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R.I. 267 *November 2007*

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November 20, 2007

Abstract

We study the homogenization of an unsteady fluid-structure interaction problem with a scaling corresponding to a long time asymptotic regime. We consider oscillating initial data which are Bloch wave packets corresponding to tubes vibrating in opposition of phase. We prove that the initial displacements follow the rays of geometric optics and that the envelope function evolves according to a Schrödinger equation which can be interpreted as an effect of dispersion.

1 Introduction

In this paper we revisit the homogenization of fluid-structure interaction model proposed by Planchard [15, 16]. It corresponds to a periodic bundle of rigid tubes immersed in a perfect incompressible fluid. Each tube can vibrate around its equilibrium position but their displacements are coupled through the pressure force exerted by the fluid flow. In the time harmonic regime, the homogenization of this model has been extensively studied [1, 4, 9, 12] (see the books [10, 11] for more references). There are however fewer works on the unsteady or time dependent problem, most notably Chapter 3 of [10]. This last work is concerned with tubes that all have the same

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macroscopic behavior. In other words the tubes vibrate in phase. From a mathematical point of view the homogenized limit of the tubes displacement is a macroscopic function which does not depend on the microscopic periodic variable. In some sense the results of Chapter 3 of [10] can be viewed as classical homogenization results in the spirit of the well known books [7, 8, 18]. However, let us explain why these results cannot be complete and fully satisfactory. Indeed, it is known for the time harmonic problem that most of the vibration frequencies and modes correspond to tubes which vibrate, at least partly, in opposition of phase. As shown by numerical evidence in [1], in particular the fundamental (i.e, smallest) frequency corresponds to tubes which vibrate periodically in one direction but anti-periodically in the other direction (in two space dimensions). From a mathematical point of view the homogenization of the time harmonic problem relies on the theory of Bloch waves, an ingredient which is absent in Chapter 3 of [10]. Therefore, the goal of the present paper is to fill this obvious gap and to homogenize the unsteady fluid-structure interaction model in a regime corresponding to tubes in opposition of phase (described by Bloch waves). As we shall see in Section 2, this new regime arises from a different time scaling from Chapter 3 of [10], which amounts to consider much longer times.

Before we explain the origin of our scaling and make a comparison with Chapter 3 of [10], we now describe the model considered in this work. The rigid and parallel tubes are assumed to be distributed on a periodic squared array throughout the space. By translation invariance in the tube axis direction we restrict ourselves to a cross-section of the bundle. Although the physical problem is then two-dimensional, we more generally consider the N-th dimensional case. As usual we denote by $\epsilon > 0$ the period of the tube bundle and by $Y = (0,1)^N$ the unit cube in $\mathbb{R}^N$. The unit tube cross-section is a smooth connected open set $T \subset Y$ and we define the fluid domain $Y^* := Y \setminus T$. The collection of tubes are then defined as $T_j^\epsilon := \epsilon(j + T)$, where j is a vector in $\mathbb{Z}^N$. The space occupied by the fluid is denoted by

$$\Omega_\epsilon := \mathbb{R}^N \setminus \bigcup_{j \in \mathbb{Z}^N} T_j^\epsilon.$$

Each tube displacement is a function $\mathbf{r}_j^\epsilon(t) : \mathbb{R}^+ \to \mathbb{C}^N$ and the fluid potential is $u_\epsilon(t,x) : \mathbb{R}^+ \times \Omega_\epsilon \to \mathbb{C}$. The fluid incompressibility, the continuity of the normal velocity at the tube boundaries and the momentum equation for each tubes yield the following system of equations

$$\left\{\begin{array}{ll} \Delta u_\epsilon(t,x) = 0 & \text{for } t \in \mathbb{R}^+, x \in \Omega_\epsilon, \\ \dfrac{\partial u_\epsilon}{\partial \mathbf{n}}(t,x) = \epsilon\, \dot{\mathbf{r}}_j^\epsilon(t) \cdot \mathbf{n}(x) & \text{for } t \in \mathbb{R}^+, x \in \partial T_j^\epsilon, j \in \mathbb{Z}^N, \\ m\ddot{\mathbf{r}}_j^\epsilon(t) + \dfrac{k}{\epsilon^4}\mathbf{r}_j^\epsilon(t) = -\dfrac{\rho}{\epsilon^{N+1}} \displaystyle\int_{\partial T_j^\epsilon} \dot{u}_\epsilon(t,x)\mathbf{n}d\sigma(x) & \text{for } t \in \mathbb{R}^+, j \in \mathbb{Z}^N, \end{array}\right. \tag{1}$$

where $\mathbf{n}$ is the inward normal to ∂T_j^ϵ (or outward normal to Ω_ϵ). System (1) is complemented with initial data that are Bloch wave packets (see (20)), i.e. eigenmodes of the tube displacements multiplied by an envelope function. Under a technical assumption (14), we prove in Theorem 4.2 that the solution of (1) is approximately given by

$$\begin{array}{l} \mathbf{r}_j^\epsilon(t) \approx e^{2i\pi\theta\cdot j}\mathbf{s}_n(\theta)\left(e^{i\frac{\omega_n(\theta)}{\epsilon^2}t}v^+\left(t, \epsilon j + \dfrac{\mathcal{V}}{\epsilon}t\right) + e^{-i\frac{\omega_n(\theta)}{\epsilon^2}t}v^-\left(t, \epsilon j - \dfrac{\mathcal{V}}{\epsilon}t\right)\right) \\ u_\epsilon(t,x) \approx i\omega_n(\theta)e^{2i\pi\theta\cdot\frac{x}{\epsilon}}\psi_n\left(\dfrac{x}{\epsilon},\theta\right)\left(e^{i\frac{\omega_n(\theta)}{\epsilon^2}t}v^+\left(t, x + \dfrac{\mathcal{V}}{\epsilon}t\right) - e^{-i\frac{\omega_n(\theta)}{\epsilon^2}t}v^-\left(t, x - \dfrac{\mathcal{V}}{\epsilon}t\right)\right) \end{array} \tag{2}$$

where the envelope functions $v^\pm(t,x)$ are solutions of two homogenized problems

$$\pm i\frac{\partial v^\pm}{\partial t} - \text{div}(A^*\nabla v^\pm) = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+ \tag{3}$$

(see (19),(21)-(23) for the definitions of the effective coefficients). In (2) θ is the Bloch parameter or reduced wave number which quantifies the phase of tube oscillations, $\omega_n(\theta)$ is the time frequency, $\mathcal{V} = \nabla\omega_n(\theta)/2\pi$ is the group velocity, $A^* = \nabla\nabla\omega_n(\theta)/8\pi^2$ is the dispersion effective tensor, $\mathbf{s}_n(\theta)$ and $\psi_n(y,\theta)$ are the corresponding eigenmodes for the tube and the fluid potential, respectively (see Section 3).

The interpretation of (2) goes as follows. The solution is approximately the sum of two waves going into opposite directions. The phase factors $\exp(2i\pi\theta \cdot j \pm i\omega_n(\theta)\epsilon^{-2}t)$ and $\exp(2i\pi\theta \cdot x/\epsilon \pm i\omega_n(\theta)\epsilon^{-2}t)$ are predicted by geometric optic like in the case of high frequency solutions for the wave equation (see e.g. [8], [5]). Similarly, the large drift with velocity $\pm\epsilon^{-1}\mathcal{V}$ of the envelope functions is again a consequence of geometric optics and simply means that high frequency tube displacements propagate along straight rays in a periodic medium. Eventually, the Schrödinger equation (3) for the envelope functions is a manifestation of the dispersive character, for long times, of this propagation. In other words, even if the wave packet is initially well localized in space and propagates along straight rays, it will inevitably diffract and spread in space because of the dispersive character of the Schrödinger equation. A similar situation has been studied for the wave equation in periodic media in [5], from which we heavily borrow. Other examples, like parabolic systems, were also studied in [3].

The content of this paper is the following. Section 2 explains the scaling of (1) and recalls, for the sake of comparison, the classical results of homogenization for this model in chapter 3 of [10]. In Section 3 we recall some results on Bloch theory applied to the case at hand. In section 4 we rigorously prove the convergence of the homogenization process by the Bloch waves method. Section 5 is devoted to the derivation of the homogenized Schrödinger equation by the formal method of two-scale asymptotic expansions. It turns out that this well-known (heuristic) method is quite delicate to apply and in particular requires a subtle technical result in Lemma 5.4. Surprisingly, the (rigorous) method of Bloch waves is much simpler, nevertheless we keep section 5, partly to be more convincing on the advantages of the Bloch wave method, and partly because the two-scale asymptotic expansions is conceptually simpler even if it leads to more involved calculations. Clearly, knowing the correct result from the Bloch wave method beforehand was an invaluable information for completing the asymptotic expansions.

Notations. We denote by $d\sigma(x)$ the surface measure on the holes boundaries ∂T_j^ϵ and by $d\sigma(y)$ the surface measure on the unit hole boundary ∂T. The same letter C denotes various positive constants which are all independent of ϵ but whose precise value may change from place to place. Vector valued functions are written with bold letters.

2 Classical homogenization

The goal of this section is twofold. First, it gives a justification of the scaling of (1). Second, it recalls the previously known results of classical homogenization for this system, as discussed in [10]. Before adimensionalization the Planchard model [15, 16] of a coupled system of solid tubes immersed in a perfect incompressible fluid reads as

$$\begin{cases} \Delta u(\tilde{t},\tilde{x}) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}}(\tilde{t},\tilde{x}) = \dot{\mathbf{r}}_j(\tilde{t}) \cdot \mathbf{n} & \text{on } \partial T_j,\ j \in \mathbb{Z}^N, \\ \tilde{m}\ddot{\mathbf{r}}_j(\tilde{t}) + \tilde{k}\mathbf{r}_j(\tilde{t}) = -\tilde{\rho} \displaystyle\int_{\partial T_j} \dot{u}(\tilde{t},\tilde{x})\mathbf{n} d\sigma(\tilde{x}) & \text{on } T_j,\ j \in \mathbb{Z}^N, \end{cases} \tag{4}$$

where $(T_j)_{j\in\mathbb{Z}^N}$ is the periodic collection of tubes, $\Omega = \mathbb{R}^N \setminus \bigcup_{j\in\mathbb{Z}^N} T_j$ is the fluid domain and $\mathbf{n}$ the inward normal to ∂T_j. System (4) is completed by initial data for the tubes displacements and velocities

$$\mathbf{r}_j(0) = \mathbf{r}_j^0 \quad \text{and} \quad \dot{\mathbf{r}}_j(0) = \mathbf{r}_j^1,\ j \in \mathbb{Z}^N.$$

No initial data is required for the fluid potential since $u(0,\tilde{x})$ can be computed in terms of the $(\mathbf{r}_j^1)_{j\in\mathbb{Z}^N}$ only.

Remark 2.1 *The existence and uniqueness theory for (4) is simple [10], [11]. We introduce auxiliary vector-valued functions $\boldsymbol{\chi}_l(\tilde{x})$ which are the unique solutions (up to an additive constant) in $D^{1,2}(\Omega;\mathbb{R}^N)$ of*

$$\begin{cases} \Delta \boldsymbol{\chi}_l(\tilde{t},\tilde{x}) = 0 & \textit{in } \Omega, \\ \dfrac{\partial \boldsymbol{\chi}_l}{\partial \mathbf{n}}(\tilde{t},\tilde{x}) = \delta_{jl}\mathbf{n} & \textit{on } \partial T_j,\ j \in \mathbb{Z}^N. \end{cases} \tag{5}$$

(Recall that $D^{1,2}(\Omega)$ is the space of functions ϕ such that $\nabla\phi \in L^2(\Omega;\mathbb{R}^N)$.) The fluid potential can be written as

$$u(\tilde{t},\tilde{x}) = \sum_{j\in\mathbb{Z}^N} \dot{\mathbf{r}}_j(\tilde{t}) \cdot \boldsymbol{\chi}_j(\tilde{x}). \tag{6}$$

We introduce the added mass matrix M which is an infinite matrix whose entries are $N \times N$ matrices M_{jl}, $j,l \in \mathbb{Z}^N$, defined by

$$M_{jl} = \int_\Omega \nabla\boldsymbol{\chi}_j \cdot \nabla\boldsymbol{\chi}_l = \int_{\partial T_j} \boldsymbol{\chi}_l \otimes \mathbf{n} = \int_{\partial T_l} \boldsymbol{\chi}_j \otimes \mathbf{n}.$$

Plugging (6) in (4) yields

$$\Big(\tilde{m}\mathrm{Id} + \tilde{\rho}M\Big)\ddot{\mathbf{r}}(\tilde{t}) + \tilde{k}\mathrm{Id}\mathbf{r}(\tilde{t}) = 0,$$

which is a standard discrete wave equation for the unknown $\mathbf{r}(\tilde{t}) = \{\mathbf{r}_j(\tilde{t})\}_{j\in\mathbb{Z}^N}$, which admits a unique solution in $C^2(\mathbb{R}^+;\ell_2(\mathbb{Z}^N))$ for suitable initial data.

