ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES

UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 41 50. Fax: 01 69 33 30 11

http://www.cmap.polytechnique.fr/

The volume preserving crystalline mean curvature flow of convex sets in $\mathbb{R}^N$

G. Bellettini, V. Caselles, A. Chambolle and M. Novaga

R.I. 622 *August 2007*

*

The volume preserving crystalline mean curvature flow of convex sets in $\mathbb{R}^N$

G. Bellettini [*] , **V. Caselles** [†] , **A. Chambolle** [‡] , **M. Novaga** [§]

Abstract

We prove the existence of a volume preserving crystalline mean curvature flat flow starting from a compact convex set $C \subset \mathbb{R}^N$ and its convergence, modulo a time-dependent translation, to a Wulff shape with the corresponding volume. We also prove that if C satisfies an interior ball condition (the ball being the Wulff shape), then the evolving convex set satisfies a similar condition for some time. To prove these results we establish existence, uniqueness and short-time regularity for the crystalline mean curvature flat flow with a bounded forcing term starting from C, showing the convergence of the Almgren-Taylor-Wang's algorithm in this case. Next we study the evolution of the volume and anisotropic perimeter, needed for the proof of the convergence to the Wulff shape as $t \to +\infty$.

Key words: crystalline mean curvature, volume preserving ϕ-regular and flat flows, convex bodies

AMS (MOS) subject classification: 53C44 35J60 49N60

1 Introduction

Mean curvature flow, which corresponds to the gradient flow of the area functional

$$E \to \int_{\partial E} 1 \, d\mathcal{H}^{N-1},$$

[*]Dipartimento di Matematica, Università di Roma "Tor Vergata" via della Ricerca Scientifica, 00133 Roma, Italy, e-mail: Giovanni.Bellettini@lnf.infn.it

[†]Departament de Tecnologies de la Informació i les Comunicacions, Universitat Pompeu-Fabra, Passeig de Circumval.lació 8, 08003 Barcelona, Spain, e-mail: vicent.caselles@upf.edu

[‡]CMAP, Ecole Polytechnique, CNRS, 91128 Palaiseau, France,
e-mail: antonin.chambolle@polytechnique.fr

[§]Dipartimento di Matematica, Università di Pisa, largo Bruno Pontecorvo 5, 56127 Pisa, Italy, e-mail: novaga@dm.unipi.it

is the evolution of a hypersurface ∂E with velocity in the direction of the unit normal ν^E at a point $x \in \partial E$ given by the sum of its principal curvatures at x. Such a flow has been studied by many authors since the works of Brakke [11], Huisken [21], and Gage-Hamilton [20], and several results have been obtained in the last two decades on the subject. For our purposes, we just recall here that in [21] it has been proved that a convex compact hypersurface shrinks to a point in finite time, while its shape approaches the shape of a sphere. Moreover, under the additional constraint that the volume enclosed by the hypersurface remains constant, the flow turns out to be defined for all times $t > 0$ and asymptotically converges to a sphere with exponential rate as $t \to +\infty$ [22].

More recently, Andrews [5] extended this result to the smooth anisotropic mean curvature flow. Namely, let us consider the anisotropic area functional P_ϕ, defined as

$$P_\phi(E) := \int_{\partial E} \phi^\circ(\nu^E)\, d\mathcal{H}^{N-1},$$

where $\phi^\circ : \mathbb{R}^N \to [0, +\infty)$ (the surface tension) is an even positively one-homogeneous function such that $\{\phi^\circ \le 1\}$ is a smooth compact uniformly convex set with nonempty interior. Then anisotropic mean curvature flow is the gradient flow of P_ϕ, and becomes the evolution of a hypersurface with normal velocity given by

$$\kappa_\phi^E := \operatorname{div} n_\phi, \qquad n_\phi := \phi^o(\nu^E)\nabla\phi^o(\nu^E) \qquad \text{on } \partial E, \tag{1.1}$$

and n_ϕ is sometimes called the Cahn-Hoffman vector field. In [5] it is proved that a convex hypersurface evolving by anisotropic mean curvature flow with constant volume (and with a quite arbitrary mobility) converges to the Wulff shape as $t \to +\infty$. The Wulff shape W_ϕ is defined as (a rescaled of) the solution of the minimum problem

$$\inf\{P_\phi(E) : |E| = \text{const}\}$$

and it turns out that $W_\phi = \{\phi \le 1\}$, where $\phi(\xi) := \sup\{\langle \eta, \xi\rangle : \phi^o(\eta) \le 1\}$ for any $\xi \in \mathbb{R}^N$.

In this paper we are interested in the case when $N \ge 3$ and $\{\phi^\circ \le 1\}$ is neither strictly convex nor smooth; in this respect, we say that the anisotropy ϕ° is *crystalline* if $\{\phi^\circ \le 1\}$ is a polyhedron.

Due to the lack of differentiability and strict convexity of the surface tension, many of the techniques employed in [22, 5] are not available in this case, therefore we adopt a completely different approach, which is based more on the variational nature of the flow than on the direct analysis of the evolution equation. Such a variational approach has been introduced by Almgren-Taylor-Wang [1] and Luckhaus-Sturzenhecker [25], where a general existence result for weak evolutions is established.

We show that the volume preserving crystalline mean curvature flow starting from a convex set (with a "natural" mobility) converges to the Wulff shape of the same volume as $t \to +\infty$, modulo a time-dependent translation. Let us observe that it is not true in general that the crystalline convex mean curvature flow (which disappears in finite time) converges to the Wulff shape after an appropriate rescaling,

as can be shown by explicit computations [29, 27].

Let us describe in detail the content and the results of the paper. In Section 2 we introduce the notion of rW_ϕ-regular flows (see also [9, 8]), which correspond to regular evolutions, in the general setting of this paper. The first part of the paper is devoted to prove the existence and uniqueness of rW_ϕ-regular and flat crystalline mean curvature ϕ-flows with forcing term and is the purpose of Sections 3 and 4. Following some ideas from [9, 17], in Section 3 we show that, if an rW_ϕ-regular flow with a time dependent forcing term $c \in L^\infty((0, +\infty))$ exists, then it is *unique* (Theorem 3.1 and Corollary 3.2). We remark that this result is valid without any convexity assumption on the initial data. The uniqueness property is a consequence of some stability estimates (Proposition 3.4 and Corollary 3.5), that allow to establish also the comparison principle. The proof of Theorem 3.1 is based on the use of the time discrete operator T_h^c defined in (3.1), (3.3), introduced by Merriman, Bence and Osher in [26], and developed further in [19], [23]. We adapt in particular some ideas from [17] to treat the case when the forcing term is present. The object of Section 4 is to prove the *existence* of a convex rW_ϕ-regular flow with forcing, which is more involved, and the *existence and uniqueness of convex flat ϕ-flows*, also with forcing term, for initial compact convex sets. In Theorem 4.5 we state the local existence of an $\frac{r}{2}W_\phi$-regular flow with forcing term starting from a compact convex set C satisfying the rW_ϕ-interior condition. Together with the results of the previous section, we therefore can conclude that such a flow is unique. The proof of Theorem 4.5 is based on a weak formulation of the evolution problem (and this is the reason for which the existence part is more involved) and is the same as in [8], with the minor modification of the presence of the time dependent forcing term c, and therefore is not presented here. Also, in Section 4, using the approach of Almgren-Taylor-Wang [1], we define the convex flat ϕ-flow with forcing, by means of the discrete operator S_h^c. Once convex flat ϕ-flows exist, they provide weak evolutions defined for all times. Theorem 4.3 shows that the algorithm of Almgren-Taylor-Wang converges along a subsequence, under the convexity assumption of the initial datum C. In addition, the convex rW_ϕ-regular flow is also obtained as the limit of the algorithm based on the operators S_h^c. Therefore, in the convex case, the flat ϕ-flow has the *consistency property*, namely it coincides with the rW_ϕ-regular flow for all times till the latter exists. Also the proof of Theorem 4.3 is essentially the same as in the one in [13], and is omitted.

Theorem 4.9 shows that the flat ϕ-flow with forcing is unique, and therefore the discrete algorithm has a unique limit. We also show in Lemma 4.7 that two such flows stay close to each other if the corresponding forcing terms are close. Uniqueness and stability are proved in Theorems 4.9 and 4.11 respectively.

In Section 5 we study the evolution of the volume $|C(t)|$ and the anisotropic perimeter $P_\phi(C(t))$ for a convex flat ϕ-flow $C(t)$ with forcing. In Proposition 5.4 we give an estimate on the rate of change of P_ϕ, which is also used for characterizing the asymptotic limit of convex volume preserving flat ϕ-flows in Section 7.1. Formula (5.27) of Theorem 5.12 gives the evolution equation for $|C(t)|$, and allows to express the volume preserving ϕ-flow as a crystalline mean curvature flow with a suitable forcing term.

In Section 6 we study the convex volume preserving crystalline mean curvature flow which is defined via the discrete algorithm considered in Section 4, where however now the forcing term depends on the evolving set itself, see (6.1). In Theorem 6.2, valid without the assumption of the interior ball condition on the initial datum, the existence of a flat ϕ-curvature flow with preserved volume is given. A uniqueness result for volume preserving convex rW_ϕ-regular flows is given in Theorem 6.5. Finally, in Section 7 we prove that a volume preserving convex flat ϕ-flow starting from a compact convex set C converges to the Wulff shape of volume $|C|$ as $t \to +\infty$ modulo a time-dependent translation.

Appendix A contains three equivalent ways of expressing the property that a convex body has bounded crystalline mean curvature. This result is essentially contained in [14], though not explicitly stated there, and we include it here for the sake of completeness.

2 Notation and setting

2.1 Anisotropies and ϕ-distance function

Let $\phi : \mathbb{R}^N \to [0, +\infty)$ be a convex function satisfying the one-homogeneity condition

$$\phi(\lambda\xi) = |\lambda|\phi(\xi) \qquad \forall \xi \in \mathbb{R}^N,\ \forall \lambda \in \mathbb{R}, \tag{2.1}$$

and the nondegenerate condition

$$m|\xi| \le \phi(\xi) \qquad \forall \xi \in \mathbb{R}^N, \tag{2.2}$$

for some $m > 0$. We let $W_\phi := \{\phi \le 1\}$ (Wulff shape) and $rW_\phi := \{\phi \le r\}$ when $r > 0$. The dual function ϕ° of ϕ (called surface tension) is defined as $\phi^o(\xi) := \sup\{\langle \eta, \xi\rangle : \phi(\eta) \le 1\}$ for any $\xi \in \mathbb{R}^N$, and turns out to be convex; moreover, it is one-homogeneous, nondegenerate and $(\phi^\circ)^\circ = \phi$. ϕ^o (and ϕ) is sometimes called anisotropy.

We write $\phi \in \mathcal{C}^\infty_+$ if ϕ^2 is of class $\mathcal{C}^\infty(\mathbb{R}^N \setminus \{0\})$ and there exists a constant $\alpha > 0$ such that $\nabla^2(\phi^2) \ge \alpha \,\mathrm{Id}$ in $\mathbb{R}^N \setminus \{0\}$.

The ball condition property reads as follows.

Definition 2.1. *Let $C \subset \mathbb{R}^N$ be a set with $\mathrm{int}(C) \ne \emptyset$ and $r > 0$. We say that C satisfies the interior (resp. exterior) rW_ϕ-condition if, for any $x \in \partial C$, there exists $y \in \mathbb{R}^N$ such that*

$$rW_\phi + y \subseteq \overline{C} \qquad \text{and} \qquad x \in \partial\,(rW_\phi + y)$$
$$\Big(\text{resp.}\quad rW_\phi + y \subseteq \overline{\mathbb{R}^N \setminus C} \qquad \text{and} \qquad x \in \partial\,(rW_\phi + y)\Big).$$

We denote by $\partial\phi(\xi)$ the subdifferential of ϕ at $\xi \in \mathbb{R}^N$. If ϕ is differentiable at ξ, we write $\nabla\phi(\xi)$ in place of $\partial\phi(\xi)$.

Given a nonempty set $C \subseteq \mathbb{R}^N$, we let

$$d^\phi(x, C) := \inf_{y \in C} \phi(x - y), \qquad x \in \mathbb{R}^N,$$

and for $\delta > 0$ we set

$$C^+_\delta := \{x \in \mathbb{R}^N : d_\phi(x, C) < \delta\}, \qquad C^-_\delta := \{x \in \mathbb{R}^N : d_\phi(x, \mathbb{R}^N \setminus C) < \delta\}.$$

We denote by d^ϕ_C the signed ϕ–distance function to ∂C negative inside C, that is

$$d^\phi_C(x) := d_\phi(x, C) - d_\phi(x, \mathbb{R}^N \setminus C), \qquad x \in \mathbb{R}^N. \tag{2.3}$$

Observe that $|d^\phi_C(x)| = d_\phi(x, \partial C)$.

The function d^ϕ_C is Lipschitz and at each point x where it is differentiable we have $\phi^\circ(\nabla d^\phi_C(x)) = 1$. We set

$$\nu^C_\phi := \nabla d^\phi_C \qquad \text{on } \partial C, \tag{2.4}$$

at those points where ∇d^ϕ_C exists. Note that $\langle \nabla d^\phi_C, n\rangle = 1$ when $n \in \partial\phi^o(\nabla d^\phi_C)$.

Observe that the signed ϕ-distance d^ϕ_C from a compact set C is convex if and only if C is convex.

For $A, B \subseteq \mathbb{R}^N$ we let $d_\phi(A, B) := \inf\{\phi(x-y) \,:\, x \in A, y \in B\}$ the ϕ-distance between A and B.

Definition 2.2. *Let $t_1 < t_2$, $c \in L^\infty((t_1, t_2))$ and $r > 0$. An rW_ϕ-regular mean curvature flow with forcing term c in $[t_1, t_2]$ is a map $t \in [t_1, t_2] \to E(t) \subset \mathbb{R}^N$ satisfying the following properties:*

(i) $E(t)$ is closed, has compact Lipschitz boundary, and satisfies the interior and exterior rW_ϕ-condition;

(ii) there exists an open neighborhood A of $\cup_{t\in[t_1,t_2]}(\partial E(t) \times \{t\})$ in $\mathbb{R}^N \times [t_1, t_2]$ such that, if we set

$$d(x,t) := d^\phi_{E(t)}(x), \qquad (x,t) \in \mathbb{R}^N \times [0, +\infty), \tag{2.5}$$

then $d \in \mathrm{Lip}(A)$;

(iii) there exists a vector field $n : A \to \mathbb{R}^N$ such that $n \in \partial\phi^\circ(\nabla d)$ almost everywhere in A, and $\mathrm{div}\, n \in L^\infty(A)$;

(iv) there exists $\lambda > 0$ such that

$$\left|\frac{\partial d}{\partial t}(x,t) - \mathrm{div}\, n(x,t) + c(t)\right| \;\le\; \lambda |d(x,t)| \qquad \text{for a.e. } (x,t) \in A. \tag{2.6}$$

2.2 ϕ-total variation and anisotropic perimeter

Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a (vector valued) Radon measure with finite total variation $|Du|(\Omega)$ in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$, see [4]. We denote by $BV_{\mathrm{loc}}(\Omega)$ the space of functions $w \in L^1_{\mathrm{loc}}(\Omega)$ such that $w\varphi \in BV(\Omega)$ for any smooth function φ with compact support in Ω.

A measurable set $E \subseteq \mathbb{R}^N$ is said to be of finite perimeter in Ω if $|D\chi_E|(\Omega) < \infty$. The (euclidean) perimeter of E in Ω is defined as $P(E, \Omega) := |D\chi_E|(\Omega)$, and we have $P(E, \Omega) = P(\mathbb{R}^N \setminus E, \Omega)$. We shall use the notation $P(E) := P(E, \mathbb{R}^N)$.

Let $u \in BV(\Omega)$. We define the anisotropic total variation of u with respect to ϕ in Ω [3] as

$$\int_\Omega \phi^\circ(Du) := \sup\left\{\int_\Omega u \,\mathrm{div}\,\sigma \; dx : \sigma \in \mathcal{C}^1_c(\Omega; \mathbb{R}^N), \phi(\sigma(x)) \le 1 \;\forall x \in \Omega\right\}. \tag{2.7}$$

If $E \subseteq \mathbb{R}^N$ has finite perimeter in Ω, we set

$$P_\phi(E, \Omega) := \int_\Omega \phi^\circ(D\chi_E)$$

and we have [3]

$$P_\phi(E,\Omega) = \int_{\Omega\cap\partial^* E} \phi^\circ(\nu^E)\, d\mathcal{H}^{N-1} =: \int_{\Omega\cap\partial^* E} 1\ dP_\phi, \tag{2.8}$$

where $\partial^* E$ is the reduced boundary of E and ν^E the (generalized) outer unit normal to E at points of $\partial^* E$. We shall use the notation $P_\phi(E) := P_\phi(E, \mathbb{R}^N)$.

