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CHAOTICITY FOR MULTI-CLASS SYSTEMS AND EXCHANGEABILITY WITHIN CLASSES

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Abstract

Under the natural partial exchangeability assumption for multi-class interacting particle systems, we prove that these converge to an independent system with infinite i.i.d. classes if and only if the empirical measure of each class satisfies a weak law of large numbers. This extension of a classical result for exchangeable systems (related to the de Finetti Theorem) is somewhat surprising, since then convergence of *each* class to infinite i.i.d. particles implies asymptotic independence of particles of *different* classes.

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1. Introduction

Families of exchangeable random variables (r.v.) are common, for instance in statistical sampling procedures, or interacting particle models in statistical mechanics. See *e.g.* Aldous [1] for many examples and results for finite and infinite families.

The related notion of chaoticity (convergence in law to an i.i.d. sequence) arises in many contexts, such as estimation in statistics, asymptotic models in statistical mechanics, and approximations for invariant laws for communication networks. It is at the basis of many pertinent heuristics; for instance, Ludwig Boltzmann derived the Boltzmann equation from particle dynamics by introducing the "molecular chaos assumption" (*Stosszahlansatz*) of independence of particles entering a collision. This

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is patently false, since particles coming out of a collision are not independent, but may nevertheless be asymptotically true in the large system limit. For a modern rigorous perspective, see Cercignani, Illner and Pulvirenti [3], in particular Sections 2 and 4.

For exchangeable r.v. with Polish state space, chaoticity is equivalent to a weak law of large numbers for the empirical measures. A.S. Sznitman used this to obtain rigorous propagation of chaos results for many varied models of interest, introducing compactness-uniqueness methods to prove the required law of large numbers; see [9] for a survey and bibliography, and *e.g.* [7, 5, 4] for some developments.

Many interacting systems in statistical mechanics, chemistry, communication networks, algorithmics, biology, etc., involve dissimilar objects which we call particles, which are classified in a finite number of types, particles of a class being similar and numerous; see *e.g.* [2, 4, 6, 8] in just one recent monograph.

The scope of this short communication is to extend the above notions and results to multi-class systems, in order to better understand their structure and to be able to apply to them the compactness-uniqueness methods of Sznitman [9].

2. Classical notions for indistinguishable particles

A system $(X_n)_{1\le n\le N}$ of random variables with state space $\mathcal{S}$ is *exchangeable* if the law of $(X_n)_{1\le n\le N}$ is invariant under permutation of the indices: for every permutation σ of $\{1,\dots,N\}$ we have $\mathcal{L}(X_{\sigma(1)},\dots,X_{\sigma(N)}) = \mathcal{L}(X_1,\dots,X_N)$. This expresses that the r.v. are statistically indistinguishable.

The systems $(X_n^N)_{1\le n\le N}$ for $N\ge 1$ are *P-chaotic* if $\lim_{N\to\infty}\mathcal{L}(X_1^N,\dots,X_k^N) = P^{\otimes k}$ for all $k\ge 1$. The systems converge in law to an i.i.d. system of law P.

The following classical result is related to the de Finetti Theorem, see Aldous [1, Prop. 7.20 p. 55]. We shall give a direct proof in a more involved setting.

Theorem 1. *Let $(X_n^N)_{1\le n\le N}$ be an exchangeable system of random variables on a Polish state space $\mathcal{S}$, for $N\ge 1$, and P be a law on $\mathcal{S}$. Then these systems are P-chaotic if and only if $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N \delta_{X_n^N} = P$ in law, hence in probability.*

Convergence in law and in probability are equivalent for a deterministic limit. The last property constitutes a (weak) law of large numbers for the empirical measures.

3. The extension to multi-class systems

We consider multi-class interacting systems $(X_{n,i}^N)_{1\le n\le N_i,\,1\le i\le C}$, where $X_{n,i}^N$ with state space $\mathcal{S}_i$ is the n-th particle of class i. We use $N = N_1 + \cdots + N_C$ as the main parameter, even though the true one is $(N_1, \ldots, N_C)$.

Such a multi-class system is *class-exchangeable* if its law is invariant under permutation of the indices *within* classes: for all permutations σ_i of $\{1, \ldots, N_i\}$,

$$\mathcal{L}(X_{\sigma_i(n),i}^N : 1 \le n \le N_i,\, 1 \le i \le C) = \mathcal{L}(X_{n,i}^N : 1 \le n \le N_i,\, 1 \le i \le C)\,.$$

It is equivalent to have this property when all σ_i but one are the identity. This expresses that particles within one class are statistically indistinguishable, and implies that $(X_{n,i}^N)_{1\le n\le N_i}$ is exchangeable for $1 \le i \le C$.