If the ratio ϵ of the period of the structure with a macroscopic dimension of Ω is small, the system (4) is adimensionalized as follows. The mass and stiffness of each tube is proportional to its cross-section, so we take

$$\tilde{m} := m\epsilon^N, \quad \tilde{k} := k\epsilon^N.$$

The time and space variables are also adimensionalized

$$\tilde{x} := \frac{x}{\epsilon}, \quad \tilde{t} := \frac{t}{\epsilon^\alpha} \text{ for } \alpha \geq 0.$$

If $\alpha = 0$ the time scale is not changed, while, if $\alpha > 0$, we consider a longer time scale. With these changes of variables, (4) becomes

$$\begin{cases} \Delta u_\epsilon(t,x) = 0 & \text{in } \Omega_\epsilon, \\ \dfrac{\partial u_\epsilon}{\partial \mathbf{n}}(t,x) = \epsilon^{\alpha-1}\dot{\mathbf{r}}_j^\epsilon(t) \cdot \mathbf{n}(x) & \text{on } \partial T_j^\epsilon,\ j \in \mathbb{Z}^N, \\ m\ddot{\mathbf{r}}_j^\epsilon(t) + \dfrac{k}{\epsilon^{2\alpha}}\mathbf{r}_j^\epsilon(t) = -\dfrac{\rho}{\epsilon^{N+\alpha-1}} \displaystyle\int_{\partial T_j^\epsilon} \dot{u}_\epsilon(t,x)\mathbf{n}\,d\sigma(x) & \text{on } T_j^\epsilon,\ j \in \mathbb{Z}^N, \end{cases} \tag{7}$$

System (4) admits a conservation of energy: multiplying its first and third equation respectively by $\dot{u}_\epsilon(t,x)$ and $\dot{\mathbf{r}}_j^\epsilon(t)$, integrating with respect to t, we obtain that the following energy

$$E_\epsilon(t) = \frac{1}{2}\sum_{j\in\mathbb{Z}^N}\left[m|\dot{\mathbf{r}}_j^\epsilon(t)|^2 + \frac{k}{\epsilon^{2\alpha}}|\mathbf{r}_j^\epsilon(t)|^2\right] + \frac{\rho}{2\epsilon^N}\int_{\Omega_\epsilon}|\nabla u_\epsilon(t)|^2dx$$

is constant in time. For $\alpha = 2$, (7) is exactly the considered system (1). The case $\alpha = 1$ corresponds to the so-called geometric optics scaling (see [8]) which turns out to be a sub-regime of our analysis for $\alpha = 2$ (see Remark 4.3). For $\alpha = 0$, changing the scale of the fluid potential, i.e., replacing ϵu_ϵ by u_ϵ, (7) yields

$$\begin{cases} \Delta u_\epsilon(t,x) = 0 & \text{in } \Omega_\epsilon, \\ \dfrac{\partial u_\epsilon}{\partial \mathbf{n}}(t,x) = \dot{\mathbf{r}}_j^\epsilon(t) \cdot \mathbf{n}(x) & \text{on } \partial T_j^\epsilon,\ j \in \mathbb{Z}^N, \\ m\ddot{\mathbf{r}}_j^\epsilon(t) + k\mathbf{r}_j^\epsilon(t) = -\dfrac{\rho}{\epsilon^N} \displaystyle\int_{\partial T_j^\epsilon} \dot{u}_\epsilon(t,x)\mathbf{n}\,d\sigma(x) & \text{on } T_j^\epsilon,\ j \in \mathbb{Z}^N, \end{cases} \tag{8}$$

which is precisely the scaling studied in Chapter 3 of [10]. By means of formal two-scale asymptotic expansions the authors of [10] obtained the following limiting behavior for (8):

$$u_\epsilon(t,x) \approx u(t,x), \quad \mathbf{r}_j^\epsilon(t) \approx \mathbf{r}(t,\epsilon j), \ j \in \mathbb{Z}^N, \tag{9}$$

where $u(t,x)$ and $\mathbf{r}(t,x)$ are solutions of a homogenized system for (8) which reads as

$$\left\{ \begin{array}{ll} -\mathrm{div}(A\nabla u) = \mathrm{div}\Big((\mathrm{Id} - A)\dot{\mathbf{r}}\Big) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ \Big(m\mathrm{Id} + \rho(|Y^*|\mathrm{Id} - A)\Big)\ddot{\mathbf{r}} + k\mathbf{r} = \rho(\mathrm{Id} - A)\nabla\dot{u} & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \end{array} \right. \tag{10}$$

where A is a symmetric positive definite homogenized matrix defined by

$$A_{ij} = \int_{Y^*} (\nabla_y w_i + e_i) \cdot (\nabla_y w_j + e_j)\, dy$$

where $w_i(y)$, for $1 \le i \le N$, is the solution of the cell problem

$$\left\{ \begin{array}{ll} \Delta_y w_i = 0 & \text{in } Y^*, \\ (\nabla_y w_i + e_i) \cdot \mathbf{n} = 0 & \text{on } \partial T. \end{array} \right.$$

One can prove that, in the sense of quadratic forms, $0 < A < |Y^*|\mathrm{Id}$. Therefore, the matrix $\rho(|Y^*|\mathrm{Id} - A)$ in (10) is positive definite and can be interpreted as the added mass of the fluid moved by the tubes. In other words, (10) is a sort of non-local wave equation for $\mathbf{r}$.

The asymptotic expansions of [10] could easily be justified by means of two-scale convergence as was done for time harmonic solutions in [4]. The difference with our result (2) is obvious: the microscopic periodic variable y does not appear in the homogenized solution $u(t,x)$ and $\mathbf{r}(t,x)$. In other words, all tubes move locally in phase which is in contrast with our result.

3 Bloch spectral cell problem

The Bloch wave decomposition is well known for the Schrödinger equation [17] but it has also been extended to the problem at hand in [1], [11]. Consider the Bloch spectral cell equation

$$\left\{ \begin{array}{ll} (\mathrm{div}_y + 2i\pi\theta)[(\nabla_y + 2i\pi\theta)\psi] = 0 & \text{in } Y^*, \\ \lambda(\theta)[(\nabla_y + 2i\pi\theta)\psi] \cdot \mathbf{n} = e^{-2i\pi\theta\cdot y} \left(\displaystyle\int_{\partial T} \psi e^{2i\pi\theta\cdot y} \mathbf{n} \right) \cdot \mathbf{n} & \text{in } \partial T. \end{array} \right. \tag{11}$$

The dual parameter θ is called the Bloch frequency and it runs in the dual cell of $\mathbb{T}^N$, i.e. by periodicity it is enough to consider $\theta \in \mathbb{T}^N$.

For any fixed value of θ, to obtain the eigensolution of (11), we introduce a finite-dimensional operator S mapping $\mathbb{C}^N$ into itself, whose eigenvalues coincide with the eigenvalues of (11). This operator is defined, for any $\mathbf{s}$ in $\mathbb{C}^N$, by

$$S(\theta)\mathbf{s} := \int_{\partial T} u(y) e^{2i\pi\theta\cdot y} \mathbf{n} d\sigma(y),$$

where u is the unique solution in $\mathbf{H}^1_\#(Y^*)$ of the problem

$$\begin{cases} (\mathrm{div}_y + 2i\pi\theta)[(\nabla_y + 2i\pi\theta)u] = 0 & \text{in } Y^*, \\ [(\nabla_y + 2i\pi\theta)u] \cdot \mathbf{n} = e^{-2i\pi\theta\cdot y}\mathbf{s}\cdot\mathbf{n} & \text{in } \partial T. \end{cases} \tag{12}$$

(Actually the solution of (12) is unique if $\theta \neq 0$ and unique up to an additive constant when $\theta = 0$.)

The following result was proved in [1].

Proposition 3.1 *The operator $S(\theta)$ is self-adjoint and positive-definite. Therefore it admits N positive real eigenvalues $0 < \lambda_1(\theta) \leq \cdots \leq \lambda_N(\theta)$ and N non-zero eigenvectors $\mathbf{s}_1(\theta), \ldots, \mathbf{s}_N(\theta)$ in $\mathbb{C}^N$. These eigenvalues coincide with those of (11) and the eigenfunctions $\psi_n(\theta)$, $n = 1, \ldots, N$, in $\mathbf{H}^1_\#(Y^*)$ of (11) are defined as the solutions of (12) for $\mathbf{s} = \mathbf{s}_n(\theta)$. The pairs $(\lambda_n(\theta), \psi_n(\theta))$, $n = 1, \ldots, N$, are thus all the independent solutions of (11).*

Remark that the eigenvectors $\mathbf{s}_n(\theta)$ can be recovered from the eigenfunctions $\psi_n(\theta)$ by

$$\mathbf{s}_n(\theta) = \lambda_n^{-1}(\theta) \int_{\partial T} \psi_n(\theta) e^{2i\pi\theta\cdot y} \mathbf{n}. \tag{13}$$

Remark 3.2 *To apply the Bloch theory to the cell problem (11) its coefficients should be Y-periodic functions. However, the function $y \to e^{2i\pi\theta\cdot y}$ is not Y-periodic in $\mathbb{R}^N$. Instead, the function $e_\theta(y) = e^{2i\pi\theta\cdot(y-E(y))}$, where $E(\cdot)$ is the integer part function, is indeed Y-periodic in $\mathbb{R}^N$. Of course, $E(y) = 0$ in the unit cell $(0,1)^N$, which explains our slight abuse of notations in (11): although the coefficients of (11) are meant to involve $e_\theta(y)$, we prefer to write $e^{2i\pi\theta\cdot y}$ for simplicity.*

We now recall some results on the Bloch decomposition associated with the spectral problem (11) (see e.g. [1, 11]).

Lemma 3.3 *Let $(\mathbf{r}_j)_{j\in\mathbb{Z}^N} \in \ell^2(\mathbb{Z}^N;\mathbb{C}^N)$. Define its Bloch coefficients $\hat{r}_n(\theta) \in L^2(\mathbb{T}^N;\mathbb{C})$, $n = 1,\cdots,N$, by*

$$\hat{r}_n(\theta) = \sum_{j\in\mathbb{Z}^N} \mathbf{r}_j \cdot \overline{\mathbf{s}}_n(\theta) e^{-2i\pi\theta\cdot j},$$

where $\mathbf{s}_n(\theta)$ are the eigenvectors of the operator $S(\theta)$ defined in Proposition 3.1. Then,

$$\mathbf{r}_j = \sum_{n=1}^{N} \int_{\mathbb{T}^N} \hat{r}_n(\theta)\mathbf{s}_n(\theta)e^{2i\pi\theta\cdot j}d\theta,$$

and it satisfies the Parseval equality

$$\sum_{j\in\mathbb{Z}^N} |\mathbf{r}_j|^2 = \sum_{n=1}^{N} \int_{\mathbb{T}^N} |\hat{r}_n(\theta)|^2\, d\theta.$$

In the sequel we make the fundamental assumption that there exist a rank n_0 and a Bloch frequency θ_0 such that

$$\lambda_{n_0}(\theta_0) \text{ is a simple eigenvalue.} \tag{14}$$

Such an assumption is not always true but it is known to be generic [2]. We shall choose initial data concentrating at those values n_0 and θ_0. To simplify the notations, we drop the 0 indexes and denote by n and θ these values.

This assumption of simplicity has two important consequences. First, if $\lambda_n(\theta)$ is simple, then it is infinitely differentiable in a vicinity of θ [13]. Second, if $\lambda_n(\theta)$ is simple, then the limit problem of (1) is going to be a single (Schrödinger) equation. The case of a multiple eigenvalue is much more delicate and it is expected to yield a system of homogenized equations, the rank of which is precisely the multiplicity (see Section 6 in [6]).

