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R.I. 630 *January 2008*

Exit problems related to the persistence of solitons for the Korteweg-de Vries equation with small noise

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January 25, 2008

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Abstract

We consider two exit problems for the Korteweg-de Vries equation perturbed by an additive white in time and colored in space noise of amplitude ϵ. The initial datum gives rise to a soliton when $\epsilon = 0$. It has been proved recently that the solution remains in a neighborhood of a randomly modulated soliton for times at least of the order of ϵ^{-2}. We prove exponential upper and lower bounds for the small noise limit of the probability that the exit time from a neighborhood of this randomly modulated soliton is less than T, of the same order in ϵ and T. We obtain that the time scale is exactly the right one. We also study the similar probability for the exit from a neighborhood of the deterministic soliton solution. We are able to quantify the gain of eliminating the secular modes to better describe the persistence of the soliton.

Key Words: Stochastic partial differential equations, Korteweg-de Vries equation, soliton, large deviations.

AMS 2000 Subject Classification: 35Q53, 60F10, 60H15, 76B25, 76B35.

1 Introduction

The Korteweg-de Vries (KdV) equation is a model for the evolution of weakly nonlinear, shallow water, unidirectional long waves. It is of the form

$$\partial_t u + \partial_x^3 u + \partial_x(u^2) = 0 \tag{1.1}$$

where the space variable x is in $\mathbb{R}$. The results of this paper could easily be extended to generalized subcritical KdV equations for which the nonlinearity is $\partial_x(u^p)$ for $p < 5$, but we consider throughout this article the $p = 2$ case for simplicity. The KdV equation is famous for its soliton solutions confirming the observation of the solitary wave propagating on a channel by Russell in 1844. These solitons are traveling waves of the form $u_{c,x_0}(t,x) = \varphi_c(x-ct+x_0)$ where c is the constant velocity, $x_0 \in \mathbb{R}$ is the initial phase and

$$\varphi_c(x) = \frac{3c}{2\cosh^2(\sqrt{c}x/2)} \tag{1.2}$$

These waves are localized, *i.e.* they decay exponentially to zero as x goes to infinity. Their shape is stable against perturbations of the initial state. A first notion of stability, for initial data close to φ_c, which takes into account the symmetries of the evolution equation, is that of orbital stability. This notion of stability was first considered, for the solution $\varphi_c(x - ct + x_0)$ of the KdV equation, by Benjamin [1]. The set $\{\varphi_c(\cdot - s),\ s \in \mathbb{R}\}$ is the orbit of φ_c. The functional $Q_c(u) = \mathbf{H}(u)+c\mathbf{M}(u)$ is used as a Lyapunov functional in the proof. It involves two important invariant quantities of the evolution equation (1.1): the Hamiltonian, defined for u in $\mathrm{H}^1(\mathbb{R})$, the space of square integrable functions with square integrable first order derivatives, by

$$\mathbf{H}(u) = \frac{1}{2}\int_{\mathbb{R}} (\partial_x u(x))^2\, dx - \frac{1}{3}\int_{\mathbb{R}} u^3(x)dx \tag{1.3}$$

and the mass defined by

$$\mathbf{M}(u) = \frac{1}{2}\int_{\mathbb{R}} u^2(x)dx. \tag{1.4}$$

The space $\mathrm{H}^1(\mathbb{R})$ is the energy space, and it is a natural space for the solutions of (1.1) : indeed, if $u \in C([0,T];\mathrm{H}^1(\mathbb{R}))$ is a solution of (1.1), then $\mathbf{H}(u(t)) = \mathbf{H}(u_0)$ and $\mathbf{M}(u(t)) = \mathbf{M}(u_0)$ for any $t \in [0,T]$, where u_0 is the initial datum in H^1. The shape of the soliton φ_c is a solution of the constrained variational problem which consists in minimizing the Hamiltonian for a constant mass. Orbital stability means that when the initial datum is close to φ_c in H^1 then the solution remains close to the orbit of φ_c. The second stronger notion of stability is that of asymptotic stability. It states that, for initial datum close in H^1 to φ_c, the solution converges in some sense as time goes to infinity to a soliton where the velocity and phase have been shifted. Convergence may correspond to weak convergence in H^1, see [15] or [16], or strong convergence in some weighted Sobolev space, see [17], for less general perturbations of the initial datum, when the solution is written in the soliton reference frame. Note that strong convergence in H^1 is not expected due to the possibility of a dispersive tail moving away from the soliton as time goes to infinity.

It is often physically relevant to consider random perturbations of equation (1.1), see [19]. It is also interesting from a theoretical perspective to study the

stability of the soliton shape under these random perturbations. We consider as in [19] the case of an additive noise which could model a random pressure at the surface of the water. The corresponding stochastic partial differential equation (SPDE for short) written in Itô form is the following

$$du + \left(\partial_x^3 u + \partial_x(u^2)\right) dt = \epsilon dW \tag{1.5}$$

where $(W(t))_{t\geq 0}$ is a Wiener process and ϵ is the small noise amplitude. As we work in infinite dimensions, and in the absence of global smoothing property of the group $S(t)$ on H^1 associated to the unbounded operator $-\partial_x^3$, W needs to be a proper Wiener process on H^1. Thus, the components of $W(1)$ need to be correlated for the law of $W(1)$ to be a *bona-fide* Gaussian measure. It can then always be seen as the direct image via a Hilbert-Schmidt self-adjoint mapping Φ of a cylindrical measure and we assume that Φ is a mapping from L^2 into H^1. Recall that Φ is Hilbert-Schmidt from L^2 into H^1 if it is a bounded linear operator and for a complete orthonormal system $(e_i)_{i\in\mathbb{N}}$ of L^2, $\sum_{i\in\mathbb{N}} \|\Phi e_i\|_{\mathrm{H}^1}^2$ is finite. The sum does not depend on the complete orthonormal system and, endowed with its square root as a norm, the space of Hilbert-Schmidt operators $\mathcal{L}_2(\mathrm{L}^2, \mathrm{H}^1) = \mathcal{L}_2^{0,1}$ is a Hilbert space. As a consequence, the Wiener process could be written as

$$W(t,x) = \sum_{i\in\mathbb{N}} \beta_i(t)\Phi e_i(x), \quad t \geq 0,\ x \in \mathbb{R} \tag{1.6}$$