The multi-class systems $(X_{n,i}^N)_{1\le n\le N_i,\,1\le i\le C}$ for $N \ge 1$ are $(P_1, \ldots, P_C)$-*chaotic* if $\lim N_i = \infty$ and P_i is a law on $\mathcal{S}_i$ and

$$\lim \mathcal{L}(X_{n,i}^N : 1 \le n \le k,\, 1 \le i \le C) = P_1^{\otimes k} \otimes \cdots \otimes P_C^{\otimes k} \ \text{ for all } \ k \ge 1\,.$$

Any fixed finite sub-system of the multi-class system is asymptotically independent with particles of class i having law P_i. In particular $(X_{n,i}^N)_{1\le n\le N_i}$ is P_i-chaotic.

Class-exchangeability is a much weaker property than exchangeability, in particular its symmetry order is $N_1! \cdots N_C! \ll (N_1 + \cdots + N_C)! = N!$. Nevertheless, if it holds true, laws of large numbers for empirical measures *within* classes force asymptotic independence between particles in *different* classes, and thus, P_i-chaoticity for all classes i is equivalent to $(P_1, \ldots, P_C)$-chaoticity, which is somewhat unexpected.

Theorem 2. *Let $(X_{n,i}^N)_{1\le n\le N_i,\,1\le i\le C}$ be class-exchangeable systems, and for every i the $X_{n,i}^N$ have Polish state space $\mathcal{S}_i$. Then these multi-class systems are $(P_1, \ldots, P_C)$-chaotic if and only if $\lim \frac{1}{N_i}\sum_{n=1}^{N_i} \delta_{X_{n,i}^N} = P_i$ in law, hence in probability, for $1 \le i \le C$. This happens if and only if $(X_{n,i}^N)_{1\le n\le N_i}$ is P_i-chaotic for $1 \le i \le C$.*

Proof. Let $\Lambda_i^N = \frac{1}{N_i}\sum_{n=1}^{N_i} \delta_{X_{n,i}^N}$. For f in $C_b(\mathcal{S}_i)$, exchangeability within class i and developing the square yields

$$\begin{aligned}\mathbf{E}\left(\langle f, \Lambda_i^N - P_i\rangle^2\right) &= \mathbf{E}\left(\left(\frac{1}{N_i}\sum_{n=1}^{N_i}(f(X_{n,i}^N) - \langle f, P_i\rangle)\right)^2\right) \\ &= \frac{1}{N_i}\mathbf{E}\left((f(X_{1,i}^N) - \langle f, P_i\rangle)^2\right) \\ &\quad + \frac{N_i - 1}{N_i}\mathbf{E}\left((f(X_{1,i}^N) - \langle f, P_i\rangle)(f(X_{2,i}^N) - \langle f, P_i\rangle)\right)\end{aligned}$$

which has limit 0 considering the P_i-chaoticity assumption for $k = 2$. A separability argument (the state space is Polish) implies $\lim \Lambda_i^N = P_i$.

Reciprocally, let $k \geq 1$ and f_i be in $C_b(\mathcal{S}_i^k)$, and $(m)_k = m(m-1)\cdots(m-k+1)$. Class-exchangeability yields

$$\begin{aligned}&\mathbf{E}\left(\prod_{i=1}^C f_i(X_{1,i}^N, \ldots X_{k,i}^N)\right) \\ &= \frac{1}{(N_1)_k \cdots (N_C)_k} \sum_{\substack{1\leq n_{1,1},\ldots,n_{k,1}\leq N_1 \text{ distinct} \\ \cdots \\ 1\leq n_{1,C},\ldots,n_{k,C}\leq N_C \text{ distinct}}} \mathbf{E}\left(\prod_{i=1}^C f_i(X_{n_{1,i},i}^N, \ldots X_{n_{k,i},i}^N)\right) \\ &= \mathbf{E}\left(\prod_{i=1}^C \frac{1}{(N_i)_k} \sum_{\substack{1\leq n_1,\ldots,n_k\leq N_i \\ \text{distinct}}} f_i(X_{n_1,i}^N, \ldots, X_{n_k,i}^N)\right) \\ &= \mathbf{E}\left(\prod_{i=1}^C \left\langle f_i, \frac{1}{(N_i)_k} \sum_{\substack{1\leq n_1,\ldots,n_k\leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}^N,\ldots,X_{n_k,i}^N}\right\rangle\right) .\end{aligned}$$

