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Abstract

A limiting one-dimensional Poisson-Nernst-Planck (PNP) equations is considered, when the three-dimensional domain shrinks to a line segment, to describe the flows of positively and negatively charged ions through open ion channel. The new model comprises the usual drift diffusion terms and takes into account for each phase, the bulk velocity defined by (4) including the water bath for ions (see [14]). The existence of global weak solution to this problem is shown. The proof relies on the use of certain embedding theorem of weighted sobolev spaces together with Hardy inequality.

Key words: Poisson-Nernst-Planck system, ion channel, unbounded domain, weighted spaces
Mathematics subject classification (MSC 2000): 35K 55, 35K 65, 47H 10, 92B 05

1 Introduction

Ion channels are proteins embedded in the cell membranes that surround all living cells; one of the interesting properties of ionic channels is their selectivity to different ions, they conduct ions of one type much better than ions of another type, this ionic movement allows to conduct electrical signals down nerves. Ion channels control many important biological processes that involve rapid changes in cells such as the coordination of muscle's contraction including cardiac muscle.

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In this paper, we analyze a nonlinear electro-diffusion model for flows of two types of ions through open ion channel of biological membrane and in surrounding electrolyte bath. The unknowns are the electric potential $\phi = \phi(t, x)$, the concentration of positively charged ions $p = p(t, x)$ and the concentration of negatively charged ions $n = n(t, x)$. Since the bath and channel in practice always include water characterized by the concentration $w = w(t, x)$, if we suppose that the channel is saturated, we take

$$h_n n + h_p p + w = 1 \tag{1}$$

where h_n is the valence of negatively charged ions and h_p is the valence of positively charged ions.

To make the paper reasonably self-contained, we present briefly the derivation of the model following the PNP (Poisson-Nernst-Planck) theory as has been used many times in the literature (e.g; [1], [3], [5], [12]). More precisely, we adopt the model proposed by Giles Richardson in[14].

We suppose that the electric potential is governed by the Poisson's equation with a source term equal to the charge generated by the ions including the permanent charge along the interior wall of the channel $q = q(x)$

$$\nabla_X . (\varepsilon_0(X)\nabla_X \phi(t, X)) = h_n n(t, X) - h_p p(t, X) + q(X) \tag{2}$$

with electrical permittivity ε_0. Nernst-Planck equations are used to describe the migration and diffusion of ions which will be treated as a continuous charge distribution. The continuity equations for the two types of ions are

$$\frac{\partial n}{\partial t} + \nabla_X . J_n = 0 \text{ and } \frac{\partial p}{\partial t} + \nabla_X . J_p = 0. \tag{3}$$

The movement of ions depends on the electrical potential across the membrane, then the flux densities J_n and J_p are given by the following relations

$$J_n = k_n(\nabla_X n - n\nabla_X \phi) - nv \quad , \quad J_p = k_p(\nabla_X p + p\nabla_X \phi) - pv$$

where k_n and k_p are the diffusion coefficients of the respective ions and v is the bulk velocity given by

$$v = h_n(k_n - k_w)\nabla_X n + h_p(k_p - k_w)\nabla_X p + (h_p k_p p - h_n k_n n)\nabla_X \phi \tag{4}$$

where k_w is the diffusion coefficient of the water.

The problem (2)-(3) will be considered in $\mathbb{R}^+ \times \Omega$, where Ω is the domain occupied by the channel. We suppose that the channel can be modelled by

$$\Omega = \left\{(X = (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } 0 < x_1 < 1 \text{ and } \ x_2^2 + x_3^2 < A^2(x_1, \mu)\right\}$$

where A is a smooth function satisfying $A(x_1, 0) = 0$ and $\frac{\partial A}{\partial \mu}(x_1, 0) = g(x_1)$ and the parameter μ measures the maximal radius of the cross-section of the ionic channel. We follow the idea of [1] and [12] to derive a one dimensional approximation of (2)-(3). We set the following change of coordinates

$$x = x_1, \;\; y = \frac{x_2}{A(x_1, \mu)}, \;\; z = \frac{x_3}{A(x_1, \mu)}$$

when the radius of the cross-section μ approaches zero, the open channel form a nearly one dimensional path for electro-diffusion even if the electric field is not. To solve the problem we have to prescribe the boundary conditions at the ends of the channel but unfortunately the values of the electric potential and concentrations are unknowns there; they are only known at a macroscopic distance away from the channel, then we take into consideration the ionic transport in the surrounding electrolyte bath. Therefore we suppose that the following PDEs are considered in the whole real line

$$\begin{cases} g^2 \partial_t n = \partial_x \left[g^2((k_n - h_n(k_n - k_w)n)\partial_x n - h_p(k_p - k_w)n\partial_x p - k_n n(1 + h_p k_p p - h_n k_n n)\partial_x \phi)\right] \\ g^2 \partial_t p = \partial_x \left[g^2(-h_n(k_n - k_w)p\, \partial_x n + (k_p - h_p(k_p - k_w)p)\partial_x p - k_p p(1 - h_p k_p p + h_n k_n n)\partial_x \phi)\right] \end{cases}$$

where ϕ satisfies (2). At the ends of the baths we impose the conditions

$$\lim_{x \to \pm\infty} n(t, x) = n_\pm \in \mathbb{R}^+, \;\; \lim_{x \to \pm\infty} p(t, x) = p_\pm \in \mathbb{R}^+, \quad t \in \mathbb{R}^+$$

$$\lim_{x \to \pm\infty} \phi(t, x) = \phi_\pm \in \mathbb{R} \;\; , \;\; t \in \mathbb{R}^+$$

and we prescribe initial conditions (n_0, p_0) for (n, p). We suppose without loss of generality that $h_n = h_p = 1$, else we perform the change of unknowns $n_1 = h_n n$ and $p_1 = h_p p$. For simplicity, we suppose also that $\varepsilon_0(x) = 1$, $k_n = k_p = k$ and we set $k' = k - k_\omega$. A remark is given at the end of the paper concerning the general case $k_n \neq k_p$. The function $g(x)$ grows indefinitely with x into the baths, we take as in [1] and [12]

$$g(x) = 1 + x^2.$$

In summary our model equations reads

$$\begin{cases} g^2 \partial_t n = \partial_x \left[g^2((k - k'n)\partial_x n - k'n\partial_x p - kn(1 + p - n)\partial_x \phi)\right] & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ g^2 \partial_t p = \partial_x \left[g^2(-k'p\partial_x n + (k - k'p)\partial_x p + kp(1 - p + n)\partial_x \phi)\right] & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ \partial_x \left(g^2 \partial_x \phi \right) = g^2(n - p + q(x)) & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ \lim_{x \to \pm\infty} n(t, x) = n_\pm, \;\; \lim_{x \to \pm\infty} p(t, x) = p_\pm, \;\; \lim_{x \to \pm\infty} \phi(t, x) = \phi_\pm & \text{in } \mathbb{R}^+ \\ n(0, x) = n_0(x), \;\; p(0, x) = p_0(x) & \text{in } \mathbb{R}. \end{cases} \tag{5}$$

The two first equations in (5) can be rewritten in the compact form as

$$g^2 \partial_t (n, p) = \partial_x (g^2 [A(n, p)\partial_x (n, p) + \partial_x \phi B(n, p)]) \tag{6}$$

where the diffusion matrix is given by

$$A(n,p) = \begin{pmatrix} k - k'n & -k'n \\ \\ -k'p & k - k'p \end{pmatrix}$$

and the vector field B by $B_1(n,p) = -kn(1+p-n)$ and $B_2(n,p) = kp(1-p+n)$. In view of (1) we will look for solutions satisfying $n, p \geq 0,\ n+p \leq 1$. Note that in this case $A(n,p)$ is positive definite, that is

$$A(n,p)\xi.\ \xi \geq \alpha \left\| \xi \right\|^2 \ ,\ \forall \xi \in \mathbb{R}^2 \tag{7}$$

for some $\alpha > 0$.