where $(\beta_i)_{i\in\mathbb{N}}$ is a collection of independent standard real valued Brownian motions and $(e_i)_{i\in\mathbb{N}}$ a complete orthonormal system of L^2. Existence of path-wise mild solutions, almost surely continuous in time for all t positive with values in H^1, of the SPDE supplemented with the initial datum $u(0) = u_0 \in \mathrm{H}^1$ and uniqueness among those having almost surely paths in some subspace $X_T \subset \mathrm{C}\left([0,T];\mathrm{H}^1\right)$ has been obtained in [3]. In [6, 7] global well posedness is obtained for rougher noises and less regular solutions. It should be noticed that in the physics literature the space-time white noise, corresponding to $\Phi = I$ is often considered, which we are not able to treat mathematically. For simplicity, we consider the sequence of operators

$$\Phi_n = \left(I - \Delta + \frac{1}{n}(x^2 I - \Delta)^k\right)^{-1/2}, \tag{1.7}$$

which are Hilbert-Schmidt from L^2 into H^1 for k large enough, in order to prove lower bounds on exit times. As n goes to infinity the Hilbert-Schmidt assumption tends to be relaxed and the noise mimics a spatially homogeneous noise with covariance $(I-\Delta)^{-1}$, which is a white noise in the Hilbert space H^1. It should also be noted that these operators are uniformly bounded in the space $\mathcal{L}^{0,1}$ of bounded operators from L^2 into H^1; indeed we have for every integer n, $\|\Phi_n\|_{\mathcal{L}^{0,1}} \leq 1$. It is possible to work with a more general approximating sequence, see for example the kind of assumptions made in [8].

The linearized operator around the soliton is particularly interesting to study the stability. It has a general null-space spanned by the two secular modes $\partial_x \varphi_c$ and $\partial_c \varphi_c$. These modes are associated with infinitesimal changes in the velocity and location of the solitary wave. In Remark 2.3 in [5], the following heuristic argument then implies that the random solution should at most remain close to the deterministic solution up to times of the order $\epsilon^{-2/3}$. It is based on an analogy with the behavior of a linear system of SDEs such that 0 is a degenerate simple eigenvalue corresponding to a Jordan block

$$\begin{cases} dX_1 = X_2 dt + \epsilon dW_1(t) \\ dX_2 = \epsilon dW_2(t) \end{cases}$$

with Brownian motions W_1 and W_2. In such a case,

$$X_1(T) = \epsilon \int_0^T W_2(s) ds + \epsilon W_1(T)$$

has variance of the order of $\epsilon^2 T^3$ for large T. Thus, for a first approximation of the solution $u^{\epsilon,\varphi_{c_0}}$ of (1.5) with initial datum φ_{c_0}, of the form

$$u^{\epsilon,\varphi_{c_0}}(t,x) = \varphi_{c_0}(x - c_0 t) + \epsilon \tilde{\eta}^\epsilon(t, x - c_0 t), \tag{1.8}$$

and with an exit time defined for a neighborhood of the soliton in H^1

$$B(\varphi_{c_0}, \alpha) = \left\{ f \in \mathrm{H}^1 : \ \|f - \varphi_{c_0}\|_{\mathrm{H}^1} < \alpha \right\}$$

by

$$\tilde{\tau}_\alpha^\epsilon = \inf \left\{ t \in [0,\infty) : \ u^{\epsilon,\varphi_{c_0}}(t, \cdot + c_0 t) \in B(\varphi_{c_0}, \alpha)^c \right\},$$

exit is expected to occur on a time scale of the order of $\epsilon^{-2/3}$. However, it is believed that the soliton shape is preserved over a longer time scale. A general approach which works for the deterministic equation, see [15, 17], is to introduce a description by a soliton ansatz where the parameters of the soliton fluctuate with time. In the case of an additive noise physicists use an approximation of the solution by a soliton ansatz of the form $\varphi_{c^\epsilon(t)}(x - x^\epsilon(t))$ where $c^\epsilon(t)$ and $x^\epsilon(t)$ are random scalar processes evolving according to a system of coupled SDEs. In the case where the noise is the time derivative of a one dimensional standard Brownian motion, it is easily seen (see [20]) that the solution can be written as a modulated soliton plus a Brownian motion. However, proving such a result is more involved when the noise is a function of space as well. A mathematical justification is given in [5] where the following result is proved.

Theorem 1.1 *For $\epsilon > 0$ and $c_0 > 0$, there exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0]$ there exists a stopping time $\tau_\alpha^\epsilon > 0$ a.s. and semi-martingales $c^\epsilon(t)$ and $x^\epsilon(t)$ defined a.s. for $t \le \tau_\alpha^\epsilon$ with values in $(0,\infty)$ and $\mathbb{R}$ respectively such that if we set*

$$\epsilon \eta^\epsilon(t) = u^{\epsilon,\varphi_{c_0}}(t, \cdot + x^\epsilon(t)) - \varphi_{c^\epsilon(t)}$$

then

$$\int_{\mathbb{R}} \eta^\epsilon(t,x) \varphi_{c_0}(x) dx = (\eta^\epsilon, \varphi_{c_0}) = 0, \quad \forall t \le \tau_\alpha^\epsilon \quad a.s., \tag{1.9}$$

$$\int_{\mathbb{R}} \eta^\epsilon(t,x)\partial_x\varphi_{c_0}(x)dx = (\eta^\epsilon, \partial_x\varphi_{c_0}) = 0, \quad \forall t \le \tau_\alpha^\epsilon \quad a.s. \tag{1.10}$$

and for all $t \le \tau_\alpha^\epsilon$,

$$\|\epsilon\eta^\epsilon(t)\|_{\mathrm{H}^1} \le \alpha \tag{1.11}$$

and

$$|c^\epsilon(t) - c_0| \le \alpha. \tag{1.12}$$

Moreover, there exists $C > 0$ *such that for all* $T > 0$ *and* $\alpha \le \alpha_0$ *there exists* $\epsilon_0 > 0$ *such that for all* $\epsilon < \epsilon_0$,

$$\mathbb{P}\left(\tau_\alpha^\epsilon \le T\right) \le \frac{C\epsilon^2 T\|\Phi\|_{\mathcal{L}_2^{0,1}}}{\alpha^4}. \tag{1.13}$$

The proof uses the Lyapunov functional Q_c as a central tool. The equations (1.9) and (1.10) are such that restricted to this subspace, the Lyapunov functional is coercive, *i.e.* the operator Q''_{c_0} is positive. It allows to keep $|c^\epsilon(t)-c_0|$ and $\|\epsilon\eta^\epsilon(t)\|_{\mathrm{H}^1}$ small on a longer time interval. Also, these two conditions together with the implicit function theorem allow to obtain c^ϵ and x^ϵ. Other results in [5] give the asymptotic distribution of η^ϵ as ϵ goes to zero as well as coupled equations for the evolution of the random scalar parameters. The parameters and remainder do not depend on $\alpha \le \alpha_0$. In the upper bound (1.13) the product $\epsilon^2 T$ appears. The theorem says that the solution stays in a neighborhood of the randomly modulated soliton $\varphi_{c^\epsilon(t)}(x - x^\epsilon(t))$ with high probability at least for times small compared to ϵ^{-2}. The time spent in a neighborhood of a soliton-like wave, when the initial datum gives rise to a soliton for $\epsilon = 0$, is called the persistence time, see [9] for numerical confirmations that the above order is the right order.