We introduce the operator $\mathbb{A}_n(\theta)$, defined for any ψ and ϕ in $H^1(\mathbb{T}^N)$ by

$$< \mathbb{A}_n(\theta)\psi, \phi >:= \lambda_n \int_{Y^*} (\nabla_y + 2i\pi\theta)\psi\cdot(\nabla_y - 2i\pi\theta)\overline{\phi} - \int_{\partial T} \psi e^{2i\pi\theta\cdot y}\mathbf{n}\cdot \int_{\partial T} \overline{\phi} e^{-2i\pi\theta\cdot y}\mathbf{n} \tag{15}$$

so that the spectral cell problem (11) for ψ_n is equivalent to $< \mathbb{A}_n(\theta)\psi_n, \phi >= 0$ for any ϕ in $\mathbf{C}^\infty(\mathbb{T}^N)$. Under assumption (14) it is easy to differentiate (11) with respect to θ. Denoting by $(e_k)_{1\le k\le N}$ the canonical basis of $\mathbb{R}^N$ and by $(\theta_k)_{1\le k\le N}$ the components of θ, we define two operators $\mathbb{B}_k(\theta)$ and $\mathbb{C}_k(\theta)$ by

$$< \mathbb{B}_k(\theta)\psi, \phi >:= \int_{Y^*} (\nabla_y + 2i\pi\theta)\psi \cdot \mathbf{e}_k\overline{\phi}, \quad \forall\psi,\phi \in H^1(\mathbb{T}^N), \tag{16}$$

$$< \mathbb{C}_k(\theta)\psi, \phi >:= \int_{\partial T} \psi y_k e^{2i\pi\theta\cdot y}\mathbf{n} \cdot \int_{\partial T} \overline{\phi} e^{-2i\pi\theta\cdot y}\mathbf{n}, \quad \forall \psi, \phi \in H^1(\mathbb{T}^N), \quad (17)$$

where $<,>$ denotes the hermitian product on either $L^2(Y^*)$ or on $L^2(\partial T)$. The first derivative of $\psi_n(\theta)$ satisfies

$$\begin{aligned} < \mathbb{A}_n(\theta)\frac{\partial \psi_n}{\partial \theta_k}, \phi >= \quad & \lambda_n 2i\pi < \mathbb{B}_k(\theta)\psi_n, \phi > -\lambda_n 2i\pi < \psi_n, \mathbb{B}_k(\theta)\phi > \\ & +2i\pi < \mathbb{C}_k(\theta)\psi_n, \phi > -2i\pi < \psi_n, \mathbb{C}_k(\theta)\phi > \\ & -\frac{\partial \lambda_n}{\partial \theta_k}\int_{Y^*}(\nabla_y + 2i\pi\theta)\psi_n \cdot (\nabla_y - 2i\pi\theta)\overline{\phi}, \end{aligned} \quad (18)$$

for any test function ϕ, and the second derivative verifies

$$\begin{aligned} < \mathbb{A}_n(\theta)\frac{\partial^2 \psi_n}{\partial \theta_k \partial \theta_j}, \phi >= \quad & \lambda_n 2i\pi < \mathbb{B}_j(\theta)\frac{\partial \psi_n}{\partial \theta_k}, \phi > -\lambda_n 2i\pi < \frac{\partial \psi_n}{\partial \theta_k}, \mathbb{B}_j(\theta)\phi > \\ & +\lambda_n 2i\pi < \mathbb{B}_k(\theta)\frac{\partial \psi_n}{\partial \theta_j}, \phi > -\lambda_n 2i\pi < \frac{\partial \psi_n}{\partial \theta_j}, \mathbb{B}_k(\theta)\phi > \\ & +\frac{\partial \lambda_n}{\partial \theta_k} 2i\pi < \mathbb{B}_j(\theta)\psi_n, \phi > -\frac{\partial \lambda_n}{\partial \theta_k} 2i\pi < \psi_n, \mathbb{B}_j(\theta)\phi > \\ & +\frac{\partial \lambda_n}{\partial \theta_j} 2i\pi < \mathbb{B}_k(\theta)\psi_n, \phi > -\frac{\partial \lambda_n}{\partial \theta_j} 2i\pi < \psi_n, \mathbb{B}_k(\theta)\phi > \\ & -\lambda_n \int_{Y^*} \psi_n 4\pi^2 \mathbf{e}_k \cdot \mathbf{e}_j \overline{\phi} - \lambda_n \int_{Y^*} \psi_n 4\pi^2 \mathbf{e}_j \cdot \mathbf{e}_k \overline{\phi} \\ & -\frac{\partial^2 \lambda_n}{\partial \theta_k \partial \theta_j}\int_{Y^*}(\nabla_y + 2i\pi\theta)\psi_n \cdot (\nabla_y - 2i\pi\theta)\overline{\phi} \\ & -\frac{\partial \lambda_n}{\partial \theta_k}\int_{Y^*}(\nabla_y + 2i\pi\theta)\frac{\partial \psi_n}{\partial \theta_j} \cdot (\nabla_y - 2i\pi\theta)\overline{\phi} \\ & -\frac{\partial \lambda_n}{\partial \theta_j}\int_{Y^*}(\nabla_y + 2i\pi\theta)\frac{\partial \psi_n}{\partial \theta_k} \cdot (\nabla_y - 2i\pi\theta)\overline{\phi} \\ & +2i\pi < \mathbb{C}_j(\theta)\frac{\partial \psi_n}{\partial \theta_k}, \phi > +2i\pi < \mathbb{C}_k(\theta)\frac{\partial \psi_n}{\partial \theta_j}, \phi > \\ & -2i\pi < \frac{\partial \psi_n}{\partial \theta_k}, \mathbb{C}_j(\theta)\phi > -2i\pi < \frac{\partial \psi_n}{\partial \theta_j}, \mathbb{C}_k(\theta)\phi > \\ & -4\pi^2 < \psi_n, \mathbb{C}_k(\theta)\phi y_j > -4\pi^2 < \mathbb{C}_k(\theta)\psi_n y_j, \phi > \\ & +4\pi^2 < \psi_n y_j, \mathbb{C}_k(\theta)\phi > +4\pi^2 < \mathbb{C}_k(\theta)\psi_n, \phi y_j > . \end{aligned} \quad (19)$$

Remark that the two last lines of (19) are symmetric in j and k as they should be. By assumption (14) we know that the map $\theta \to \psi_n(\theta, \cdot)$ is analytic so necessarily (18) and (19) admit solutions. This implies, in particular, that the right hand sides of (18) and (19) satisfy the required compatibility condition or Fredholm alternative (i.e. they are orthogonal to ψ_n). The Fredholm alternative for (18) yields a formula for $\nabla_\theta \lambda_n(\theta)$ (which shall be interpreted as a group velocity) and that for (19) yields a formula for the Hessian matrix $\nabla_\theta \nabla_\theta \lambda_n(\theta)$ (which shall be interpreted as the homogenized tensor). In any case the solution is unique up to the addition of a multiple of ψ_n.

4 Rigorous homogenization using Bloch waves

In this section we give a rigorous derivation of the homogenized problem of (1) by using the method of Bloch waves. This approach turns out to be, not only mathematically rigorous, but also much simpler to perform than the two-scale asymptotic expansions of Section 5.

We consider the following initial data

$$
\begin{aligned}
\mathbf{r}_j^\epsilon(0) &= \int_{\epsilon^{-1}\mathbb{T}^N} \hat{r}^0(\eta) \mathbf{s}_n(\theta + \epsilon\eta) e^{2i\pi(\theta+\epsilon\eta)\cdot j} d\eta, \\
\dot{\mathbf{r}}_j^\epsilon(0) &= \epsilon^{-2} \int_{\epsilon^{-1}\mathbb{T}^N} \hat{r}^1(\eta) \mathbf{s}_n(\theta + \epsilon\eta) e^{2i\pi(\theta+\epsilon\eta)\cdot j} d\eta,
\end{aligned}
\tag{20}
$$

where $\hat{r}^0(\eta)$ and $\hat{r}^1(\eta)$ are smooth functions with compact support in $\mathbb{R}^N$. The advantage of (20) is twofold. First, upon the change of variables $\tilde{\theta} = \theta + \epsilon\eta$, the initial data is already written as a Bloch decomposition (see Lemma 3.3) which is useful when we shall diagonalize the equation (1) by means of the Bloch transform. Second, thanks to the assumption on the compact support of $\hat{r}^0(\eta)$ and $\hat{r}^1(\eta)$, the integrals on $\epsilon^{-1}\mathbb{T}^N$ can be replaced by integrals on $\mathbb{R}^N$ (for sufficiently small ϵ) which yields a connection with the usual Fourier transform. Specifically, let us define the inverse Fourier transforms of $\hat{r}^0(\eta)$ and $\hat{r}^1(\eta)$

$$
v^0(x) = \int_{\mathbb{R}^N} \hat{r}^0(\eta) e^{2i\pi\eta\cdot x} d\eta, \quad \text{and } v^1(x) = \int_{\mathbb{R}^N} \hat{r}^1(\eta) e^{2i\pi\eta\cdot x} d\eta,
$$

then, by a simple Taylor expansion of $\mathbf{s}_n$ in (20), we deduce the following

Lemma 4.1 *Under assumption (14) on the simplicity of $\lambda_n(\theta)$, we have*

$$\sup_{j\in\mathbb{Z}^N} \left|\mathbf{r}_j^\epsilon(0) - \mathbf{s}_n(\theta)\, e^{2i\pi\theta\cdot j} v^0(\epsilon j)\right| \le C\epsilon,$$
$$\sup_{j\in\mathbb{Z}^N} \left|\epsilon^2 \dot{\mathbf{r}}_j^\epsilon(0) - \mathbf{s}_n(\theta)\, e^{2i\pi\theta\cdot j} v^1(\epsilon j)\right| \le C\epsilon,$$

where we recall that $j = E(\frac{x}{\epsilon})$ for $x \in Y_j^\epsilon$ and ϵj is the position of the tube T_j^ϵ.

We introduce a so-called dispersion relation which defines the time frequency $\omega_n(\theta)$ in terms of the eigenvalue $\lambda_n(\theta)$, where θ is the Bloch frequency or reduced wave number

$$\omega_n(\theta) = \sqrt{\frac{k}{m + \rho\lambda_n(\theta)}}. \tag{21}$$

By differentiating this time frequency with respect to θ we obtain first the the group velocity $\mathcal{V}$ defined by

$$\mathcal{V} = \frac{1}{2\pi}\nabla\omega_n(\theta)\,, \tag{22}$$

and the homogenized tensor, or dispersion coefficient, A^* defined by

$$A^* = \frac{1}{8\pi^2}\nabla\nabla\omega_n(\theta)\,. \tag{23}$$

Our main result is

Theorem 4.2 *Under assumptions (14) and (20), the solution of system* (1) *is given by*

$$\begin{aligned} \mathbf{r}_j^\epsilon(t) \;=\; e^{2i\pi\theta\cdot j}\mathbf{s}_n(\theta) \quad & \left(e^{i\frac{\omega_n(\theta)}{\epsilon^2}t} v^+\left(t, \epsilon j + \frac{\mathcal{V}}{\epsilon}t\right) \right. \\ & \left. + e^{-i\frac{\omega_n(\theta)}{\epsilon^2}t} v^-\left(t, \epsilon j - \frac{\mathcal{V}}{\epsilon}t\right)\right) + \rho_\epsilon^1(t, \epsilon j) \end{aligned} \tag{24}$$

$$\begin{aligned} u_\epsilon(t,x) \;= i\omega_n(\theta) e^{2i\pi\theta\cdot\frac{x}{\epsilon}}\psi_n\left(\frac{x}{\epsilon},\theta\right) \quad & \left(e^{i\frac{\omega_n(\theta)}{\epsilon^2}t} v^+\left(t, x + \frac{\mathcal{V}}{\epsilon}t\right) \right. \\ & \left. - e^{-i\frac{\omega_n(\theta)}{\epsilon^2}t} v^-\left(t, x - \frac{\mathcal{V}}{\epsilon}t\right)\right) + \rho_\epsilon^2(t,x) \end{aligned} \tag{25}$$

where ρ_ϵ^1 and ρ_ϵ^2 are two small remainder terms such that, for any positive finite time T, there exists a constant C and

$$\|\rho_\epsilon^1(t,x)\|_{L^\infty((0,T)\times\mathbb{R}^N))} \le C\epsilon, \qquad \|\rho_\epsilon^2(t,x)\|_{L^\infty((0,T)\times\mathbb{R}^N))} \le C\epsilon,$$

and where $v^\pm \in C\left([0,T]; L^2(\mathbb{R}^N)\right)$ are the solutions of the two homogenized problems

$$\begin{cases} \pm i\dfrac{\partial v^\pm}{\partial t} - \operatorname{div}(A^*\nabla v^\pm) = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ v^\pm(t=0,x) = \dfrac{1}{2}\Big(v^0(x) \pm \dfrac{1}{i\omega_n(\theta)} v^1(x)\Big) & \text{in } \mathbb{R}^N. \end{cases} \tag{26}$$

Remark 4.3 *The outcome of Theorem 4.2 is the following: the exact solution is approximately equal to two wave packets traveling in opposite directions at the large velocity $\epsilon^{-1}\mathcal{V}$ with macroscopic profiles $v^\pm$ which have a dispersive behavior since they obey a Schrödinger equation. The oscillating phase in (25), $\exp(2i\pi\theta\cdot j \pm \omega_n(\theta)\epsilon^{-2}t)$, is precisely the term which is predicted by the geometric optics method. The novelty in our result is the dispersion of the envelope functions which can be detected for longer times.*

Proof We use the Bloch decomposition to diagonalize the system of equations (1). By Lemma 3.3 we know that the family of tube displacements can be written

$$\mathbf{r}_j^\epsilon(t) = \sum_{n'=1}^{N} \int_{\epsilon^{-1}\mathbb{T}^N} \hat{r}_{n'}^\epsilon(t,\eta)\mathbf{s}_{n'}(\theta+\epsilon\eta)e^{2i\pi(\theta+\epsilon\eta)\cdot j}\,d\eta.$$

We seek a solution of (1) where the fluid potential is decomposed as

$$u_\epsilon(t,x) = \sum_{n'=1}^{N} \int_{\epsilon^{-1}\mathbb{T}^N} \hat{u}_{n'}^\epsilon(t,\eta)\psi_{n'}\left(\frac{x}{\epsilon},\theta+\epsilon\eta\right)e^{2i\pi(\theta+\epsilon\eta)\cdot\frac{x}{\epsilon}}\,d\eta.$$

(Note that not all functions necessarily admit the above decomposition.) Plugging these expression of u_ϵ and $\mathbf{r}_\epsilon$ in system (1), and using the spectral cell problem (11), as well as formula (13), we obtain that the equation $\Delta u_\epsilon =$

