is thus an improvement of the quoted result.

We end this section by gathering some results on birth–death processes that resemble our findings.

Let Y be a birth–death process with birth rate λ_n and death rate μ_n when in state n. Assume that $\lambda_0 = \mu_0 = 0$ and that extinction (absorption at 0) occurs with probability 1. Let

$$S = \sum_{i\geq 1} \pi_i + \sum_{n\geq 1} (\lambda_n \pi_n)^{-1} \sum_{i\geq n+1} \pi_i \,,$$

where

$$\pi_n = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \,.$$

We may state

Proposition 7.11. *For a birth–death process Y absorbed at 0 with probability 1, the following are equivalent:*

(i) *Y comes down from infinity*
(ii) *There is one and only one QSD*
(iii) $\lim_{n\uparrow\infty} \uparrow \mathbb{E}_n(T_0) < \infty$
(iv) $S < \infty$.

Proof. In [8, Theorem 3.2], it is stated that either $S = \infty$ and there is no or infinitely many QSD's, or $S < \infty$, and there is a unique QSD, that is (ii) and (iv) are equivalent. Let us now examine how this criterion is related to the nature of the boundary at $+\infty$. Set

$$U_n = \sum_{k=1}^{n-1} \frac{\mu_1 \cdots \mu_k}{\lambda_1 \cdots \lambda_k} \,.$$

According to basic theory of stochastic processes [1], extinction has probability 1 if and only if the sequence $(U_n)_n$ converges to $+\infty$. Also extinction times have finite first-order moment if and only if the sequence $(\pi_n)_n$ is summable. In addition, the expected time to extinction starting from n can be shown to be equal to

$$\mathbb{E}_n(T_0) = \sum_{k\geq 1} \pi_k (1 + U_{n\wedge k}) \,.$$

Then by Beppo Levi's theorem, this quantity converges as $n \to \infty$ to $\sum_{k\geq 1} \pi_k(1+U_k)$, which after elementary transformations, can be seen to equal S:

$$S = \lim_{n\uparrow\infty} \uparrow \mathbb{E}_n(T_0) \,.$$

Hence (iv) and (iii) are equivalent. It is clear that (iii) implies (i), this shows in particular that if $S < \infty$ then Y comes down from infinity.

Now other elementary transformations on the expression given for S yield

$$S = \sum_{n\geq 1} \frac{1}{\mu_{n+1}} \left(1 + \frac{\lambda_n}{\mu_n} + \cdots + \frac{\lambda_n \cdots \lambda_1}{\mu_n \cdots \mu_1} \right) \,.$$

Furthermore, Theorem II.2.3. in [1] states that the solutions to Kolmogorov forward equations associated with birth–death rate matrices are not unique if and only if S is finite. This is precisely the case when the birth–death process comes down from infinity (and the rate matrix is non conservative), since in that case both the minimal process and the minimal process resurrected at infinity at each killing time have transition functions which solve the forward equations. Hence (i) implies (iv) and the proof is completed. $\square$

Appendix A. Back to the biological models

We start with the proof of Proposition 1.1 given in the introduction.

Proof of Proposition 1.1. Recall that $u(x) = \mathbb{P}_x(\text{Extinction}) = \mathbb{P}(\lim_{t\to\infty} Z_t = 0)$. Thanks to the properties of h, one can easily prove that u is a smooth function satisfying $u(0) = 1$ and $\lim_{x\to+\infty} u(x) = 0$ (cf. [14]), so that

$$\frac{\gamma}{2}xu''(x) + h(x)u'(x) = 0 \qquad \forall x \geq 0.$$

Introducing

$$H(x) := \int_0^x \frac{2h(z)}{\gamma z}dz,$$

(well defined since $h \in C^1((0, +\infty))$ with $h(0) = 0$), the harmonic equation yields

$$u(x) = a\int_x^\infty e^{-H(z)}dz,$$

with $a = (\int_0^\infty e^{-H(z)}dz)^{-1}$ (well defined because h tends to $+\infty$). Also because u is harmonic, $u(Z_t)$ is a martingale, and it is straightforward that

$$d\mathbb{P}^\star_{x|\mathcal{F}_t} = \frac{u(Z_t)}{u(x)}d\mathbb{P}_{x|\mathcal{F}_t},$$