In the last few years, Poisson-drift-diffusion models have been drawing great attention, but up to now, only partial results are available in the literature concerning the well-posedness of such problems. For example in [6], [7] and [8] Poisson-drift-diffusion model for semiconductors with linear diffusion was treated in a bounded domain of $\mathbb{R}^n$ for $n \leq 3$, there the diffusion matrix is constant and diagonal. In [4] the solvability and uniqueness of solution were established for a degenerate Poisson-drift-diffusion problem with a non linear diffusion describing semiconductors device, there the problem is considered in a bounded domain of $\mathbb{R}^n$, $n \leq 3$ and the diffusion matrix is diagonal. In [2] the existence of weak solution was shown, for a non linear degenerate drift-diffusion problem with full diffusion matrix, also there the problem is considered in a bounded domain of $\mathbb{R}^n$ for $n \leq 3$. The case at hand differs from the preceding models in the fact that our problem will be solved in the unbounded domain $\mathbb{R}$, then the situation becomes more complicated. Moreover the diffusion matrix A is not diagonal and there are values for n, p for which the diffusion matrix is not positive.

The rest of the paper is organized as follows. In section 2, we precise the functional frame of our work together with the asumptions and we give an a priori estimate which is the key of the existence proof of a solution. In section 3, we define our notion of weak solution and give the main result of this paper, that is the existence theorem of a weak solution to the PNP problem. The proof is based on an approximating method via an introduction of a small parameter $\varepsilon > 0$ and a regularization of the diffusion matrix, we set the approximated problems in section 4 solve them and give a maximum principle satisfied by the approximated solutions. This allows to obtain a solution of our problem by letting $\varepsilon \to 0$ in section 5.

2 Assumptions and preliminary results

We begin with the notations that we will use throughout this paper. For any $T > 0$, we set $Q_T = (0,T) \times \mathbb{R}$. We define the positive and negative parts of a real number s by $s^+ := \max\{s, 0\}$, $s^- := \max\{-s, 0\}$ respectively. The symbol C will denote positive constants and sometimes we will write $C(a_1, a_2, ..., a_m)$ to precise the arguments on which depend C.

The norm in a Banach space E will be denoted by $\|.\|_2$ if $E = L^2(\mathbb{R})$ and $\|.\|_E$ otherwise. The Banach space $F = L^2(0,T;E)$ will be endowed with the norm $\|u\|_F^2 = \int_0^T \|u(t)\|_E^2 dt$.

To take into account the behavior of the solutions in the baths $x \to \pm\infty$ we introduce the functions $\theta(x)$, $\vartheta(x)$ satisfying the stationary equations

$$\begin{cases} \dfrac{d}{dx}(g^2\dfrac{d\theta}{dx}) = \dfrac{d}{dx}(g^2\dfrac{d\vartheta}{dx}) = 0 & \text{in } \mathbb{R} \\ \lim\limits_{x\to\pm\infty} \theta(x) = n_\pm, \ \lim\limits_{x\to\pm\infty} \vartheta(x) = \phi_\pm \end{cases} \tag{8}$$

where $n_\pm, \phi_\pm$ are real numbers satisfying assumption **(H1)** below, we easily verify the following

Lemma 1. *(θ, ϑ), are given by*

$$\theta(x) = n_- + \frac{a}{2}(\arctan x + \frac{x}{1+x^2} + \frac{\pi}{2}) \quad , \quad \vartheta(x) = \phi_- + \frac{b}{2}(\arctan x + \frac{x}{1+x^2} + \frac{\pi}{2})$$

with $a = \frac{2}{\pi}(n_+ - n_-)$, $b = \frac{2}{\pi}(\phi_+ - \phi_-)$ and satisfy $g\frac{d\theta}{dx}, g\frac{d\vartheta}{dx} \in L^2(\mathbb{R})$ together with

$$\min(n_+, n_-) \le \theta(x) \le \max(n_+, n_-), \ \forall x \in \mathbb{R}.$$

Note that θ and ϑ belong to $L^\infty(\mathbb{R})$ but not to $L^2(\mathbb{R})$.

Hypotheses: We will make use of the following hypotheses

$(H1)$ $\quad n_\pm = p_\pm \in \mathbb{R}^+, \ n_\pm \le 1/2 \ , \ \phi_\pm \in \mathbb{R}$

$(H2)$ $\quad (3 - 2\sqrt{2})k < k_w < (3 + 2\sqrt{2})k \ , \ k > 0$

$(H3)$ $\quad q$ has a compact support in $\mathbb{R}, q \in L^2(\mathbb{R})$

$(H4)$ $\quad g\,(n_0 - \theta)\,, g\,(p_0 - \theta) \in L^2(\mathbb{R})$

$(H5)$ $\quad n_0 - p_0 + q = 0$ on $\mathbb{R}$ and $n_0, p_0 \ge 0, \ n_0 + p_0 \le 1.$

The main result of this paper is the global solvability of our problem in certain weighted spaces. The key of the proof is the observation that the system (5) possesses an energy functional which is uniformly bounded if $n, p \ge 0$ and $n + p \le 1$. Indeed, multiplying the first equation of (5) by $n - \theta$ and the second one by $p - \theta$ and adding the resulting equations lead to the following formal equality

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} g^2(|n-\theta|^2 + |p-\theta|^2)dx + \int_{\mathbb{R}} g^2 A(n,p)\partial_x(n-\theta, p-\theta)\cdot\partial_x(n-\theta, p-\theta)dx = -(I_1 + I_2 + I_3)$$

where

$$\begin{cases} I_1 = \displaystyle\int_{\mathbb{R}} g^2(A_{11}(n,p) + A_{12}(n,p))\partial_x\theta \ \partial_x(n-\theta)\,dx \\ I_2 = \displaystyle\int_{\mathbb{R}} g^2(A_{21}(n,p) + A_{22}(n,p))\partial_x\theta \ \partial_x(p-\theta)\,dx \\ I_3 = \displaystyle\int_{\mathbb{R}} g^2(B_1(n,p)\ \partial_x(n-\theta) + B_2(n,p)\ \partial_x(p-\theta))\ \partial_x\phi\,dx. \end{cases}$$

(7) with the help of Young inequality, implies that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} g^2(|n-\theta|^2+|p-\theta|^2)dx+\alpha\int_{\mathbb{R}} g^2\left(|\partial_x(n-\theta)|^2+|\partial_x(p-\theta)|^2\right)dx \le C+C\int_{\mathbb{R}} g^2|\partial_x\phi|^2dx$$

In the other hand, deriving Poisson equation with respect to t leads to

$$\partial_x\left(g^2\ \partial^2_{x,t}\phi\right) = g^2\left(\partial_t p - \partial_t n\right)$$

thus multiplying this relation by ϕ, using the equations of n and p, we obtain thanks to Young inequality