Recall that, since ϕ° is homogeneous, $\phi^\circ(Du)$ coincides with the nonnegative Radon measure in $\mathbb{R}^N$ given by

$$\phi^\circ(Du) = \phi^\circ(\nabla u(x))\, dx + \phi^\circ\left(\frac{D^s u}{|D^s u|}\right) |D^s u|,$$

where $\nabla u(x)\, dx$ is the absolutely continuous part of Du, and $D^s u$ its singular part.

3 Stability of rW_ϕ-regular flows with forcing

In this section we derive some stability estimates, comparison and uniqueness for the flows of Definition 2.2. These will be deduced from estimates for the Merriman-Bence-Osher [26] approximation algorithms, which is shown to converge to the flow.

3.1 The Merriman-Bence-Osher algorithm

As in [17], we introduce the anisotropic generalization of the Merriman-Bence-Osher algorithm with a forcing term. Following [24], the forcing term is enforced by thresholding at a suitable level the solution of a heat-type partial differential inclusion at time h, h the time discretization step.

Given a constant $\bar{c} \in \mathbb{R}$, a closed set $E \subset \mathbb{R}^N$ with compact boundary and $h > 0$ sufficiently small, define

$$T_h^{\bar{c}}(E) := \left\{ x \in \mathbb{R}^N \ :\ u(x,h) \geq \frac{1}{2} - \frac{\bar{c}}{2\sqrt{\pi}}\sqrt{h} \right\}, \tag{3.1}$$

where $u : \mathbb{R}^N \times [0,+\infty) \to [0,1]$ is the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} \ \in \ \mathrm{div}\left(\phi^\circ(\nabla u)\partial\phi^\circ(\nabla u)\right) & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(\cdot,0) \ = \ \chi_E(\cdot) & \text{in } \mathbb{R}^N \times \{t=0\}, \end{cases} \tag{3.2}$$

and χ_E is the characteristic function of E. The function u is well defined and unique by classical results on contraction semigroups [12]: if E is compact, it corresponds to the flow in $L^2(\mathbb{R}^N)$ of the subdifferential of the functional $u \mapsto \frac{1}{2}\int_{\mathbb{R}^N} (\phi^\circ(\nabla u))^2\, dx$ if $u \in H^1(\mathbb{R}^N)$, and extended to $+\infty$ otherwise. On the other hand, if $\overline{\mathbb{R}^N \setminus E}$ is compact, one defines u by letting $u := 1 + v$ where v solves the same equation with initial datum $\chi_E - 1$.

The idea is that an evolution $t \to E(t)$ starting from $E(t_1)$ can be approximated with

$$\begin{aligned} \mathcal{E}_h(t) \;&:=\; T_h^{c_h^{n-1}}\, T_h^{c_h^{n-2}} \cdots T_h^{c_h^0}(E(t_1)), \qquad n \geq 1, \\ \text{where } \; n &:= \left[\frac{t - t_1}{h}\right] \;\text{ and }\; c_h^i := \frac{1}{h}\int_{t_1+ih}^{t_1+(i+1)h} c(s)\,ds \end{aligned} \tag{3.3}$$

(here $[\ell]$ denotes the integer part of $\ell \in [0, +\infty)$). In particular, our theorem states that the anisotropic Merriman-Bence-Osher scheme is consistent with the evolutions given by Definition 2.2.

Theorem 3.1. *Let $t \in [t_1, t_2] \to E(t)$ be an rW_ϕ-regular flow with forcing term $c \in L^\infty((t_1, t_2))$. Then, for any $t \in [t_1, t_2]$, $\partial\mathcal{E}_h(t)$ converges to $\partial E(t)$ in the Hausdorff distance $d_{\mathcal{H}}$, as $h \to 0$.*

The proof of this theorem relies on an estimate for the approximate flow which is computed in Section 3.2, and is given in Section 3.3. We can deduce the following corollaries.

A first corollary, also proven in [9], shows that if an rW_ϕ-regular flow exists, then it is unique.

Corollary 3.2. *Let $t \in [t_1, t_2] \to E(t), F(t)$ be two rW_ϕ-regular flows with forcing term $c \in L^\infty((t_1, t_2))$. Assume $E(t_1) \subseteq F(t_1)$. Then $E(t) \subseteq F(t)$ for all $t \in [t_1, t_2]$. In particular, if $E(t_1) = F(t_1)$, then $E(t) = F(t)$ for all $t \in [t_1, t_2]$.*

An additional, more precise stability property will be shown at the end of Section 3.3 (Corollary 3.5). The next corollary follows, with a standard proof [7], from the monotonicity and consistency of the scheme.

Corollary 3.3. *Assume that ϕ, $\phi^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$ and are uniformly convex. Let $E \subset \mathbb{R}^N$ be a closed set with compact boundary and denote by $E_{\mathrm{ls}}(t)$ the level set ϕ-curvature flow starting from E on a time interval $[0, T)$; assume in addition that no fattening occurs [18]. Then $\partial\mathcal{E}_h(t) \to \partial E_{\mathrm{ls}}(t)$ in the Hausdorff distance for any $t < T$, as $h \to 0$.*

Let us observe that Corollary 3.3 follows from Theorem 3.1 when evolutions according to Definition 2.2 are known to exist.

3.2 Estimate for the time-discrete flow $T_h^{\bar{c}}$

In this section we consider an rW_ϕ-regular flow with forcing, and we show (Proposition 3.4) an estimate on one step of the algorithm applied to $E(t)$ or a neighboring set.

Let $E(t)$ be an rW_ϕ-regular flow on $[t_1, t_2]$ with forcing term $c \in L^\infty((t_1, t_2))$. Possibly choosing a smaller A and reducing r, we may assume that A is of the form $A' \times [t_1, t_2]$ (A' open subset of $\mathbb{R}^N$), and that $\{|d(\cdot, t)| \le r\} \subset A$ for any $t \in [t_1, t_2]$, where we recall that d is defined in (2.5).

Proposition 3.4. *Fix $\delta \in [0, r/2]$ and $t \in [t_1, t_2)$. Then, for any $\alpha \in \mathbb{R}$ and $\varepsilon \in (0, r/2)$, there exists $h_0 > 0$ depending only on $\varepsilon, |\alpha|, \|c\|_{L^\infty((t_1,t_2))}$, such that if $h \in (0, \min(h_0, t_2 - t)]$ we have*

$$T_h^{c_h+\alpha}\big(\{d(\cdot, t) \le \delta\}\big) \ \subseteq \ \big\{d(\cdot, t+h) \le (1+\lambda h)\delta + \alpha h + (1+2\lambda)\varepsilon h\big\}, \qquad (3.4)$$

where

$$c_h := \frac{1}{h}\int_t^{t+h} c(s)\, ds, \qquad (3.5)$$

$T_h^{c_h+\alpha}$ is defined in (3.1) *and λ is as in* (2.6).

Proof. Let u be the solution of the anisotropic heat inclusion with initial datum $\chi_{\{d(\cdot,t)\le\delta\}}$:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} \ \in \ \operatorname{div}\Big(\phi^\circ(\nabla u)\partial\phi^\circ(\nabla u)\Big) & \text{in } \mathbb{R}^N \times (0, t_2 - t_1), \\ u(\cdot, 0) \ = \ \chi_{\{x\in\mathbb{R}^N : d(x,t)\le\delta\}}(\cdot) & \text{in } \mathbb{R}^N \times \{\tau = 0\}. \end{cases} \qquad (3.6)$$

We estimate $u(\cdot, h)$ for small h with a suitable supersolution v of (3.6). Define

$$g(\tau) := \int_t^{t+\tau} c(s)\, ds, \qquad \tau \in [0, t_2 - t],$$

and

$$v(x, \tau) := \gamma\big(-d(x, t+\tau) + \delta - g(\tau) + \lambda\eta\tau, \tau\big) + h, \qquad \forall x\ \mathbb{R}^N, \forall \tau \in [0, t_2 - t], \quad (3.7)$$

where $\eta > 0$ is a small parameter which will be fixed later on (see (3.15)), and $\gamma : \mathbb{R} \times [0, +\infty) \to [0, 1]$ is the solution of the one-dimensional heat equation starting from the Heaviside function:

$$\begin{cases} \dfrac{\partial \gamma}{\partial \tau}(\xi, \tau) \ = \ \gamma_{\xi\xi}(\xi, \tau)\,, & \xi \in \mathbb{R},\ \tau > 0, \\ \gamma(\cdot, 0) = \chi_{[0,+\infty)}(\cdot)\,, & \tau = 0, \end{cases} \qquad (3.8)$$

where we shorthand $\gamma_\xi = \frac{\partial\gamma}{\partial\xi}$ and $\gamma_{\xi\xi} = \frac{\partial^2\gamma}{\partial\xi^2}$, and we recall that

$$\gamma(\xi, \tau) \ = \ \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\xi} e^{-\frac{s^2}{4\tau}}\, ds = \gamma\left(\frac{\xi}{\sqrt{\tau}}, 1\right) \ =: \ \gamma_1\left(\frac{\xi}{\sqrt{\tau}}\right).$$

We first observe that

$$v(x,0) = \chi_{[0,+\infty)}(-d(x,t)+\delta) + h = \chi_{\{d(\cdot,t)\le\delta\}}(x) + h > \chi_{\{d(\cdot,t)\le\delta\}}(x) = u(x,0).$$

Furthermore, for almost every $(x,\tau) \in A' \times [t_1,t_2]$,

$$\frac{\partial v}{\partial \tau}(x,\tau) = \left(-\frac{\partial d}{\partial t}(x,t+\tau) - c(t+\tau)\right)\gamma_\xi(\bullet) + \lambda\eta\gamma_\xi(\bullet) + \frac{\partial \gamma}{\partial \tau}(\bullet) \tag{3.9}$$

where $(\bullet)$ means $\big(-d(x,t+\tau)+\delta-g(\tau)+\lambda\eta\tau,\tau\big)$, and

$$\nabla v(x,\tau) \;=\; -\gamma_\xi(\bullet)\nabla d(x,t+\tau). \tag{3.10}$$

Since $\gamma_\xi > 0$, from (2.1) we have, $\phi^\circ(\nabla v) = \gamma_\xi$ while $\partial\phi^\circ(\nabla v(x,\tau)) = -\partial\phi^\circ(\nabla d(x,t+\tau)) \ni -n(x,t+\tau)$, where n is as in Definition 2.2 (iii). Hence, the vector field Z defined as

$$Z(x,\tau) := -n(x,t+\tau)\gamma_\xi(\bullet) \qquad \forall (x,\tau) \in A' \times [t_1,t_2],$$

is such that $Z \in \phi^\circ(\nabla v)\partial\phi^\circ(\nabla v)$ almost everywhere in $A' \times [t_1,t_2]$. Moreover, recalling that $\langle n(x,t+\tau), \nabla d(x,t+\tau)\rangle = 1$, we also have

$$\operatorname{div} n(x,t+\tau)\gamma_\xi(\bullet) = -\operatorname{div} Z(x,\tau) + \gamma_{\xi\xi}(\bullet). \tag{3.11}$$

From (2.6) it follows

$$-\frac{\partial d}{\partial t}(x,t+\tau) - c(t+\tau) \ge -\operatorname{div} n(x,t+\tau) - \lambda|d(x,t+\tau)|.$$

Therefore, using (3.11),

$$\left(-\frac{\partial d}{\partial t}(x,t+\tau) - c(t+\tau)\right)\gamma_\xi(\bullet) \ge \operatorname{div} Z(x,\tau) - \gamma_{\xi\xi}(\bullet) - \lambda|d(x,t+\tau)|\gamma_\xi(\bullet). \tag{3.12}$$

From (3.9), (3.12) and (3.8) we deduce

$$\frac{\partial v}{\partial \tau}(x,\tau) \;\ge\; \operatorname{div} Z(x,\tau) + \lambda(\eta - |d(x,t+\tau)|)\gamma_\xi(\bullet) \qquad \text{a.e. in } A' \times [0,t_2-t_1]. \tag{3.13}$$

Therefore v is a supersolution of (3.6) in $\{|d| \le \eta\}$, provided we show it is also above u on its parabolic boundary.

We claim that $v \ge u$ on the parabolic boundary of the set

$$B \;:=\; \{(x,\tau) \in A' \times [0,h] : d(x,t) \le \delta+\varepsilon, d(x,t+\tau) \ge \delta - 2\varepsilon\},$$

provided h is less than some h_0 depending only on ϵ and $c_{L^\infty((t_1,t_2))}$. Let $x \in A'$ be such that $d(x,t) = \delta+\varepsilon \le r$. Then, as in [17, Lemma 3.2], one shows that there exists $\tau_1 = \tau_1(\epsilon)$ independent of δ, such that

$$u(x,\tau) \le \tau \qquad \forall \tau \le \tau_1(\varepsilon).$$

This is obtained by comparison with the evolution starting from $1 - \chi_{\{y\in\mathbb{R}^N:\phi(y-x)\le\varepsilon\}}(\cdot)$, which is above $u(\cdot,0)$. Indeed, since $E(t)$ satisfies the rW_ϕ-condition and $\delta < r/2$, it follows that $\{d(\cdot,t) < \delta\}$ satisfies the $\frac{r}{2}W_\phi$-condition. From this and $\epsilon < r/2$ it follows that $\{d(\cdot,t) < \delta\} \cap \{y \in \mathbb{R}^N : \phi(y-x) \le \varepsilon\} = \emptyset$, which in turn implies

$$1 - \chi_{\{y\in\mathbb{R}^N:\phi(y-x)\le\varepsilon\}}(\cdot) \ge u(\cdot,0).$$

Hence, if $h \le \tau_1$, one has $v(x,\tau) \ge h \ge u(x,\tau)$ as long as $\tau \le h$.
Choose now (x,τ) with $d(x,t+\tau) = \delta - 2\varepsilon \ge -r$. Then

$$-d(x,t+\tau) + \delta - g(\tau) + \lambda\eta\tau \;=\; 2\varepsilon - g(\tau) + \lambda\eta\tau \;\ge\; \varepsilon \tag{3.14}$$

as long as $\tau \le \tau_2 := \varepsilon/\|c\|_{L^\infty((t_1,t_2))}$. We now recall that in [17, Lemma 3.1], it is proved that for any $\varepsilon > 0$, there exists $\tau_0 > 0$ such that $\gamma(\varepsilon,\tau) \ge 1-\tau$ for any $\tau \in [0,\tau_0]$. Therefore, using (3.14), one finds that $v(x,\tau) \ge \gamma(\epsilon,\tau) + h \ge 1 - \tau + h$ as long as $\tau \le \min\{\tau_0,\tau_2\}$. In particular, if

$$h \le h_0 := \min\{\tau_0,\tau_1,\tau_2\},$$

we also have $v(x,\tau) \ge 1 \ge u(x,\tau)$ as long as $\tau \le h$. The proof of the claim is concluded.

The claim, together with (3.13), imply that v is a supersolution of (3.6) in B, provided $\{|d| \le \eta\} \supseteq B$, hence as soon as $\eta \ge \delta + 2\varepsilon$. We therefore let

$$\eta := \delta + 2\varepsilon. \tag{3.15}$$

By standard parabolic estimates, we deduce that $v(x,h) \ge u(x,h)$ if $d(x,t) \le \delta + \varepsilon$, $\delta - 2\varepsilon \le d(x,t+h)$, as soon as $h \le h_0$ (and $t + h \le t_2$).