In this paper we first study the exit time $\tilde{\tau}_\alpha^\epsilon$ and obtain that the time scale on which the solution stays close to the deterministic soliton is indeed at most of the order of $\epsilon^{-2/3}$. We then revisit the upper bound (1.13) and prove a sharper exponential bound. This bound is supplemented with an exponential lower bound of the same order in the parameters T and ϵ. We thus obtain the right order of the cumulative distribution function (CDF) of the exit time off neighborhoods of the randomly modulated soliton. This gives a confirmation that the time scale on which the approximation of the solution by a randomly modulated soliton is valid is of the order ϵ^{-2}. Our main tools are large deviations along with a study of the associated variational problems. Similarly, factors T and T^3 have also been obtained in the study of the tails of the mass and arrival time for stochastic nonlinear Schrödinger equations (NLS) in [8, 13] with the same techniques. These quantities are the main processes impairing soliton transmission in optical fibers. In that setting, physicists again use the approximation by a randomly modulated soliton. An analogue of Theorem 1.1 for stochastic NLS equations would allow to tell up to what length of the fiber line the approximation is licit. Large deviations are also known to be a useful

tool to study the exit problem from an asymptotic equilibrium point or noised induced transition between several equilibrium points in the small noise limit (see [12], and [14] for an exit problem for stochastic weakly damped nonlinear Schrödinger equation). Here we however study a simpler problem than the escape from the asymptotically stable central manifold for the KdV equation which we hope to study in future works.

2 Large deviations and escape from a neighborhood of the soliton

We use sample path large deviations in this article in order to obtain lower bounds of the asymptotic as n goes to infinity and ϵ goes to zero of probabilities $\mathbb{P}\left(\tau^{n,\epsilon} \leq T\right)$ where $\tau^{n,\epsilon}$ is the exit time of a neighborhood of either the deterministic soliton or the randomly modulated soliton. The n recalls that we consider a sequence of operators Φ_n, see (1.7). Large deviation techniques, see for example [10, 11], allow to quantify convergence to zero of rare events. For example, it is easy to check that on a finite time interval $[0,T]$, the paths of the solution of 1.5 starting form φ_{c_0} converge in probability to the paths of the deterministic soliton solution. The probability that exit from a neighborhood of the soliton occurs before T goes to zero as ϵ goes to zero. Large deviations quantify the convergence to zero of such probabilities. Following Varadhan's formalism, large deviations could be stated as a sequence of inequalities called a large deviation principle (LDP for short). The convergence to zero of the logarithm of the probabilities of rare events is characterized by a speed, here ϵ^2, and a deterministic functional I^n depending on the operator Φ_n considered, called rate function, to be minimized on the closure and interior of the set defining the rare event in the state space. In the small noise asymptotics and for sample path large deviations, the rate function could be expressed in terms of the mild solution of the control problem

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^2) = \Phi_n h, \\ u(0) = \varphi_{c_0} \text{ and } h \in \mathrm{L}^2\left(0,T;\mathrm{L}^2\right). \end{cases} \tag{2.1}$$

We denote the solution by $\mathbf{S}^{n,\varphi_{c_0}}(h)$. The mapping $h \to \mathbf{S}^{n,\varphi_{c_0}}(h)$ is called the control map and (2.1) the control equation. We recall that a rate function I on the sample space (here the paths space $\mathrm{C}([0,T];\mathrm{H}^1)$) is lower semicontinuous and that a good rate function is such that $I^{-1}([0,R])$ is compact for every R positive.

Theorem 2.1 *The laws $\left(\mu^{u^{n,\epsilon,\varphi_{c_0}}}\right)_{\epsilon>0}$ of the paths of the solutions of (1.5) for the operator $\Phi = \Phi_n$ on $\mathrm{C}([0,T];\mathrm{H}^1)$ with initial datum φ_{c_0} satisfy a LDP of speed ϵ^2 and good rate function*

$$I^n(w) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2):\ w = \mathbf{S}^{n,\varphi_{c_0}}(h)} \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.$$

It means that for every Borel set B of $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$, we have the lower bound

$$-\inf_{w\,\in\,\overset{\circ}{B}} I^n(w) \leq \underline{\lim}_{\epsilon\to 0}\epsilon^2 \log \mathbb{P}\left(u^{n,\epsilon,\varphi_{c_0}} \in B\right)$$

and the upper bound

$$\overline{\lim}_{\epsilon\to 0}\epsilon^2 \log \mathbb{P}\left(u^{n,\epsilon,\varphi_{c_0}} \in B\right) \leq -\inf_{w\in\overline{B}} I^n(w).$$

The proof uses the LDP for the laws of the stochastic convolution ϵZ where $Z(t) = \int_0^t S(t-s)dW(s)$ on the Banach path space X_T; it is a subspace of $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$ where the fixed point argument proving the local well-posedness is used, see [3]. The stochastic convolution appears when we write the equation satisfied by the mild solution of (1.5). These laws are Gaussian measures and the LDP is a consequence of the general result on LDP for centered Gaussian measures on real Banach spaces, see [11]. The second step is to prove the continuity of the mapping which, to the perturbation Z in X_T assigns the solution $u^{n,1,\varphi_{c_0}} := \mathcal{G}(Z)$ in $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$. It is obtained noting that $\mathcal{G}(Z) = v(Z) + Z$ where $v(x)$ denotes the solution of

$$\begin{cases} \partial_t v + \partial_x^3 v + \partial_x\left((v+Z)^2\right) = 0 \\ v(0) = \varphi_{c_0}. \end{cases}$$

Then the continuity of $\mathcal{G}$ is a consequence of the continuity of v with respect to the perturbation Z. It could be proved as in [4] where the stochastic NLS equation is considered. LDP for the paths of the mild solution of the SPDE is then obtained by a direct application of the contraction principle which states that we can push forward LDP for measures on a Hausdorff topological space to a LDP for direct image measures on another Hausdorff topological space when the mapping is continuous. More details on the proof of such LDP are given in [13] where the stochastic NLS equation with additive noise is considered. The control map $\mathbf{S}^{H,\varphi_{c_0}}(h)$ for the theoretical stochastic equation with spatially homogeneous noise is defined as the mild solution of

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^2) = \Phi_H h, \\ u(0) = \varphi_{c_0} \text{ and } h \in \mathrm{L}^2\left(0,T;\mathrm{L}^2\right), \end{cases} \tag{2.2}$$

where $\Phi_H = (I-\Delta)^{-1/2}$. Note that though we cannot give a mathematical meaning to the stochastic equation with such homogeneous noise, the corresponding control map is well defined.