where $\mathbb{P}^\star_x := \mathbb{P}_x(\cdot \mid \text{Extinction})$. As a consequence, the generator $L^\star$ of the diffusion Y, which is by definition the diffusion Z conditioned on extinction, is given by

$$L^\star f(x) = \frac{1}{u(x)}L(uf)(x) \qquad x \geq 0,$$

where L is the generator of Z, and f sufficiently smooth. The last displayed equation is well-known, but we prove it again. Consider a function f such that uf belongs to the domain of the extended generator of Z, and uf, $L(uf)$ are non-negative or bounded. Then for $s < t$ and B in $\sigma(Z_u, u \leq s)$,

$$\mathbb{E}_x\bigg(\mathbb{1}_B\big((uf)(Z_t) - (uf)(Z_s) - \int_s^t L(uf)(Z_s)ds\big)\bigg) = 0.$$

Writing $\frac{L(uf)(Z_s)}{u(x)}$ as $\frac{L(uf)(Z_s)}{u(Z_s)}\frac{u(Z_s)}{u(x)}$, we deduce that

$$\mathbb{E}^*_x\bigg(\mathbb{1}_B\big(f(Z_t) - f(Z_s) - \int_s^t L^* f(Z_s)ds\big)\bigg) = 0,$$

which ensures the result.

Let us compute the generator $L^\star$ more explicitly. Because $Lu = 0$, it is easy to get

$$L^\star f(x) = \frac{\gamma}{2}xf''(x) + \bigg(h(x) + \gamma x\frac{u'(x)}{u(x)}\bigg) \qquad x \geq 0,$$

which ends the first part of the proposition.

For the second part, notice that H is strictly increasing after some x_0, so we consider its inverse φ on $[H(x_0), +\infty)$. Next observe that for $x > x_0$,

$$-\frac{u}{u'}(x) = e^{H(x)} \int_x^\infty e^{-H(z)} dz = e^{H(x)} \int_{H(x)}^\infty e^{-v} \varphi'(v) dv,$$

with the change $v = H(z)$. As a consequence, we can write for $y > H(x_0)$

$$-\frac{u}{u'}(\varphi(y)) = e^y \int_y^\infty e^{-v} \varphi'(v) dv = \int_0^\infty e^{-v} \varphi'(y+v) dv.$$

Because h tends to $+\infty$, $H(x) \geq (1+\varepsilon)\log(x)$ for x sufficiently large, so that $\varphi(y) \leq \exp(y/(1+\varepsilon))$, and $\varphi(y)\exp(-y)$ vanishes as $y \to \infty$. Now since $\varphi' = \gamma\varphi/2h \circ \varphi = o(\varphi)$, $\varphi'(y)\exp(-y)$ also vanishes. Since h is differentiable, H is twice differentiable, and so is φ, so performing an integration by parts yields

$$-\frac{u}{u'}(\varphi(y)) = \varphi'(y) + \int_0^\infty e^{-v} \varphi''(y+v) dv.$$

By the technical assumption given in the statement of the proposition,

$$\varphi'' \circ H(x) = \varphi' \circ H(x) \left(\frac{1}{H'}\right)'(x) = \frac{\gamma}{2} \varphi' \circ H(x) \left(\frac{1}{h(x)} - \frac{xh'(x)}{h(x)^2}\right) = o\left(\varphi' \circ H(x)\right).$$

Then the fact that $\varphi'' = o(\varphi')$ entails that

$$-\frac{u}{u'}(\varphi(y)) \sim_{y\to\infty} \varphi'(y).$$

This is equivalent to

$$\gamma x \frac{u'}{u}(x) \sim_{x\to\infty} -\gamma x H'(x) = -2h(x),$$

which ends the proof. $\square$

Finally we prove Theorem 1.3, which amounts to checking all hypotheses for the generalized Feller diffusion.

Proof of Theorem 1.3. According to what precedes, we may limit ourselves to the case $h \to -\infty$ at ∞.