Recalling that $c_h = g(h)/h$ and that

$$T_h^{c_h+\alpha}(\{d(\cdot,t) \le \delta\}) = \left\{x \in \mathbb{R}^N : u(x,h) \ge \frac{1}{2} - \frac{c_h+\alpha}{2\sqrt{\pi}}\sqrt{h}\right\},$$

we deduce, using (3.15),

$$\begin{aligned} &T_h^{c_h+\alpha}\left(\{d(\cdot,t) \le \delta\}\right) \subseteq \\ &\qquad \left\{\gamma\left(-d(\cdot,t+h) + \delta - hc_h + \lambda(\delta+2\varepsilon)h, h\right) \ge \frac{1}{2} - (c_h+\alpha)\frac{\sqrt{h}}{2\sqrt{\pi}} - h\right\}. \end{aligned} \tag{3.16}$$

As shown in [17], we have that $\gamma(\cdot,h)^{-1}(1/2 - (c_h+\alpha)\sqrt{h}/(2\sqrt{\pi}) - h) = -(c_h+\alpha)h + o(h)$ where the infinitesimal $o(h)$ only depends on $|\alpha| + \|c\|_{L^\infty((t_1,t_2))}$. Hence (3.16) becomes

$$\begin{aligned} &T_h^{c_h+\alpha}\left(\{d(\cdot,t) \le \delta\}\right) \subseteq \\ &\qquad \left\{x \in \mathbb{R}^N : -d(x,t+h) + \delta - hc_h + \lambda(\delta+2\varepsilon)h \ge -(c_h+\alpha)h + o(h)\right\}. \end{aligned} \tag{3.17}$$

Possibly reducing h_0 (still depending only on $\varepsilon, |\alpha|, \|c\|_{L^\infty((t_1,t_2))}$), we have $o(h) \le \varepsilon h$ so that (3.4) is deduced from (3.17). $\square$

3.3 Consistency of the algorithm and stability of rW_ϕ-regular flows

We are now in the position to prove Theorem 3.1. Let us fix $\delta > 0$ and $\alpha \in \mathbb{R}$. Let $\mathcal{E}_h^{\pm\delta,\alpha}(t)$ be the time discrete evolution with step h, as given by (3.3), but starting from the set $\{d(\cdot, t_1) \le \pm\delta\}$, and with a speed given by $c(t) + \alpha$:

$$\mathcal{E}_h^{\pm\delta,\alpha}(t) \;:=\; T_h^{c_h^{n-1}+\alpha} T_h^{c_h^{n-2}+\alpha} \cdots T_h^{c_h^{0}+\alpha}(\{d(\cdot,t_1) \le \pm\delta\}),$$

where c_h is defined in (3.5). From estimate (3.4), it follows that for any $\varepsilon > 0$, if $h > 0$ is small enough, one has for any $i \ge 0$ with $t_1 + ih \le t_2$ and $t_1 + (n-1)h < t \le t_1 + nh$

$$\mathcal{E}_h^{\delta,\alpha}(t_1 + ih) \;\subseteq\; \{d(\cdot, t_1 + ih) \le \delta_i\}, \tag{3.18}$$

as long as $0 \le \delta_i \le r/2$, where δ_i is defined as follows: $\delta_0 := \delta$ and

$$\delta_{i+1} \;:=\; (1 + \lambda h)\delta_i + \alpha h + (1 + 2\lambda)\varepsilon h\,.$$

By induction, we find

$$\delta_i \;=\; (1+\lambda h)^i \delta \;+\; \big(\alpha + (1+2\lambda)\,\varepsilon\big)\frac{(1+\lambda h)^i - 1}{\lambda}. \tag{3.19}$$

Letting $\delta_h(t) := \delta_{i(t)}$ with $i(t) = [(t - t_1)/h]$, we have that $\delta_h(t)$ converges, as $h \to 0$, to the function

$$\delta(t) := e^{\lambda(t-t_1)}\delta \;+\; \big(\alpha + (1+2\lambda)\,\varepsilon\big)\frac{e^{\lambda(t-t_1)} - 1}{\lambda}, \qquad t \ge t_1\,. \tag{3.20}$$

Let us observe that, by symmetry of the scheme, we also have

$$\mathcal{E}_h^{-\delta,-\alpha}(t_1 + ih) \;\supseteq\; \{d(\cdot, t_1 + ih) \le -\delta_i\}, \tag{3.21}$$

as long as $0 \le \delta_i \le r/2$. The proof of Theorem 3.1 then follows from (3.18) and (3.21), choosing $\delta = \alpha = 0$: indeed for any $\varepsilon > 0$, we find that any Hausdorff limit of $\partial\mathcal{E}_h(t)$, as $h \to 0$, lies in $\{(x,t) : |d(x,t)| \le \delta(t) \le r/2\}$ with $\delta(t) = \varepsilon(1+2\lambda)(\exp(\lambda(t-t_1)) - 1)/\lambda$. Letting $\varepsilon \to 0$ we get the convergence. $\square$

In a similar way, we derive from Theorem 3.1 and estimates (3.18), (3.21) the following result:

Corollary 3.5. *let $t \in [t_1, t_2] \to E_1(t), E_2(t)$ be two rW_ϕ-regular flows defined in the same open set A, with forcing terms c_1, c_2, respectively. Let $r > 0$ be such that the flow $E_1(t)$ satisfies the rW_ϕ-condition for any $t \in [t_1, t_2]$ and $\{(x,t) : |d^\phi_{E_1(t)}(x)| \le r\} \subset A$. Letting $\delta := \mathrm{dist}_{\mathcal{H}}(\partial E_1(t_1), \partial E_2(t_1))$, we have for all $t \in [t_1, t_2]$ with $\delta(t) \le r/2$,*

$$\mathrm{dist}_{\mathcal{H}}\big(\partial E_1(t), \partial E_2(t)\big) \le \delta(t), \tag{3.22}$$

where $\delta(t)$ is defined as in (3.20) *with $\varepsilon = 0$ and $\alpha = \|c_1 - c_2\|_{L^\infty((t_1,t_2))}$.*

4 Convex flat ϕ-flows with forcing

If the initial set is convex, it happens that the flow remains convex for subsequent times, whatever the anisotropy. This strong regularity property allows to build unique flows in the convex, rW_ϕ-regular case, and by comparison to define convex flows starting from an arbitrary compact convex set. This section relies on two previously released papers where the situation with no forcing term was investigated [13, 8]. These papers are quite long and, for the second, very technical and we cannot recall all the results in all details. The construction for showing existence relies on an implicit time-discretization scheme first proposed by Almgren, Taylor and Wang [1] (see also Luckhaus-Sturzenhecker [25]).

4.1 Existence and uniqueness of convex flat ϕ-flows

Let us shortly recall the basic ingredients of the approach in [13]. Let $C \subset \mathbb{R}^N$ be a compact convex set and $\bar{c} \in \mathbb{R}$. Let us consider the equation

$$u - h \operatorname{div} \partial\phi^\circ(\nabla u) \ + h\bar{c} - \ d^\phi_C \ \ni \ 0 \ \text{ in } \mathbb{R}^N, \tag{4.1}$$

which has to be understood in the sense that $u \in BV_{\mathrm{loc}}(\mathbb{R}^N) \cap L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and there exists a vector field $\xi \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ with $\xi(x) \in \partial\phi^\circ(\nabla u(x))$ almost everywhere in $\mathbb{R}^N$ such that $\xi \cdot Du = \phi^\circ(Du)$ as measures in any bounded set of $\mathbb{R}^N$ (see [6, 13]) and

$$u - h \operatorname{div} \xi \ + h\bar{c} - \ d^\phi_C \ = \ 0 \ \text{ in } \mathbb{R}^N. \tag{4.2}$$

The following result was proved in [13] when $\bar{c} = 0$. The same proof applies to the present case.

Theorem 4.1. *Problem* (4.1) *admits a unique solution u in the class of functions in $BV_{\mathrm{loc}}(\mathbb{R}^N) \cap L^2_{\mathrm{loc}}(\mathbb{R}^N)$ with bounded sub-levels. This function u is convex, Lipschitz, and* *each sub-level $\{u < s\}$ is a solution of*

$$(P_s) \qquad \min_{F \subseteq \mathbb{R}^N} \left\{ P_\phi(F) + \frac{1}{h} \int_{F \triangle C_s} |d_C^\phi - s| \, dx - \bar{c}|F| \right\}, \tag{4.3}$$

where $C_s := \{d_\phi^C < s\}$ and $F \triangle C_s$ is the symmetric difference between F and C_s. Moreover, if $s < s'$ and F_s and $F_{s'}$ are solutions of (P_s) and $(P_{s'})$ respectively, then $F_s \subseteq F_{s'}$. Hence, for any $s \in \mathbb{R}$ there exists a minimal and a maximal solution of (P_s), and this solution is unique for almost any $s \in \mathbb{R}$.

Remark 4.2. As in [13], we observe that the vector field ξ associated with the solution u of (4.1) is such that $\operatorname{div} \xi \geq 0$.

Taking $s = 0$ in (4.3) we may define [1, 13, 16]

$$S_h^{\bar{c}}(C) := \arg\min_{F \subseteq \mathbb{R}^N} \left\{ P_\phi(F) + \int_{F \triangle C} \frac{|d_C^\phi|}{h} dx - \bar{c}|F| \right\}. \tag{4.4}$$

In case there are multiple solutions, we define $S_h^{\bar{c}}(C)$ as the smallest one which coincides with $\{u < 0\}$ where u denotes the solution of (4.1) (see Theorem 4.1 in [13]). Observe that

$$\begin{aligned} S_h^{\bar{c}}(C) &= \arg\min_{F \subseteq \mathbb{R}^N} \left\{ P_\phi(F) + \int_{F \triangle C} \left(\frac{|d_C^\phi|}{h} - \bar{c} \operatorname{sgn}(d_C^\phi) \right) dx \right\} \\ &= \arg\min_{F \subseteq \mathbb{R}^N} \left\{ P_\phi(F) + \int_F \left(\frac{d_C^\phi - h\bar{c}}{h} \right) dx \right\}. \end{aligned}$$

Observe that if $\bar{c} \geq 0$ and u is a solution of (4.2), then u is a solution of

$$u - h \operatorname{div} \xi = d_{C + h\bar{c}W_\phi}^\phi \quad \text{in } \mathbb{R}^N,$$

that is

$$S_h^0(C) = \{u + h\bar{c} < 0\} \qquad \text{and} \qquad S_h^0(C + h\bar{c}W_\phi) = \{u < 0\} = S_h^{\bar{c}}(C).$$

Hence, if $x \in \partial S_h^{\bar{c}}(C)$, then

$$d_{S_h^0(C)}^\phi(x) \geq u(x) + h\bar{c} = h\bar{c}.$$

Thus

$$d_\phi(\partial S_h^{\bar{c}}(C), \partial S_h^0(C)) \geq h\bar{c}. \tag{4.5}$$

The same is true if $\bar{c} < 0$ and $(C + h\bar{c}W_\phi) + h|\bar{c}|W_\phi = C$.

For $h > 0$, let $c_h \in L^\infty((0,+\infty))$ be a piecewise constant function, constant on each interval $(ih,(i+1)h]$, $i \in \mathbb{N}$, and such that $\sup_h \|c_h\|_{L^\infty((0,+\infty))} < +\infty$. We then define a discrete (in time) evolution by letting for any $t \geq 0$

$$C_h(t) \ := \ S_h^{c_h(nh)} \, S_h^{c_h((n-1)h)} \cdots S_h^{c_h(h)}(C), \qquad n := [t/h]. \tag{4.6}$$

Notice that $C_h(t)$ coincides with $\{u^n < 0\}$ where u^n is the solution of

$$u - h \operatorname{div} \partial\phi^\circ(\nabla u) \ + h c_h(nh) - \ d^\phi_{C_h^{n-1}} \ \ni \ 0 \text{ in } \mathbb{R}^N .$$

where $C_h^{n-1} := S_h^{c_h((n-1)h)} \cdots S_h^{c_h(h)}(C)$

Denote by $\mathcal{K}$ the class of all compact convex subsets of $\mathbb{R}^N$, endowed with the Hausdorff distance $d_{\mathcal{H}}$.

Theorem 4.3. *Let $C \in \mathcal{K}$. There exists a sequence $\{h_k\}$ converging to 0 as $k \to \infty$ and a continuous function $C : [0,+\infty) \to \mathcal{K}$ with $C(0) = C$ such that*

$$\lim_{k\to+\infty} d_{\mathcal{H}}\left(Q_{C_{h_k}}, Q_C\right) = 0,$$

where $Q_{C_{h_k}}$ and Q_C are the space-time tracks defined as

$$Q_{C_{h_k}} := \bigcup_{t\geq 0} \left(C_{h_k}(t) \times \{t\}\right), \qquad Q_C := \bigcup_{t\geq 0} \left(C(t) \times \{t\}\right). \tag{4.7}$$

Proof. The proof is the same as in [13, 8] and is omitted. □

Definition 4.4. *We call the evolution $C(t)$ of Theorem 4.3 a convex flat ϕ-flow with forcing term $c \in L^\infty((0,+\infty))$ (the weak-$*$ limit of c_h) starting from C.*

The next two results (existence of convex rW_ϕ-regular flows, and comparison for convex flat ϕ-flows), can be proven following the same lines as in, respectively, [8, Theorem 6.1] and [8, Theorem 7.4]; in particular, the local existence proof of Theorem 4.5 is based on the weak formulation given by the flat ϕ-flow. Together with the uniqueness result of Theorem 3.2, we can conclude the existence and uniqueness of an $\frac{r}{2}W_\phi$-regular flow starting from a compact convex set satisfying the rW_ϕ-condition.

Theorem 4.5. *Let $c \in L^\infty((0,+\infty))$. Let $C \in \mathcal{K}$ satisfy an interior rW_ϕ-condition for some $r > 0$, and let $t_1 \geq 0$. Then there exist $t_2 > t_1$ and a unique convex $\frac{r}{2}W_\phi$-regular flow $t \in [t_1,t_2] \to C(t)$ with forcing term c such that $C(t_1) = C$, where $t_2 - t_1$ depends only on r and $\|c\|_{L^\infty}$. Moreover, if $c_h \rightharpoonup c$ weakly-$*$ in $L^\infty((0,+\infty))$, then $C(t)$ is obtained as the Hausdorff limit of the discretized evolutions $C_h(t)$ defined by (4.6).*

Proposition 4.6. *Let $C, C' \in \mathcal{K}$ with $C \subset \mathrm{int}(C')$. Let $t \in [0,+\infty) \to C(t), C'(t)$ be two convex flat ϕ-flows with forcing term $c \in L^\infty((0,+\infty))$ starting from C and C' respectively. Then $C(t) \subset C'(t)$ for any $t \geq 0$.*

Then, we provide with a lemma to compare flows with different (close) forcing terms.

Lemma 4.7. *Let $C_1 \subset C_2$ be two compact convex sets, $c_1, c_2 \in L^\infty((0,+\infty))$, and let $C_i(t)$ be a convex flat ϕ-flow with forcing term c_i starting from C_i for any $i = 1, 2$. Define*

$$\delta(t) := d_\phi(\partial C_1(t), \partial C_2(t)) \qquad \forall t \in [0,+\infty)$$

and assume that $\delta(0) > 0$. Let $T_{\text{contact}} = T_{\text{contact}}(C_1, C_2, c_1, c_2) \in (0,+\infty]$ be the first contact time (if any) between $\partial C_1(t)$ and $\partial C_2(t)$. Then

$$\delta(t) \geq \delta(0) - \int_0^t (c_2 - c_1)ds \qquad \forall t \in [0, T_{\text{contact}}). \tag{4.8}$$

Proof. Set $\delta := \delta(0)$. Choose $\widetilde{C}_1 := C_1 + \delta/3W_\phi$, $\widetilde{C}_2 := C_1 + 2\delta/3W_\phi$, and let $\widetilde{C}_i(t)$ be the rW_ϕ-regular flows starting from $\widetilde{C}_i$ with forcing term c_i, given by Theorem 4.5, in a suitable common time interval $[0,T)$. Let $\widetilde{\delta}(t) := \mathrm{dist}(\partial\widetilde{C}_1(t), \partial\widetilde{C}_2(t))$ for any $t \in [0,T)$. By approximating the sets $W_\phi, \widetilde{C}_1, \widetilde{C}_2$ with smooth sets as in [8, Remark 12, Section 6], it is possible to prove that $\widetilde{\delta}(t) \geq \widetilde{\delta}(0) - \int_0^t (c_2 - c_1)ds$ for $t \in [0,T)$, see also [8, Section 8, Lemma 13]. This also follows by combining the observations leading to (4.5) and the proof of [8, Section 8, Lemma 13]. Finally, since the distance between $\partial C_1(t)$ and $\partial\widetilde{C}_1(t)$ (resp. $\partial C_2(t)$ and $\partial\widetilde{C}_2(t)$) is nondecreasing (see [8, Section 8, Lemma 13]), estimate (4.8) follows. $\square$

Definition 4.8. *Given a compact convex set C and a a convex flat ϕ-curvature flow with forcing term $c \in L^\infty((0,+\infty))$ starting from C, we define*

$$t_{C,c} := \sup\{t \geq 0 : |C(\tau)| > 0 \text{ for any } \tau \in [0,t)\} \in [0,+\infty], \tag{4.9}$$

$$Q'_C := \bigcup_{0 \leq t < t_{C,c}} (C(t) \times \{t\}).$$

Theorem 4.9. *Let $C_1 \subseteq C_2$ be two compact convex sets and let $C_1(t)$ and $C_2(t)$ be two convex flat ϕ-curvature flows with forcing term $c \in L^\infty((0,+\infty))$, starting from C_1 and C_2 respectively. Then*

$$C_1(t) \subseteq C_2(t) \qquad \forall t \in [0, t_{C_2,c}).$$

In particular, the convex flat ϕ-flow starting from a compact convex set is unique, as long as the enclosed volume remains positive.