Let us now consider the exit times $\tilde{\tau}_\alpha^{n,\epsilon}$. Note that we only consider here a lower bound of the probability since it is enough to prove the heuristic of Remark 2.3 in [5]. Recall that we want to check that the time scale on which the approximation by the deterministic soliton is licit is at most of the order of $\epsilon^{-2/3}$. When studying the exit times $\tau_\alpha^{n,\epsilon}$, however, we give both upper and lower bounds of the tail probabilities of the same order in the parameters ϵ and T.

Proposition 2.2 *Take T, c_0 positive; then for α_0 small enough, for every $\alpha < \alpha_0$, there exists a constant $C(\alpha, c_0)$ which depends on α and c_0 but not on T such that*

$$\underline{\lim}_{n\to\infty}\underline{\lim}_{\epsilon\to 0}\epsilon^2 \log \mathbb{P}\left(\tilde{\tau}_\alpha^{n,\epsilon} \leq T\right) \geq -\frac{C(\alpha, c_0)}{T^3}.$$

Proof. For fixed n, using Theorem 2.1, $\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\tilde{\tau}_\alpha^{n,\epsilon} \leq T\right)$ is larger than

$$-\frac{1}{2}\inf\left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)},\ h:\ \|\mathbf{S}^{n,\varphi_{c_0}}(h)(T) - \varphi_{c_0}(\cdot - c_0 T)\|_{\mathrm{H}^1} > \alpha\right\}.$$

In a first step, we consider the preceding variational problem in which the operator Φ_n is replaced by the operator $\Phi_H = (I-\Delta)^{-1/2}$ and α is replaced by 2α. We give upper bounds on the infimum by minimizing on smaller and smaller sets of controls, until we are able to handle the variational problem. Note that up to now, the problem is a control problem for the KdV equation that we cannot handle. We will show that we can work on more restrictive classes of controls and still obtain a nice qualitative order. Using the Sobolev embedding of H^1 into L^∞, with norm C_∞, the infimum is found to be less than

$$\inf\left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)},\ h:\ \left\|\mathbf{S}^{H,\varphi_{c_0}}(h)(T) - \varphi_{c_0}(\cdot - c_0 T)\right\|_{\mathrm{L}^\infty} > 2C_\infty\alpha\right\}$$

since we then minimize on a smaller set. We consider controls h giving rise to modulated solitons of the form

$$\varphi_{c(t)}\left(x - \int_0^t c(s)\right)$$

in the homogeneous case. They are such that $c(0) = c_0$ and

$$\Phi_H h(t,x) = c'(t)\ \partial_c\varphi_c|_{c=c(t)}\left(x - \int_0^t c(s)ds\right),$$

since the soliton profile φ_c satisfies the equation

$$-c\partial_x\varphi_c + \partial_x^3\varphi_c + \partial_x(\varphi_c)^2 = 0.$$

Again, taking the infimum on a smaller set of controls, we obtain the lower bound

$$-\frac{1}{2}\inf\left\{\int_0^T \left\|c'(t)\Phi_H^{-1}\left[\partial_c\varphi_c|_{c=c(t)}\left(x - \int_0^t c(s)ds\right)\right]\right\|^2_{\mathrm{L}^2} dt,\right.$$
$$\left. c\in \mathrm{C}^1([0,T];(0,+\infty)):\ c(0) = c_0,\ \left|\varphi_{c(T)}\left(\int_0^T (c_0 - c(s))ds\right) - \frac{3c_0}{2}\right| > 2C_\infty\alpha\right\},$$

where we have bounded from below the L^∞ norm of the function by its value at c_0T. This is in turn bigger than

$$-\frac{1}{2}\inf\left\{\int_0^T (c'(t))^2\left\|\Phi_H^{-1}\left(\partial_c\varphi_c|_{c=c(t)}\right)\right\|^2_{\mathrm{L}^2} dt,\right.$$
$$\left. c\in \mathrm{C}^1([0,T];(0,+\infty)):\ c(0) = c_0,\ \frac{3c_0}{2} - \varphi_{c(T)}\left(\int_0^T (c_0 - c(s))ds\right) > 2C_\infty\alpha\right\},$$

due to the fact that Φ_H commutes with spatial translations. Let us fix $\alpha_0 < \frac{3c_0}{4C_\infty}$ so that $c_0 - \frac{4C_\infty \alpha}{3} > 0$ for $0 < \alpha < \alpha_0$. A sufficient condition for the constraint on the terminal value to hold is

$$c_0 - \frac{4C_\infty \alpha}{3} > \frac{4c(T)}{\exp\left(\sqrt{c(T)} \int_0^T (c_0 - c(s))ds\right)}.$$

Noticing that the function λ defined by $\lambda(x) = 4x^2 \exp\left(-x \int_0^T (c_0 - c(s))ds\right)$ attains its maximum at $x = 2/\int_0^T (c_0 - c(s))ds$ for $x \geq 0$, if $\int_0^T (c_0 - c(s))ds \geq 0$, we obtain that it is enough to have

$$\int_0^T (c_0 - c(s))ds > \frac{4}{e\sqrt{c_0 - 4C_\infty \alpha/3}} := \delta(c_0, \alpha).$$

As in [8] for the tails of the arrival time of a pulse driven by a stochastic nonlinear Schrödinger equation where we obtained the order $-CT^{-3}$, the boundary condition is in integrated form. The integral to be minimized is of the form

$$\int_0^T (c'(t))^2 \, g(c(t)) \, dt$$

where $g(c) = \|(I - \Delta)^{1/2} \partial_c \varphi_c\|_{\mathrm{L}^2}^2$. Instead of solving the problem of the calculus of variations with a nonstandard boundary condition, we make a guess and look for solutions of the form $c(t) = c_0 - 2\gamma t/T^2$ for some positive γ with $c_0 - 2\gamma/T > 0$. Note that if $\gamma = \inf\{\frac{3}{2}\delta(c_0, \alpha), \frac{c_0}{4}\}$, then the boundary condition is satisfied for $T \geq 1$. Also the term $g(c(t))$ in the integral is then such that there exists a constant $C(c_0)$ with

$$\int_0^T (c'(t))^2 \, g(c(t)) \, dt \leq C(c_0) \int_0^T (c'(t))^2 \, dt, \text{ for } T \geq 1,$$

since $c_0/2 \leq c(t) \leq c_0$ for any $T \in [0, T]$. Thus, for a new constant $C(\alpha, c_0)$, we obtain

$$\int_0^T (c'(t))^2 \, g(c(t)) \, dt \leq \frac{C(\alpha, c_0)}{T^3}, \text{ for } T \geq 1.$$