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} g^2|\partial_x\phi|^2dx \le \frac{\alpha}{2}\int_{\mathbb{R}} g^2\left(|\partial_x(n-\theta)|^2+|\partial_x(p-\theta)|^2\right)dx + C(\alpha)\int_{\mathbb{R}} g^2|\partial_x\phi|^2dx + C(\alpha)$$

therefore Gronwall inequality provides the energy estimate

$$\int_{\mathbb{R}} g^2(|n-\theta|^2+|p-\theta|^2+|\partial_x\phi|^2)dx+\alpha\int_0^t\int_{\mathbb{R}} g^2\left(|\partial_x(n-\theta)|^2+|\partial_x(p-\theta)|^2\right)dxds \le C. \quad (9)$$

In order to make energy estimate rigorous, we have to prove that $n,p \ge 0$ and $n+p \le 1$ (see section 3). Observe that unfortunately, (9) does not provide an $L^2(0,T;H^1(\mathbb{R}))$ estimate for ϕ since Poincaré inequality is not valid in the unbounded domain $\mathbb{R}$. We overcome this difficulty by using functions which decrease towards 0 at infinity so we introduce the following weighted Sobolev spaces. For $m = 0,1,2$ and positive functions σ_i, $i = 0,1,2$, we set

$$H^m_{\sigma_0,\ldots,\sigma_m}(\mathbb{R}) = \left\{\text{functions } u:\mathbb{R}\to\mathbb{R};\ \sigma_i\frac{d^iu}{dx^i}\in L^2(\mathbb{R}),\, i=0,...m\right\}$$

endowed with the norm

$$\|u\|^2_{H^m_{\sigma_0,\ldots,\sigma_m}} = \Sigma_{i=0}^m\left\|\sigma_i\frac{d^iu}{dx^i}\right\|_2^2.$$

If all the weight functions are equal to σ, we will denote the corresponding space by $H^m_\sigma(\mathbb{R})$ if $m \ge 1$ and by $L^2_\sigma(\mathbb{R})$ if $m = 0$. Note that $u \in H^1_g(\mathbb{R})$ if and only if $gu \in H^1(\mathbb{R})$ with $C_1\|gu\|_{H^1} \le \|u\|_{H^1_g} \le C_2\|gu\|_{H^1}$, ($C_1, C_2$ positive constants) and it holds

Lemma 2. *(i)The dual space $(H_g^1(\mathbb{R}))'$ of $H_g^1(\mathbb{R})$ is characterized by $\Psi \in (H_g^1(\mathbb{R}))'$ iff $g^{-1}\Psi \in H^{-1}(\mathbb{R})$ and*

$$C_2^{-1}\|g^{-1}\Psi\|_{H^{-1}} \leq \|\Psi\|_{(H_g^1)'} \leq C_1\|g^{-1}\Psi\|_{H^{-1}}$$

(ii) $H_g^1(\mathbb{R})$ is continuously and compactly embedded in $L^2(\mathbb{R})$
(iii) (Hardy inequality) There exists a constant $C > 0$ such that for every $u \in H_{1,g}^1(\mathbb{R})$

$$\|u\|_2 \leq C\left\|\frac{du}{dx}\right\|_{L_g^2}.$$

Proof. The first point is a consequence of the precedent remark. For the second one, the continuity of the embedding is obvious while the compactness is a direct consequence of [13], theorem 2.1 when choosing the different data $p = q = 2$, $b_0(x) = 1$, $b_1(x) = (1 + g(x))^2$, $w(x) = 1$, $v_0(x) = v_1(x) = g^2(x)$ and $r(x) = 1$. See also [10] (theorems 18.12 and 20.5). For the last point, we refer the reader to [10] (theorem 21.8) or [13] (theorem 2.3 and example 2.4). □

3 The main result

First let us specify our notion of weak solution

Definition 1. *Under hypotheses (H1)-(H5), (n, p, ϕ) is a weak solution of (5) in Q_T if the following properties hold*

$$(i) \qquad n-\theta, p-\theta \in L^\infty(0,T;L_g^2(\mathbb{R})) \cap L^2(0,T;H_g^1(\mathbb{R})),\ g^2\partial_t n, g^2\partial_t p \in L^2(0,T;(H_g^1(\mathbb{R}))')$$

$$\phi - \vartheta \in L^\infty\left(0,T;H_{1,g,g}^2(\mathbb{R})\right),\ 0 \leq n,\ p,\ n+p \leq 1 \ a.e.\ in\ Q_T$$

$$(ii) \qquad n(0,x) = n_0(x),\ p(0,x) = p_0(x)\ a.e.\ in\ \mathbb{R}.$$

$$(iii) \qquad \int_0^T \left\langle g^2\partial_t(n,p),(\xi,\zeta)\right\rangle dt + \int_{Q_T} g^2 A(n,p)\partial_x(n-\theta,p-\theta)\cdot\partial_x(\xi,\zeta)dxdt + \int_{Q_T} g^2 B(n,p)\partial_x(\phi-\vartheta)\cdot\partial_x(\xi,\zeta)dxdt = 0$$

for all $\xi, \zeta \in L^2(0,T;H_g^1(\mathbb{R}))$ where $\langle\ ,\ \rangle$ is the dual product between $(H_g^1(\mathbb{R}))' \times (H_g^1(\mathbb{R}))'$ and $H_g^1(\mathbb{R}) \times H_g^1(\mathbb{R})$ and for all $\eta \in L^1(0,T;H_g^1(\mathbb{R}))$

$$\int_{Q_T} g^2\partial_x\phi\partial_x\eta dxdt + \int_{Q_T} g^2\left(n-p+q\right)\eta dxdt = 0$$

The main result of the paper can be stated as follows

Theorem 1. *Let $T > 0$, under the assumptions (H1)- (H5), there exists (at least) a weak solution (n, p, ϕ) to the system (5) in Q_T. Moreover the following energy estimate holds for $t \in (0, T)$*

$$\int_{\mathbb{R}} g^2(|n-\theta|^2 + |p-\theta|^2 + |\partial_x(\phi - \vartheta)|^2)dx + \int_0^t \int_{\mathbb{R}} g^2(|\partial_x(n-\theta)|^2 + |\partial_x(p-\theta)|^2)dxds \leq C$$

The proof will be done in several steps. We consider an approximation of problem (5) involving the additional term $\varepsilon g^4 \phi$ in the equation of ϕ and we prove the existence of weak solution of the approximate problem using decoupling mapping and Leray Shauder's fixed point theorem. Finally, uniform bounds with respect to ε are obtained and the limit $\varepsilon \to 0$ can be performed.