Proof. We argue as in [8, Theorem 7.4]. Assume $C_1 \subset \mathrm{int}(\theta C_2)$, with $\theta > 1$, and let $\delta_\theta(t) := \mathrm{dist}(\partial C_1(t), \theta\partial C_2(t/\theta^2))$. Applying Lemma 4.7 with $c_1 = c$, $c_2 = \frac{1}{\theta}c(t/\theta^2)$, and with C_2 replaced by θC_2, we get

$$\delta_\theta(t) \geq \delta_\theta(0) - \int_0^t \left(\frac{1}{\theta} c\left(\frac{s}{\theta^2}\right) - c(s) \right) ds \qquad \forall t \in (0, t_\theta), \tag{4.10}$$

where $t_\theta := T_{\mathrm{contact}}(C_1, \theta C_2, c, c_2)$ is the first contact time between $\partial C_1(t)$ and $\theta\partial C_2(t/\theta^2)$. Now, if R is the radius of a ball inside C_2, we must have $\delta_\theta(0) \geq (\theta - 1)R$. On the other hand,

$$\begin{aligned} \int_0^t \left(\frac{1}{\theta} c\left(\frac{s}{\theta^2}\right) - c(s) \right) ds &= (\theta - 1) \int_0^{t/\theta^2} c(s)\, ds + \int_{t/\theta^2}^t c(s)\, ds \\ &\leq 3t(\theta - 1)\|c\|_{L^\infty((0,+\infty))}\,. \end{aligned}$$

Hence, we find that $t_\theta \geq t_1 := R/(3\|c\|_\infty)$. Letting $\theta \to 1^+$ and recalling that $\partial C_2(\cdot)$ is continuous from the left (see Theorem 4.3 and [13, Lemma 7.2]), we deduce that $C_1(t) \subseteq C_2(t)$ as long as $t \leq t_1$. Now, we may start again from t_1 and push further the inclusion as long as $R > 0$ (i.e., for C_2 having nonempty interior). $\square$

Under the assumptions of Theorem 4.9, we cannot exclude the existence of a contact time between $\partial C_1(t)$ and $\partial C_2(t)$ when the volumes of $C_1(t)$ and $C_2(t)$ vanish.

Remark 4.10. If $|C_2| = 0$, it is not clear whether the comparison remains true. Indeed, if for instance, $C_2 = \{x^2 + y^2 \leq R, z = 0\} \subset \mathbb{R}^3$ and $c \equiv 1$, it is likely that for R large enough, a solution with positive volume may evolve starting from C, while other approximations of c will yield an empty flat flow. Hence, we cannot expect uniqueness in this situation.

We have shown that the convex flat ϕ-flow with forcing define a continuous semigroup up to extinction of the interior. The next result is a slightly stronger stability result.

Theorem 4.11 (stability of the convex flat ϕ-flow)**.** *Let $C_n, C \in \mathcal{K}$, and assume that C has nonempty interior and $\lim_{n\to+\infty} d_{\mathcal{H}}(C_n, C) = 0$. Let $c_n, c \in L^\infty_{\mathrm{loc}}([0,+\infty))$ and suppose that $c_n \rightharpoonup c$ weakly-$*$ as $n \to +\infty$. Let $C_n(t)$, $0 \leq t < t_{C_n,c_n}$ and $C(t)$, $0 \leq t < t_{C,c}$ be the convex flat ϕ-flows with forcing terms c_n and c starting from C_n and C, respectively. Then*

$$t_{C,c} \leq \liminf_{n\to+\infty}\ t_{C_n,c_n}$$

and

$$\lim_{n\to+\infty} d_{\mathcal{H}}(C_n(t), C(t)) = 0$$

locally uniformly.

Proof. We combine the previous proofs. Let $\theta > 1$. If n is large enough, from the assumption $\lim_{n\to+\infty} d_{\mathcal{H}}(C_n, C) = 0$ we have $C_n \subset\subset \theta C$. Define $\delta_\theta^n(t) := \mathrm{dist}(\partial C_n(t), \theta\partial C(t/\theta^2))$; by Lemma 4.7 we have

$$\delta_\theta^n(t) \ \geq \ \delta_\theta^n(0) - \int_0^t \left(\frac{1}{\theta} c\left(\frac{s}{\theta^2}\right) - c_n(s)\right) ds \tag{4.11}$$

for all t before the first contact time between $\partial C_n(t)$ and $\theta\partial C(t/\theta^2)$. As $n \to +\infty$, (4.11) converges to (4.10), with $\delta_\theta(0)$ given by $\mathrm{dist}(\partial C, \theta\partial C)$ and estimated from below by $(\theta - 1)R$ where R is a ball inside C. For n large enough, we therefore get, as in the proof of Theorem 4.9, that as long as $t \leq t_1 := \max(R/(6\|c\|_{L^\infty}, 1)$ and $C(t)$ does not vanish, $C_n(t) \subset \theta C(t/\theta^2)$. Sending $n \to +\infty$, we find that any Hausdorff limit of $C_n(t)$ is inside $\theta C(t/\theta^2)$ for any $\theta > 1$ and $t \leq t_1$. Letting then $\theta \to 1^+$, we get that $C(t)$ is a bound from above for the Hausdorff limits of $C_n(t)$. The same argument with now $\theta < 1$ and $\theta C \subset\subset C_n$ (for n large enough) will yield the same bound from below. Hence, $\lim_{n\to+\infty} d_{\mathcal{H}}(C_n(t) \to C(t)) = 0$ on $(0, t_1)$ if $|C(t)|$ does not vanish. It is then possible to bootstrap and show that this must happen up to $t_{C,c}$, by contradiction. $\square$

We also mention that the crystalline flow of convex sets may be approximated with smooth anisotropies: to state the result we recall that in [28, 8] it is proved the following approximation lemma.

Lemma 4.12. *Let $\phi : \mathbb{R}^N \to [0, +\infty)$ be a convex function satisfying* (2.1) *and* (2.2)*, and let C be a compact convex set satisfying the rW_ϕ-condition for some $r > 0$. Then there exist a sequence $\{\phi_\epsilon\} \subset \mathcal{C}_+^\infty$ of convex functions satisfying* (2.1), (2.2) *and $\phi_\epsilon^o \in \mathcal{C}_+^\infty$, and a sequence $\{C_\epsilon\}$ of compact smooth uniformly convex sets satisfying the $r\mathcal{W}_{\phi_\epsilon}$-condition for any $\epsilon > 0$, such that*

$$\lim_{\epsilon\to 0} \phi_\epsilon = \phi \text{ uniformly in } \mathbb{R}^N, \qquad \lim_{\epsilon\to 0} d_{\mathcal{H}}(C_\epsilon, C) = 0.$$

Proposition 4.13 (stability with respect to the anisotropy)**.** *Let $\phi_\varepsilon \to \phi$, and let $C(t)$, $C_\varepsilon(t)$ be the convex flat flows corresponding to the anisotropies ϕ and ϕ_ε respectively, starting from the same initial convex set C, see Lemma 4.12. Assume that $|C(t)| \geq \eta > 0$ for all $t \in [0, T]$. Then*

$$\lim_{\epsilon\to 0^+} d_{\mathcal{H}}\left(\partial C_\varepsilon(t), \partial C(t)\right) = 0, \tag{4.12}$$

uniformly on $[0, T]$.

Proof. If $C(t)$ is an rW_ϕ-regular flow, the thesis has been proved in [8, Remark 12, Th. 12]. Therefore, we can use rW_ϕ-regular flows to compare $\lim_{\varepsilon\to 0} C_\varepsilon(t)$ with appropriate dilations of $C(t)$ (and viceversa), as in Lemma 4.7 and in Theorem 4.9, using the fact that $|C(t)| > 0$. □

5 Evolution of volume and perimeter for a convex flat ϕ-flow with forcing

Let $C \subset \mathbb{R}^N$ be a compact convex set with nonempty interior. Let $C_h(t)$ be defined by (4.6). By Theorem 4.9, we know that there exists a time $t_{C,c} \in (0, +\infty]$ (defined in (4.9)) and a unique convex flat ϕ-flow $C(t)$, of positive volume as long as $t < t_{C,c}$, such that the space-time tracks defined in (4.7) satisfy $\lim_{h\to+\infty} d_{\mathcal{H}}(Q_{C_h}, Q_C) = 0$ (locally in time, if $t_{C,c} = +\infty$). Reasoning as in [13, Th. 5], we obtain also the following properties:

Proposition 5.1. *The function $t \to \partial C(t) \in \mathcal{K}$ is continuous, the Hausdorff convergence of $\partial C_h(t)$ to $\partial C(t)$ is locally uniform in time, and, letting*

$$d(x, t) := d^{\phi}_{C(t)}(x), \tag{5.1}$$

we can find $z \in L^\infty(\mathbb{R}^N \times (0, t_{C,c}); \mathbb{R}^N)$, with $z \in \partial\phi^o(\nabla d)$ almost everywhere, and such that $\operatorname{div} z$ *is a nonnegative Radon measure in* $\mathbb{R}^N \times (0, t_{C,c})$*, with*

$$-\mathrm{div} z + c + \frac{\partial d}{\partial t} \geq 0 \qquad \textit{out of } Q'_C, \tag{5.2}$$

$$-\mathrm{div} z + c + \frac{\partial d}{\partial t} \leq 0 \qquad \textit{in } \mathrm{int}(Q'_C), \tag{5.3}$$

in the sense of measures. Moreover, out of Q'_C the measure $\operatorname{div} z$ *is represented by a locally bounded function, and more precisely*

$$0 \leq \operatorname{div} z \leq \frac{N-1}{\delta} \tag{5.4}$$

almost everywhere in $\{d \geq \delta\}$, for all $\delta > 0$.

Notice that (5.4) follows by comparison with the (discrete) evolution of a Wulff shape of radius δ.

Remark 5.2. As a consequence, $\partial d/\partial t$ is a Radon measure on $\mathbb{R}^N \times (0, t_{C,c}) \setminus \partial Q'_C$. In fact, one can show from the construction that $d(t+h) \geq d(t) - \int_t^{t+h} c(s)\, ds$, so that $\partial d/\partial t \geq -c \in L^\infty((0, t_{C,c}))$ is a Radon measure in $\mathbb{R}^N \times (0, t_{C,c})$.

Definition 5.3. *Given a convex set C, with the symbol $V_2^{\phi}(C)$ we indicate the second mixed volume of order N, $V(W_\phi, W_\phi, C, \ldots, C)$ (see [28]), multiplied by $N(N-1)$.*

The mixed volume $V_2^\phi(C)$ can be defined by the relationship

$$\lim_{\delta\to 0^+} \frac{P_\phi(C+\delta W_\phi)-P_\phi(C)}{\delta} \ = \ V_2^\phi(C)$$

and is a nondecreasing, continuous function of $C \in \mathcal{K}$ ([28, proof of Theorem 5.1.6, (5.1.23)]). If $\phi, \phi^\circ \in \mathcal{C}_+^\infty$ and C is of class $\mathcal{C}^{1,1}$, we have $V_2^\phi(C) = \int_{\partial C} \kappa_\phi^C \, dP_\phi$ where $\kappa_\phi^C :=$ $\mathrm{div}\, n_\phi^C$ and $n_\phi^C = \nabla\phi^\circ(\nabla d_C^\phi)$ (recall (1.1)). In the same way, we have $|C| = V(C,\ldots,C)$ whereas $P_\phi(C) = NV(W_\phi, C, \ldots, C)$.

Proposition 5.4. *Let $C \in \mathcal{K}$, $t \in [0,+\infty) \to C(t)$ be a convex flat ϕ-flow with forcing term $c \in L^\infty((0,+\infty))$ starting from C, let d be defined as in* (5.1)*, and let z be the vector field given by Proposition 5.1. Then,*

$$\sup_{\delta>0} \frac{1}{\delta} \int_0^{t_{C,c}} \int_{\{0<d(\cdot,t)<\delta\}} (\mathrm{div}\, z)^2 \, dx\, dt \ < \ +\infty\,. \tag{5.5}$$

Moreover, for any $0 \le t_1 < t_2 < +\infty$,

$$\begin{aligned}
&P_\phi(C(t_2)) - P_\phi(C(t_1)) \\
\le& \ -\limsup_{\delta\to 0} \int_{t_1}^{t_2} \frac{1}{\delta} \int_{\{0<d(\cdot,t)<\delta\}} (\mathrm{div}\, z)^2 \, dx\, dt + \int_{t_1}^{t_2} c(t) V_2^\phi(C(t))\, dt \\
\le& \ \int_{t_1}^{t_2} \left(-\frac{(V_2^\phi(C(t)))^2}{P_\phi(C(t))} \ + \ c(t)V_2^\phi(C(t)) \right) dt
\end{aligned} \tag{5.6}$$

and in particular

$$\frac{d}{dt} P_\phi(C(t)) \ \le \ -\frac{(V_2^\phi(C(t)))^2}{P_\phi(C(t))} \ + \ c(t)V_2^\phi(C(t)) \qquad \text{in } \mathcal{D}'((0,t_{C,c})). \tag{5.7}$$

Remark 5.5. We observe that (5.5) gives a sort of $W^{2,2}$-regularity of $\partial C(t)$.

Proof. Let $b > a > 0$, and let $T_{ab}(r) := \max\{a, \min\{b, r\}\}$ for any $r \in \mathbb{R}$. Recall from (5.4) that $\mathrm{div}\, z$ is a nonnegative bounded function in $\{d \ge a\}$. From the first inequality in (5.2), we get, for almost every $t \in (0, t_{C,c})$,

$$\frac{\partial T_{ab}(d)}{\partial t} \ge T'_{ab}(d)(\mathrm{div}\, z - c) \tag{5.8}$$

in the sense of measures. We now test (5.8) with a sequence of time-dependent test functions supported in $[t, t+h]$ and increasing to $\chi_{[t,t+h]}$; integrating by parts, passing to the limit and using the fact that $d(x,\cdot)$ is continuous, we get

$$T_{ab}(d(x,t+h)) - T_{ab}(d(x,t)) \ge \int_t^{t+h} T'_{ab}(d(x,s))(\mathrm{div}\, z(x,s) - c(s))\, ds$$

almost everywhere in $\mathbb{R}^N$.

We compute

$$
\begin{aligned}
& \int_{\mathbb{R}^N} \phi^\circ(\nabla(T_{ab}(d(x,t+h))))\,dx - \int_{\mathbb{R}^N} \phi^\circ(\nabla(T_{ab}(d(x,t))))\,dx \\
\leq \quad & \int_{\mathbb{R}^N} \langle \eta, \nabla(T_{ab}(d(x,t+h)) - T_{ab}(d(x,t)))\rangle \,dx,
\end{aligned}
$$

where η is any vector in $\partial\phi^o(\nabla(T_{ab}(d(x,t+h))))$. Therefore

$$
\begin{aligned}
& \int_{\mathbb{R}^N} \phi^\circ(\nabla(T_{ab}(d(x,t+h))))\,dx - \int_{\mathbb{R}^N} \phi^\circ(\nabla(T_{ab}(d(x,t))))\,dx \\
\leq \quad & \int_{\mathbb{R}^N} -\mathrm{div}\, z(x,t+h)(T_{ab}(d(x,t+h)) - T_{ab}(d(x,t)))\,dx \\
\leq \quad & -\int_{\mathbb{R}^N}\left(\int_t^{t+h} \chi_{\{a<d(\cdot,s)<b\}}(\mathrm{div}\, z(x,s) - c(s))\,ds\right)\mathrm{div}\, z(x,t+h)dx
\end{aligned}
$$

since $\mathrm{div}\, z \geq 0$ almost everywhere out of Q'_C. Dividing all terms by h, we use the fact that d is continuous in time and that all the functions appearing in the integrals are uniformly bounded, for h small enough, we can pass to the limit as $h \to 0^+$, and we obtain

$$
\begin{aligned}
& \frac{d}{dt}|\{a < d(\cdot,t) < b\}| \\
\leq \; & -\int_{\{a<d(\cdot,t)<b\}} (\mathrm{div}\, z(x,t))^2\,dx + c(t)\int_{\{a<d(\cdot,t)<b\}} \mathrm{div}\, z(x,t)\,dx
\end{aligned}
$$

in $\mathcal{D}'((0,t_{C,c}))$. If $0 \leq t_1 < t_2 < t_{C,c}$, we find

$$
\begin{aligned}
& \int_{t_1}^{t_2}\int_{\{a<d(\cdot,t)<b\}} (\mathrm{div}\, z(x,t))^2 dx\,dt \\
\leq \; & -|\{a < d(\cdot,t_2) < b\}| + |\{a < d(\cdot,t_1) < b\}| + \int_{t_1}^{t_2} c(t)\left(P_\phi\left(C(t)^+_b\right) - P_\phi\left(C(t)^+_a\right)\right)\,dt,
\end{aligned}
$$

where we recall that $C(t)^+_\rho := C(t) + \rho W_\phi$, $\rho \in \{a,b\}$.