Let us now consider the case where the square root of the covariance operator of the noise is Φ_n, and let us start from the path $c(t)$ exhibited in the homogeneous noise case; we denote the corresponding control by

$$h_c(t, x) = c'(t) \, (I - \Delta)^{1/2} \left(\partial_c \varphi_c|_{c=c(t)} \left(\cdot - \int_0^t c(s)ds \right) \right).$$

Then, since for such h_c

$$\left\| \mathbf{S}^{H, \varphi_{c_0}}(h_c)(T) - \varphi_{c_0}(\cdot - c_0 T) \right\|_{\mathrm{H}^1} > 2\alpha,$$

we deduce from the continuity of the mild-solution of the control map with respect to the convolution of the semi-group with the control, used to prove the LDP, and the continuity of this last quantity with respect to the control, that for sufficiently large n

$$\begin{aligned}
&\|\mathbf{S}^{n,\varphi_{c_0}}(h_c)(T) - \varphi_{c_0}(\cdot - c_0 T)\|_{\mathrm{H}^1} \\
&= \left\| \mathbf{S}^{H,\varphi_{c_0}} \left((I-\Delta)^{1/2} (I - \Delta + \frac{1}{n}(x^2 I - \Delta)^k)^{-1/2} h_c \right)(T) - \varphi_{c_0}(\cdot - c_0 T) \right\|_{\mathrm{H}^1} \\
&> \alpha.
\end{aligned}$$

This ends the proof. □

As a consequence, the time scale on which an exit from a neighborhood of the soliton occurs is at most $\epsilon^{-2/3}$. In the next section we prove that the time scale on which the solution remains close to the randomly modulated soliton is exactly of the much longer order ϵ^{-2}. We provide upper and lower bounds for this result.

3 Escape from a neighborhood of the randomly modulated soliton

In the proof of Theorem 2.1 in [5] a local parametrization $u \mapsto (\mathcal{C}(u), \mathcal{X}(u))$ is used, in order to obtain parameters of the soliton wave form such that $u = \varphi_{\mathcal{C}(u)}(\cdot - \mathcal{X}(u)) + \mathcal{R}(u)$ with $\mathcal{R}$ satisfying some adequate orthogonality conditions. This parametrization is obtained using the implicit function theorem, imposing that the constraints (1.9) and (1.10) hold. Such a parametrization holds as long as u remains in a proper neighborhood of the spatial translates of φ_{c_0}; thus, setting $c^\epsilon(t) = \mathcal{C}(u^{\epsilon,\varphi_{c_0}}(t))$ and $x^\epsilon(t) = \mathcal{X}(u^{\epsilon,\varphi_{c_0}}(t))$, with $u^{\epsilon,\varphi_{c_0}}$ a solution of (1.5) with paths a.s. in $\mathrm{C}^1(\mathbb{R}^+; \mathrm{H}^1)$ and initial datum φ_{c_0}, the processes $c^\epsilon(t)$ and $x^\epsilon(t)$ are well defined adapted processes, up to a stopping time of the form

$$\overline{\tau}^\epsilon_\alpha = \inf \left\{ t \geq 0,\ |c^\epsilon(t) - c_0| \geq \alpha \text{ or } \|u^{\epsilon,\varphi_{c_0}}(t, \cdot + x^\epsilon(t)) - \varphi_{c_0}\|_{\mathrm{H}^1} \geq \alpha \right\}.$$

We can indeed always replace a function u by the function $u(\cdot + \mathcal{X}(u))$ and come back to the case where $x^\epsilon(t)$ is close to 0, and it is not necessary to include a condition of the form $|x^\epsilon(t)| \geq \alpha$. It is also shown that this stopping time could be bounded above and below by

$$\tau^\epsilon_{C\alpha} = \inf \left\{ t \geq 0,\ |c^\epsilon(t) - c_0| \geq \alpha \text{ or } \|u^{\epsilon,\varphi_{c_0}}(t, \cdot + x^\epsilon(t)) - \varphi_{c^\epsilon(t)}\|_{\mathrm{H}^1} \geq \alpha \right\}$$

for some constants C depending solely on c_0 and α_0, with $0 < \alpha \leq \alpha_0$. Hence, the two stopping times are equivalent and the qualitative behavior of $\mathbb{P}(\tau^\epsilon_\alpha \leq T)$ with respect to ϵ and T should be the same as the original $\mathbb{P}(\overline{\tau}^\epsilon_\alpha \leq T)$. These new stopping times τ^ϵ_α prove to be more convenient to work with. Note that the implicit function theorem says that each function in H^1 sufficiently close to

a translate of the soliton could be written as a translated soliton with slightly different parameters plus a remainder in the subspace of H^1 orthogonal in L^2 to φ_{c_0} and $\partial_x\varphi_{c_0}$. Therefore the exit in terms of the H^1 norm is the exit of a proper open subset.

Let us first revisit the upper bound given in [5] and prove that the upper bound is indeed exponential. The proof relies on exponential tail estimates. We denote by $\mathcal{L}^{-1,0}$ and $\mathcal{L}^{0,1}$ the spaces of bounded operators from H^{-1} to L^2 (respectively from L^2 to H^1).

Proposition 3.1 *For $T > 0$ and $0 < \alpha \leq \alpha_0$ fixed, there exists a constant $C(\alpha, c_0)$, depending on α and c_0 but not on T, and $\epsilon_0 > 0$ with $\epsilon_0^2 T$ sufficiently small depending on $\|\Phi\|_{\mathcal{L}_2^{0,1}}$ and α, such that for every positive $\epsilon < \epsilon_0$,*

$$\mathbb{P}\left(\tau_\alpha^\epsilon \leq T\right) \leq \exp\left(-\frac{C(\alpha, c_0)}{\epsilon^2 T \|\Phi\|^2_{\mathcal{L}^{0,1}}}\right). \tag{3.1}$$