0 is indeed satisfied and the boundary conditions on each tube T_j^ϵ are

$$\left\{\begin{array}{l} \epsilon^{-1}\displaystyle\sum_{n'=1}^{N}\int_{\epsilon^{-1}\mathbb{T}^N}\hat{u}_{n'}^\epsilon(t,\eta)\frac{1}{\lambda_{n'}}\mathbf{s}_{n'}(\theta+\epsilon\eta)e^{2i\pi(\theta+\epsilon\eta)\cdot j}d\eta\cdot\mathbf{n} \\ \qquad\qquad = \epsilon\displaystyle\sum_{n'=1}^{N}\int_{\epsilon^{-1}\mathbb{T}^N}\dot{\hat{r}}_{n'}^\epsilon(t,\eta)\mathbf{s}_{n'}(\theta+\epsilon\eta)e^{2i\pi(\theta+\epsilon\eta)\cdot j}d\eta\cdot\mathbf{n}, \\ \epsilon\displaystyle\sum_{n'=1}^{N}\int_{\epsilon^{-1}\mathbb{T}^N}[m\ddot{\hat{r}}_{n'}^\epsilon(t,\eta)+\epsilon^{-4}k\hat{r}_{n'}^\epsilon(t,\eta)]\mathbf{s}_{n'}(\theta+\epsilon\eta)e^{2i\pi(\theta+\epsilon\eta)\cdot j}d\eta \\ \quad = -\rho\displaystyle\sum_{n'=1}^{N}\int_{\epsilon^{-1}\mathbb{T}^N}\dot{\hat{u}}_{n'}^\epsilon(t,\eta)\lambda_{n'}(\theta+\epsilon\eta)\mathbf{s}_{n'}(\theta+\epsilon\eta)e^{2i\pi(\theta+\epsilon\eta)\cdot j}d\eta. \end{array}\right.$$

By the Bloch decomposition of Lemma 3.3 we can identify the coefficients which yields

$$\left\{\begin{array}{l} \hat{u}_{n'}^\epsilon(t,\eta)=\epsilon^2\dot{\hat{r}}_{n'}^\epsilon(t,\eta), \\ \epsilon[m\ddot{\hat{r}}_{n'}^\epsilon(t,\eta)+\epsilon^{-4}k\hat{r}_{n'}^\epsilon(t,\eta)]=-\rho\lambda_{n'}\dot{\hat{u}}_{n'}^\epsilon(t,\eta). \end{array}\right.$$

(The same result can be obtained if we use the variational formulation (33) of system (1) with test functions written as Bloch decompositions.) Taking into account the initial data we deduce that the Bloch coefficients are solutions of the following o.d.e.'s

$$\left\{\begin{array}{ll} \epsilon^4(m+\lambda_{n'}\rho)\ddot{\hat{r}}_{n'}^\epsilon+k\hat{r}_{n'}^\epsilon=0 & \text{in } (0,T), \\ \hat{u}_{n'}^\epsilon=\epsilon^2\dot{\hat{r}}_{n'}^\epsilon, & \\ \hat{r}_{n'}^\epsilon(0)=\hat{r}^0(\eta)\delta_{n'n}, & \\ \dot{\hat{r}}_{n'}^\epsilon(0)=\epsilon^{-2}\hat{r}^1(\eta)\delta_{n'n}. & \end{array}\right. \tag{27}$$

For $n'\neq n$ we immediately find that $\hat{r}_{n'}^\epsilon\equiv 0$ and $\hat{u}_{n'}^\epsilon\equiv 0$. Introducing the notations

$$\hat{r}^{\pm}(\xi)=\frac{1}{2}\left(\hat{r}^0(\eta)\pm\frac{\hat{r}^1(\eta)}{i\omega_n(\xi)}\right),$$

an explicit formula for the solutions is therefore

$$\begin{aligned} \mathbf{r}_j^\epsilon(t) &= \int_{\epsilon^{-1}\mathbb{T}^N}\hat{r}^+(\theta+\epsilon\eta)\mathbf{s}_n(\theta+\epsilon\eta)\,e^{2i\pi(\theta+\epsilon\eta)\cdot j+i\frac{\omega_n(\theta+\epsilon\eta)}{\epsilon^2}t}d\eta \\ &\quad+\int_{\epsilon^{-1}\mathbb{T}^N}\hat{r}^-(\theta+\epsilon\eta)\mathbf{s}_n(\theta+\epsilon\eta)\,e^{2i\pi(\theta+\epsilon\eta)\cdot j-i\frac{\omega_n(\theta+\epsilon\eta)}{\epsilon^2}t}d\eta, \end{aligned}$$

$$
\begin{aligned}
u_\epsilon(t,x) \quad &= i\omega_n(\theta+\epsilon\eta)\int_{\epsilon^{-1}\mathbb{T}^N}\hat{r}^+(\theta+\epsilon\eta)\psi_n\left(\frac{x}{\epsilon},\theta+\epsilon\eta\right)e^{2i\pi(\theta+\epsilon\eta)\cdot\frac{x}{\epsilon}+i\frac{\omega_n(\theta+\epsilon\eta)}{\epsilon^2}t}d\eta \\
&-i\omega_n(\theta+\epsilon\eta)\int_{\epsilon^{-1}\mathbb{T}^N}\hat{r}^-(\theta+\epsilon\eta)\psi_n\left(\frac{x}{\epsilon},\theta+\epsilon\eta\right)e^{2i\pi(\theta+\epsilon\eta)\cdot\frac{x}{\epsilon}-i\frac{\omega_n(\theta+\epsilon\eta)}{\epsilon^2}t}d\eta,
\end{aligned}
\tag{28}
$$

where ω_n is defined by (21) and thus satisfies $\lambda_n = (k - m\omega_n^2)/(\rho\omega_n^2)$. We then perform a Taylor expansion, up to second order, of $\omega_n(\theta)$:

$$
\begin{aligned}
\omega_n(\theta+\epsilon\eta) \quad &= \omega_n(\theta) + \nabla\omega_n(\theta)\cdot\epsilon\eta + \frac{1}{2}\nabla\nabla\omega_n(\theta)\epsilon^2\eta\cdot\eta + \mathcal{O}(\epsilon^3) \\
&= \omega_n(\theta) + 2\pi\mathcal{V}\cdot\epsilon\eta + 4\pi^2A^*\epsilon^2\eta\cdot\eta + \mathcal{O}(\epsilon^3).
\end{aligned}
\tag{29}
$$

Plugging (29) into (28) and using a zero order Taylor expansion of $\mathbf{s}_n(\theta)$ and $\psi_n(\theta)$ we obtain

$$
\begin{aligned}
\mathbf{r}_j^\epsilon(t) \quad &= \mathbf{s}_n(\theta)e^{2i\pi\theta\cdot j+i\frac{\omega_n(\theta)}{\epsilon^2}t}\int_{\epsilon^{-1}\mathbb{T}^N}\hat{r}^+(\theta)e^{2i\pi(\epsilon j+\frac{\mathcal{V}}{\epsilon}t)\cdot\eta+4i\pi^2A^*\eta\cdot\eta t+\mathcal{O}(\epsilon)t}d\eta \\
&+\mathbf{s}_n(\theta)e^{2i\pi\theta\cdot j-i\frac{\omega_n(\theta)}{\epsilon^2}t}\int_{\epsilon^{-1}\mathbb{T}^N}\hat{r}^-(\theta)e^{2i\pi(\epsilon j-\frac{\mathcal{V}}{\epsilon}t)\cdot\eta-4i\pi^2A^*\eta\cdot\eta t+\mathcal{O}(\epsilon)t}d\eta \\
&+R_\epsilon^1(t,x) \\
u_\epsilon(t,x) \quad &= i\omega_n(\theta)\psi_n\Big(\frac{x}{\epsilon},\theta\Big)e^{2i\pi\theta\cdot\frac{x}{\epsilon}+i\frac{\omega_n(\theta)}{\epsilon^2}t}\int_{\epsilon^{-1}\mathbb{T}^N}\hat{r}^+(\theta)e^{2i\pi(x+\frac{\mathcal{V}}{\epsilon}t)\cdot\eta+4i\pi^2A^*\eta\cdot\eta t+\mathcal{O}(\epsilon)t}d\eta \\
&-i\omega_n(\theta)\psi_n\Big(\frac{x}{\epsilon},\theta\Big)e^{2i\pi\theta\cdot\frac{x}{\epsilon}-i\frac{\omega_n(\theta)}{\epsilon^2}t}\int_{\epsilon^{-1}\mathbb{T}^N}\hat{r}^-(\theta)e^{2i\pi(x-\frac{\mathcal{V}}{\epsilon}t)\cdot\eta-4i\pi^2A^*\eta\cdot\eta t+\mathcal{O}(\epsilon)t}d\eta \\
&+R_\epsilon^2(t,x)
\end{aligned}
\tag{30}
$$

where R_ϵ^1 and R_ϵ^2 are the sum of all higher order terms. Since the functions $\hat{r}^0$ and $\hat{r}^1$ are compactly supported, for ϵ sufficiently small we can replace the integrals over $\epsilon^{-1}\mathbb{T}^N$ by integrals over the whole space $\mathbb{R}^N$ and replace the factor $e^{\mathcal{O}(\epsilon)t}$ by $1+\mathcal{O}(\epsilon)$ since we consider finite times $t \le T$. To see that we therefore obtain precisely formula (25), we consider the Fourier transform of the homogenized problem (26)

$$
\begin{cases}
\pm i\dfrac{\partial\hat{v}^\pm}{\partial t} + 4\pi^2A^*\eta\cdot\eta\,\hat{v}^\pm = 0 & \text{in } \mathbb{R}^N\times(0,T), \\
\hat{v}^\pm(t=0,\eta) = \hat{r}^\pm(\theta) = \dfrac{1}{2}\Big(\hat{v}^0(\eta) \pm \dfrac{\hat{v}^1(\eta)}{i\omega_n(\theta)}\Big) & \text{in } \mathbb{R}^N.
\end{cases}
\tag{31}
$$

The explicit formula for the solution to (31) is given by

$$\hat{v}^{\pm}(t,\eta) = \hat{r}^{\pm}(\theta) e^{\pm 4i\pi^2 A^* \eta\cdot\eta\, t}.$$

Therefore (30) can be rewritten as

$$\begin{aligned}
\mathbf{r}_j^{\epsilon}(t) &= \mathbf{s}_n(\theta) e^{2i\pi\theta\cdot j + i\frac{\omega_n(\theta)}{\epsilon^2}t} \int_{\mathbb{R}^N} \hat{v}^{+}(t,\eta) e^{2i\pi(\epsilon j + \frac{\mathcal{V}}{\epsilon}t)\cdot\eta} d\eta \\
&+\mathbf{s}_n(\theta) e^{2i\pi\theta\cdot j - i\frac{\omega_n(\theta)}{\epsilon^2}t} \int_{\mathbb{R}^N} \hat{v}^{-}(t,\eta) e^{2i\pi(\epsilon j - \frac{\mathcal{V}}{\epsilon}t)\cdot\eta} d\eta + \rho_\epsilon^1(t,\epsilon j) \\
&= \mathbf{s}_n(\theta) e^{2i\pi\theta\cdot j} \left(e^{i\frac{\omega_n(\theta)}{\epsilon^2}t} v^{+}\left(t, \epsilon j + \frac{\mathcal{V}}{\epsilon}t\right) \right. \\
&\qquad \left. + e^{-i\frac{\omega_n(\theta)}{\epsilon^2}t} v^{-}\left(t, \epsilon j - \frac{\mathcal{V}}{\epsilon}t\right)\right) + \rho_\epsilon^1(t,\epsilon j) \\
u_\epsilon(t,x) &= i\omega_n(\theta)\psi_n\left(\frac{x}{\epsilon},\theta\right) e^{2i\pi\theta\cdot\frac{x}{\epsilon} + i\frac{\omega_n(\theta)}{\epsilon^2}t} \int_{\mathbb{R}^N} \hat{v}^{+}(t,\eta) e^{2i\pi(x + \frac{\mathcal{V}}{\epsilon}t)\cdot\eta} d\eta \\
&-i\omega_n(\theta)\psi_n\left(\frac{x}{\epsilon},\theta\right) e^{2i\pi\theta\cdot\frac{x}{\epsilon} - i\frac{\omega_n(\theta)}{\epsilon^2}t} \int_{\mathbb{R}^N} \hat{v}^{-}(t,\eta) e^{2i\pi(x - \frac{\mathcal{V}}{\epsilon}t)\cdot\eta} d\eta + \rho_\epsilon^2(t,x) \\
&= i\omega_n(\theta)\, e^{2i\pi\theta\cdot\frac{x}{\epsilon}} \psi_n\left(\frac{x}{\epsilon},\theta\right) \left(e^{i\frac{\omega_n(\theta)}{\epsilon^2}t} v^{+}\left(t, x + \frac{\mathcal{V}}{\epsilon}t\right) \right. \\
&\qquad \left. - e^{-i\frac{\omega_n(\theta)}{\epsilon^2}t} v^{-}\left(t, x - \frac{\mathcal{V}}{\epsilon}t\right)\right) + \rho_\epsilon^2(t,x)
\end{aligned}$$

where ρ_ϵ^1 and ρ_ϵ^2 take into account the terms R_ϵ^1 and R_ϵ^2 in (30) and the approximation we have done by replacing $e^{\mathcal{O}(\epsilon)t}$ by 1.