Letting $a \to 0^+$ we deduce that $\mathrm{div}\, z \in L^2(\{0 < d < \delta\})$. Then, we divide by $\delta := b$ and send $\delta \to 0^+$, and get

$$
\begin{aligned}
& \limsup_{\delta\to 0^+} \frac{1}{\delta}\int_{t_1}^{t_2}\int_{\{0<d(\cdot,t)<\delta\}} (\mathrm{div}\, z(x,t))^2 dx\,dt \\
\leq \; & -P_\phi(C(t_2)) + P_\phi(C(t_1)) + \int_{t_1}^{t_2} c(t)V_2^\phi(C(t))\,dt
\end{aligned}
$$

showing (5.5), as well as the first inequality in (5.6).

The second inequality (and (5.7)) follows by noticing that

$$\frac{1}{\delta}\int_{\{0<d(\cdot,t)<\delta\}}(\operatorname{div} z)^2\,dx \geq \frac{1}{\delta|\{0<d(\cdot,t)<\delta\}|}\left(\int_{\{0<d(\cdot,t)<\delta\}}\operatorname{div} z\,dx\right)^2$$

$$= \frac{|\{0<d(\cdot,t)<\delta\}|}{\delta}\left(\frac{P_\phi(C(t)_\delta)-P_\phi(C(t))}{|\{0<d(\cdot,t)<\delta\}|}\right)^2 \to \frac{(V_2^\phi(C(t)))^2}{P_\phi(C(t))} \qquad \text{as } \delta\to 0^+.$$

□

Remark 5.6. In general (5.7) is not optimal even when $\phi,\phi^o\in\mathcal{C}_+^\infty$, $C(t)$ is smooth, and $c\equiv 0$: indeed, in this case it is well known that

$$\frac{d}{dt}P_\phi(C(t)) = -\int_{\partial C(t)}(\kappa_\phi^{C(t)})^2 dP_\phi \leq -\frac{(V_2^\phi(C(t)))^2}{P_\phi(C(t))}$$

and the inequality may be strict. However, we point out that the first inequality in (5.6) is always optimal.

5.1 On convex sets having ϕ-mean curvature in L^2

Definition 5.7. *Assume that $\phi\in\mathcal{C}_+^\infty$. Let C be a convex set. We say that C has ϕ-mean curvature in $L^2(\partial C)$ if the vector field*

$$z(x) := \partial\phi^\circ(\nabla d_C^\phi(x)) \qquad \text{for a.e. } x\in\mathbb{R}^N, \tag{5.9}$$

satisfies

$$\overline{h}_{\partial C} := \liminf_{\delta\to 0^+}\frac{1}{\delta}\int_{\{0<d_C^\phi<\delta\}}(\operatorname{div} z)^2\,dx \;<\; +\infty\,. \tag{5.10}$$

Proposition 5.8. *Assume that $\phi\in\mathcal{C}_+^\infty$ and let C be a compact convex set with ϕ-mean curvature in $L^2(\partial C)$. Then the problem*

$$\min\left\{\int_C(\operatorname{div}\xi)^2\,dx\ :\quad \xi: C\to W_\phi,\ \operatorname{div}\xi\in L^2(C),\int_C\operatorname{div}\xi\,dx = P_\phi(C)\right\} \tag{5.11}$$

has a solution $\overline{z}$, and two solutions of (5.11) have the same divergence. Moreover there exists a constant $\kappa>0$ such that

$$\int_C(\operatorname{div}\overline{z})^2\,dx \;\leq\; \kappa(P_\phi(C)+\overline{h}_{\partial C}). \tag{5.12}$$

Proof. Step 1. We build a competitor $z^*: C \to \mathbb{R}^N$ for the minimization problem in (5.11), hence satisfying

$$\int_C (\operatorname{div} z^*)^2\, dx \ \le\ \kappa(P_\phi(C) + \overline{h}_{\partial C}) \tag{5.13}$$

for some $\kappa > 0$ independent of z^*.
Let z be defined as in (5.9), and let $\varepsilon_k \in (0,1)$ be such that $\varepsilon_k \downarrow 0$ and

$$\lim_{k\to+\infty} \int_{\partial C^+_{\varepsilon_k}} (\operatorname{div} z)^2\, dP_\phi \ =\ \overline{h}_{\partial C}\,. \tag{5.14}$$

Observe that near $\partial C^+_{\varepsilon_k}$ the vector field z is Lipschitz. Indeed, by [8, Eq. (16)], one has

$$|\nabla z| \ \le\ \frac{\Lambda}{\lambda}|\operatorname{div} z| \qquad \text{a.e. outside of } C, \tag{5.15}$$

where λ, Λ are the ellipticity constants of ϕ°, defined by [8, Eq. (7)]:

$$\lambda\, \mathrm{Id} \ \le\ \phi^\circ D^2\phi^\circ + \nabla\phi^\circ \otimes \nabla\phi^\circ \ \le\ \Lambda\, \mathrm{Id}\,.$$

Let $B(\overline{x}, R) \subset C$ a maximal ball contained in C; without loss of generality we may assume $\overline{x} = 0$. Denote by $h_k : \mathbb{R}^N \to [0, +\infty)$ the convex, one-homogeneous function such that $C^+_{\varepsilon_k} = \{h_k \le 1\}$. In particular, ∇h_k is zero-homogeneous, bounded by $1/R$, and $\nabla h_k(x) = |\nabla h_k(x)|\nu^C(x)$ for $\mathcal{H}^{N-1}$-almost every $x \in \partial C$, where we recall that ν^C is the outward unit normal to ∂C. We define

$$z^k(x) \ :=\ h_k(x)\, z\left(\frac{x}{h_k(x)}\right), \qquad x \in C^+_{\varepsilon_k}.$$

Then $\phi^\circ(z^k(x)) = h_k(x)\phi^\circ(z(x/h_k(x))) \le 1$ for almost every $x \in C^+_{\varepsilon_k}$, and

$$\operatorname{div} z^k(x) \ = \langle \nabla h_k(x), z\left(\frac{x}{h_k(x)}\right)\rangle \ +\ \sum_{i,l=1}^N \partial_l z_i \left(\frac{x}{h_k(x)}\right)\left(\delta_{i,l} - \frac{x_l}{h_k(x)}\partial_i h_k(x)\right)$$

where $\delta_{i,l}$ is the Kronecker symbol. We deduce, using (5.15),

$$|\operatorname{div} z^k(x)| \ \le\ \frac{\phi^\circ(\nu^C(x/h_k(x)))}{R} \ +\ \left(1 + \frac{\Lambda}{\lambda R}\right)\left|\operatorname{div} z\left(\frac{x}{h_k(x)}\right)\right|\,. \tag{5.16}$$

We now employ the co-area formula to write

$$\begin{aligned}\int_{C^+_{\varepsilon_k}} (\operatorname{div} z^k)^2\, dx \ &= \int_0^1 \left(\int_{\{h_k=s\}} (\operatorname{div} z^k)^2\, \frac{d\mathcal{H}^{N-1}}{|\nabla h_k|}\right) ds \\ &= \frac{1}{N}\int_{\partial C^+_{\varepsilon_k}} (\operatorname{div} z^k)^2\, \frac{d\mathcal{H}^{N-1}}{|\nabla h_k|},\end{aligned} \tag{5.17}$$

where we used that ∇h_k and $\operatorname{div} z^k$ are zero-homogeneous. Since $|\nabla h_k|$ is estimated from below by the inverse of (ε_k+ the diameter of C), we deduce from (5.16) and (5.17) that

$$\int_{C^+_{\varepsilon_k}} (\operatorname{div} z^k)^2\, dx \;\le\; \kappa \left(P_\phi(C^+_{\varepsilon_k}) \,+\, \int_{\partial C^+_{\varepsilon_k}} (\operatorname{div} z)^2\, dP_\phi \right), \tag{5.18}$$

where κ depends on R, Λ/λ, N, the diameter of C and $\max_{\nu\in\mathbb{S}^{N-1}} \phi^\circ(\nu)$. As $k \to +\infty$, z^k converge weakly-$*$ in $L^\infty(C;\mathbb{R}^N)$ to some $z^* : C \to W_\phi$, and passing to the limit in (5.18) (using (5.14)) we get estimate (5.13). To conclude the proof of step 1, we need to show that

$$\int_C \operatorname{div} z^*\, dx = P_\phi(C). \tag{5.19}$$

Since

$$\int_C \operatorname{div} z^k\, dx \;=\; P_\phi(C^+_{\varepsilon_k}) - \int_{C^+_{\varepsilon_k}\setminus C} \operatorname{div} z^k\, dx\,,$$

sending $k \to +\infty$ formula (5.19) follows.

As a consequence of *step 1*, the class of competitors in the minimum problem (5.11) is nonempty.

Step 2. We build a solution of (5.11).

For all $\lambda > 0$ denote by u_λ the unique solution of the problem

$$\min_{u\in BV(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} \phi^\circ(Du) \,+\, \frac{\lambda}{2}\int_{\mathbb{R}^N} (u-\chi_C)^2\, dx \right\}. \tag{5.20}$$

One can show that $0 \le u_\lambda \le 1$, and that $u_\lambda = 0$ almost everywhere outside of C (by showing, for instance, that $u\chi_C$ has an energy lower than u, because of the co-area formula and the convexity of C). Therefore problem (5.20) is equivalent to

$$\min_{u\in BV(C)} \left\{ \int_C \phi^\circ(Du) + \int_{\partial C} |u|\, dP_\phi \,+\, \frac{\lambda}{2}\int_C (u-1)^2\, dx \right\}. \tag{5.21}$$

It is shown in [14, 2] that for any $\mu > 0$, as soon as $\lambda > \mu$, the set

$$E_\mu := \{u_\lambda > 1 - \mu/\lambda\} \tag{5.22}$$

is the unique solution of

$$\min_{E\subseteq C} \{P_\phi(E) \,-\, \mu|E|\}\,, \tag{5.23}$$

and does not depend on λ. Moreover, for $\lambda > 0$ large enough, $\mu^* := \lambda(1 - \|u_\lambda\|_\infty)$ does not depend on λ and coincides with the *ϕ-Cheeger constant* of C. In addition, it is shown that u_λ is concave (hence locally Lipschitz) in $E_\lambda = \{u_\lambda > 0\}$, so that E_μ is convex for all $\mu > \mu^*$. For $\mu = \mu^*$, (5.23) has at least two solution, $\emptyset$ and the convex set $\{u_\lambda = \|u_\lambda\|_\infty\}$, while it has been shown in [15, 2] that, at least in the isotropic case $\phi(\cdot) = |\cdot|$, there is no other solution. If $\mu < \mu^*$ then $\emptyset$ is the only solution of (5.23).

The Euler-Lagrange equation for (5.21) is

$$\begin{cases} -\mathrm{div}\, z_\lambda \ = \lambda(1 - \ u_\lambda) & \text{a.e. in } C\,, \\ \phi(z_\lambda) \le 1\ ,\ \langle z_\lambda, \nabla u_\lambda\rangle = \ \phi^\circ(\nabla u_\lambda) & \text{a.e. in } C\,, \\ z_\lambda \cdot \nu^C = -\phi^\circ(\nu^C) & \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial C \text{ with } u(x) > 0. \end{cases} \tag{5.24}$$

In particular, $-\mathrm{div}\, z_\lambda = \mu$ on $\partial E_\mu \setminus \partial C$ (which expresses the fact that ∂E_μ has ϕ-curvature μ inside C), and $\mathrm{div}\, z_\lambda = \mathrm{div}\, z_{\lambda'}$ almost everywhere in E_λ for any $\lambda' > \lambda$. As $\lambda \to +\infty$, $-z_\lambda$ converges weakly-$*$ in $L^\infty(C;\mathbb{R}^N)$ to a vector field $\overline{z} : C \to W_\phi$ with $\mathrm{div}\,\overline{z} = \mu$ on $\partial E_\mu \setminus \partial C$, $\mu > \mu^*$, and $\mathrm{div}\,\overline{z} = \mu^*$ in E_{μ^*}.
Now, convex duality shows that z_λ is also a solution of the problem

$$\min\left\{\int_C (\lambda + \mathrm{div}\, n)^2\, dx :\quad n : C \to W_\phi,\ \mathrm{div}\, n \in L^2(C)\right\}.$$

In particular, if ξ is an admissible vector field for problem (5.11), one has for any $\lambda > 0$

$$\int_C (\lambda + \mathrm{div}\, z_\lambda)^2\, dx \le \int_C (\lambda + \mathrm{div}\,(-\xi))^2\, dx\,,$$

that is, using $\int_C \mathrm{div}\, \xi\, dx = P_\phi(C)$,

$$\int_C (\mathrm{div}\, z_\lambda)^2\, dx + 2\lambda\left(P_\phi(C) + \int_C \mathrm{div}\, z_\lambda\, dx\right) \ \le\ \int_C (\mathrm{div}\, \xi)^2\, dx.$$

Passing to the limit, we find that $\int_C (\mathrm{div}\,\overline{z})^2\, dx \le \int_C (\mathrm{div}\,\xi)^2\, dx$, and $\int_C \mathrm{div}\,\overline{z} = P_\phi(C)$, so that $\overline{z}$ is a solution of (5.11). Since problem (5.11) is strictly convex in the divergence, we deduce the uniqueness of $\mathrm{div}\,\overline{z}$. In particular, we have shown that given any solution $\overline{z}$ of (5.11), one has $\mathrm{div}\,\overline{z} = \mu$ on $\partial E_\mu \setminus \partial C$ for any $\mu > \mu^*$, and $\mathrm{div}\,\overline{z} = \mu^*$ in E_{μ^*}. □

Proposition 5.9. *Let ϕ, C and $\overline{z}$ be as in Proposition 5.8. For $\delta > 0$ small enough, define*

$$C^\delta := (C^-_\delta)^+_\delta \subseteq C.$$

Then

$$0 \ \le\ P_\phi(C) - P_\phi(C^\delta) \ \le\ \delta \int_{C\setminus C^\delta} (\mathrm{div}\,\overline{z})^2\, dx\,. \tag{5.25}$$

Proof. We observe that that E_μ in (5.22) is of class $\mathcal{C}^{1,1}$ and $\kappa_\phi^{E_\mu} \le \mu$ (otherwise we easily contradict the minimality in (5.23)) and $\kappa_\phi^{E_\mu} = \mu$ on $\partial E_\mu \setminus \partial C$. Hence, it satisfies an interior ball condition [8, Remark 4]. Moreover, by [8, Corollary 2], E_μ satisfies the interior $\frac{1}{\mu}W_\phi$-condition. Therefore, if $\delta < 1/\mu$, one has $\{x \in E_\mu \,:\, x+\delta W_\phi \subseteq E_\mu\} + \delta W_\phi = E_\mu$, hence $E_\mu \subseteq C^\delta$. In particular, since $\cup_\mu E_\mu = C$, we have that $\operatorname{div} \overline{z} \geq 1/\delta$ almost everywhere in $C \setminus C^\delta$. Therefore,

$$\begin{aligned} P_\phi(C) - P_\phi(C^\delta) \quad &\leq \quad \int_{\partial C} \langle \overline{z}, \nu^C\rangle \, d\mathcal{H}^{N-1} - \int_{\partial C^\delta} \langle \overline{z}, \nu^C\rangle \, d\mathcal{H}^{N-1} \\ &= \quad \int_{C\setminus C^\delta} \operatorname{div} \overline{z} \, dx \;\leq\; \delta \int_{C\setminus C^\delta} (\operatorname{div} \overline{z})^2 \, dx \,. \end{aligned}$$

□

In particular, we deduce the following corollary.