For Z solution of (1.3), recall that $X_t = 2\sqrt{Z_t/\gamma}$, so that X satisfies the SDE

$$dX_t = dB_t - q(Z_t)dt,$$

with

$$q(x) = \frac{1}{2x} - \frac{2h(\gamma x^2/4)}{\gamma x} \qquad x > 0,$$

so that

$$q'(x) = -\frac{1}{2x^2} + \frac{2h(\gamma x^2/4)}{\gamma x^2} - h'(\gamma x^2/4)$$

and

$$q^2(x) - q'(x) = \frac{3}{4x^2} + h(\gamma x^2/4)\left(\frac{4}{\gamma^2 x^2} h(\gamma x^2/4) - \frac{4}{\gamma x^2}\right) + h'(\gamma x^2/4).$$

We recall

$$Q(x) = \int_1^x 2q(y)dy, \qquad \Lambda(x) = \int_1^x e^{Q(y)} dy,$$

and

$$\kappa(x)=\int_1^x e^{Q(y)}\left(\int_1^y e^{-Q(z)}dz\right)dy.$$

Straightforward calculations show that under Assumption (HH),

$$\forall a>0,\ \lim_{x\to\infty} x^a e^{-Q(x)}=0 \qquad \text{and} \qquad \lim_{x\to 0^+} Q(x)-\log(x)\in(-\infty,+\infty),$$

In particular, $\Lambda(\infty)=\infty$, and the integrand in the definition of κ is equivalent to $y\log(y)$, which ensures $\kappa(0^+)<+\infty$, so that X, and subsequently Z, is absorbed at 0 with probability 1.

In addition, check that under Assumption (HH), we have $q(x)\sim_{x\to 0^+} 1/2x$, as well as

$$q^2(x)-q'(x)\sim_{x\to 0^+}\frac{3}{4x^2} \qquad \text{and} \qquad (q^2-q')(2\sqrt{x/\gamma})\sim_{x\to\infty}\frac{h(x)^2}{x}\left(\frac{1}{\gamma}+\frac{xh'(x)}{h(x)^2}\right).$$

Then Assumption (H2), which ensures the discreteness of the spectrum L, is implied by (HH)(i) and (ii).

Next, Assumption (H3), ensuring the existence of a quasi stationary probability measure, always holds under (HH).

Recall that Assumption (H5) holds if and only if the process comes down from infinity. Thanks to Remark 7.6, there is a simple sufficient condition for (H5) to hold, which has three components. The first one is fulfilled thanks to (HH)(i). The second one can be shown to be equivalent to

$$\limsup_{x\to\infty}\frac{xh'(x)}{h(x)^2}<2/\gamma, \tag{A.1}$$

which is obviously true when h is non-increasing, and holds under (HH)(ii). The third one is equivalent to

$$\int_1^\infty \frac{dx}{h(x)}>-\infty. \tag{A.2}$$

In conclusion, all assumptions necessary for our results to hold, are fulfilled under (HH), except (H5). For (H5) to hold, one has to make the additional assumption (A.2), which, in particular, does not hold for pure continuous-state branching processes, but holds for logistic Feller diffusions. □

Appendix B. Proof of Lemma 4.4

We first prove the second bound. For any non negative and continuous function f with support in $\mathbb{R}^+$ we have from hypothesis (H2)

$$\int \tilde{p}_1(x,y)\,f(y)dy=\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\,\mathbb{1}_{1<T_0}(\omega)\,\exp\left(-\frac{1}{2}\int_0^1(q^2-q')(\omega_s)ds\right)\right]$$
$$\leq e^{C/2}\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\,\mathbb{1}_{1<T_0}(\omega)\right].$$

The estimate (4.3) follows at once by a limiting argument (letting f tend to the Dirac measure in y).