4 The approximated problems

First, we set the change of unknowns $N = n - \theta$, $P = p - \theta$, $\Phi = \phi - \vartheta$ which transforms (5) into

$$\begin{cases} g^2 \partial_t(N, P) = \partial_x[g^2(\tilde{A}(N,P)\partial_x(N+\theta, P+\theta) + \partial_x(\Phi + \vartheta)\tilde{B}(N,P))] & \text{in } Q_T \\ \partial_x(g^2 \partial_x \Phi) = g^2(N - P + q(x)) & \text{in } Q_T \\ \lim\limits_{x\to\pm\infty} N(t,x) = \lim\limits_{x\to\pm\infty} P(t,x) = \lim\limits_{x\to\pm\infty} \Phi(t,x) = 0 & \text{in } \ (0,T) \\ N(0,x) = n_0(x) - \theta(x), \ P(0,x) = p_0(x) - \theta(x) & \text{in } \mathbb{R} \end{cases} \tag{10}$$

where the matrix $\tilde{A}$ is defined by $\tilde{A}(N,P) = A(N+\theta, P+\theta)$ and the vector $\tilde{B}$ by $\tilde{B}(N,P) = B(N+\theta, P+\theta)$. To solve the first equation, we need a definite positive matrix diffusion and bounded coefficients, therefore we replace the diffusion matrix $\tilde{A}$ by $\tilde{A}^+$ and the vector field $\tilde{B}$ by $\tilde{B}^+$ defined by

$$\begin{cases} \tilde{A}^+_{11}(r,s) = \tilde{A}^+_{22}(s,r) = \dfrac{k - k' \min\left((r+\theta)^+, 1-(s+\theta)^+\right) + k_w\left((r+\theta)^+ + (s+\theta)^+ - 1\right)^+}{1 + ((r+\theta)^+ + (s+\theta)^+ - 1)^+} \\ \tilde{A}^+_{12}(r,s) = \tilde{A}^+_{21}(,r) = -\dfrac{k'(r+\theta)^+}{1 + ((r+\theta)^+ + (s+\theta)^+ - 1)^+} \\ \tilde{B}^+_1(r,s) = \tilde{B}^+_2(s,r) = -\dfrac{k(r+\theta)^+\left[\max((r+\theta)^+ + (s+\theta)^+, 1) + (s+\theta)^+ - (r+\theta)^+\right]}{1 + ((r+\theta)^+ + (s+\theta)^+ - 1)^+}. \end{cases}$$

Then we define the approximate problem

$$\begin{cases} g^2\partial_t(N,P) = \partial_x[g^2(\tilde{A}^+(N,P)\partial_x(N+\theta,P+\theta) + \partial_x(\Phi+\vartheta)\tilde{B}^+(N,P))] & \text{in } Q_T \\ \partial_x(g^2\partial_x\Phi) - \varepsilon g^4\Phi = g^2(N-P+q(x)) & \text{in } Q_T \\ \lim_{x\to\pm\infty} N(t,x) = \lim_{x\to\pm\infty} P(t,x) = \lim_{x\to\pm\infty} \Phi(t,x) = 0 & \text{in } \ (0,T) \\ N(0,x) = n_0(x) - \theta(x), \ \ P(0,x) = p_0(x) - \theta(x) & \text{in } \mathbb{R} \end{cases} \tag{11}$$

where $\varepsilon > 0$ is a small parameter. We uncouple the problem (11) and consider two linear problems. First, we solve for given $\left(\overline{N},\overline{P}\right) \in L^2\left(Q_T\right) \times L^2\left(Q_T\right)$ the problem

$$\begin{cases} -\partial_x(g^2\partial_x\Phi) + \varepsilon g^4\Phi = -g^2(\overline{N} - \overline{P} + q) & \text{in } Q_T \\ \lim_{x\to\pm\infty} \Phi(t,x) = 0 & \text{in } (0,T) \end{cases} \tag{12}$$

then, we solve the linear problem

$$\begin{cases} g^2\partial_t(N,P) = \partial_x[g^2(\tilde{A}^+\left(\overline{N},\overline{P}\right)\partial_x(N+\theta,P+\theta) + \partial_x(\Phi+\vartheta)\tilde{B}^+\left(\overline{N},\overline{P}\right)] & \text{in } Q_T \\ \lim_{x\to\pm\infty} (N,P) = (0,0) & \text{in } (0,T) \\ (N,P)(0) = (n_0-\theta, p_0-\theta) & \text{in } \mathbb{R} \end{cases} \tag{13}$$

where Φ is the solution of (12).

4.1 Solving problem (12)

We have

Lemma 3. *Let $\left(\overline{N},\overline{P}\right) \in \left(L^2\left(Q_T\right)\right)^2$ be given, for all $\varepsilon > 0$ the problem (12) has a unique solution $\Phi \in L^2(0,T;H^2_{g^2,g,1}(\mathbb{R}))$ and it holds for $t \in (0,T)$*

$$\|\Phi(t)\|_{H^2_{g^2,g,1}} \leq C(\varepsilon)\|\overline{N}(t) - \overline{P}(t) + q\|_2. \tag{14}$$

Proof. We use the variational method so we set

$$b(\Phi,\Psi) = \int_{Q_T} g^2\partial_x\Phi\,\partial_x\Psi\,dxdt + \varepsilon\int_{Q_T} g^4\Phi\,\Psi\,dxdt\ ,\ \Phi,\Psi \in L^2(0,T;H^1_{g^2,g}(\mathbb{R}))$$

$$l(\Psi) = -\int_{Q_T} g^2(\overline{N} - \overline{P} + q)\,\Psi\,dxdt\ ,\ \Psi \in L^2(0,T;H^1_{g^2,g}).$$

Applying Lax-Milgram theorem, we get a unique $\Phi \in L^2(0,T;H^1_{g^2,g})$ such that

$$b(\Phi,\Psi) = l(\Psi) \text{ for all } \Psi \in L^2(0,T;H^1_{g^2,g}). \tag{15}$$

Taking $\Psi = \Phi$ in (15), we obtain

$$\int_{\mathbb{R}} g^2 \left|\partial_x \Phi(t,x)\right|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} g^4 |\Phi(t,x)|^2 dx \le C(\varepsilon) \int_{\mathbb{R}} |\overline{N}(t,x) - \overline{P}(t,x) + q(x)|^2 dx. \tag{16}$$

As usual, the regularity of $\partial_x^2 \Phi$ comes from the equation since $\partial_x^2 \Phi = -2\dfrac{g'}{g^2} g\partial_x \Phi + \varepsilon g^2 \Phi + (\overline{N} - \overline{P} + q)$ and we get its L^2 estimate using (16). □

Moreover, the estimate (16) leads to

Lemma 4. *The operator $\mathcal{P} : (L^2(Q_T))^2 \longrightarrow L^2(0,T; H^2_{g^2,g,1}(\mathbb{R}))$ defined by $\mathcal{P}(\overline{N}, \overline{P}) = \Phi$ where Φ is the solution of (12) is continuous.*

Proof. Indeed let $(N_1, P_1), (N_2, P_2) \in \left(L^2(Q_T)\right)^2$, $\Phi_i = \mathcal{P}(N_i, P_i)$, $i = 1, 2$, we set $N = N_1 - N_2$, $P = P_1 - P_2$ and $\Phi = \Phi_1 - \Phi_2$. Since $-\partial_x(g^2 \partial_x \Phi) + \varepsilon g^4 \Phi = -g^2(N - P)$, we get using (14)

$$\int_0^T \|\Phi(t)\|^2_{H^1_{g^2,g}} dt \le C(\varepsilon) \int_0^T \|N(t) - P(t)\|_2^2 dt.$$

□

4.2 Solving problem (13)

In order to solve problem (13), we need

Lemma 5. *The matrix A^+ is positive definite, that is there exists $\alpha > 0$ such that $A^+(u,v)\,\xi.\,\xi \ge \alpha \left\|\xi\right\|^2$ for all $(u,v) \in \mathbb{R}^2$ and $\xi \in \mathbb{R}^2$. The same holds for $\tilde{A}^+$.*

The proof is technical and will be given in appendix at the end of the paper. Let us prove the following existence result for problem (13)