Corollary 5.10. *Let C be a compact convex set with ϕ-mean curvature in $L^2(\partial C)$. Then*

$$\lim_{\delta\to 0^+} \frac{P_\phi(C) - P_\phi(C^-_\delta)}{\delta} \;=\; V_2^\phi(C)\,. \tag{5.26}$$

Proof. Let $C^\delta := (C^-_\delta)^+_\delta$. By [28], we know that

$$P_\phi(C^\delta) = P_\phi(C^-_\delta) \;+\; \delta V_2^\phi(C^-_\delta) \;+\; O(\delta^2)$$

as $\delta \downarrow 0$. Since $C^\delta \subseteq C$ and V_2^ϕ is continuous, we deduce $P_\phi(C) - P_\phi(C^-_\delta) \geq \delta V_2^\phi(C) + o(1)$. The reverse inequality follows from (5.25). □

Remark 5.11. Observe that (5.26) is not true if, for instance, C is a square in the plane and $\phi = |\cdot|$. Indeed, in this case the right hand side of (5.26) equals 2π, while the left hand side is equal to 8. This is due to the fact that $\operatorname{div} z$ is not in $L^2(\mathbb{R}^2 \setminus C)$.

We are finally in the position to compute the evolution equation for the enclosed volume.

Theorem 5.12. *Let $C(t)$ be a convex flat ϕ-flow with forcing term $c \in L^\infty((0,+\infty))$. Then*

$$\frac{d}{dt}|C(t)| = -V_2^\phi(C(t)) + c(t)P_\phi(C(t)) \qquad \text{in } \mathcal{D}'((0,t_{C,c})). \tag{5.27}$$

Proof. Assume first that $\phi \in \mathcal{C}^\infty_+$. Let $a < b < 0$. Let $Q^\varepsilon_{ab}(r) = 1$ if $a+\varepsilon \leq r \leq b-\varepsilon$, 0 if $r < a$ or $r > b$, and let $Q^\varepsilon_{ab}(r)$ be equal to the linear interpolation between 0 and 1 if $r \in [a, a+\varepsilon] \cup [b-\varepsilon, b]$. Let $T^\varepsilon_{ab}(r)$ be the primitive of Q^ε_{ab} with $T^\varepsilon_{ab}(r) = a$ for $r \leq a$.

Then, using the fact that d is continuous and $\partial d/\partial t$ is a Radon measure in $\mathbb{R}^N \times (0, t_{C,c})$, we find, recalling (5.3)

$$
\begin{aligned}
\frac{d}{dt}\int_B T^{\varepsilon}_{ab}(d)\,dx &= \int_B Q^{\varepsilon}_{ab}(d)\frac{\partial d}{\partial t} \le \int_B Q^{\varepsilon}_{ab}(d)(\operatorname{div} z - c)\,dx \\
&= -\int_B \langle z, DQ^{\varepsilon}_{ab}(d)\rangle \,-\, c(t)\int_B Q^{\varepsilon}_{ab}(d)\,dx \qquad (5.28)\\
&= -\frac{1}{\varepsilon}|\{a \le d \le a+\varepsilon\}| \,+\, \frac{1}{\varepsilon}|\{b-\varepsilon \le d \le b\}| \,-\, c(t)\int_B Q^{\varepsilon}_{ab}(d)\,dx,
\end{aligned}
$$

where B is a ball containing $\{d(\cdot,t) \le 0\}$ for any $t \in [0, t_{C,c}]$ in its interior. Observing that

$$
\int_B T^{\varepsilon}_{ab}(d)\,dx \;=\; (b-\epsilon)|B| - \int_a^{b-\epsilon} |\{T^{\varepsilon}_{ab}(d) < s\}|\,ds\,,
$$

and letting $\varepsilon \to 0^+$ we obtain, from (5.28),

$$
-\frac{d}{dt}\int_a^b |\{d(t) \le s\}|\,ds \le\; P_\phi(\{d(\cdot,t) \le b\}) - P_\phi(\{d(\cdot,t) \le a\}) - c(t)|\{a \le d(\cdot,t) \le b\}|.
$$

Dividing the above expression by $b-a$, and letting $b \to 0^-$ we get

$$
-\frac{d}{dt}\frac{1}{|a|}\int_a^0 |\{d(t) \le s\}|\,ds \le\; \frac{P_\phi(C(t)) - P_\phi(\{d(\cdot,t) \le a\})}{|a|} - c(t)\frac{|\{a \le d(\cdot,t) \le 0\}|}{|a|}\,.
$$

As $a \to 0^-$, the first and last term of this inequality converge respectively to $-(d/dt)|C(t)|$ and $-c(t)P_\phi(C(t))$ in $\mathcal{D}'((0, t_{C,c}))$.

We now need to find the limit of the quotient $\frac{P_\phi(C(t)) - P_\phi(\{d(\cdot,t)\le a\})}{|a|}$ as $a \to 0^-$, which requires a rather delicate argument. In view of Remark 5.11, we already know that, if we want this term to converge to $-V_2^{\phi}(C(t))$, we need to exploit some regularity property of $C(t)$; and this will be provided by estimate (5.5). Consider for almost every $t \in (0, t_{C,c})$ the vector field $\overline{z}(x,t)$ obtained by solving problem (5.11) in $C(t)$ ($C(t)$ is continuous in time so that $\operatorname{div}\overline{z}$ is measurable). We have

$$
\int_0^{t_{C,c}}\int_{C(t)} (\operatorname{div}\overline{z}(x,t))^2\,dxdt \le \kappa\int_0^{t_{C,c}}\left(P_\phi(C(t)) + \liminf_{\delta\to 0^+}\frac{1}{\delta}\int_{\{0<d(\cdot,t)<\delta\}} (\operatorname{div} z(x,t))^2\,dx\right)dt,
$$

which is finite by (5.5). Denoting by $C^{|a|}(t)$ the set $\{d(\cdot,t) < a\} + |a|W_\phi$, we deduce from

Proposition 5.8 that for any nonnegative $\psi \in \mathcal{D}((0, t_{C,c}))$, one has

$$\begin{aligned}
& \int_0^{t_{C,c}} \psi(t) \frac{P_\phi(C(t)) - P_\phi(\{d(\cdot,t) \le a\})}{|a|}\, dt \\
= & \int_0^{t_{C,c}} \psi(t) \left(\frac{P_\phi(C(t)) - P_\phi(C^{|a|}(t))}{|a|} + \frac{P_\phi(C^{|a|}(t)) - P_\phi(\{d(\cdot,t) \le a\})}{|a|} \right) dt \\
\le & \int_0^{t_{C,c}} \psi(t) \left(\int_{C(t)\setminus C^{|a|}(t)} (\operatorname{div} \overline{z}(x,t))^2\, dx + V_2^\phi(C(t)) + o(1) \right) dt \\
\le & \int_0^{t_{C,c}} \psi(t) V_2^\phi(C(t))\, dt + o(1)\,,
\end{aligned}$$

from which we deduce

$$-\frac{d}{dt}|C(t)| \le V_2(C(t)) - c(t)P_\phi(C(t)) \qquad \text{in } \mathcal{D}'((0, t_{C,c})).$$

The opposite inequality is obtained almost in the same way using the first inequality in (5.2) and letting first $a \to 0^-$ and then $b \to 0^-$: the main difference is that this time, passing to the limit in the expression $(P_\phi(C(t) + bW_\phi) - P_\phi(C(t)))/b$ does not raise any difficulty.

In the general case, we approximate ϕ with $\phi_\varepsilon \in \mathcal{C}_+^\infty$ and pass to the limit in (5.27), recalling Proposition 4.13, and the continuity of the mixed volumes [28]. □

6 Convex volume preserving ϕ-curvature flows

6.1 Existence of a convex volume preserving flat ϕ-flow

Let C be a compact convex set in $\mathbb{R}^N$; we define

$$\bar{c} := V_2^\phi(C)/P_\phi(C), \tag{6.1}$$

and

$$\Sigma_h(C) := S_h^{\bar{c}}(C)$$

where $S_h^{\bar{c}}$ is introduced in (4.4), and *depends on C*. Then, we define a discrete (in time) evolution by letting for any $t \ge 0$

$$C_h(t) \ := \ \Sigma_h^{[t/h]}(C), \tag{6.2}$$

namely we iterate the operator Σ_h for $[t/h]$ times.

Remark 6.1. For $h > 0$ fixed, the volume of $C_h(t)$ is in general not equal to the volume of $C_0(t)$. This property however becomes true in the limit, as a particular consequence of the next theorem.

Theorem 6.2. *Let $C \subset \mathbb{R}^N$ be a compact convex set. Let $C_h(t)$ be defined by (6.2). Then there exist a sequence $\{h_k\}$ converging to 0 as $k \to \infty$ and a continuous function $C(t) : [0, +\infty) \to \mathcal{K}$ such that $C(0) = C$,*

$$|C(t)| = |C(0)| \qquad \forall t \geq 0,$$

and

$$\lim_{k \to +\infty} d_{\mathcal{H}} \left(Q_{C_{h_k}}, Q_C \right) = 0.$$

Proof. This follows from Theorem 4.3, provided we can show that the piecewise constant function $c_h(t) := V_2^{\phi}(C_h(t))/P_{\phi}(C_h(t)) \geq 0$ remains uniformly bounded for all times as $h \to 0^+$. From standard inequalities between mixed volumes [28, Theorem 6.31], we have

$$\frac{V_2^{\phi}(C_h(t))}{P_{\phi}(C_h(t))} \leq \frac{N-1}{N} \frac{P_{\phi}(C_h(t))}{|C_h(t)|}. \tag{6.3}$$

Let r, R and $x \in C$ be such that $x + rW_{\phi} \subseteq C \subseteq RW_{\phi}$. Since $c_h \geq 0$, $C_h(t) \supseteq (S_h^0)^{[t/h]}(x + rW_{\phi})$ which in turns contains $x + (r/2)W_{\phi}$ for t less than or equal to some $T_0 > 0$ of order r^2. Hence

$$|C_h(t)| \geq |rW_{\phi}| \qquad \forall t \in [0, T_0].$$

On the other hand, $P_{\phi}(C_h(t)) \leq P_{\phi}(2RW_{\phi})$, so that $c_h(t) \leq (1 - 1/N) P_{\phi}(2RW_{\phi})/|rW_{\phi}| =: \bar{c}$ as long as $t \leq T_0$ and $C_h(t) \subseteq 2RW_{\phi}$, which will happen (by induction) if we also choose $T_0 \leq R/\bar{c}$. Hence on $[0, T_0]$, c_h remains uniformly bounded with respect to h, and we may apply Theorem 4.3 to get existence of a flow on $[0, T_0]$.

Let now

$$T^* := \sup \left\{ T \geq 0 : \quad \exists h_T > 0 : \sup_{h \leq h_T} \|c_h\|_{L^\infty((0,T))} < +\infty \right\} \geq T_0. \tag{6.4}$$

By a diagonal procedure, from Theorem 4.3 we can find a sequence $\{h_k\}_k$ converging to zero as $k \to +\infty$, and a convex evolution $C(t)$ such that $C_{h_k}(t) \to C(t)$ in the Hausdorff distance, locally uniformly in $[0, T^*)$. In particular, by continuity we have $c_{h_k}(t) \to c(t) = V_2^{\phi}(C(t))/P_{\phi}(C(t))$ locally uniformly in $[0, T^*)$. Using (5.7) and (5.27), we deduce that $|C(t)| = |C|$, and that $t \mapsto P_{\phi}(C(t))$ is nonincreasing. In particular, by inequality (6.3) applied to $c(t)$, we find that

$$c(t) = \frac{V_2^{\phi}(C(t))}{P_{\phi}(C(t))} \leq \frac{N-1}{N} \frac{P_{\phi}(C)}{|C|} \leq \bar{c},$$

for $t < T^*$. Hence $C(t)$ is contained in $(R + t\overline{c})W_\phi$, and must contain some Wulff shape $x(t)+r(t)W_\phi$ where $r(t) > 0$ depends only on $|C(t)| = |C|$ and $R+t\overline{c}$ (see for instance [28, Eq. (6.2.13)]).

It remains to show that $T^* = +\infty$. Assume this is not true. Then

(*) the sets $C(t)$ obtained above are contained in R^*W_ϕ, with $R^* = R+T^*\overline{c}$, and contain a small Wulff shape $x(t) + r^*W_\phi$ where r^* depends only on $|C|$ and R^*.

Reasoning as in the beginning of this proof, we can find $\tau(R^*, r^*) > 0$ such that for any convex K contained in $2R^*W_\phi$ and containing a Wulff shape of radius $r^*/2$, the forcing term of the motion $\Sigma_h^{[t/h]}K$ remains uniformly bounded for $h < \tau$, as long as $t \le \tau$.

Let us show that if h is small enough, $C_h(T^* - \tau/2)$ satisfies (*); by (6.4), this will yield $T^* \ge T^* + \tau/2$, a contradiction. If it is not true, there must exist a sequence $\{h_k\}$ converging to zero as $k \to +\infty$ such that either $C_{h_k}(T^* - \tau/2) \not\subset 2RW_\phi$ for all k, or $C_{h_k}(T^* - \tau/2) \subset 2RW_\phi$ but does not contain any Wulff shape of radius $r^*/2$ for all k. Extracting a further subsequence, we may assume that $C_{h_k}(t) \to C(t)$ locally uniformly on $[0, T^*)$ and in this case we have seen that for some x, $x+r^*W_\phi \subseteq C(T^*-\tau/2) \subseteq R^*W_\phi$, a contradiction. This shows that $T^* = +\infty$. □

Remark 6.3. If $d(x,t) = d^\phi_{C(t)}(x)$, then d satisfies (5.2) out of Q_C and (5.3) in $\mathrm{int}\,(Q_C)$.

6.2 Convex volume preserving rW_ϕ-regular flows

6.2.1 Existence

The following result is a consequence of Theorems 4.5 and 6.2.

Theorem 6.4. *Let $C \subset \mathbb{R}^N$ be a compact convex set satisfying an interior rW_ϕ-condition, for some $r > 0$. Then there exists a convex volume preserving $\frac{r}{2}W_\phi$-regular flow $C(t)$, for $t \in [t_1, t_2]$, such that $C(t_1) = C$.*

6.2.2 Uniqueness

We will prove that the forcing term of a convex volume preserving rW_ϕ-regular flow $E(t)$ depends only on the initial set. In particular, we will obtain a comparison result similar to Corollary 3.5.

Theorem 6.5. *Let $t \in [t_1, t_2] \to E_1(t)$, $E_2(t)$ be two convex volume preserving rW_ϕ-regular flows, defined in the same open set A. Let $\eta := d_{\mathcal{H}}\big(\partial E_1(t_1), \partial E_2(t_1)\big)$ and assume* *that $\eta < r/2$. Then*

$$d_{\mathcal{H}}\big(\partial E_1(t), \partial E_2(t)\big) \le e^{(\lambda+K)t}\eta \qquad \forall t \in [t_1, t_2] \text{ with } e^{(\lambda+K)t}\eta < r/2, \tag{6.5}$$

where λ is defined in (2.6) and K depends only on N and on the radius of a ball contained in $E_1(t)$ for all $t \in [t_1, t_2]$.

We begin with the following lemma.

Lemma 6.6. *Let $\phi \in \mathcal{C}^\infty_+$. Let $C_1 \subseteq C_2$ be two compact convex sets satisfying the interior rW_ϕ-condition, and let $R > 0$ be the radius of a ball contained in C_1. Define*

$$\eta := d_{\mathcal{H}}\big(\partial C_1, \partial C_2\big).$$

Then

$$\begin{aligned} P_\phi(C_1) \le P_\phi(C_2) &\le \Big(1 + \frac{\eta}{R}\Big)^{N-1} P_\phi(C_1), \\ V_2^\phi(C_1) \le V_2^\phi(C_2) &\le \Big(1 + \frac{\eta}{R}\Big)^{N-2} V_2^\phi(C_1)\,. \end{aligned} \tag{6.6}$$

Proof. The inequalities in (6.6) immediately follow from the observation that (assuming the origin is the center of a ball of radius R contained in C_1) $C_2 \subseteq (1 + \eta/R)C_1$ and the monotonicity of P_ϕ and V_2^ϕ with respect to the inclusion of convex sets (which is a consequence of the fact that these quantities are multiples of mixed volumes (see [28], [14])). □

Proof of Theorem 6.5. Let us assume that $\phi \in \mathcal{C}^\infty_+$. We have

$$c_i(t) \;=\; \frac{1}{P_\phi(E_i(t))} V_2^\phi(E_i(t)) \qquad \forall t \in [t_1, t_2].$$

Hence, from Lemma 6.6, we deduce that if $R > 0$ is the radius of a ball contained in $E_1(t)$ and $\eta(t) := \mathrm{dist}_{\mathcal{H}}(\partial E_1(t), \partial E_2(t))$,

$$\left(1 + \frac{\eta(t)}{R}\right)^{-(N-1)} c_1(t) \;\le\; c_2(t) \;\le\; \left(1 + \frac{\eta(t)}{R}\right)^{N-2} c_1(t). \tag{6.7}$$

Indeed

$$P_\phi(E_1)c_1 \le P_\phi(E_2)c_2 \le (1 + \frac{\eta}{R})^{N-1} P_\phi(E_1)c_2$$

so that $(1 + \frac{\eta}{R})^{-(N-1)}c_1 \le c_2$. Similarly,

$$P_\phi(E_2)c_2 \le (1 + \frac{\eta}{R})^{N-2} P_\phi(E_1)c_1 \le (1 + \frac{\eta}{R})^{N-2} P_\phi(E_2)c_1$$

From (6.7) we deduce

$$|c_1(t) - c_2(t)| \le K\eta(t),$$

where the constant K depends on N and R.