Proof. Fix T positive. The estimate (1.13) relies on the two following inequalities. Let $\tau = \tau_\alpha^\epsilon \wedge T$; then for α_0 sufficiently small there exists a positive constant C independent of T such that

$$|c^\epsilon(\tau) - c_0|^2 \leq C\left[\|\epsilon\eta^\epsilon(\tau)\|^4_{\mathrm{L}^2} + 4\epsilon^2\left|\int_0^\tau (u^{\epsilon,\varphi_{c_0}}(s), dW(s))_{\mathrm{L}^2}\right|^2 + \epsilon^4\tau^2\|\Phi\|^4_{\mathcal{L}_2^{0,1}}\right]$$

and

$$\begin{aligned}\|\epsilon\eta^\epsilon(\tau)\|^2_{\mathrm{H}^1} \leq & C\left[\|\epsilon\eta^\epsilon(\tau)\|^4_{\mathrm{L}^2} + 4\epsilon^2\left|\int_0^\tau (u^{\epsilon,\varphi_{c_0}}(s), dW(s))_{\mathrm{L}^2}\right|^2 + \epsilon^4\tau^2\|\Phi\|^4_{\mathcal{L}_2^{0,1}}\right.\\ & + \epsilon\int_0^\tau (\partial_x u^{\epsilon,\varphi_{c_0}}(s), \partial_x dW(s))_{\mathrm{L}^2} - \epsilon\int_0^\tau \left((u^{\epsilon,\varphi_{c_0}}(s))^2, dW(s)\right)_{\mathrm{L}^2}\\ & + c_0\epsilon\int_0^\tau (u^{\epsilon,\varphi_{c_0}}(s), dW(s))_{\mathrm{L}^2} + \frac{\epsilon^2}{2}\tau\|\Phi\|^2_{\mathcal{L}_2^{0,1}}\\ & \left.+\epsilon^2\|\Phi\|^2_{\mathcal{L}_2^{0,1}}\int_0^\tau \|u^{\epsilon,\varphi_{c_0}}(s)\|_{\mathrm{L}^2} ds + c_0\frac{\epsilon^2}{2}\tau\|\Phi\|^2_{\mathcal{L}_2^{0,1}}\right].\end{aligned}$$

These are obtained from several manipulations of the Lyapunov functional and evolution equations for the mass and Hamiltonian evaluated on the solution of the stochastic KdV equation (see [5]). The evolution of these quantities is obtained using the Itô formula and a smoothing procedure. We do not reproduce the proof here. We may also write

$$\begin{aligned}\mathbb{P}\left(\tau_\alpha^\epsilon \leq T\right) &\leq \mathbb{P}\left(|c^\epsilon(\tau) - c_0|^2 \geq \alpha^2 \text{ or } \|\epsilon\eta^\epsilon(\tau)\|^2_{\mathrm{H}^1} \geq \alpha^2\right)\\ &\leq \mathbb{P}\left(|c^\epsilon(\tau) - c_0|^2 \geq \alpha^2\right) + \mathbb{P}\left(\|\epsilon\eta^\epsilon(\tau)\|^2_{\mathrm{H}^1} \geq \alpha^2\right).\end{aligned}$$

Note that when $|c^\epsilon(\tau) - c_0|^2 \geq \alpha^2$ we have $|c^\epsilon(\tau) - c_0|^2 = \alpha^2$ and $\|\epsilon\eta^\epsilon(\tau)\|_{\mathrm{L}^2} \leq \alpha$. Thus for ϵ_0 sufficiently small, depending on $\|\Phi\|_{\mathcal{L}_2^{0,1}}$, T (so that $\epsilon_0^2 T$ is small)

and α, and for $\epsilon < \epsilon_0$,

$$\mathbb{P}\left(|c^\epsilon(\tau) - c_0|^2 \geq \alpha^2\right) \leq \mathbb{P}\left(\epsilon \left|\int_0^\tau \left(u^{\epsilon,\varphi_{c_0}}(s), dW(s)\right)_{\mathrm{L}^2}\right| \geq \frac{\alpha}{4}\right)$$
$$\leq \mathbb{P}\left(\epsilon \sup_{t\in[0,T]} \left|\int_0^t \left(u^{\epsilon,\varphi_{c_0},\tau_\alpha^\epsilon}(s), dW(s)\right)_{\mathrm{L}^2}\right| \geq \frac{\alpha}{4}\right). \tag{3.2}$$

where $u^{\epsilon,\varphi_{c_0},\tau_\alpha^\epsilon}$ is the process stopped at time τ_α^ϵ. Similarly, using as well the following property

(P) When $|c^\epsilon(\tau) - c_0| \leq \alpha \leq \alpha_0$, then for some C depending only on c_0 and α_0, $\|\varphi_{c^\epsilon(t)} - \varphi_{c_0}\|_{\mathrm{H}^1} \leq C\alpha$, for all $t \leq \tau$,

we obtain that there exists ϵ_0 sufficiently small, depending on $\|\Phi\|_{\mathcal{L}_2^{0,1}}$, T (with $\epsilon_0^2 T$ small) and α, such that for all $\epsilon < \epsilon_0$,

$$\begin{aligned}
&\mathbb{P}\left(\|\epsilon\eta^\epsilon(\tau)\|_{\mathrm{H}^1}^2 \geq \alpha^2\right)\\
\leq&\mathbb{P}\left(\epsilon \sup_{t\in[0,T]} \left|\int_0^t \left(u^{\epsilon,\varphi_{c_0},\tau_\alpha^\epsilon}(s), dW(s)\right)_{\mathrm{L}^2}\right| \geq \frac{1}{2}\sqrt{\frac{\alpha^2}{10}}\right)\\
+&\mathbb{P}\left(\epsilon \sup_{t\in[0,T]} \left|\int_0^t \left(\partial_x u^{\epsilon,\varphi_{c_0}\tau_\alpha^\epsilon}(s), \partial_x dW(s)\right)_{\mathrm{L}^2}\right| \geq \frac{\alpha^2}{10}\right)\\
+&\mathbb{P}\left(\epsilon \sup_{t\in[0,T]} \left|\int_0^t \left(\left(u^{\epsilon,\varphi_{c_0}\tau_\alpha^\epsilon}(s)\right)^2, dW(s)\right)_{\mathrm{L}^2}\right| \geq \frac{\alpha^2}{10}\right)\\
+&\mathbb{P}\left(\epsilon \sup_{t\in[0,T]} \left|\int_0^t \left(u^{\epsilon,\varphi_{c_0},\tau_\alpha^\epsilon}(s), dW(s)\right)_{\mathrm{L}^2}\right| \geq \frac{\alpha^2}{10c_0}\right).
\end{aligned} \tag{3.3}$$