This features the empirical measure for distinct k-uplets corresponding to sampling without replacement, which factorizes nicely in the limit, in which it is equivalent to sampling with replacement: we have

$$(\Lambda_i^N)^{\otimes k} = \frac{1}{N_i^k} \sum_{\substack{1\leq n_1,\ldots,n_k\leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}^N,\ldots,X_{n_k,i}^N} + \frac{1}{N_i^k} \sum_{\substack{1\leq n_1,\ldots,n_k\leq N_i \\ \text{not distinct}}} \delta_{X_{n_1,i}^N,\ldots,X_{n_k,i}^N}$$

so that, in total variation norm $\|\mu\| = \sup\{\, \langle \phi, \mu\rangle : \|\phi\|_\infty \leq 1 \,\}$,

$$\left\| (\Lambda_i^N)^{\otimes k} - \frac{1}{(N_i)_k} \sum_{\substack{1\leq n_1,\ldots,n_k\leq N_i \\ \text{distinct}}} \delta_{X_{n_1,i}^N,\ldots,X_{n_k,i}^N} \right\| \leq 2\frac{N_i^k - (N_i)_k}{N_i^k} \leq \frac{k(k-1)}{N_i}$$

where we bound $N_i^k - (N_i)_k$ by counting $k(k-1)/2$ possible positions for two identical indices with N_i choices and N_i^{k-2} choices for the other positions. We conclude that

$$\lim\left\langle f_i, \frac{1}{(N_i)_k} \sum_{\substack{1\le n_1,\dots,n_k\le N_i \\ \text{distinct}}} \delta_{X^N_{n_1,i},\dots,X^N_{n_k,i}} \right\rangle = \lim\langle f_i, (\Lambda_i^N)^{\otimes k}\rangle = \langle f_i, P_i^{\otimes k}\rangle$$

and the dominated convergence Theorem yields

$$\lim \mathbf{E}\left(\prod_{i=1}^{C} f_i(X^N_{1,i},\dots X^N_{k,i})\right) = \prod_{i=1}^{C}\langle f_i, P_i^{\otimes k}\rangle$$

which implies by a density argument that

$$\lim \mathcal{L}(X^N_{n,i} : 1 \le n \le k,\, 1 \le i \le C) = P_1^{\otimes k} \otimes \cdots \otimes P_C^{\otimes k}$$

proving the reciprocal ("if") statement.

References

[1] Aldous, D.J. (1985). Exchangeability and related topics. In *École d'été de Probabilités de Saint-Flour XIII - 1983.* Lecture Notes in Mathematics 1117, Springer, Berlin, pp. 1–198.

[2] Bellomo, N. and Stöcker, S. (2000). Development of Boltzmann models in mathematical biology. In *Modelling in Applied Sciences: A Kinetic Theory Approach.* N. Bellomo and M. Pulvirenti eds., Birkhauser, Boston, pp. 225–262.

[3] Cercignani, C., Illner, and R., Pulvirenti, M. (1994). *The mathematical theory of dilute gases.* Applied mathematical sciences 106, Springer, New-York.

[4] Graham, C. (2000). Kinetic limits for large communication networks. In *Modelling in Applied Sciences: A Kinetic Theory Approach.* N. Bellomo and M. Pulvirenti eds., Birkhauser, Boston, pp. 317–370.

[5] Graham, C. and Méléard, S. (2001). Probabilistic tools and Monte-Carlo approximations for some Boltzmann equations. In *ESAIM Proceedings Vol. 10 - CEMRACS 1999.* F. Coquel and S. Cordier eds., SMAI, Paris, pp. 77–126.

[6] Grünfeld, C.P. (2000). Nonlinar kinetic models with chemical reactions. In *Modelling in Applied Sciences: A Kinetic Theory Approach.* N. Bellomo and M. Pulvirenti eds., Birkhauser, Boston, pp. 173–224.

[7] Méléard, S. (1996). Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. In: *CIME summer school Montecatini Terme 1995.* D. Talay and L. Tubaro eds., Lecture Notes in Mathematics 1627, Springer, Berlin, pp. 42–95.

[8] Struckmaier, J (2000). Numerical simulation of the Boltzmann equation by particle methods. In *Modelling in Applied Sciences: A Kinetic Theory Approach.* N. Bellomo and M. Pulvirenti eds., Birkhauser, Boston, pp. 371–419.

[9] Sznitman, A.S. (1991). Topics in propagation of chaos. In *École d'été de Probabilités de Saint-Flour XIX - 1989.* Lecture Notes in Mathematics 1464, Springer, Berlin, pp. 165–251.