Assume $\eta < r/2$ where the evolution $E_1(t)$ is rW_ϕ-regular. Let now $\varepsilon > 0$ and $\tau > 0$ be the first time in $[0, t_2 - t_1]$ at which $\eta(t_1 + \tau) = (1+\varepsilon)\eta$ (if it exists). From the previous inequality, one has $|c_1(t) - c_2(t)| \le (1+\varepsilon)K\eta$ if $t_1 \le t \le t + \tau$. Hence, from Corollary 3.5, one finds

$$\eta(t) \le e^{\lambda\tau}\eta + (1+\varepsilon)K\eta\frac{e^{\lambda\tau}-1}{\lambda}$$

for $t \le t_1 + \tau$. Hence,

$$(1+\varepsilon) \le e^{\lambda\tau} + (1+\varepsilon)K\frac{e^{\lambda\tau}-1}{\lambda}, \quad \text{so that} \quad \tau \ge \frac{1}{\lambda}\ln\left(1 + \frac{\lambda\varepsilon}{\lambda + (1+\varepsilon)K}\right).$$

We get that

$$\frac{\eta(t_1+\tau) - \eta}{\tau} = \eta\frac{\varepsilon}{\tau} \le \eta\frac{\lambda\varepsilon}{\ln\left(1 + \frac{\lambda\varepsilon}{\lambda+(1+\varepsilon)K}\right)},$$

which in the limit gives $\liminf_{\tau\to 0^+}(\eta(t_1+\tau) - \eta)/\tau \le (\lambda + K)\eta$. This argument is valid starting from any time, as long as $\eta(t) < r/2$. The thesis follows. $\square$

7 Asymptotics of the volume preserving flat ϕ-flow in the convex case

The main purpose of this section is to prove the following result.

Theorem 7.1. *Let C be compact convex set, and let $t \in [0, +\infty) \to C(t)$ be a convex volume presering flat ϕ-flow starting from C, as given by Theorem 6.2. Then, modulo a time-dependent translation, $C(t)$ converges in the Hausdorff distance as $t \to +\infty$ to a translate of the Wulff shape of volume $|C|$.*

We develop the proof along the next subsections.

7.1 Asymptotic flow

Let $t \to C(t)$ be a convex volume preserving flat ϕ-flow starting from C (Theorem 6.2). Throughout this section we assume that modulo a time-dependent translation, $C(t)$ is uniformly bounded. Therefore, upon extracting a diverging subsequence $\{t_k\}$, we may assume that

$$\lim_{k\to+\infty} d_{\mathcal{H}}(C(t_k), \tilde{C}) = 0,$$

where $\tilde{C}$ is a compact convex set with $|\tilde{C}| = |C|$. Note that, on the other hand, $P_\phi(\tilde{C}) = \inf_{t>0} P_\phi(C(t))$.

Consider the sequence of convex flat ϕ-flows

$$\tilde{C}_k(t) := C(t_k + t), \qquad \forall t \in [0,1],$$

with forcing terms

$$c_k(t) := V_2^\phi(\tilde{C}_k(t))/P_\phi(\tilde{C}_k(t)).$$

By passing to the limit as $k \to +\infty$ and invoking Theorem 4.11, we can show that $\tilde{C}_k(t)$ converges uniformly in $[0,1]$ to a flat ϕ-flow $\tilde{C}(t)$, starting from $\tilde{C}$, and with forcing term $\tilde{c}(t) = V_2^\phi(\tilde{C}(t))/P_\phi(\tilde{C}(t))$. Moreover, according to Remark 6.3 each $d_k(x,t) = d^\phi_{C_k(t)}(x)$ satisfies the PDE

$$-\text{div}\, z_k + c_k(t) + \frac{\partial d_k}{\partial t} \geq 0 \quad \text{out of } \textstyle\bigcup_{0<t<1}(C_k(t) \times \{t\}),$$

and the corresponding PDE in int $\left(\bigcup_{0<t<1}(C_k(t) \times \{t\})\right)$, where $z_k \in \partial\phi^\circ(\nabla d_k)$ almost everywhere. Therefore, by extracting a subsequence if necessary, we may assume that $d_k(x,t) \to \tilde{d}(x,t) = d^\phi_{\tilde{C}(t)}(x)$ uniformly in $\mathbb{R}^N \times [0,1]$, $z_k \rightharpoonup \tilde{z}$ weakly-$*$ in $L^\infty(\mathbb{R}^N \times (0,1); \mathbb{R}^N)$, and

$$-\text{div}\, \tilde{z} + \tilde{c}(t) + \frac{\partial \tilde{d}}{\partial t} \geq 0 \quad \text{out of } \textstyle\bigcup_{0<t<1}(\tilde{C}(t) \times \{t\}), \tag{7.1}$$

and $\tilde{z}$ satifies the corresponding PDE inside int $\left(\bigcup_{0<t<1}(\tilde{C}(t) \times \{t\})\right)$, where $\tilde{z} \in \partial\phi^\circ(\nabla\tilde{d})$ almost everywhere. A full account of this passage to the limit can be found in [13].

Since

$$P_\phi(\tilde{C}(t)) = P_\phi(\tilde{C}) = \inf_{s>0} P_\phi(C(s)) \qquad \forall t \in [0,1],$$

by (5.6) we deduce that for any $t_1 < t_2$

$$\limsup_{\delta\to 0^+} \int_{t_1}^{t_2} \frac{1}{\delta} \int_{\{0<\tilde{d}(\cdot,t)<\delta\}} (\text{div}\, \tilde{z}(x,t))^2 \, dxdt \leq \int_{t_1}^{t_2} \frac{V_2^\phi(\tilde{C}(t))^2}{P_\phi(\tilde{C}(t))} \, dt. \tag{7.2}$$

On the other hand, for almost every t,

$$\begin{aligned}
&\liminf_{\delta\to 0^+} \frac{1}{\delta} \int_{\{0<\tilde{d}(\cdot,t)<\delta\}} (\text{div}\, \tilde{z}(x,t))^2 \, dx \\
\geq\ & \liminf_{\delta\to 0^+} \left(\frac{1}{\delta} \int_{\{0<\tilde{d}(\cdot,t)<\delta\}} \text{div}\, \tilde{z}(x,t) \, dx \right)^2 \frac{\delta}{|\{0<\tilde{d}(\cdot,t)<\delta\}|} \qquad (7.3) \\
=\ & \lim_{\delta\to 0^+} \left(\frac{P_\phi(\tilde{C}(t) + \delta W_\phi) - P_\phi(\tilde{C}(t))}{\delta} \right)^2 \frac{\delta}{|\{0<\tilde{d}(\cdot,t)<\delta\}|} = \frac{V_2^\phi(\tilde{C}(t))^2}{P_\phi(\tilde{C}(t))},
\end{aligned}$$

from which we deduce that all inequalities in (7.3) and (7.2) are in fact equalities, and

$$\begin{aligned}\lim_{\delta\to 0^+}\frac{1}{\delta}\int_{t_1}^{t_2}\int_0^{\delta}\int_{\partial(\tilde C(t)+sW_\phi)}\Big(\mathrm{div}\,\tilde z(x,t)\\ -\tfrac{1}{P_\phi(\tilde C(t)+sW_\phi)}\int_{\partial(\tilde C(t)+sW_\phi)}\mathrm{div}\,\tilde z(y,t)\,dP_\phi(y)\Big)^2\,dP_\phi(x)ds\,dt \;=\; 0.\end{aligned} \tag{7.4}$$

for any $t_1 < t_2$.

7.2 The limit flow is stationary and rW_ϕ-regular

The following proposition concerns flat ϕ-flows satisfying suitable properties.

Proposition 7.2. *Let $\tilde C(t)$ be a convex volume preserving flat ϕ-flow starting from a compact convex set $\tilde C$ satisfying* (7.1) *and assume $P_\phi(\tilde C(t))$ is independent of time, so that* (7.4) *holds. Then*

(i) $\tilde C(t) = \tilde C$ for any $t \geq 0$,

(ii) $\tilde C$ satisfies the interior rW_ϕ-condition.

Proof. Let $\varepsilon > 0$ and let $F_n : \mathbb{R} \to [0, +\infty)$ be a smooth non-increasing function with $F_n(r) = 1$ when $r \leq \frac{1}{n} < \varepsilon$, $F_n(r) = 0$ if $r \geq \varepsilon$, converging uniformly to $F : \mathbb{R} \to [0, +\infty)$ where $F(r) = 1 - \frac{r}{\varepsilon}$ when $r \in [0, \varepsilon]$. Let us consider a nonnegative, bounded continuous function $\varphi : \mathbb{R}^N \to \mathbb{R}^+$ and a nonnegative $\psi \in C_0^\infty(0, +\infty)$. Then,

$$\begin{aligned}-\int_0^{+\infty}\int_{\mathbb{R}^N}\psi'(t)F_n(\tilde d(t))\varphi(x)\,dx\,dt &= \int_0^{+\infty}\int_{\mathbb{R}^N}\psi(t)\varphi(x)\frac{\partial F_n(\tilde d)}{\partial t}(t)\,dx\,dt\\ &= \int_0^{+\infty}\int_{\mathbb{R}^N}\psi(t)F_n'(\tilde d(t))\varphi(x)\frac{\partial \tilde d}{\partial t}(t)\,dx\,dt\,.\end{aligned}$$

Here, both $\partial F_n(\tilde d))/\partial t$ and $\partial\tilde d/\partial t$ are measures (on $\{\tilde d > 1/n\}$, where the other terms are not zero), but the last equality is shown by first mollifying $\tilde d$ and then passing to the limit. Using (5.2) and $F_n' \leq 0$, we find

$$\int_0^{+\infty}\int_{\mathbb{R}^N}\psi(t)F_n'(\tilde d(t))\varphi(x)\frac{\partial\tilde d}{\partial t}\,dx\,dt \;\leq\; \int_0^{+\infty}\int_{\{0<\tilde d(\cdot,t)\leq\varepsilon\}}\psi(t)F_n'(\tilde d(t))(\mathrm{div}\,\tilde z - \tilde c)\varphi(x)\,dx\,dt$$

Letting $n \to +\infty$ we obtain

$$\begin{aligned}&-\int_0^{+\infty}\int_{\mathbb{R}^N}\psi'(t)F(\tilde d(t))\varphi(x)\,dx\,dt\\ &\leq -\int_0^{+\infty}\psi(t)\frac{1}{\varepsilon}\int_{\{0<\tilde d(\cdot,t)\leq\varepsilon\}}(\mathrm{div}\,\tilde z - \tilde c)\varphi(x)\,dx\,dt\\ &\leq \left(\frac{1}{\varepsilon}\int_0^{+\infty}\psi(t)\int_{\{0<\tilde d(\cdot,t)\leq\varepsilon\}}\varphi(x)^2\,dx\,dt\right)^{\frac12}\left(\frac{1}{\varepsilon}\int_0^{+\infty}\psi(t)\int_{\{0<\tilde d(\cdot,t)\leq\varepsilon\}}(\mathrm{div}\,\tilde z(x,t) - \tilde c)^2\,dx\,dt\right)^{\frac12}.\end{aligned}$$

Since, using (7.4), the right hand side tends to zero as $\varepsilon \to 0^+$, we deduce that

$$-\int_0^{+\infty}\int_{\tilde{C}(t)} \psi'(t)\varphi(x)\,dx\,dt \;\le\; 0 \tag{7.5}$$

for any nonnegative $\psi \in C_0^\infty(0,+\infty)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}_+$ bounded and continuous.

Since $|\tilde{C}(t)|$ is constant, we may add any real constant to the above inequality and we deduce that (7.5) holds for any bounded $\varphi \in C(\mathbb{R}^N)$, and since it holds also for $-\varphi$, we deduce that the left hand side of (7.5) vanishes. This implies that $\tilde{C}(t)$ is independent of t, and the flow is stationary.

We deduce from (7.1) that

$$0 = \frac{\partial \tilde{d}}{\partial t} \ge \operatorname{div} \tilde{z}(x,t) - \tilde{c}, \qquad \text{in } \mathbb{R}^N \setminus \tilde{C}.$$

In particular, $\operatorname{div} \tilde{z} \in L^\infty(\{\tilde{d}(\cdot,\cdot) > 0\})$. Using Proposition A.1 in Appendix A, we deduce that $\tilde{C}$ satisfies the rW_ϕ-condition for some radius $r > 0$. This radius is at least given by $\min\{1/\tilde{c}, |\tilde{C}|/P_\phi(\tilde{C})\}$. However, since $\tilde{c} = V_2^\phi(\tilde{C})/P_\phi(\tilde{C})$, by (6.3) we find that $r \ge |\tilde{C}|/P_\phi(\tilde{C})$. □

7.3 The limit shape is the Wulff shape

The remaining of this section is devoted to the proof that the stationary limit flow $\tilde{C}$ can only be the (invariant) Wulff shape of volume $|C|$. If $\phi, \phi^\circ \in \mathcal{C}_+^\infty$, this was proved in [5]. We adapt the proof when ϕ is not smooth. Let us first show the following lemma.

Lemma 7.3. *Let K be a convex set, $d := d_K^\phi$, $\delta > 0$ small enough, and write $\Sigma_\delta := \{0 < d < \delta\}$. Assume that there exists $z \in L^\infty(\Sigma_\delta)$ such that $z \in \partial\phi^\circ(\nabla d)$ almost everywhere and $\operatorname{div} z \in L^2(\Sigma_\delta)$. Let $\sigma(x) := \langle x, \nabla d(x)\rangle$. Then*

$$\int_{\partial K_s^+} \sigma \; dP_\phi \;=\; N|K_s^+|\,, \tag{7.6}$$

$$\int_{\partial K_s^+} \sigma \operatorname{div} z \; dP_\phi \;=\; (N-1)P_\phi(K_s^+)\,, \tag{7.7}$$

for almost every $s \in (0,\delta)$.

Proof. Equation (7.6) is standard, since the integral reduces to $\int_{\partial K}\langle x, \nu^K\rangle d\mathcal{H}^{N-1}$, which is $N|K|$ by Green's formula. To show (7.7), we prove that

$$\operatorname{div}(\sigma z) \;=\; \sigma \operatorname{div} z \,+\, 1 \tag{7.8}$$

in the sense of distributions in Σ_δ. Let $w \in W^{1,\infty}(\Sigma_\delta)$ be a function with compact support. Since $\langle z, \nabla d\rangle = 1$ almost everywhere, we have

$$\begin{aligned}
-\int_{\Sigma_\delta} \langle x, \nabla d(x)\rangle \langle z, \nabla w\rangle\, dx \;&=\; -\frac{d}{d\lambda}\int_{\Sigma_\delta} d(\lambda x)\langle z, \nabla w\rangle\, dx\Big|_{\lambda=1} \\
= \frac{d}{d\lambda}\int_{\Sigma_\delta} w\; d(\lambda x)\; \operatorname{div} z + \lambda\langle \nabla d(\lambda x), z(x)\rangle)\, dx\Big|_{\lambda=1} \\
= \int_{\Sigma_\delta} \sigma\, \operatorname{div} z\; w\, dx + \int_{\Sigma_\delta} w\, dx + \frac{d}{d\lambda}\int_{\Sigma_\delta} w\, \langle \nabla d(\lambda x), z(x)\rangle)\, dx\Big|_{\lambda=1}.
\end{aligned}$$

It is enough to show that the last term is zero. First of all, $\lambda \mapsto \int_{\Sigma_\delta} w\, \langle \nabla d(\lambda\cdot), z\rangle)\, dx$ (which is well defined if $\lambda \sim 1$ since w has compact support) is differentiable at 1, as a sum of terms which are all differentiable. Then, since $z \in \partial\phi^\circ(\nabla d)$ almost everywhere, we have, almost everywhere in Σ_δ, $\langle z, \nabla d\rangle = \phi^\circ(\nabla d) = 1$, while $\langle z(x), \nabla d(\lambda x)\rangle \le \phi^\circ(\nabla d(\lambda x)) = 1$ if $\lambda \neq 1$. Hence, if for instance $w \ge 0$ almost everywhere,

$$\int_{\Sigma_\delta} w\, \langle \nabla d(\lambda x), z(x)\rangle\, dx \;\le\; \int_{\Sigma_\delta} w\, \langle \nabla d, z\rangle\, dx\,.$$

This yields

$$\frac{d}{d\lambda}\int_{\Sigma_\delta} w\, \langle \nabla d(\lambda x), z(x)\rangle\, dx\Big|_{\lambda=1} \;=\; 0.$$

If $w \le 0$ almost everywhere, $\lambda = 1$ is now a minimum and the derivative is, again, 0. If w changes sign, it suffices to compute the derivative separately for the positive and negative parts of w. We have shown (7.8).