Remark 4.4 *If, as in (29), we perform a second order Taylor expansion of* $\mathbf{s}_n$ *and* ψ_n *we can improve the error estimate and show that*

$$\begin{aligned}
\mathbf{r}_j^{\epsilon}(t) \approx\ & e^{2i\pi\theta\cdot j + i\frac{\omega_n(\theta)}{\epsilon^2}t} \left((v^+)^{\epsilon}\mathbf{s}_n + \frac{\epsilon}{2i\pi}\sum_{k=1}^{N} \frac{\partial \mathbf{s_n}}{\partial\theta_k}\left(\frac{\partial v^+}{\partial x_k}\right)^{\epsilon} \right. \\
&\qquad \left. - \frac{\epsilon^2}{4\pi^2}\sum_{k,l=1}^{N} \frac{\partial^2 \mathbf{s}_n}{\partial\theta_k\partial\theta_l}\left(\frac{\partial^2 v^+}{\partial x_k\partial x_l}\right)^{\epsilon}\right) \\
&+ e^{2i\pi\theta\cdot j - i\frac{\omega_n(\theta)}{\epsilon^2}t} \left((v^-)^{\epsilon}\mathbf{s}_n + \frac{\epsilon}{2i\pi}\sum_{k=1}^{N} \frac{\partial \mathbf{s_n}}{\partial\theta_k}\left(\frac{\partial v^-}{\partial x_k}\right)^{\epsilon} \right. \\
&\qquad \left. - \frac{\epsilon^2}{4\pi^2}\sum_{k,l=1}^{N} \frac{\partial^2 \mathbf{s}_n}{\partial\theta_k\partial\theta_l}\left(\frac{\partial^2 v^-}{\partial x_k\partial x_l}\right)^{\epsilon}\right),
\end{aligned}$$

up to a remainder term of order $\mathcal{O}(\epsilon^3)$ in the L^∞-norm,

$$
\begin{aligned}
u_\epsilon(t,x) \approx\ & i\omega_n(\theta)e^{2i\pi\theta\cdot\frac{x}{\epsilon}+i\frac{\omega_n(\theta)}{\epsilon^2}t}\left(\psi_n\left(\frac{x}{\epsilon}\right)(v^+)^\epsilon+\frac{\epsilon}{2i\pi}\sum_{k=1}^N\frac{\partial\psi_n}{\partial\theta_k}\left(\frac{x}{\epsilon}\right)\left(\frac{\partial v^+}{\partial x_k}\right)^\epsilon\right.\\
&\left.-\frac{\epsilon^2}{4\pi^2}\sum_{k,l=1}^N\frac{\partial^2\psi_n}{\partial\theta_k\partial\theta_l}\left(\frac{x}{\epsilon}\right)\left(\frac{\partial^2 v^+}{\partial x_k\partial x_l}\right)^\epsilon\right)\\
&-i\omega_n(\theta)e^{2i\pi\theta\cdot\frac{x}{\epsilon}-i\frac{\omega_n(\theta)}{\epsilon^2}t}\left(\psi_n\left(\frac{x}{\epsilon}\right)(v^-)^\epsilon+\frac{\epsilon}{2i\pi}\sum_{k=1}^N\frac{\partial\psi_n}{\partial\theta_k}\left(\frac{x}{\epsilon}\right)\left(\frac{\partial v^-}{\partial x_k}\right)^\epsilon\right.\\
&\left.-\frac{\epsilon^2}{4\pi^2}\sum_{k,l=1}^N\frac{\partial^2\psi_n}{\partial\theta_k\partial\theta_l}\left(\frac{x}{\epsilon}\right)\left(\frac{\partial^2 v^-}{\partial x_k\partial x_l}\right)^\epsilon\right),
\end{aligned}
\tag{32}
$$

up to a remainder term of order $\mathcal{O}(\epsilon^3)$ in the L^∞-norm. Here we used the notation $(v^\pm)^\epsilon = v^\pm\left(t, x \pm \frac{\mathcal{V}}{\epsilon}t\right)$ and similarly for their derivatives. Of course, (32) *are nothing but the beginning of the two-scale asymptotic expansion of $\mathbf{r}_j^\epsilon$ and u_ϵ.*

5 Formal two-scale asymptotic expansions

The aim of this section is to obtain by a formal method of two-scale asymptotic expansions the homogenized problem for (1) (we gave a rigorous derivation of it in section 4). It turns out that a standard two-scale asymptotic expansion in the strong from of the equations (1), as in [7], [8] or [18], may yield wrong results since we work in a periodically perforated open set with non-local boundary conditions. Indeed, the boundary conditions give rise to a special type of boundary integral on $\partial\Omega_\epsilon$, the limit of which is as usual an integral on the product $\Omega\times\partial T$ but with some small correctors (see Lemma 5.1) which are not negligible in the homogenized limit. Therefore, following an idea of J.-L. Lions [14] it is much safer to use two-scale asymptotic expansions in the variational formulation of the problem which is:

$$
\left\{
\begin{aligned}
&\int_{\Omega_\epsilon}\nabla u_\epsilon\cdot\nabla\overline{\Phi}_\epsilon dx = \epsilon\sum_{j\in\mathbb{Z}^N}\int_{\partial T_j^\epsilon}\dot{\mathbf{r}}_j^\epsilon\cdot\mathbf{n}\overline{\Phi}_\epsilon d\sigma(x),\\
&\epsilon^{N+1}\sum_{j\in\mathbb{Z}^N}[m\ddot{\mathbf{r}}_j^\epsilon\cdot\overline{\boldsymbol{\gamma}}_j^\epsilon + k\epsilon^{-4}\mathbf{r}_j^\epsilon\cdot\overline{\boldsymbol{\gamma}}_j^\epsilon] = -\sum_{j\in\mathbb{Z}^N}\rho\int_{\partial T_j^\epsilon}\dot{u}_\epsilon\mathbf{n}\cdot\overline{\boldsymbol{\gamma}}_j^\epsilon d\sigma(x),
\end{aligned}
\right.
\tag{33}
$$

for any test functions $\Phi_\epsilon \in \mathcal{D}([0,T) \times \mathbb{R}^N; \mathbb{C})$, $\boldsymbol{\gamma}_j^\epsilon \in \mathcal{D}([0,T); \mathbb{C}^N)$. To pass to the limit in the variational formulation (33) we shall need the following result.

Lemma 5.1 *Let $\mu(y)$ and $\nu(y)$ be two periodic non-negative measures on the unit cube Y. Let $u(x,y)$ and $v(x,y)$ be two Y-periodic functions which are smooth with respect to x and decay fast enough at infinity when $|x| \to \infty$ (so that all integrals below are finite). As ϵ tends to 0, we have*

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}^N} \epsilon^{-N} \left(\int_{Y_j^\epsilon} u\left(x, \frac{x}{\epsilon}\right) d\mu(\frac{x}{\epsilon}) \right) \left(\int_{Y_j^\epsilon} v\left(x, \frac{x}{\epsilon}\right) d\nu(\frac{x}{\epsilon}) \right) \\
&= \int_{\mathbb{R}^N} \left(\int_Y u\,(x,y)\, d\mu(y) \right) \left(\int_Y v\,(x,y)\, d\nu(y) \right) dx \\
&\quad +\epsilon \left[\int_{\mathbb{R}^N} \left(\int_Y y \cdot \nabla_x u(x,y) d\mu(y) \right) \left(\int_Y v(x,y) d\nu(y) \right) dx \right. \\
&\qquad \left. + \int_{\mathbb{R}^N} \left(\int_Y u(x,y) d\mu(y) \right) \left(\int_Y y \cdot \nabla_x v(x,y) d\nu(y) \right) dx \right] \\
&\quad +\frac{\epsilon^2}{2} \left[\int_{\mathbb{R}^N} \left(\int_Y y \cdot \nabla_x \nabla_x u(x,y) \cdot y\, d\mu(y) \right) \left(\int_Y v(x,y) d\nu(y) \right) dx \right. \\
&\qquad +2 \int_{\mathbb{R}^N} \left(\int_Y y \cdot \nabla_x u(x,y) d\mu(y) \right) \left(\int_Y y \cdot \nabla_x v(x,y) d\nu(y) \right) dx \\
&\qquad \left. + \int_{\mathbb{R}^N} \left(\int_Y u(x,y) d\mu(y) \right) \left(\int_Y y \cdot \nabla_x \nabla_x v(x,y) \cdot y\, d\nu(y) \right) dx \right] + \mathcal{O}(\epsilon^3).
\end{aligned}$$

Remark 5.2 *In the sequel we shall use Lemma 5.1 with the following examples of periodic non-negative measures: $d\mu(y) = dy$ is just the Lebesgue measure in Y and $\nu(y)$ is the surface measure on the hole's boundary ∂T, i.e., $\int_Y v(y)\, d\nu(y) = \int_{\partial T} v(y)\, d\sigma(y)$.*

Remark 5.3 *Lemma 5.1 has to be compared with the well-known convergence result which says that any smooth and Y-periodic functions $u(x,y)$ satisfies*

$$\sum_{j\in\mathbb{Z}^N} \int_{Y_j^\epsilon} u\left(x, \frac{x}{\epsilon}\right) dx = \int_{\mathbb{R}^N} \int_Y u\,(x,y)\, dy\, dx + \mathcal{O}(\epsilon^k) \quad \textit{for any } k \geq 1. \quad (34)$$

On the contrary, Lemma 5.1 gives corrector result which are not negligibly small. The equivalent of formula (34) for Riemann sums is that any smooth *function $u(x)$ (with a fast enough decay at infinity) satisfies*

$$\sum_{j\in\mathbb{Z}^N} \epsilon^N u(x_j^\epsilon) = \int_{\mathbb{R}^N} u(x)\,dx + \mathcal{O}(\epsilon^k) \quad \textit{for any } k \geq 1. \tag{35}$$

Proof For $x \in Y_j^\epsilon$ consider the Taylor expansion of $x \to u(x,y)$ around the center x_j^ϵ of Y_j^ϵ:

$$u\left(x,\frac{x}{\epsilon}\right) = u\left(x_j^\epsilon,\frac{x}{\epsilon}\right) + (x-x_j^\epsilon)\cdot\nabla_x u\left(x_j^\epsilon,\frac{x}{\epsilon}\right) + \frac{1}{2}(x-x_j^\epsilon)\cdot\nabla_x\nabla_x u\left(x_j^\epsilon,\frac{x}{\epsilon}\right)\cdot(x-x_j^\epsilon) + \mathcal{O}(\epsilon^3)$$

and an analogous expansion for $x \to v(x,y)$. We deduce

$$\begin{aligned}
&\left(\int_{Y_j^\epsilon} u\left(x,\frac{x}{\epsilon}\right) d\mu(\frac{x}{\epsilon})\right)\left(\int_{Y_j^\epsilon} v\left(x,\frac{x}{\epsilon}\right) d\nu(\frac{x}{\epsilon})\right) \\
&= \left[\int_{Y_j^\epsilon} u\left(x_j^\epsilon,\frac{x}{\epsilon}\right) d\mu(\frac{x}{\epsilon}) + \int_{Y_j^\epsilon} (x-x_j^\epsilon)\cdot\nabla_x u\left(x_j^\epsilon,\frac{x}{\epsilon}\right) d\mu(\frac{x}{\epsilon})\right. \\
&\qquad \left. +\frac{1}{2}\int_{Y_j^\epsilon} (x-x_j^\epsilon)\cdot\nabla_x\nabla_x u\left(x_j^\epsilon,\frac{x}{\epsilon}\right)\cdot(x-x_j^\epsilon) d\mu(\frac{x}{\epsilon}) + \mathcal{O}(\epsilon^3)\right] \\
&\times \left[\int_{Y_j^\epsilon} v\left(x_j^\epsilon,\frac{x}{\epsilon}\right) d\nu(\frac{x}{\epsilon}) + \int_{Y_j^\epsilon} (x-x_j^\epsilon)\cdot\nabla_x v\left(x_j^\epsilon,\frac{x}{\epsilon}\right) d\nu(\frac{x}{\epsilon})\right. \\
&\qquad \left. +\frac{1}{2}\int_{Y_j^\epsilon} (x-x_j^\epsilon)\cdot\nabla_x\nabla_x v\left(x_j^\epsilon,\frac{x}{\epsilon}\right)\cdot(x-x_j^\epsilon) d\nu(\frac{x}{\epsilon}) + \mathcal{O}(\epsilon^3)\right] \\
&= \epsilon^{2N}\left[\int_Y \left(u\left(x_j^\epsilon,y\right) + \epsilon y\cdot\nabla_x u\left(x_j^\epsilon,y\right) + \frac{\epsilon^2}{2} y\cdot\nabla_x\nabla_x u\left(x_j^\epsilon,y\right)\cdot y\right) d\mu(y) + \mathcal{O}(\epsilon^3)\right] \\
&\quad \times\left[\int_Y \left(v\left(x_j^\epsilon,y\right) + \epsilon y\cdot\nabla_x v\left(x_j^\epsilon,y\right) + \frac{\epsilon^2}{2} y\cdot\nabla_x\nabla_x v\left(x_j^\epsilon,y\right)\cdot y\right) d\nu(y) + \mathcal{O}(\epsilon^3)\right] \\
&= \epsilon^{2N}\left\{\int_Y u\left(x_j^\epsilon,y\right) d\mu(y)\int_Y v\left(x_j^\epsilon,y\right) d\nu(y)\right. \\
&\quad +\epsilon\left[\int_Y y\cdot\nabla_x u\left(x_j^\epsilon,y\right) d\mu(y)\int_Y v\left(x_j^\epsilon,y\right) d\nu(y) + \int_Y u\left(x_j^\epsilon,y\right) d\mu(y)\int_Y y\cdot\nabla_x v\left(x_j^\epsilon,y\right) d\nu(y)\right] \\
&\quad +\frac{\epsilon^2}{2}\left[\int_Y y\cdot\nabla_x\nabla_x u\left(x_j^\epsilon,y\right)\cdot y d\mu(y)\int_Y v\left(x_j^\epsilon,y\right) d\nu(y)\right. \\
&\qquad +2\int_Y y\cdot\nabla_x u\left(x_j^\epsilon,y\right) d\mu(y)\int_Y y\cdot\nabla_x v\left(x_j^\epsilon,y\right) d\nu(y) \\
&\qquad \left.\left. +\int_Y u\left(x_j^\epsilon,y\right)\cdot y d\mu(y)\int_Y y\cdot\nabla_x\nabla_x v\left(x_j^\epsilon,y\right) d\nu(y)\right] + \mathcal{O}(\epsilon^3)\right\}
\end{aligned} \tag{36}$$

Summing over the cells Y_j^ϵ yields a Riemann sum which converges to an integral over $\mathbb{R}^N$ with infinite speed (as in Remark 5.3). We thus obtain the desired result.