Let us denote by $Z_i(t)$ for $i = \{1, 2, 3\}$ the stochastic integrals arising in the right hand sides of (3.2) and (3.3). We obtain exponential tail estimates for each of the above probabilities in the usual way, see for example [18], Theorem 2.1. We introduce the function $f_l(x) = \sqrt{1 + lx^2}$, where l is a positive parameter. We then apply the Itô formula to $f_l(Z_i(t))$ and each process decomposes into $1 + E_{l,i}(t) + R_{l,i}(t)$ where

$$E_{l,i}(t) = \int_0^t \frac{lZ_i(t)}{\sqrt{1 + lZ_i(t)^2}} dZ_i(t) - \frac{1}{2}\int_0^t \left(\frac{lZ_i(t)}{\sqrt{1 + lZ_i(t)^2}}\right)^2 d < Z_i >_t,$$

and

$$R_{l,i}(t) = \frac{1}{2}\int_0^t \left(\frac{lZ_i(t)}{\sqrt{1 + lZ_i(t)^2}}\right)^2 d < Z_i >_t + \frac{1}{2}\int_0^t \frac{d < Z_i >_t}{(1 + lZ_i(t)^2)^{3/2}}.$$

Let us for example consider Z_2. Given $(e_j)_{j\in\mathbb{N}}$ a complete orthonormal system of L^2,

$$< Z_2 >_t = \int_0^t \sum_{j\in\mathbb{N}} \left(\partial_x u^{\epsilon,\varphi_{c_0}\tau_\alpha^\epsilon}, \partial_x \Phi e_j\right)^2_{\mathrm{L}^2}(s)ds;$$

thus, using the Hölder inequality and the property (P) we have, for some constant $C(\alpha, c_0) = C\alpha^2 + \|\partial_x \varphi_{c_0}\|_{\mathrm{H}^1}$, and any t,

$$< Z_2 >_t \leq C(\alpha, c_0)\|\Phi^*\|^2_{\mathcal{L}^{-1,0}} t.$$

Then

$$|R_{l,2}(t)| \leq lC(\alpha, c_0)\|\Phi\|^2_{\mathcal{L}^{0,1}} T.$$

The same bound also holds for Z_1 and Z_3. We may thus write for any i and constants $\delta_i > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]} |Z_i(t)| \geq \frac{\delta_i}{\epsilon}\right) \\
&= \mathbb{P}\left(\sup_{t\in[0,T]} \exp\left(f_l(Z_i(t))\right) \geq \exp\left(f_l\left(\frac{\delta_i}{\epsilon}\right)\right)\right) \\
&\leq \mathbb{P}\left(\sup_{t\in[0,T]} \exp\left(E_{l,i}(t)\right) \geq \exp\left(f_l\left(\frac{\delta_i}{\epsilon}\right) - 1 - lC(\alpha, c_0)\|\Phi\|^2_{\mathcal{L}^{0,1}} T\right)\right).
\end{aligned}$$

The Novikov condition is also satisfied and $E_{l,i}(t)$ is such that $(\exp(E_{l,i}(t)))_{t\geq 0}$ is a uniformly integrable martingale. The Doob inequality then gives

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]} \exp\left(E_{l,i}(t)\right) \geq \exp\left(f_l\left(\frac{\delta_i}{\epsilon}\right) - 1 - lC(\alpha, c_0)\|\Phi\|^2_{\mathcal{L}^{0,1}} T\right)\right) \\
&\leq \exp\left(-f_l\left(\frac{\delta_i}{\epsilon}\right) + 1 + lC(\alpha, c_0)\|\Phi\|^2_{\mathcal{L}^{0,1}} T\right) \mathbb{E}\left[\exp\left(E_{l,i}(T)\right)\right].
\end{aligned}$$

Since $\exp(E_{l,i}(T))$ is an exponential martingale $\mathbb{E}[\exp(E_{l,i}(T))] = \mathbb{E}[\exp(E_{l,i}(0))] = 1$. For ϵ_0 small enough we have for $\epsilon < \epsilon_0$

$$C(\alpha, c_0)\|\Phi\|^2_{\mathcal{L}^{0,1}} T < \frac{\delta_i^2}{2\epsilon^2}$$

which implies that the l-derivative at 0 of the function in the exponential bound is negative. Then, optimizing on the parameter l, we obtain the minimum in l of the upper bound, which has the form

$$\exp\left(1 - \frac{C(\alpha, c_0)}{\epsilon^2 T\|\Phi\|^2_{\mathcal{L}^{0,1}}}\right),$$

with possibly another constant $C(\alpha, c_0)$. Using the largest of all constants in the exponentials for each tail probabilities, and taking a constant slightly bigger and ϵ_0 smaller if necessary, the multiplicative constant in front of the exponential decay disappears and the result follows. □

Remark 3.2 *As will appear elsewhere, Theorem 1.1 also holds for a noise of multiplicative type, and an exponential upper bound holds as well.*

If we now consider the sequence of operators Φ_n mimicking the spatially homogeneous noise with covariance operator $(I-\Delta)^{-1}$, denoting the exit times $\tau_\alpha^{n,\epsilon}$, we obtain the following statement:

Corollary 3.3 *For $T>0$, $0<\alpha<\alpha_0$ and n fixed, there exists a constant $C(\alpha,c_0)$, depending on α and c_0, and there exists $\epsilon_0>0$ with $\epsilon_0^2 T$ sufficiently small with respect to $\|\Phi_n\|_{\mathcal{L}_2^{0,1}}$ and α, such that for every $\epsilon<\epsilon_0$,*

$$\mathbb{P}\left(\tau_\alpha^{n,\epsilon}\leq T\right)\leq \exp\left(-\frac{C(\alpha,c_0)}{\epsilon^2 T}\right). \tag{3.4}$$

In particular, we have the following double asymptotic result

$$\overline{\lim}_{n\to\infty}\overline{\lim}_{\epsilon\to 0}\epsilon^2\log\mathbb{P}\left(\tau_\alpha^{n,\epsilon}\leq T\right)\leq -\frac{C(\alpha,c_0)}{T}.$$

Let us now prove that when considering the sequence of operators Φ_n as above, one can obtain a lower bound of the same order in the parameters T and ϵ as the upper bound in Corollary 3.3. We make use of the approximation via a modulated soliton together with the LDP obtained in Theorem 2.1, and again minimize the rate function on a smaller set of controls giving rise to a set of parameterized exit paths.