We are now in the position to show (7.7). For almost every $s \in (0, \delta)$, we have, using (7.8),

$$\begin{aligned}
\int_{\partial K_s^+} \sigma \operatorname{div} z dP_\phi \;&=\; \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{K_{s+\varepsilon}^+\setminus K_s^+} \sigma \operatorname{div} z\, dx = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{K_{s+\varepsilon}^+\setminus K_s^+} \operatorname{div}(\sigma z) - 1\, dx \\
= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}&\left(\int_{\partial K_{s+\varepsilon}^+} \sigma\langle z, \nu^{K_{s+\varepsilon}^+}\rangle\, d\mathcal{H}^{N-1} - \int_{\partial K_s^+} \sigma\langle z, \nu^{K_s^+}\rangle\, d\mathcal{H}^{N-1} - |K_{s+\varepsilon}^+ \setminus K_s^+|\right).
\end{aligned}$$

Thanks to (7.6), the quantity inside the limit is $(N-1)|\{s < d < s+\varepsilon\}|/\varepsilon$, which converges to $(N-1)P_\phi(K_s^+)$ as $\varepsilon \to 0$. □

We apply Lemma 7.3 to $K = \tilde{C}$, $z = \tilde{z}$, $d = \tilde{d}$. Since we also have for almost every $s \in (0, \delta)$ that $V_2^\phi(K_s^+) = \int_{\partial K_s^+} \operatorname{div} z\, dP_\phi$, we obtain, as in [5, Cor. 4.2], that

$$\begin{aligned}
0 \;&\le\; (N-1)P_\phi(K_s^+)^2 - N|K_s^+|V_2^\phi(K_s^+) \\
&=\; P_\phi(K_s^+)\int_{\partial K_s^+} \sigma \operatorname{div} z\, dP_\phi - \int_{\partial K_s^+} \sigma\, dP_\phi \int_{\partial K_s^+} \operatorname{div} z\, dP_\phi \\
&=\; P_\phi(K_s^+)\int_{\partial K_s^+} \sigma\left(\operatorname{div} z - \frac{1}{P_\phi(\partial K_s^+)}\int_{\partial K_s^+} \operatorname{div} z\right) dP_\phi.
\end{aligned}$$

Letting $s \to 0^+$ and using (7.4), we deduce that for almost every t,

$$(N-1)P_\phi(\tilde{C})^2 - N|\tilde{C}|V_2^\phi(\tilde{C}) \;=\; 0 \tag{7.9}$$

In particular, for the velocity $\tilde{c}$, we have $\tilde{c} = (1-1/N)P_\phi(\tilde{C})/|\tilde{C}|$ (we get again the stationarity of the limiting velocity).

If $\phi \in \mathcal{C}_+^\infty$, Andrews [5] uses (7.9) together with [28, Th. 6.6.8] to conclude that $\tilde{C}$ is a translate and homothetic of the Wulff shape (after proving that $\tilde{C}$ has also smooth boundary). In two dimension, we can deduce that $\tilde{C}$ is the Wulff shape, without any further assumption, since (7.9) reduces to the isoperimetric inequality.

In higher dimension, the situation is not so simple. Thanks to the regularity proven in Proposition 7.2, we show again that the limit shape is the Wulff shape, but the proof is more involved.

First of all, recalling (7.9), we may invoke [28, Th. 6.6.18] to conclude that $\tilde{C}$ is a "$(N-2)$-tangential body of a homothetic translate of W_ϕ", according to the following definitions [28, pp. 74, 75]:

Definition 7.4. *Let K be a compact convex set in $\mathbb{R}^N$, $\nu \in \mathbb{R}^N$, $|\nu| = 1$, $\nu^\perp = \{y \in \mathbb{R}^N : \langle y, \nu\rangle = 0\}$. A hyperplane $P = x + \nu^\perp$, with $x \in \partial K$, is a 1-extreme support plane of K if ν belongs to the relative interior of a face F of the exterior normal cone N to K at a point $y \in \mathrm{relint}(K \cap P)$, and $\dim F \le 2$.*

Definition 7.5. *Given two compact convex sets $L \subseteq K$ in $\mathbb{R}^N$, K is a $(N-2)$-tangential body of L if each 1-extreme support plane of K is a support plane of L.*

To clarify the situation, we mention the following characterization [28, Theorem 2.2.7]:

Theorem 7.6. *If P is a 1-extreme support plane, it is limit of support planes whose normal cone has dimension at most 2.*

Notice that, if $N = 2$, every support plane is a 1-extreme support plane, so that L is the only 0-tangential body of itself. In general, if a $(N-2)$-tangential body K of L has smooth boundary, or more generally if the dimension of the normal cone at each point of ∂K does not exceed 2, then $K = L$. We now show:

Proposition 7.7. *If a convex body K is a $(N-2)$-tangential body of L, and satisfies the rL-condition for some $r > 0$, then $K = L$.*

Proof. Let $y \in \partial L \cap \mathrm{int}(K)$ such that ∂L is differentiable at y, and let $x \in \partial K$ such that $P = x + \nu^L(y)^\perp$ is a support plane of K. Since K satisfies the rL-condition, it follows that $x \in (z + rL) \subset K$ for some $z \in \mathbb{R}^N$. In particular, P is a support plane of $z + rL$, hence it contains the whole face of $z + rL$ normal to $\nu^L(y)$, and in particular the point $\bar{x} = z + ry \in \partial K$. Thus, ∂K is differentiable at $\bar{x}$, so that P is a 1-extreme support plane (even, 0-extreme); by assumption, we deduce it is the support plane of L of normal $\nu^L(y)$. Hence $y \in P$, and then $y \in \partial K$, a contradiction. $\square$

It follows from Proposition 7.7, the identity (7.9) and Proposition 7.2, that $\tilde{C}$ is the Wulff shape. Thus Theorem 7.1 is proved. $\square$

A Convex sets with bounded crystalline mean curvature

The following result, which shows the equivalence between three different ways of expressing the fact that a convex set has bounded crystalline curvature, is essentially contained in [14], though not explicitly stated there.

Proposition A.1. *Let ϕ be an anisotropy, and let C be a convex body in $\mathbb{R}^N$. Let $\lambda_C := \frac{P_\phi(C)}{|C|}$. The following assertions are equivalent:*

(i) there exist $\delta_0 > 0$ and a vector field $z \in L^\infty(\{0 < d_\phi^C < \delta_0\}; \mathbb{R}^N)$, with $z \in \partial\phi^\circ(\nabla d_\phi^C)$ almost everywhere, such that $0 \le \operatorname{div} z \le \kappa$ in $\{0 < d_\phi^C < \delta_0\}$, $\kappa > 0$;

(ii) C satisfies the rW_ϕ-condition with $r = \max(\kappa, \lambda_C)^{-1}$;

(iii) C is rW_ϕ-regular, that is there exist δ_1 and a vector field $z \in L^\infty(\{|d_\phi^C| < \delta_1\}; \mathbb{R}^N)$, with $z \in \partial\phi^\circ(\nabla d_\phi^C)$ almost everywhere, such that $0 \le \operatorname{div} z \le \tilde{\kappa}$ in $\{|d_\phi^C| < \delta_1\}$ for some $\tilde{\kappa} > 0$.

Proof. $(i) \Rightarrow (ii)$: Let $\lambda > \max(\kappa, \lambda_C)$. We notice that for $0 < \delta < \delta_0$ small enough we have that

$$0 \le \operatorname{div} z \le \kappa$$

in a neighborhood of ∂C_δ^+ and $\lambda > \max(\kappa, \lambda_{C_\delta^+})$, where $\lambda_{C_\delta^+} := \frac{P_\phi(C_\delta^+)}{|C_\delta^+|}$. By Theorem 7.3 in [14] we know that C_δ^+ is the unique solution of

$$(P)_{\lambda,\delta} \qquad \min_{F \subseteq C_\delta^+} \{P_\phi(F) - \lambda|F|\}. \tag{A.1}$$

Let $\{\phi_\varepsilon\} \subset C_+^\infty$ be a sequence of anisotropies converging to ϕ as $\varepsilon \to 0$, locally uniformly (so that $W_{\phi_\varepsilon} \to W_\phi$ in the Hausdorff distance), and C_ε^δ be smooth convex sets converging to C_δ^+ in the Hausdorff distance. Let $\lambda^{\varepsilon,\delta} := \frac{P_{\phi_\varepsilon}(C_\varepsilon^\delta)}{|C_\varepsilon^\delta|}$, $\bar\lambda^{\varepsilon,\delta} := \inf_{X\subseteq C_\varepsilon^\delta} \frac{P_{\phi_\varepsilon}(X)}{|X|}$. Observe that $\lambda^{\varepsilon,\delta} \geq \bar\lambda^{\varepsilon,\delta}$.

One can show that $P_{\phi_\varepsilon}(C_\varepsilon^\delta) \to P_\phi(C_\delta^+)$, hence, $\lambda^{\varepsilon,\delta} \to \lambda_{C_\delta^+}$ as $\varepsilon \to 0$. Hence choosing ε small enough we know that $\lambda > \lambda^{\varepsilon,\delta} \geq \bar\lambda^{\varepsilon,\delta}$. Now, we consider the problem

$$(P)_{\lambda,\varepsilon,\delta} \qquad \min_{F\subseteq C_\varepsilon^\delta} \{P_{\phi_\varepsilon}(F) - \lambda|F|\}. \tag{A.2}$$

Let $D_{\varepsilon,\delta}$ be a minimizer of $(P)_{\lambda,\varepsilon,\delta}$. Since C_ε^δ is of class $\mathcal{C}^{1,1}$ and $\phi_\varepsilon \in \mathcal{C}_+^\infty$, we know that C_ε^δ is Lipschitz $\phi_\varepsilon-$regular and satisfies the $\tau\mathcal{W}_{\phi_\varepsilon}$-condition for some $\tau > 0$ ([10, Lemmas 3.4,3.5], see also [8, Remark 4]). By Theorems 6.3 and 7.2 in [14], moreover, this minimum is unique and it is a convex set, with $\mathcal{C}^{1,1}$ boundary. Let $n_{\varepsilon,\delta}$ be the Cahn-Hoffman vector field of $D_{\varepsilon,\delta}$. Since it solves (A.2), we have that $\operatorname{div} n_{\varepsilon,\delta} \leq \lambda$ on $\partial D_{\varepsilon,\delta}$ [10]. Let $d_{\varepsilon,\delta} := d_{\phi_\varepsilon}^{D_{\varepsilon,\delta}}$. By [8, Theorem 4] we have that $d_{\varepsilon,\delta} \in \mathcal{C}^{1,1}_{\mathrm{loc}}(\{|d_{\varepsilon,\delta}| < \lambda^{-1}\})$, and we deduce [8, Corollary 1] that it satisfies the $\lambda^{-1}W_{\phi_\varepsilon}$-condition.

As $\varepsilon \to 0$, the solution of $(P)_{\lambda,\varepsilon,\delta}$ goes to the solution of $(P)_{\lambda,\delta}$, in other words, $D_{\varepsilon,\delta} \to C_\delta^+$ (in L^1, but since these sets are convex and uniformly bounded, equivalently in the Hausdorff distance). In the limit, we find that C_δ^+ satisfies the $\lambda^{-1}W_\phi$-condition. Letting $\lambda \to r$ we deduce (ii).

The implication $(ii) \Rightarrow (iii)$ follows from [8, Proposition 2]. The implication $(iii) \Rightarrow (i)$ follows from the definition of a ϕ-regular set. $\square$

Acknowledgement. V. Caselles acknowledges partial support by PNPGC project, reference MTM2006-14836. A. Chambolle is partially supported by ANR project “MICA”.

References

[1] F. Almgren, J. E. Taylor, and L.-H. Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31(2):387–438, 1993.

[2] F. Alter and V. Caselles. Uniqueness of the cheeger set of a convex body. in preparation.

[3] M. Amar and G. Bellettini. A notion of total variation depending on a metric with discontinuous coefficients. *Ann. Inst. Henri Poincaré*, 11:91–133, 1994.

[4] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[5] B. Andrews. Volume-preserving anisotropic mean curvature flow. *Indiana Univ. Math. J.*, 50(2):783–827, 2001.

[6] G. Anzellotti. Pairings between measures and bounded functions and compensated compactness. *Ann. Mat. Pura Appl.*, 135:293–318, 1983.

[7] G. Barles and P. E. Souganidis. A new approach to front propagation problems: theory and applications. *Arch. Rational Mech. Anal.*, 141(3):237–296, 1998.

[8] G. Bellettini, V. Caselles, A. Chambolle, and M. Novaga. Crystalline mean curvature flow of convex sets. *Arch. Ration. Mech. Anal.*, 179(1):109–152, 2006.

[9] G. Bellettini and M. Novaga. Approximation and comparison for nonsmooth anisotropic motion by mean curvature in $\mathbf{R}^N$. *Math. Models Methods Appl. Sci.*, 10(1):1–10, 2000.

[10] G. Bellettini, M. Novaga, and M. Paolini. On a crystalline variational problem. I. First variation and global L^∞ regularity. *Arch. Ration. Mech. Anal.*, 157(3):165–191, 2001.

[11] K. A. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1978.

[12] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland Publishing Co., Amsterdam, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[13] V. Caselles and A. Chambolle. Anisotropic curvature-driven flow of convex sets. *Nonlinear Anal.*, 65(8):1547–1577, 2006.

[14] V. Caselles, A. Chambolle, S. Moll, and M. Novaga. A characterization of convex calibrable sets in $\mathbb{R}^N$ with respect to anisotropic norms. Preprint Università di Pisa, 2005.

[15] V. Caselles, A. Chambolle, and M. Novaga. Uniqueness of the cheeger set of a convex body. *Pacific J. Math.*, 2007.

[16] A. Chambolle and M. Novaga. Implicit time discretization of the mean curvature flow with a discontinuous forcing term. preprint CMAP #571, March 2005.

[17] A. Chambolle and M. Novaga. Convergence of an algorithm for the anisotropic and crystalline mean curvature flow. *SIAM J. Math. Anal.*, 37(6):1978–1987 (electronic), 2006.

[18] Y. G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *Proc. Japan Acad. Ser. A Math. Sci.*, 65(7):207–210, 1989.

[19] L.C. Evans. Convergence of an algorithm for mean curvature motion. *Indiana Univ. Math. J.*, 42(2):533–557, 1993.

[20] M.E. Gage and R.S. Hamilton. The heat equation shrinking convex plane curves. *J. Differential Geom.*, 23:69–96, 1986.

[21] G. Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20:237–266, 1984.

[22] G. Huisken. The volume preserving mean curvature flow. *J. Reine Angew. Math.*, 382:35–48, 1987.

[23] H. Ishii. A generalization of the bence, merriman and osher algorithm for motion by mean curvature. In *Curvature flows and related topics (Levico, 1994)*, pages 111–127. GAKUTO Internat. Ser. Math. Sci. Appl. 5, Tokyo, 1995.

[24] H. Ishii, G. E. Pires, and P. E. Souganidis. Threshold dynamics type approximation schemes for propagating fronts. *J. Math. Soc. Japan*, 51(2):267–308, 1999.

[25] S. Luckhaus and T. Sturzenhecker. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations*, 3(2):253–271, 1995.

[26] B. Merriman, J. K. Bence, and S. J. Osher. Diffusion generated motion by mean curvature. In J. E. Taylor, editor, *"Computational Crystal Growers Workshop"*, Selected Lectures in Math. Amer. Math. Soc., Providence, Rhode Island, 1992.

[27] M. Novaga and E. Paolini. Stability of crystalline evolutions. *Math. Models Methods Appl. Sci.*, 15(6):921–937, 2005.

[28] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[29] A. Stancu. Asymptotic behavior of solutions to a crystalline flow. *Hokkaido Math. J.*, 27(2):303–320, 1998.