Lemma 6. *Let $\varepsilon > 0$, $\left(\overline{N}, \overline{P}\right) \in (L^2(Q_T))^2$ be given and $\Phi = \mathcal{P}\left(\overline{N}, \overline{P}\right)$. The problem (13) possesses a unique solution (N, P) such that $N, P \in L^2\left(0, T; H^1_g(\mathbb{R})\right)$ and it holds*

$$\begin{aligned} &\|N(t)\|^2_{L^2_g} + \|P(t)\|^2_{L^2_g} + \alpha \int_0^t (\|\partial_x N(s)\|^2_{L^2_g} + \|\partial_x P(s)\|^2_{L^2_g}) ds \le \\ &C(\alpha, T) + \|n_0 - \theta\|^2_{L^2_g} + \|p_0 - \theta\|^2_{L^2_g} + C(\alpha, \varepsilon) \int_0^t \|\overline{N}(s) - \overline{P}(s) + q\|_2^2 ds. \end{aligned} \tag{17}$$

Moreover $g\partial_t N, g\partial_t P \in L^2(0,T; H^{-1}(\mathbb{R}))$ and satisfy

$$\|g\partial_t N\|^2_{L^2(0,T;H^{-1}(\mathbb{R}))}, \|g\partial_t P\|^2_{L^2(0,T;H^{-1}(\mathbb{R}))} \le C(\alpha, \varepsilon, T) + C(\alpha, \varepsilon) \int_0^T \|\overline{N}(s) - \overline{P}(s) + q\|_2^2 ds. \tag{18}$$

Proof. We introduce the change of unknowns $\tilde{N} = \exp(-\lambda t)\, N$, $\tilde{P} = \exp(-\lambda t)\, P$ where $\lambda > 0$, so $\tilde{V} = \left(\tilde{N}, \tilde{P}\right)$ satisfies

$$\begin{cases} g^2\partial_t\tilde{V} + \lambda g^2\tilde{V} = \partial_x[g^2(\tilde{A}^+\left(\overline{N},\overline{P}\right)\partial_x(\tilde{V}+\tilde{\Theta}) + \tilde{B}^+\left(\overline{N},\overline{P}\right)\partial_x(\tilde{\Phi}+\tilde{\vartheta}))] & \text{in } Q_T \\ \lim\limits_{x\to\pm\infty}\tilde{V} = (0,0) & \text{in } (0,T) \\ \tilde{V}(0) = (n_0-\theta, p_0-\theta) & \text{in } \mathbb{R} \end{cases} \tag{19}$$

where $\tilde{\Theta} = \left(\tilde{\theta},\tilde{\theta}\right)$, $\tilde{\theta}(t,x) = \exp(-\lambda t)\,\theta(x)$, $\tilde{\vartheta}(t,x) = \exp(-\lambda t)\,\vartheta(x)$ and $\tilde{\Phi} = \exp(-\lambda t)\,\Phi$. We set $X = L^2(0,T;H^1_g(\mathbb{R}))$ and we introduce the space $Y = \mathcal{D}([0,T[\times\mathbb{R})$ endowed with the norm $\|u\|_Y^2 = \|u\|_X^2 + \frac{1}{2}\int_{\mathbb{R}} g^2(x)u^2(0,x)dx$, such as the injection of Y into X is continuous. Then we define the continuous bilinear form $\tilde{a}$ on $X^2\times Y^2$ and the continuous linear functional $\tilde{l}$ on Y^2 by

$$\tilde{a}(V_1,V_2) = -\int_{Q_T} g^2V_1\cdot\partial_tV_2dxdt + \lambda\int_{Q_T} g^2V_1\cdot V_2dxdt + \int_{Q_T} g^2\tilde{A}^+\left(\overline{N},\overline{P}\right)\partial_xV_1\cdot\partial_xV_2dxdt$$

$$\tilde{l}\,(V) = \int_{\mathbb{R}} g^2\,(n_0-\theta,p_0-\theta)\cdot V\,(0,x)\,dx - \int_{Q_T} g^2[(\tilde{A}^+(\overline{N},\overline{P})\partial_x\tilde{\Theta} + \tilde{B}^+(\overline{N},\overline{P})\partial_x(\tilde{\Phi}+\tilde{\vartheta})]\cdot\partial_xV\,dxdt.$$

Lemma 5 implies that for some real $\tilde{\alpha} > 0$ we have

$$\tilde{a}(V,V) \geq \tilde{\alpha}\|V\|^2_{Y^2},\ \forall V \in Y^2. \tag{20}$$

Indeed

$$\begin{aligned} \tilde{a}(V,V) &= \int_{Q_T} g^2(-\frac{1}{2}\partial_t|V|^2 + \lambda|V|^2 + \tilde{A}^+(\overline{N},\overline{P})\partial_xV\cdot\partial_xV)dxdt \\ &\geq \frac{1}{2}\int_{\mathbb{R}} g^2|V(0,x)|^2dx + \lambda\int_{Q_T} g^2|V|^2dxdt + \alpha\int_{Q_T} g^2|\partial_xV|^2dxdt \end{aligned}$$

so thanks to theorem of Lions [11], there exists $\tilde{V} = (\tilde{N},\tilde{P}) \in X^2$ satisfying $\tilde{a}(\tilde{V},V) = \tilde{l}(V), \forall V \in Y^2$. Then we get $g^2\partial_t\tilde{V} + \lambda g^2\tilde{V} = \partial_x[g^2(\tilde{A}^+\left(\overline{N},\overline{P}\right)\partial_x(\tilde{V}+\tilde{\Theta}) + \tilde{B}^+\left(\overline{N},\overline{P}\right)\partial_x(\tilde{\Phi}+\tilde{\vartheta}))]$ in the sense of distributions and we deduce that $g\partial_t\tilde{N}, g\partial_t\tilde{P} \in L^2(0,T;H^{-1}(\mathbb{R}))$. Indeed we have $U \equiv g(\tilde{A}^+_{11}(\overline{N},\overline{P})\partial_x(\tilde{N}+\tilde{\theta}) + \tilde{A}^+_{12}(\overline{N},\overline{P})\partial_x(\tilde{P}+\tilde{\theta}) + \tilde{B}^+_1\left(\overline{N},\overline{P}\right)\partial_x(\tilde{\Phi}+\tilde{\vartheta})) \in L^2\left(0,T;L^2\left(\mathbb{R}\right)\right)$ then $g\partial_t\tilde{N} = \frac{1}{g}\partial_x\left(gU\right) = \frac{g'}{g}U + \partial_xU \in L^2\left(0,T;H^{-1}\left(\mathbb{R}\right)\right)$ and similarly $g\partial_t\tilde{P} \in L^2\left(0,T;H^{-1}\left(\mathbb{R}\right)\right)$. Therefore $g\tilde{N}, g\tilde{P} \in L^2(0,T;H^1(\mathbb{R})) \cap H^1(0,T;H^{-1}(\mathbb{R}))$. Consequently $g\tilde{N}(0,\cdot), g\tilde{P}(0,\cdot)$ are well defined in $L^2(\mathbb{R})$ and the initial conditions are satisfied in (19). Hence $(N,P) = \exp(\lambda t)(\tilde{N},\tilde{P})$ is a solution of (13) and we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} g^2(|N|^2+|P|^2)dx + \frac{\alpha}{2}\int_{\mathbb{R}} g^2(|\partial_xN|^2+|\partial_xP|^2)dx \leq C(\alpha)\int_{\mathbb{R}} g^2(|\theta'|^2+|\vartheta'|^2+|\partial_x\Phi|^2)dx$$