Let us now prove the upper bound in (4.2). Let B_1 be the function defined by

$$B_1(u)=\inf_{u\geq z}\left(q^2(u)-q'(u)\right).$$

We have

$$\int \tilde{p}_1(x,y)\ f(y)dy = \mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\ 1\!\!1_{1<T_0} 1\!\!1_{1<T_{x/3}}\ \exp\left(-\frac{1}{2}\int_0^1 (q^2-q')(\omega_s)ds\right)\right]$$

$$+\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\ 1\!\!1_{1<T_0}\ 1\!\!1_{1\geq T_{x/3}} \exp\left(-\frac{1}{2}\int_0^1 (q^2-q')(\omega_s)ds\right)\right]\ .$$

For the first expectation we have

$$\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\ 1\!\!1_{1<T_0} 1\!\!1_{1<T_{x/3}}\ \exp\left(-\frac{1}{2}\int_0^1 (q^2-q')(\omega_s)ds\right)\right]$$

$$\leq e^{-B_1(x/3)/2}\ \mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\ 1\!\!1_{1<T_0}\right]\ .$$

For the second expectation, we obtain

$$\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\ 1\!\!1_{1<T_0}\ 1\!\!1_{1\geq T_{x/3}} \exp\left(-\frac{1}{2}\int_0^1 (q^2-q')(\omega_s)ds\right)\right]$$

$$\leq e^{C/2}\ \mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\ 1\!\!1_{1<T_0} 1\!\!1_{1\geq T_{x/3}}\right]$$

$$= e^{C/2}\left(\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\ 1\!\!1_{1<T_0}\right] -\ \mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\ 1\!\!1_{1<T_{x/3}}\right]\right)\ .$$

Using a limiting argument as above and the invariance by translation of the law of the Brownian motion, and firstly assuming that $y/2<x<2y$, we obtain

$$\tilde{p}_1(x,y) \leq e^{-B_1(x/3)/2} p_1^D(x,y) + e^{C/2}\left(p_1^D(x,y) - p_1^D(2x/3, y-x/3)\right)\ .$$

From the explicit formula for p_1^D we have

$$p_1^D(x,y) - p_1^D(2x/3, y-x/3) = \frac{1}{\sqrt{2\pi}}\left(e^{-(y+x/3)^2/2} - e^{-(x+y)^2/2}\right) \leq \frac{1}{\sqrt{2\pi}}\ e^{-\max\{x,y\}^2/18}\ .$$

Since the function B_1 is non decreasing, we get for $y/2<x<2y$

$$\tilde{p}_1(x,y) \leq \frac{1}{\sqrt{2\pi}}\left(e^{-B_1(\max\{x,y\}/6)/2} + e^{-\max\{x,y\}^2/18}\right)\ .$$

If $x/y \notin]1/2, 2[$, we get from the estimate (4.3)

$$\tilde{p}_1(x,y) \leq \mathcal{O}(1)\ e^{-(y-x)^2/2} \leq \mathcal{O}(1) e^{-\max\{x,y\}^2/8}\ .$$

We now define the function B by

$$B(z) = \mathcal{O}(1) + \min\left\{B_1(z/6)/4\ ,\ z^2/36\right\}\ .$$

It follows from hypothesis (H2) that $\lim_{z\to+\infty} B(z) = +\infty$. Combining the previous estimates we get for any x and y in $\mathbb{R}^+$

$$\tilde{p}_1(x,y) \leq e^{-2B(\max\{x,y\})}\ .$$

The upper estimate (4.2) follows by taking the geometric average of this result and (4.3). We now prove that $\tilde{p}_1(x,y)>0$. For this purpose, let $a=\min\{x,y\}/2$ and $b=2\max\{x,y\}$. We have as above for f a non negative continuous function with support in $\mathbb{R}^+$

$$\int \tilde{p}_1(x,y)\ f(y)dy \geq \mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\ 1\!\!1_{1<T_{[a,b]}}\ \exp\left(-\frac{1}{2}\int_0^1 (q^2-q')(\omega_s)ds\right)\right]$$

where we denote by $T_{[a,b]}$ the exit time from the interval $[a,b]$. Let

$$R_{a,b} = \sup_{x\in[a,b]} \left(q^2(x) - q'(x)\right) ,$$

this quantity is finite since $q \in C^1((0,+\infty))$. We obtain immediately

$$\int \tilde{p}_1(x,y)\ f(y)dy \geq e^{-\ R_{a,b}/2} \int p_1^{[a,b]}(x,y)\ f(y)dy$$

where we denote by $p_t^{[a,b]}$ the heat kernel with Dirichlet conditions in $[a,b]$. The result follows from a limiting argument as above since $p_1^{[a,b]}(x,y) > 0$.

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