Proposition 3.4 *For every T and α positive, there exists a constant $C(\alpha,c_0)$ which depends on c_0 and α but not on T, such that*

$$\underline{\lim}_{n\to\infty}\underline{\lim}_{\epsilon\to 0}\epsilon^2\log\mathbb{P}\left(\tau_\alpha^{n,\epsilon}\leq T\right)\geq -\frac{C(\alpha,c_0)}{T}.$$

Proof. Let us denote by $\mathcal{U}_{\alpha_0}$ the open set

$$\mathcal{U}_{\alpha_0}=\left\{\varphi_c(\cdot-y)+g,\ g\in \mathrm{H}^1:\ \|g\|_{\mathrm{H}^1}<\alpha_0,\ y\in\mathbb{R},\ |c-c_0|<\alpha_0\right\}.$$

We know from [5] that the velocity is obtained via a continuous mapping $\mathcal{C}$ from $\mathcal{U}_{\alpha_0}$ to $\mathbb{R}$ such that

$$c^\epsilon(t)=\mathcal{C}\left(u^{n,\epsilon,\varphi_{c_0}}(t)\right).$$

Also for fixed n, we write that for $0<2\alpha<\alpha_0$,

$$\mathbb{P}\left(\tau_\alpha^{n,\epsilon}\leq T\right)\geq \mathbb{P}\left(u^{n,\epsilon,\varphi_{c_0}}\in B\right)$$

where

$$B=\left\{u\in \mathrm{C}([0,T];\mathrm{H}^1):\ \forall t\in[0,T],\ u(t)\in\mathcal{U}_{\alpha_0},\ |\mathcal{C}\left(u(T)\right)-c_0|>\alpha\right\}.$$

Theorem 2.1 then leads to the following lower bound for $\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\tau_\alpha^{n,\epsilon}\leq T\right)$:

$$-\frac{1}{2}\inf\left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)},\ h:\ \forall t\in[0,T],\ \mathbf{S}^{n,\varphi_{c_0}}(h)(t)\in\mathcal{U}_{\alpha_0},\ |\mathcal{C}\left(\mathbf{S}^{n,\varphi_{c_0}}(h)(T)\right)-c_0|>\alpha\right\}. \tag{3.5}$$

Let us, as in the proof of Proposition 2.2, replace in a first step the above variational problem by a variational problem for $\Phi = \Phi_H = (I-\Delta)^{-1/2}$ and α by 2α. Minimizing on a smaller set, we obtain

$$-\frac{1}{2}\inf\left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)},\ h:\ \forall t\in[0,T],\ \mathbf{S}^{H,\varphi_{c_0}}(h)(t)\in\mathcal{U}_{\alpha_0},\ \mathcal{C}\left(\mathbf{S}^{H,\varphi_{c_0}}(h)(T)\right)-c_0=3\alpha\right\}.$$

We minimize on an even smaller set, considering solutions of the controlled equation which are modulated solitons of the form

$$\varphi_{c(t)}\left(x-\int_0^t c(s)\right)$$

where the one dimensional paths c are assumed to belong to $\mathrm{C}^1([0,T];(0,+\infty))$. The boundary conditions are thus that $c(0)=c_0$ and $c(T)=c_0+2\alpha$. A control h_c associated to such a solution is given by

$$\begin{aligned} h_c(t,x) &= \Phi_H^{-1}\left(\partial_t\mathbf{S}^{H,\varphi_{c_0}}(h_c)+\partial_x^3\mathbf{S}^{H,\varphi_{c_0}}(h_c)+\partial_x\left(\mathbf{S}^{H,\varphi_{c_0}}(h_c)^2\right)\right)(t,x)\\ &= c'(t)\ \Phi_H^{-1}\partial_c\varphi_c\big|_{c=c(t)}\left(x-\int_0^t c(s)ds\right). \end{aligned}$$

Note that this control is the same as in Proposition 2.2, only the terminal boundary condition changes. We thus obtain the lower bound

$$\begin{aligned} -\frac{1}{2}\inf\Bigg\{&\int_0^T\left\|c'(t)(I-\Delta)^{1/2}\ \partial_c\varphi_c|_{c=c(t)}\left(x-\int_0^t c(s)ds\right)\right\|^2_{\mathrm{L}^2}dt,\\ &c\in\mathrm{C}^1([0,T];(0,+\infty)):\ c(0)=c_0,\ c(T)=c_0+2\alpha\Bigg\}. \end{aligned}\tag{3.6}$$

We now have to solve a problem of the calculus of variations. Our aim is to find the optimal paths c among the set of constrained paths minimizing the path integral. The integral may be written, with the same function g as in Proposition 2.2, as

$$\int_0^T (c'(t))^2\, g\left(c(t)\right)dt.$$

Using successively the change of variables $t=Tu$ and the change of unknown function $v(u)=c(Tu)$, we obtain an upper bound of the form

$$\frac{1}{T}\int_0^1 (v'(u))^2\, g(v(u))du$$

for functions v which are $\mathrm{C}^1([0,T];(0,+\infty))$ and satisfy the two boundary conditions $v(0)=c_0$ and $v(1)=c_0+2\alpha$ independent of T. We recall that $g(c)=\|(I-\Delta)^{1/2}\partial_c\varphi_c\|^2_{\mathrm{L}^2}$ so that $s\mapsto g(v(s))$ is bounded on $[0,1]$ for any $v\in\mathrm{C}^1([0,T];(0,+\infty))$. Hence we deduce that the infimum in (3.6) is bounded above by $\frac{C(\alpha,c_0)}{T}$. Now, coming back to the case where Φ_n is the square root of the covariance operator of the noise, we start from a path c obtained from v

which say minimizes the objective function in the above problem of the calculus of variations, though following the above argument it does not really matter. Then, the control h_c is such that

$$\mathcal{C}\left(\mathbf{S}^{H,\varphi_{c_0}}(h_c)(T)\right) = c_0 + 2\alpha.$$

>From the continuity of the mild-solution of the control map with respect to the convolution of the semi-group with the control, used to prove the LDP, and the continuity of this last quantity with respect to the control, and using also the continuity of $\mathcal{C}$ with respect to $u \in \mathcal{U}_{\alpha_0}$, we know that for sufficiently large n

$$\begin{aligned}
&\mathcal{C}\left(\mathbf{S}^{n,\varphi_{c_0}}(h_c)(T)\right) \\
&= \mathcal{C}\left(\mathbf{S}^{H,\varphi_{c_0}}\left((I-\Delta)^{1/2}(I-\Delta+\frac{1}{n}(x^2I-\Delta)^k)^{-1/2}h_c\right)(T)\right) \\
&> \alpha.
\end{aligned}$$

We deduce that the inf-limit as n goes to infinity of the infimum in (3.5) is again bounded above by $\frac{C(\alpha,c_0)}{T}$ and this ends the proof of Proposition 3.4. □

As a consequence of our two bounds, the typical time scale on which the solution remains in the neighborhood of the modulated soliton is indeed $1/\epsilon^2$.

Acknowledgments. The authors are grateful to the Centro di Ricerca Matematica Ennio De Giorgi in Pisa for allowing us to participate to the spring 2006 research period when this research was initiated.

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