so for all $t \in (0,T)$,

$$\int_{\mathbb{R}} g^2(|N|^2+|P|^2)dx + \alpha \int_0^t \int_{\mathbb{R}} g^2(|\partial_x N|^2+|\partial_x P|^2)dxds \leq$$
$$\int_{\mathbb{R}} g^2(|n_0-\theta|^2+|p_0-\theta|^2)dx + tC(\alpha)\int_{\mathbb{R}} g^2(|\theta'|^2+|\vartheta'|^2)dx + C(\alpha)\int_0^t \int_{\mathbb{R}} g^2|\partial_x \Phi|^2 dxds \tag{21}$$

which leads to (17) using (16) while (18) is a consequence of (17). For the uniqueness, it is sufficient to prove that the solution of the homogeneous problem

$$\begin{cases} g^2\partial_t(N,P) = \partial_x(g^2(\tilde{A}^+(\overline{N},\overline{P})\,\partial_x(N,P)) & \text{in } Q_T \\ \lim\limits_{x\to\pm\infty}(N,P) = (0,0) & \text{in } (0,T) \\ (N,P)(0) = (0,0) & \text{in } \mathbb{R} \end{cases} \tag{22}$$

is $(N,P)=(0,0)$ and this is a direct consequence of (21). □

Remark 1. *To give a meaning to the initial conditions, it was sufficient to prove that $g^{-1}\partial_t N, g^{-1}\partial_t P$ belong to $L^2(0,T;H^{-1}(\mathbb{R}))$ since $L^2\left(0,T;H^1_g(\mathbb{R})\right) \cap H^1(0,T;(H^1_g(\mathbb{R}))')$ is contained in $\mathcal{C}(0,T;L^2_g(\mathbb{R}))$.*

The solution of problem (13) satisfies the following dependance with respect to the datum $(\overline{N},\overline{P})$

Lemma 7. *The mapping $\mathcal{S} : L^2(Q_T)\times L^2(Q_T) \longrightarrow L^2(Q_T)\times L^2(Q_T)$ defined by $\mathcal{S}(\overline{N},\overline{P}) = (N,P)$ where (N,P) is the solution of (13) provided by lemma 6, is continuous and compact.*

Proof. Consider a sequence $(\overline{N}_m,\overline{P}_m)$ in $(L^2(Q_T))^2$ such that $\left(\overline{N}_m,\overline{P}_m\right) \to \left(\overline{N},\overline{P}\right)$ strongly in $(L^2(Q_T))^2$ and set $\mathcal{P}(\overline{N}_m,\overline{P}_m) = \Phi_m$, $\mathcal{S}(\overline{N}_m,\overline{P}_m) = (N_m,P_m)$. (17), (18), lemma 2 and Aubin lemma yield to the existence of a subsequence (not relabelled) and $N,P \in L^2(0,T;H^1_g(\mathbb{R})) \cap H^1(0,T;(H^1_g(\mathbb{R}))')$ such that

$$\begin{aligned} &(N_m,P_m) \rightharpoonup (N,P) \quad \text{weakly in } \left(L^2(0,T;H^1_g(\mathbb{R})) \cap H^1(0,T;(H^1_g(\mathbb{R}))')\right)^2 \\ &(N_m,P_m) \to (N,P) \quad \text{strongly in } (L^2(Q_T))^2 \end{aligned}$$

and from lemma 4, we infer that

$$\Phi_m \to \mathcal{P}(\overline{N},\overline{P}) = \Phi \text{ strongly in } L^2(0,T;H^1_{g^2,g}(\mathbb{R})).$$

Since the coefficients of the matrices $\tilde{A}^+$ and $\tilde{B}^+$ are bounded, we easily pass to the limit in the equation satisfied by (N_m,P_m) (at least in the sense of distributions) then we conclude as in proof of lemma 6 that $(N,P)=\mathcal{S}(\overline{N},\overline{P})$ thus $\mathcal{S}$ is continuous. The compactness is a direct consequence of (17), (18), lemma 2 and Aubin lemma. □

4.3 Solving the full problem (11)

Now we are ready to solve the coupling problem (11). We have

Theorem 2. *Assume (H1)-(H5), then for all $\varepsilon > 0$ the approximated problem (11) admits at least one solution $(N_\varepsilon, P_\varepsilon, \Phi_\varepsilon) \in L^2(0,T; H^1_g(\mathbb{R}) \times H^1_g(\mathbb{R}) \times H^1_{g^2,g,1}(\mathbb{R}))$.*

Proof. We use the previous results to perform a fixed point procedure. We have proved in lemma 7 that the mapping $\mathcal{S}$ is continuous and compact. Let us prove that the sets $\Lambda_\delta = \left\{(N,P) \in \left(L^2\left(Q_T\right)\right)^2 \ ; \ (N,P) = \delta\,\mathcal{S}\left(N,P\right)\right\}$ are uniformly bounded with respect to $\delta \in [0,1]$. The set $\Lambda_0 = \{(0,0)\}$; let $\delta \neq 0$, the equation $(N,P) = \delta\,\mathcal{S}\left(N,P\right)$ is equivalent to

$$\begin{cases} g^2\partial_t(N,P) = \partial_x(g^2[\tilde{A}^+\left(N,P\right)\partial_x(N+\delta\theta, P+\delta\theta) + \delta\tilde{B}^+\left(N,P\right)\partial_x(\Phi+\vartheta)]) & \text{in } Q_T \\ \lim\limits_{x\to\pm\infty}(N,P) = (0,0) & \text{in } (0,T) \\ (N,P)(0) = \delta(n_0-\theta, p_0-\theta) & \text{in } \mathbb{R} \end{cases} \tag{23}$$

Testing the equation of (23) with (N,P), we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} g^2\left(N^2+P^2\right)dx + \int_{\mathbb{R}} g^2\tilde{A}^+\left(N,P\right)\partial_x(N,P).\,\partial_x(N,P)dx = -(I_1+I_2+I_3)$$

with

$$\begin{cases} I_1 = \delta\displaystyle\int_{\mathbb{R}} g^2\tilde{A}^+\left(N,P\right)\partial_x(N,P).\partial_x(\theta,\theta)\,dx \\ I_2 = \delta\displaystyle\int_{\mathbb{R}} g^2\tilde{B}(N,P)\partial_x(N,P)\,.\partial_x(\Phi+\vartheta)dx. \end{cases}$$

We have the following inequalities

$$|I_1| \leq C(\alpha)\int_{\mathbb{R}} g^2\left|\theta'\right|^2dx + \frac{\alpha}{4}\int_{\mathbb{R}} g^2(\left|\partial_x N\right|^2 + \left|\partial_x P\right|^2)dx$$

$$|I_3| \leq C(\alpha)\int_{\mathbb{R}} g^2\left|\partial_x\Phi\right|^2dx + C(\alpha)\int_{\mathbb{R}} g^2\left|\vartheta'\right|^2dx + \frac{\alpha}{4}\int_{\mathbb{R}} g^2\left(\left|\partial_x N\right|^2 + \left|\partial_x P\right|^2\right)dx.$$

Using (16) and inserting these inequalities in (24), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} g^2(N^2+P^2)dx + \frac{\alpha}{2}\int_{\mathbb{R}} g^2(\left|\partial_x N\right|^2 + \left|\partial_x P\right|^2)dx \leq C(\alpha,\varepsilon) + C(\alpha,\varepsilon)\int_{\mathbb{R}} g^2(N^2+P^2)dx$$

then Gronwal's lemma implies the claim.