Actually we need a variant of Lemma 5.1 when the integrands are subject to a large drift in the macroscopic variable but not in the microscopic one. We emphasize that Lemma 5.4 below does not reduce to Lemma 5.1 by a change of variables.

Lemma 5.4 *Let $\mathcal{V} \in \mathbb{R}^N$ be a given velocity and $T > 0$ a finite time. Let $\mu(y)$ and $\nu(y)$ be two periodic non-negative measures on the unit cube Y. Let $u(t,x,y)$ and $v(t,x,y)$ be two Y-periodic functions which are smooth with respect to x and decay fast enough at infinity when $|x| \to \infty$ (so that all integrals below are finite). As ϵ tends to 0, we have*

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}^N} \epsilon^{-N} \int_0^T \left(\int_{Y_j^\epsilon} u\left(t, x+\mathcal{V}\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) d\mu(\frac{x}{\epsilon}) \right) \left(\int_{Y_j^\epsilon} v\left(t, x+\mathcal{V}\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) d\nu(\frac{x}{\epsilon}) \right) dt \\
&= \int_{\mathbb{R}^N} \int_0^T \left(\int_Y u\left(t,x,y\right) d\mu(y) \right) \left(\int_Y v\left(t,x,y\right) d\nu(y) \right) dxdt \\
&\quad +\epsilon \left[\int_{\mathbb{R}^N} \int_0^T \left(\int_Y y\cdot\nabla_x u\left(t,x,y\right) d\mu(y) \right) \left(\int_Y v\left(t,x,y\right) d\nu(y) \right) dxdt \right. \\
&\qquad \left. + \int_{\mathbb{R}^N} \int_0^T \left(\int_Y u\left(t,x,y\right) d\mu(y) \right) \left(\int_Y y\cdot\nabla_x v\left(t,x,y\right) d\nu(y) \right) dxdt \right] \\
&\quad +\frac{\epsilon^2}{2} \left[\int_{\mathbb{R}^N} \int_0^T \left(\int_Y y\cdot\nabla_x\nabla_x u\left(t,x,y\right)\cdot y d\mu(y) \right) \left(\int_Y v\left(t,x,y\right) d\nu(y) \right) dxdt \right. \\
&\qquad +2 \int_{\mathbb{R}^N} \int_0^T \left(\int_Y y\cdot\nabla_x u\left(t,x,y\right) d\mu(y) \right) \left(\int_Y y\cdot\nabla_x v\left(t,x,y\right) d\nu(y) \right) dxdt \\
&\qquad \left. + \int_{\mathbb{R}^N} \int_0^T \left(\int_Y u\left(t,x,y\right) d\mu(y) \right) \left(\int_Y y\cdot\nabla_x\nabla_x v\left(t,x,y\right)\cdot y d\nu(y) \right) dxdt \right] + \mathcal{O}(\epsilon^3).
\end{aligned}
$$

Proof We first perform the change of variables (a simple translation when the time variable is fixed) $x' = x + \mathcal{V}\frac{t}{\epsilon}$. We decompose $\mathcal{V}t\epsilon^{-2}$ in its integer and decimal parts

$$
\mathcal{V}\frac{t}{\epsilon^2} = j + \delta \quad \text{with} \quad j \in \mathbb{Z}^N, \delta \in Y,
$$

so that we have

$$
u\left(t, x+\mathcal{V}\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) = u\left(t, x', \frac{x'}{\epsilon} - \delta\right).
$$

Remark that δ depends on t and ϵ but not on x. We introduce the cube $Y_j^{'\epsilon} = Y_j^\epsilon + \mathcal{V}\frac{t}{\epsilon}$ of center $x_j^{'\epsilon}$, and write the following Taylor expansion for $x' \in Y_j^{'\epsilon}$

$$\begin{aligned} u\left(t, x', \frac{x'}{\epsilon} - \delta\right) &= u\left(t, x_j^{'\epsilon}, \frac{x'}{\epsilon} - \delta\right) + (x' - x_j^{'\epsilon}) \cdot \nabla_x u\left(t, x_j^{'\epsilon}, \frac{x'}{\epsilon} - \delta\right) \\ &+ \frac{1}{2}(x' - x_j^{'\epsilon}) \cdot \nabla_x \nabla_x u\left(t, x_j^{'\epsilon}, \frac{x'}{\epsilon} - \delta\right) \cdot (x' - x_j^{'\epsilon}) + \mathcal{O}(\epsilon^3). \end{aligned}$$

Thus, we deduce

$$\begin{aligned} &\int_{Y_j^\epsilon} u\left(t, x + \mathcal{V}\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) d\mu(\frac{x}{\epsilon}) = \int_{Y_j^{'\epsilon}} u\left(t, x', \frac{x'}{\epsilon} - \delta\right) d\mu(\frac{x'}{\epsilon} - \delta) \\ &= \int_{Y_j^{'\epsilon}} u\left(t, x_j^{'\epsilon}, \frac{x'}{\epsilon} - \delta\right) d\mu(\frac{x'}{\epsilon} - \delta) + \int_{Y_j^{'\epsilon}} (x' - x_j^{'\epsilon}) \cdot \nabla_x u\left(t, x_j^{'\epsilon}, \frac{x'}{\epsilon} - \delta\right) d\mu(\frac{x'}{\epsilon} - \delta) \\ &+ \frac{1}{2}\int_{Y_j^{'\epsilon}} (x' - x_j^{'\epsilon}) \cdot \nabla_x \nabla_x u\left(t, x_j^{'\epsilon}, \frac{x'}{\epsilon} - \delta\right) \cdot (x' - x_j^{'\epsilon}) d\mu(\frac{x'}{\epsilon} - \delta) + \mathcal{O}(\epsilon^{N+3}) \\ &= \epsilon^N \int_Y \left(u + \epsilon(y+\delta) \cdot \nabla_x u + \frac{\epsilon^2}{2}(y+\delta) \cdot \nabla_x \nabla_x u \cdot (y+\delta)\right)\left(t, x_j^{'\epsilon}, y\right) d\mu(y) + \mathcal{O}(\epsilon^{N+3}) \end{aligned}$$

and an analogous expansion for the function v. Performing a computation similar to that in (36) we obtain exactly the same terms plus the following ones

$$\begin{aligned} &\epsilon^{N+1}\delta \cdot \nabla_x \left[\left(\int_Y u\left(t, x_j^{'\epsilon}, y\right) d\mu(y)\right)\left(\int_Y v\left(t, x_j^{'\epsilon}, y\right) d\nu(y)\right)\right] \\ &+\epsilon^{N+2}\delta \cdot \nabla_x \left[\left(\int_Y y \cdot \nabla_x u\left(t, x_j^{'\epsilon}, y\right) d\mu(y)\right)\left(\int_Y v\left(t, x_j^{'\epsilon}, y\right) d\nu(y)\right)\right] \\ &+\epsilon^{N+2}\delta \cdot \nabla_x \left[\left(\int_Y u\left(t, x_j^{'\epsilon}, y\right) d\mu(y)\right)\left(\int_Y y \cdot \nabla_x v\left(t, x_j^{'\epsilon}, y\right) d\nu(y)\right)\right] \\ &+\epsilon^{N+2}\delta \cdot \nabla_x \nabla_x \left[\left(\int_Y u\left(t, x_j^{'\epsilon}, y\right) d\mu(y)\right)\left(\int_Y v\left(t, x_j^{'\epsilon}, y\right) d\nu(y)\right)\right] \cdot \delta. \end{aligned} \tag{37}$$

When summing (37) over all cubes $Y_j^{'\epsilon}$ we obtain a Riemann sum which converges to an integral over $\mathbb{R}^N$ with infinite speed (as in Remark 5.3). Remark that δ, which depends on ϵ but remains bounded in Y, can be factorized out of this Riemann sum. The additional terms in (37) do not contribute in the final result since

$$\int_{\mathbb{R}^N} \nabla_x \left[\left(\int_Y u(t,x,y)\, d\mu(y)\right)\left(\int_Y v(t,x,y)\, d\nu(y)\right)\right] dx = 0,$$

and the same holds true for the integrals over $\mathbb{R}^N$ of the other second order derivatives.

Having in mind the initial data (20), we postulate the following ansatz for the solutions u_ϵ and $\mathbf{r}_j^\epsilon$ of (1):

$$\begin{aligned} u_\epsilon(t,x) &= e^{2i\pi\theta\cdot\frac{x}{\epsilon}\pm i\omega_n\frac{t}{\epsilon^2}} \sum_{k=0}^{\infty} \epsilon^k u^k\left(t, x \pm \mathcal{V}\frac{t}{\epsilon}, \frac{x}{\epsilon}\right), \\ \mathbf{r}_j^\epsilon(t) &= e^{2i\pi\theta\cdot j\pm i\omega_n\frac{t}{\epsilon^2}} \sum_{k=0}^{\infty} \epsilon^k \mathbf{r}_j^k(t), \end{aligned} \tag{38}$$

where θ is the reduced wave number of the tube oscillations (moving out of phase if $\theta \neq 0$), $\omega_n := \omega_n(\theta)$ is the time frequency defined by the dispersion relation (21), $\mathcal{V}$ is the group velocity defined by (22), each term $u^k(t,x,y)$ is a Y-periodic function with respect to y, and

$$\mathbf{r}_j^k(t) := \frac{1}{|Y_j^\epsilon|}\int_{Y_j^\epsilon} \mathbf{r}^k\left(t, x \pm \mathcal{V}\frac{t}{\epsilon}\right) dx.$$

Recall that, for $x \in Y_j^\epsilon$, the integer parts of the components of x/ϵ are equal to the vector $j \in \mathbb{Z}^N$, i.e., $E(x/\epsilon) = j$. In truth, the ansatz should be the sum of the two ansatz (38) with the $+$ or $-$ sign. However, for the sake of simplicity of the exposition, we instead write a single ansatz with one of the two possible signs $\pm$.

The main (formal) result of this section is

Proposition 5.5 *The leading terms in the ansatz* (38) *are*

$$u^0(t,x,y) = v^\pm(t,x)\psi_n(y,\theta) \quad \text{and} \quad \mathbf{r}^0(t,x) = \frac{\pm 1}{i\omega_n(\theta)} v^\pm(t,x)\mathbf{s}_n(\theta),$$

where $v^\pm$ is the solution of the Schrödinger homogenized problem

$$\pm i\frac{\partial v^\pm}{\partial t} - \operatorname{div}(A^*\nabla v^\pm) = 0, \tag{39}$$

with $A^* := (8\pi^2)^{-1}\nabla_\theta\nabla_\theta\omega_n(\theta)$.

Remark 5.6 *In this section we do not discuss initial data, an issue that was addressed in the previous section. Let us simply say that the ansatz* (38) *is* *actually the sum of two ansatz, one with + and the other with −. Therefore, the initial data $v^{\pm}(0)$ for the two homogenized equations should be deduced formally from the following asymptotic behavior as ϵ goes to 0*

$$u_\epsilon(0,x) \approx e^{2i\pi\theta\cdot y}\psi_n(y,\theta)\left(v^+(0)+v^-(0)\right)$$

and

$$\mathbf{r}_j^\epsilon(0) \approx \frac{1}{i\omega_n(\theta)} e^{2i\pi\theta\cdot j}\mathbf{s}_n(\theta)\left(v^+(0)-v^-(0)\right).$$

Proof In the variational formulation (33) we choose the test functions

$$\Phi_\epsilon(t,x) := \phi\left(t, x \pm \mathcal{V}\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) e^{2i\pi\theta\cdot\frac{x}{\epsilon} \pm i\omega_n\frac{t}{\epsilon^2}},$$

and

$$\boldsymbol{\gamma}_j^\epsilon(t) := \epsilon^2 \int_{\partial T_j^\epsilon} \Phi_\epsilon(t,x)\mathbf{n}(x)d\sigma(x).$$

By introducing the Y-periodic function $e_\theta(y) = e^{2i\pi\theta\cdot(y-E(y))}$ (see Remark 3.2) such that $e_\theta(\frac{x}{\epsilon}) = e^{2i\pi\theta\cdot\left(\frac{x}{\epsilon}-j\right)}$ we can rewrite

$$\boldsymbol{\gamma}_j^\epsilon(t) = \epsilon^2 e^{2i\pi\theta\cdot j \pm i\omega_n\frac{t}{\epsilon^2}} \int_{\partial T_j^\epsilon} e_\theta(\frac{x}{\epsilon})\phi\left(t, x \pm \mathcal{V}\frac{t}{\epsilon}, \frac{x}{\epsilon}\right)\mathbf{n}(x)d\sigma(x).$$

In order to simplify the presentation and by a slight abuse of notations, we define

$$\nabla_x \mathbf{r}_j^k(t) := \frac{1}{|Y_j^\epsilon|}\int_{Y_j^\epsilon} \nabla_x \mathbf{r}^k\left(t, x \pm \mathcal{V}\frac{t}{\epsilon}\right)dx.$$

Differentiating the ansatz for u_ϵ and $\mathbf{r}_j^\epsilon$ yields

$$\begin{aligned}
\nabla u_\epsilon = \;& e^{2i\pi\theta\cdot\frac{x}{\epsilon}\pm i\omega_n\frac{t}{\epsilon^2}}\left[\epsilon^{-1}(\nabla_y + 2i\pi\theta)u^0 + (\nabla_x u^0 + (\nabla_y + 2i\pi\theta)u^1)\right. \\
& \left. + \epsilon(\nabla_x u^1 + (\nabla_y + 2i\pi\theta)u^2) + \mathcal{O}(\epsilon^2)\right], \\
\dot{u}_\epsilon = \;& e^{2i\pi\theta\cdot\frac{x}{\epsilon}\pm i\omega_n\frac{t}{\epsilon^2}}\left[\pm\epsilon^{-2} i\omega_n u^0 \pm \epsilon^{-1}(i\omega_n u^1 + \mathcal{V}\cdot\nabla_x u^0)\right. \\
& \left. + (\dot{u}^0 \pm i\omega_n u^2 \pm \mathcal{V}\cdot\nabla_x u^1) + \mathcal{O}(\epsilon)\right], \\
\dot{\mathbf{r}}_j^\epsilon = \;& e^{2i\pi\theta\cdot j\pm i\omega_n\frac{t}{\epsilon^2}}\left[\pm\epsilon^{-2} i\omega_n \mathbf{r}_j^0 \pm \epsilon^{-1}(i\omega_n \mathbf{r}_j^1 + \mathcal{V}\cdot\nabla_x \mathbf{r}_j^0)\right. \\
& \left. + (\dot{\mathbf{r}}_j^0 \pm i\omega_n \mathbf{r}_j^2 \pm \mathcal{V}\cdot\nabla_x \mathbf{r}_j^1) + \mathcal{O}(\epsilon)\right], \\
\ddot{\mathbf{r}}_j^\epsilon = \;& e^{2i\pi\theta\cdot j\pm i\omega_n\frac{t}{\epsilon^2}}\left[-\epsilon^{-4}\omega_n^2\mathbf{r}_j^0 + \epsilon^{-3}(-\omega_n^2\mathbf{r}_j^1 + 2i\omega_n\mathcal{V}\cdot\nabla_x\mathbf{r}_j^0)\right. \\
& \left. + \epsilon^{-2}(\pm 2i\omega_n\dot{\mathbf{r}}_j^0 + \mathcal{V}\cdot\nabla\nabla_x\mathbf{r}_j^0\cdot\mathcal{V} + 2i\omega_n\mathcal{V}\cdot\nabla_x\mathbf{r}_j^1 - \omega_n^2\mathbf{r}_j^2) + \mathcal{O}(\epsilon^{-1})\right].
\end{aligned}$$

We plug these ansatz in the variational formulation (33) and we get a cascade of equations which is slightly more complicated than in the "usual" cases. Let us explain these subtleties: (33) involves a sum of products of integrals and, according to Lemma 5.1, the boundary integrals do not converge with infinite speed to their limits but rather involve corrector terms. Therefore, when passing to the limit in the ϵ^k equation, these correctors yield new contributions for the limit of the ϵ^{k+p} equations for all $p \geq 1$. It turns out that the three first terms of this cascade of equation are enough for our purpose

$$\epsilon^{-2} E_\epsilon^{-2} + \epsilon^{-1} E_\epsilon^{-1} + \epsilon^0 E_\epsilon^0 + \mathcal{O}(\epsilon) = 0. \tag{40}$$

Collecting the terms of order ϵ^{-2}, we obtain the E_ϵ^{-2} equation

$$\begin{cases} \displaystyle\int_{\Omega_\epsilon} (\nabla_y + 2i\pi\theta) u^0 \cdot (\nabla_y - 2i\pi\theta)\overline{\phi} = \pm \sum_{j\in\mathbb{Z}^N} \epsilon \int_{\partial T_j^\epsilon} i\omega_n \mathbf{r}_j^0 \cdot \mathbf{n}\overline{\phi e_\theta}, \\ \displaystyle\pm \sum_{j\in\mathbb{Z}^N} \epsilon \int_{\partial T_j^\epsilon} i\omega_n \mathbf{r}_j^0 \cdot \mathbf{n}\overline{\phi e_\theta} = \frac{-i\omega_n\rho}{k - m\omega_n^2} \sum_{j\in\mathbb{Z}^N} \epsilon^{2-N} \int_{\partial T_j^\epsilon} i\omega_n u^0 e_\theta \mathbf{n} \cdot \int_{\partial T_j^\epsilon} \overline{\phi e_\theta}\mathbf{n}. \end{cases} \tag{41}$$

Eliminating the right hand side of the first line of (41) with its second line yields

$$\int_{\Omega_\epsilon} (\nabla_y + 2i\pi\theta) u^0 \cdot (\nabla_y - 2i\pi\theta)\overline{\phi} = \frac{-i\omega_n\rho}{k - m\omega_n^2} \sum_{j\in\mathbb{Z}^N} \epsilon^{2-N} \int_{\partial T_j^\epsilon} i\omega_n u^0 e_\theta \mathbf{n} \cdot \int_{\partial T_j^\epsilon} \overline{\phi e_\theta}\mathbf{n}.$$

Since $e_\theta(y) = e^{2i\pi\theta\cdot y}$ in the unit cell Y, passing to the limit $\epsilon \to 0$ with the help of Lemma 5.4 yields

$$\int_{\mathbb{R}^N} \int_{Y^*} (\nabla_y + 2i\pi\theta) u^0 \cdot (\nabla_y - 2i\pi\theta)\overline{\phi} = \frac{\rho\omega_n^2}{k - m\omega_n^2} \int_{\mathbb{R}^N} \int_{\partial T} u^0 e^{2i\pi\theta\cdot y}\mathbf{n} \cdot \int_{\partial T} \overline{\phi} e^{-2i\pi\theta\cdot y}\mathbf{n}, \tag{42}$$

which is just the weak formulation of the cell problem (11) since $\lambda_n = (k - m\omega_n^2)/(\rho\omega_n^2)$ by the dispersion relation (21). By assumption (14) on the simplicity of the eigenvalue $\lambda_n(\theta)$, we deduce that u^0 is necessarily a multiple of the eigenvector ψ_n, i.e., there exists a function $u(t, x)$ such that

$$u^0(t, x, y) = v^\pm(t, x)\psi_n(y, \theta). \tag{43}$$

From the second line of (41) we then deduce that

$$\mathbf{r}^0(t, x) = \mp \frac{i\omega_n\rho}{k - m\omega_n^2} v^\pm(t, x) \int_{\partial T} \psi_n e^{2i\pi\theta\cdot y}\mathbf{n} = \frac{\pm 1}{i\omega_n} v^\pm(t, x)\mathbf{s}_n(\theta). \tag{44}$$

Collecting the terms of order ϵ^{-1}, we obtain the E_ϵ^{-1} equation

$$\begin{aligned}
&\int_{\Omega_\epsilon}(\nabla_y+2i\pi\theta)u^0\cdot\nabla_x\overline{\phi}+\int_{\Omega_\epsilon}[\nabla_x u^0+(\nabla_y+2i\pi\theta)u^1]\cdot(\nabla_y-2i\pi\theta)\overline{\phi}\\
&=\pm\sum_{j\in\mathbb{Z}^N}\epsilon\int_{\partial T_j^\epsilon}(i\omega_n\mathbf{r}_j^1+\mathcal{V}\cdot\nabla_x\mathbf{r}_j^0)\cdot\mathbf{n}\overline{\phi e_\theta},
\end{aligned}\tag{45}$$

and

$$\begin{aligned}
&\pm\sum_{j\in\mathbb{Z}^N}\epsilon\int_{\partial T_j^\epsilon}(i\omega_n\mathbf{r}_j^1-\frac{2m\omega_n^2}{k-m\omega_n^2}\mathcal{V}\cdot\nabla_x\mathbf{r}_j^0)\cdot\mathbf{n}\overline{\phi e_\theta}dx\\
&=\frac{-i\omega_n\rho}{k-m\omega_n^2}\sum_{j\in\mathbb{Z}^N}\epsilon^{2-N}\int_{\partial T_j^\epsilon}(i\omega_n u^1+\mathcal{V}\cdot\nabla_x u^0)e_\theta\mathbf{n}\cdot\int_{\partial T_j^\epsilon}\overline{\phi e_\theta}\mathbf{n}.
\end{aligned}\tag{46}$$

Adding (45) to (46) eliminates $\mathbf{r}_j^1$, replacing $\mathbf{r}_j^0$ by its value in (44), and passing to the two-scale limit yields

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{Y^*}(\nabla_y+2i\pi\theta)u^0\cdot\nabla_x\overline{\phi}+\int_{\mathbb{R}^N}\int_{Y^*}[\nabla_x u^0+(\nabla_y+2i\pi\theta)u^1]\cdot(\nabla_y-2i\pi\theta)\overline{\phi}\\
&=\frac{\rho\omega_n^2}{k-m\omega_n^2}\int_{\mathbb{R}^N}\int_{\partial T}u^1e^{2i\pi\theta\cdot y}\mathbf{n}\cdot\int_{\partial T}\overline{\phi}e^{-2i\pi\theta\cdot y}\mathbf{n}\\
&\quad+\frac{2k}{i\omega_n(k-m\omega_n^2)}\int_{\mathbb{R}^N}\int_{Y^*}(\nabla_y+2i\pi\theta)(\mathcal{V}\cdot\nabla_x u^0)\cdot(\nabla_y-2i\pi\theta)\overline{\phi}\\
&\quad+\frac{\rho\omega_n^2}{k-m\omega_n^2}\int_{\mathbb{R}^N}\int_{\partial T}y\cdot\nabla_x u^0e^{2i\pi\theta\cdot y}\mathbf{n}\cdot\int_{\partial T}\overline{\phi}e^{-2i\pi\theta\cdot y}\mathbf{n}\\
&\quad+\frac{\rho\omega_n^2}{k-m\omega_n^2}\int_{\mathbb{R}^N}\int_{\partial T}u^0e^{2i\pi\theta\cdot y}\mathbf{n}\cdot\int_{\partial T}y\cdot\nabla_x\overline{\phi}e^{-2i\pi\theta\cdot y}\mathbf{n}
\end{aligned}\tag{47}$$

where the two last lines of (47) come from the first order correction in Lemma 5.4 for the previous E_ϵ^{-2} equation. Since

$$\frac{1}{2\pi}\nabla\lambda_n(\theta)=\frac{d\lambda_n}{d\omega_n}\mathcal{V}=\frac{-2k\lambda_n}{\omega_n(k-m\omega_n^2)}\mathcal{V},$$

and because of equation (18), we deduce from (47) that

$$u^1(t,x,y)=\frac{1}{2i\pi}\sum_{p=1}^N\frac{\partial v^\pm}{\partial x_p}(t,x)\frac{\partial\psi_n}{\partial\theta_p}(y,\theta).\tag{48}$$