We deduce from the theorem of Leray-Schauder that $\mathcal{S}$ has a fixed point $(N_\varepsilon, P_\varepsilon)$ in $(L^2(Q_T))^2$. Thus $(N_\varepsilon, P_\varepsilon, \Phi_\varepsilon)$ (with $\Phi_\varepsilon = \mathcal{P}(N_\varepsilon, P_\varepsilon)$) is a weak solution of problem (11) in Q_T. $\square$

In fact, we get more regularity for $(N_\varepsilon, P_\varepsilon, \Phi_\varepsilon)$. We have

Lemma 8. *The solutions $(N_\varepsilon, P_\varepsilon, \Phi_\varepsilon)$ of problem (11) provided by theorem 2 are such that*

$$g\partial_t N_\varepsilon, g\partial_t P_\varepsilon \in L^2(0,T;H^{-1}(\mathbb{R})) \quad , \quad \partial_t \Phi_\varepsilon \in L^2(0,T;H^1_{g^2,g}(\mathbb{R})).$$

Proof. The regularity of $\partial_t N_\varepsilon, \partial_t P_\varepsilon$ is a consequence of lemma 6 then the same proof as for lemma 3 lead to the existence and uniqueness of a solution $\Psi_\varepsilon \in L^2(0,T;H^1_{g^2,g}(\mathbb{R}))$ to the equation

$$\begin{cases} -\partial_x(g^2\partial_x\Psi_\varepsilon) + \varepsilon g^4\Psi_\varepsilon = -g^2(\partial_t N_\varepsilon - \partial_t P_\varepsilon) & \text{in } Q_T \\ \lim\limits_{x\to\pm\infty} \Psi_\varepsilon(t,x) = 0 & \text{in } (0,T) \end{cases} \tag{24}$$

But since $\partial_t\Phi_\varepsilon$ solves this equation, we get the result. $\square$

These solutions satisfy the following maximum principle

Lemma 9. *The solutions $(N_\varepsilon, P_\varepsilon)$ of (11) provided by theorem 2 satisfy*

$$N_\varepsilon + \theta, P_\varepsilon + \theta \geq 0,\ \ N_\varepsilon + P_\varepsilon + 2\theta \leq 1 \ \textit{a.e. in } Q_T.$$

Proof. First, let us show that $(N_\varepsilon+\theta)^-$, $(P_\varepsilon+\theta)^-$, $(N_\varepsilon+P_\varepsilon+2\theta-1)^+ \in L^2(0,T;H^1_g(\mathbb{R}))$. We have

$$\int_{Q_T} g^2|\,(N_\varepsilon+\theta)^-\,|^2 dxdt = \int_{N_\varepsilon\leq-\theta} g^2(N_\varepsilon^2 + 2N_\varepsilon\theta + \theta^2)dxdt \leq \int_{N_\varepsilon+\theta\leq 0} g^2 N_\varepsilon^2 dxdt < \infty.$$

Moreover

$$\int_{Q_T} g^2\left|\partial_x\left(N_\varepsilon+\theta\right)^-\right|^2 dxdt = \int_{N_\varepsilon+\theta\leq 0} g^2|\partial_x(N_\varepsilon+\theta)|^2 dxdt < \infty$$

because $g\theta' \in L^2(\mathbb{R})$. Thus $(N_\varepsilon+\theta)^- \in L^2(0,T;H^1_g(\mathbb{R}))$ and the same arguments lead to $(P_\varepsilon+\theta)^- \in L^2(0,T;H^1_g(\mathbb{R}))$. Next let $v = (N_\varepsilon+P_\varepsilon+2\theta-1)^+$, we write

$$\begin{aligned} \int_{Q_T} g^2v^2dxdt &= \int_{N_\varepsilon+P_\varepsilon\geq 1-2\theta} g^2(N_\varepsilon+P_\varepsilon+2\theta-1)^2dxdt \\ &= \int_{N_\varepsilon+P_\varepsilon\geq 1-2\theta} g^2((N_\varepsilon+P_\varepsilon)^2 - 2(N_\varepsilon+P_\varepsilon)(1-2\theta) + (1-2\theta)^2)dxdt. \end{aligned} \tag{25}$$

Using the fact that $\theta \leq \frac{1}{2}$, we get

$$\int_{Q_T} g^2v^2\,dxdt \leq \int_{N_\varepsilon+P_\varepsilon\geq 1-2\theta} g^2(N_\varepsilon+P_\varepsilon)^2dxdt < \infty.$$

Moreover

$$\int_{Q_T} g^2|\partial_x v|^2dxdt = \int_{N_\varepsilon+P_\varepsilon\geq 1-2\theta} g^2|\partial_x(N_\varepsilon+P_\varepsilon+2\theta)|^2dxdt < \infty$$

so $v \in L^2(0,T;H_g^1(\mathbb{R}))$. Now, multiplying the first equation of (11) by $((N_\varepsilon+\theta)^-, (P_\varepsilon+\theta)^-)$ and integrating by parts, we get

$$\begin{gathered}-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} g^2(\left|(N_\varepsilon+\theta)^-\right|^2+|(P_\varepsilon+\theta)^-|^2)dx = \\ \int_{\mathbb{R}} g^2(\tilde{A}_{11}^+(N_\varepsilon,P_\varepsilon)\left|\partial_x((N_\varepsilon+\theta)^-)\right|^2+\tilde{A}_{22}^+(N_\varepsilon,P_\varepsilon)\left|\partial_x((P_\varepsilon+\theta)^-)\right|^2)dx \geq 0\end{gathered} \tag{26}$$

Then since $n_0^- = p_0^- = 0$ we get $(N_\varepsilon+\theta)^- = (P_\varepsilon+\theta)^- = 0$. Similarly, adding the equations of N_ε and P_ε and testing the resulting equation with $v = (N_\varepsilon+P_\varepsilon+2\theta-1)^+$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} g^2v^2dx+\int_{\mathbb{R}} g^2[(\tilde{A}_{11}^++\tilde{A}_{21}^+)(N_\varepsilon,P_\varepsilon)\partial_xN_\varepsilon+(\tilde{A}_{12}^++\tilde{A}_{22}^+)(N_\varepsilon,P_\varepsilon)\partial_xP_\varepsilon)]\partial_xv\,dx = -L \tag{27}$$

where $L = \int_{\mathbb{R}} g^2((\tilde{B}_1^++\tilde{B}_2^+)(N_\varepsilon,P_\varepsilon)\partial_x(\Phi_\varepsilon+\vartheta)\,\partial_xv\,dx = 0$. Since

$$\begin{gathered}\int_{\mathbb{R}} g^2[(\tilde{A}_{11}^++\tilde{A}_{21}^+)(N_\varepsilon,P_\varepsilon)\partial_x\,(N_\varepsilon+\theta)+(\tilde{A}_{12}^++\tilde{A}_{22}^+)(N_\varepsilon,P_\varepsilon)\partial_x\,(P_\varepsilon+\theta)]\partial_xvdx = \\ k_w\int_{N_\varepsilon+P_\varepsilon+2\theta\geq 1} g^2(N_\varepsilon+P_\varepsilon)\partial_x(N_\varepsilon+P_\varepsilon)\,\partial_xv\,dx = k_w\int_{N_\varepsilon+P_\varepsilon+2\theta\geq 1} g^2(N_\varepsilon+P_\varepsilon)|\partial_xv|^2dx \geq 0\end{gathered}$$

we deduce that $\frac{d}{dt}\int_{\mathbb{R}} g^2v^2dx \leq 0$ then $v=0$ because $v(0,x) = (n_0+p_0-1)^+ = 0$. Therefore $N_\varepsilon+P_\varepsilon+2\theta \leq 1$. $\square$