From this value of u^1 and the two-scale limit of (45) we could deduce a formula for $\mathbf{r}^1$. However, in the sequel we shall need merely the value of $\mathcal{V}\cdot\nabla_x\mathbf{r}^1$ and $\mathcal{V}\cdot\nabla_x u^1$. Therefore, we content ourselves in giving a variational formulation for $\mathcal{V}\cdot\nabla_x\mathbf{r}^1$ and $\mathcal{V}\cdot\nabla_x u^1$, which are obtained from the two-scale limit of (45) and (47) with the test function $\mathcal{V}\cdot\nabla_x\phi$ (instead of just ϕ) and an additional integration by parts in x:

$$
\begin{aligned}
&\mathcal{V}\cdot\int_{\mathbb{R}^N}\int_{Y^*}\Big((\nabla_y+2i\pi\theta)\nabla_x u^0\cdot\nabla_x\overline{\phi}+[\nabla_x\nabla_x u^0+(\nabla_y+2i\pi\theta)\nabla_x u^1]\cdot(\nabla_y-2i\pi\theta)\overline{\phi}\Big)\\
&=\pm\int_{\mathbb{R}^N}\int_{\partial T}(i\omega_n\mathcal{V}\cdot\nabla_x\mathbf{r}^1+\mathcal{V}\cdot\nabla_x\nabla_x\mathbf{r}^0\cdot\mathcal{V})\cdot\overline{\phi}e^{-2i\pi\theta\cdot y}\mathbf{n}
\end{aligned}
\tag{49}
$$

and

$$
\begin{aligned}
&\mathcal{V}\cdot\int_{\mathbb{R}^N}\int_{Y^*}\Big((\nabla_y+2i\pi\theta)\nabla_x u^0\cdot\nabla_x\overline{\phi}+[\nabla_x\nabla_x u^0+(\nabla_y+2i\pi\theta)\nabla_x u^1]\cdot(\nabla_y-2i\pi\theta)\overline{\phi}\Big)\\
&=\frac{\rho\omega_n^2}{k-m\omega_n^2}\int_{\mathbb{R}^N}\int_{\partial T}\mathcal{V}\cdot\nabla_x u^1e^{2i\pi\theta\cdot y}\mathbf{n}\cdot\int_{\partial T}\overline{\phi}e^{-2i\pi\theta\cdot y}\mathbf{n}\\
&\quad+\frac{2k}{i\omega_n(k-m\omega_n^2)}\int_{\mathbb{R}^N}\int_{Y^*}(\nabla_y+2i\pi\theta)(\mathcal{V}\cdot\nabla_x\nabla_x u^0\cdot\mathcal{V})\cdot(\nabla_y-2i\pi\theta)\overline{\phi}\\
&\quad+\frac{\rho\omega_n^2}{k-m\omega_n^2}\int_{\mathbb{R}^N}\int_{\partial T}\mathcal{V}\cdot\nabla_x\nabla_x u^0\cdot ye^{2i\pi\theta\cdot y}\mathbf{n}\cdot\int_{\partial T}\overline{\phi}e^{-2i\pi\theta\cdot y}\mathbf{n}\\
&\quad+\frac{\rho\omega_n^2}{k-m\omega_n^2}\int_{\mathbb{R}^N}\int_{\partial T}\mathcal{V}\cdot\nabla_x u^0e^{2i\pi\theta\cdot y}\mathbf{n}\cdot\int_{\partial T}y\cdot\nabla_x\overline{\phi}e^{-2i\pi\theta\cdot y}\mathbf{n}.
\end{aligned}
\tag{50}
$$

Collecting the terms of order ϵ^0, we obtain the E^0_ϵ equation

$$
\begin{aligned}
&\int_{\Omega_\epsilon}\Big([\nabla_x u^0+(\nabla_y+2i\pi\theta)u^1]\cdot\nabla_x\overline{\phi}+[\nabla_x u^1+(\nabla_y+2i\pi\theta)u^2]\cdot(\nabla_y-2i\pi\theta)\overline{\phi}\Big)\\
&\qquad=\sum_{j\in\mathbb{Z}^N}\epsilon\int_{\partial T_j^\epsilon}(\dot{\mathbf{r}}_j^0\pm i\omega_n\mathbf{r}_j^2\pm\mathcal{V}\cdot\nabla_x\mathbf{r}_j^1)\cdot\mathbf{n}\overline{\phi e_\theta},
\end{aligned}
\tag{51}
$$

and

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}^N}\epsilon\int_{\partial T_j^\epsilon}[\pm i\omega_n\mathbf{r}_j^2+\frac{im\omega_n}{k-m\omega_n^2}(2i\omega_n\dot{\mathbf{r}}_j^0\pm\mathcal{V}\cdot\nabla_x\nabla_x\mathbf{r}_j^0\cdot\mathcal{V}\pm 2i\omega_n\mathcal{V}\cdot\nabla_x\mathbf{r}_j^1)]\cdot\mathbf{n}\overline{\phi e_\theta}\\
&=\frac{-i\omega_n\rho}{k-m\omega_n^2}\sum_{j\in\mathbb{Z}^N}\epsilon^{2-N}\int_{\partial T_j^\epsilon}(\pm\dot{u}^0+i\omega_n u^2+\mathcal{V}\cdot\nabla_x u^1)e_\theta\mathbf{n}\cdot\int_{\partial T_j^\epsilon}\overline{\phi e_\theta}\mathbf{n}.
\end{aligned}
\tag{52}
$$

Now, we add (51) and (52), which eliminates $\mathbf{r}_j^2$, and we pass to the two-scale limit. We also eliminate $\mathcal{V} \cdot \nabla_x \mathbf{r}^1$ and $\mathcal{V} \cdot \nabla_x u^1$ by using (49), (50) and we replace $\mathbf{r}^0$ by its value in terms of u^0. This yields

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \int_{Y^*} \Big([\nabla_x u^0 + (\nabla_y + 2i\pi\theta) u^1] \cdot \nabla_x \overline{\phi} + [\nabla_x u^1 + (\nabla_y + 2i\pi\theta) u^2] \cdot (\nabla_y - 2i\pi\theta) \overline{\phi} \Big) \\
&= \mp \frac{i\omega_n \rho 2k}{(k - m\omega_n^2)^2} \int_{\mathbb{R}^N} \int_{\partial T} \dot{u}^0 e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} \overline{\phi} e^{-2i\pi\theta\cdot y} \mathbf{n} \\
&\quad + \frac{\rho\omega_n^2}{k - m\omega_n^2} \int_{\mathbb{R}^N} \int_{\partial T} u^2 e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} \overline{\phi} e^{-2i\pi\theta\cdot y} \mathbf{n} \\
&\quad + \frac{2k}{i\omega_n (k - m\omega_n^2)} \mathcal{V} \cdot \Big(\int_{\mathbb{R}^N} \int_{Y^*} [(\nabla_y + 2i\pi\theta) \nabla_x u^0] \cdot \nabla_x \overline{\phi} \\
&\qquad\qquad + \int_{\mathbb{R}^N} \int_{Y^*} [\nabla_x \nabla_x u^0 + (\nabla_y + 2i\pi\theta) \nabla_x u^1] \cdot (\nabla_y - 2i\pi\theta) \overline{\phi} \Big) \\
&\quad + \frac{3k}{\omega_n^2 (k - m\omega_n^2)} \mathcal{V} \cdot \int_{\mathbb{R}^N} \int_{Y^*} (\nabla_y + 2i\pi\theta) \nabla_x \nabla_x u^0 \cdot (\nabla_y - 2i\pi\theta) \overline{\phi} \cdot \mathcal{V} \\
&\quad + \frac{\omega_n^2 \rho}{k - m\omega_n^2} \Big(\int_{\mathbb{R}^N} \int_{\partial T} y \cdot \nabla_x u^1 e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} \overline{\phi} e^{-2i\pi\theta\cdot y} \mathbf{n} \\
&\qquad\qquad + \int_{\mathbb{R}^N} \int_{\partial T} u^1 e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} y \cdot \nabla_x \overline{\phi} e^{-2i\pi\theta\cdot y} \mathbf{n} \Big) \\
&\quad + \frac{1}{2} \frac{\omega_n^2 \rho}{k - m\omega_n^2} \Big(\int_{\mathbb{R}^N} \int_{\partial T} y \cdot \nabla_x \nabla_x u^0 \cdot y e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} \overline{\phi} e^{-2i\pi\theta\cdot y} \mathbf{n} \\
&\qquad\qquad + 2 \int_{\mathbb{R}^N} \int_{\partial T} y \cdot \nabla_x u^0 e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} y \cdot \nabla_x \overline{\phi} e^{-2i\pi\theta\cdot y} \mathbf{n} \\
&\qquad\qquad + \int_{\mathbb{R}^N} \int_{\partial T} u^0 e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} y \cdot \nabla_x \nabla_x \overline{\phi} \cdot y e^{-2i\pi\theta\cdot y} \mathbf{n} \Big)
\end{aligned}
\tag{53}
$$

where the three last lines of (53) come from the second order correction in Lemma 5.4 for the E_ϵ^{-2} equation, and the two previous lines of (53) come from the first order correction in Lemma 5.4 for the E_ϵ^{-1} equation. Taking $\phi(t,x,y) = \varphi(t,x)\psi_n(y)$ in (53), where φ is a smooth function, eliminates u^2

(this is a consequence of the Fredholm alternative) and we obtain

$$
\begin{aligned}
&\lambda_n \int_{\mathbb{R}^N} \int_{Y^*} \Big([\nabla_x u^0 + (\nabla_y + 2i\pi\theta) u^1] \cdot \nabla_x (\overline{\varphi} \overline{\psi}_n) + \nabla_x u^1 \cdot (\nabla_y - 2i\pi\theta)(\overline{\varphi}\overline{\psi}_n) \Big) \\
&= \mp \frac{i}{\omega_n} \frac{2k}{k - m\omega_n^2} \int_{\mathbb{R}^N} \int_{\partial T} \dot{u}^0 e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} \overline{\varphi}\overline{\psi}_n e^{-2i\pi\theta\cdot y} \mathbf{n} \\
&\quad - \frac{1}{2i\pi} \nabla_\theta \lambda_n \cdot \Big[\int_{\mathbb{R}^N} \int_{Y^*} [(\nabla_y + 2i\pi\theta) \nabla_x u^0] \cdot \nabla_x (\overline{\varphi}\overline{\psi}_n) \\
&\qquad + \int_{\mathbb{R}^N} \int_{Y^*} [\nabla_x \nabla_x u^0 + (\nabla_y + 2i\pi\theta) \nabla_x u^1] \cdot (\nabla_y - 2i\pi\theta)(\overline{\varphi}\overline{\psi}_n) \Big] \\
&\quad + \frac{3k}{\omega_n^2 (k - m\omega_n^2)} \mathcal{V} \cdot \int_{\mathbb{R}^N} \int_{\partial T} \nabla_x \nabla_x u^0 e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} \overline{\varphi}\overline{\psi}_n e^{-2i\pi\theta\cdot y} \mathbf{n} \cdot \mathcal{V} \\
&\quad + \int_{\mathbb{R}^N} \int_{\partial T} y \cdot \nabla_x u^1 e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} \overline{\varphi}\overline{\psi}_n e^{-2i\pi\theta\cdot y} \mathbf{n} \\
&\quad + \int_{\mathbb{R}^N} \int_{\partial T} u^1 e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} y \cdot \nabla_x (\overline{\varphi}\overline{\psi}_n) e^{-2i\pi\theta\cdot y} \mathbf{n} \\
&\quad + \int_{\mathbb{R}^N} \int_{\partial T} y \cdot \nabla_x \nabla_x u^0 \cdot y e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} \overline{\varphi}\overline{\psi}_n e^{-2i\pi\theta\cdot y} \mathbf{n} \\
&\quad + \int_{\mathbb{R}^N} \int_{\partial T} y \cdot \nabla_x u^0 e^{2i\pi\theta\cdot y} \mathbf{n} \cdot \int_{\partial T} y \cdot \nabla_x (\overline{\varphi}\overline{\psi}_n) e^{-2i\pi\theta\cdot y} \mathbf{n}.
\end{aligned} \tag{54}
$$

Replacing u^0 by (43) and u^1 by (48) yields that (54) is a variational formulation of a Schrödinger type equation for $v^\pm$. The only remaining difficulty is to identify the coefficient tensor in the elliptic part of (54). This is a tedious computation that we simply summarize. Using the Fredholm alternative for (19) and assuming the normalization $\|\psi_n\|_{\mathbf{L}^2(\partial T)} = 1$, we deduce that (54) is the weak form of the following homogenized Schrödinger equation

$$
\frac{\mp 2ik}{\omega_n (k - m\omega_n^2)} \frac{\partial v^\pm}{\partial t} + \frac{3k}{\omega_n^2 (k - m\omega_n^2)} \mathcal{V} \cdot \nabla\nabla v^\pm \cdot \mathcal{V} - \frac{1}{\lambda_n} \mathrm{div}(A^* \nabla v^\pm) = 0,
$$

where $A^* = \frac{1}{8\pi^2} \nabla\nabla\lambda_n$. Multiplying by $\lambda_n \dfrac{\rho\omega_n^4}{2k}$, this equation becomes (39) because

$$
\frac{\rho\omega_n^4}{2k} \frac{\partial^2 \lambda_n}{\partial\theta_k \partial\theta_j} = 3 \frac{\partial\omega_n}{\partial\theta_k} \frac{\partial\omega_n}{\partial\theta_j} - \omega_n \frac{\partial^2 \omega_n}{\partial\theta_k \partial\theta_j} = 12\pi^2 \mathcal{V}_k \mathcal{V}_j - \omega_n \frac{\partial^2 \omega_n}{\partial\theta_k \partial\theta_j}.
$$

Acknowledgments. Part of this research has been supported by the Marie Curie MULTIMAT "Multi-scale, Modelling and Characterisation for Phase

Transformations in Advanced Materials" Research Training Network under the contract number MRTN-CT-2004-505226.

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