Thus $(N_\varepsilon,P_\varepsilon,\Phi_\varepsilon)$ also solves

$$\begin{cases} g^2\partial_t(N_\varepsilon,P_\varepsilon) = \partial_x[g^2(\tilde{A}\,(N_\varepsilon,P_\varepsilon)\,\partial_x(N_\varepsilon+\theta,P_\varepsilon+\theta)+\tilde{B}\,(N_\varepsilon,P_\varepsilon)\,\partial_x(\Phi_\varepsilon+\vartheta))] & \text{in } Q_T \\ -\partial_x(g^2\partial_x\Phi_\varepsilon)+\varepsilon g^4\Phi_\varepsilon = -g^2(N_\varepsilon-P_\varepsilon+q(x)) & \text{in } Q_T \\ \lim\limits_{x\to\pm\infty} N_\varepsilon(t,x) = \lim\limits_{x\to\pm\infty} P_\varepsilon(t,x) = \lim\limits_{x\to\pm\infty} \Phi_\varepsilon(t,x) = 0 & \text{in } (0,T) \\ N_\varepsilon(0,x) = n_0(x)-\theta(x),\ \ P_\varepsilon(0,x) = p_0(x)-\theta(x) & \text{in } \mathbb{R} \end{cases} \tag{28}$$

In order to pass to the limit in problem (28) when ε approaches 0, we need uniform estimates on $(N_\varepsilon,P_\varepsilon,\Phi_\varepsilon)$. Let us prove the following

Lemma 10. *There exists a constant $C(T)>0$ independent of ε such that for $t\in[0,T]$*

$$\|N_\varepsilon(t)\|_{L_g^2}^2+\|P_\varepsilon(t)\|_{L_g^2}^2+\|\partial_x\Phi_\varepsilon(t)\|_{L_g^2}^2+\|\Phi_\varepsilon(t)\|_2^2+\int_0^t(\|\partial_xN_\varepsilon(s)\|_{L_g^2}^2+\|\partial_xP_\varepsilon(s)\|_{L_g^2}^2)ds \leq C(T) \tag{29}$$

$$\|g^2\partial_tN_\varepsilon\|_{L^2(0,T;(H_g^1(\mathbb{R}))')}+\|g^2\partial_tP_\varepsilon\|_{L^2(0,T;(H_g^1(\mathbb{R}))')} \leq C(T). \tag{30}$$

Proof. We take back the formal calculus which have led to (9) and we use the regularity of the solutions given in theorem 2 and lemma 8. First since $(N_\varepsilon, P_\varepsilon) \in \Lambda_1$, we get as in the proof of Theorem 2

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} g^2(N_\varepsilon^2 + P_\varepsilon^2)dx + \alpha \int_{\mathbb{R}} g^2(|\partial_x N_\varepsilon|^2 + |\partial_x P_\varepsilon|^2)dx \leq$$
$$\eta \int_{\mathbb{R}} g^2(|\partial_x N_\varepsilon|^2 + |\partial_x P_\varepsilon|^2)dx + C_\eta \int_{\mathbb{R}} g^2(|\theta'|^2 + |\vartheta'|^2)\, dx + C_\eta \int_{\mathbb{R}} g^2|\partial_x \Phi_\varepsilon|^2 dx$$

where η is any positive constant. In the other hand, as $\partial_t \Phi_\varepsilon$ satisfies the equation (24) then testing it by Φ_ε, using the equations satisfied by N_ε and P_ε and integrating over $\mathbb{R}$, we get

$$\frac{1}{2}\frac{d}{dt}\Big(\varepsilon \int_{\mathbb{R}} g^4 \Phi_\varepsilon^2 dx + \int_{\mathbb{R}} g^2 |\partial_x \Phi_\varepsilon|^2 dx\Big) = J_1 + J_2$$

with

$$\begin{cases} J_1 = \displaystyle\int_{\mathbb{R}} g^2((\tilde{A}_{11} - \tilde{A}_{21})(N_\varepsilon, P_\varepsilon)\partial_x(N_\varepsilon + \theta) + (\tilde{A}_{12} - \tilde{A}_{22})(N_\varepsilon, P_\varepsilon)\partial_x(P_\varepsilon + \theta))\, \partial_x \Phi_\varepsilon \, dx \\ J_2 = \displaystyle\int_{\mathbb{R}} g^2(\tilde{B}_1 - \tilde{B}_2)(N_\varepsilon, P_\varepsilon)\, \partial_x(\Phi_\varepsilon + \vartheta) \,.\, \partial_x \Phi_\varepsilon \, dx. \end{cases}$$

Using Young inequality, we obtain

$$|J_1| \leq \eta \int_{\mathbb{R}} g^2(|\partial_x N_\varepsilon|^2 + |\partial_x P_\varepsilon|^2)dx + C_\eta \int_{\mathbb{R}} g^2|\theta'|^2 dx + C_\eta \int_{\mathbb{R}} g^2 |\partial_x \Phi_\varepsilon|^2 dx$$

$$|J_2| \leq \eta \int_{\mathbb{R}} g^2 |\partial_x \Phi_\varepsilon|^2 dx + C_\eta \int_{\mathbb{R}} g^2 |\vartheta'|^2 dx.$$

Gathering all these inequalities and choosing judiciously η, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\varepsilon \int_{\mathbb{R}} g^4 \Phi_\varepsilon^2 dx + \int_{\mathbb{R}} g^2(N_\varepsilon^2 + P_\varepsilon^2 + |\partial_x \Phi_\varepsilon|^2)dx\Big) + \frac{\alpha}{2}\int_{\mathbb{R}} g^2(|\partial_x N_\varepsilon|^2 + |\partial_x P_\varepsilon|^2)dx \leq \\ C_1 \int_{\mathbb{R}} g^2(|\theta'|^2 + |\vartheta'|^2)dx + C_2 \int_{\mathbb{R}} g^2 |\partial_x \Phi_\varepsilon|^2 dx \end{aligned} \tag{31}$$

with $C_i > 0$ independent of ε. Integrating between 0 and t, we get using Gronwall inequality, for all $t \in (0, T)$

$$\begin{aligned} \varepsilon \int_{\mathbb{R}} g^4 \Phi_\varepsilon^2(t) dx + \int_{\mathbb{R}} g^2(N_\varepsilon^2(t) + P_\varepsilon^2(t) + |\partial_x \Phi_\varepsilon(t)|^2)dx + \frac{\alpha}{2}\int_0^t \int_{\mathbb{R}} g^2(|\partial_x N_\varepsilon|^2 + |\partial_x P_\varepsilon|^2)dxds \leq \\ C_1 \exp(C_2 t) \int_{\mathbb{R}} g^2 \left[|n_0 - \theta|^2 + |p_0 - \theta|^2 + |\partial_x \Phi_\varepsilon(0)|^2 + \varepsilon g^2 |\Phi_\varepsilon(0)|^2\right] dx. \end{aligned} \tag{32}$$

From hypothesis (H5), we see that $\Phi_\varepsilon(0)$ solves the equation $-\partial x\left(g^2\partial_x\Phi_\varepsilon(0)\right)+\varepsilon g^4(\partial_x\Phi_\varepsilon(0)) = 0$ with $\lim\limits_{x\to\pm\infty}\Phi_\varepsilon(0,x)=0$ so $\Phi_\varepsilon(0)=0$ and we infer that

$$\varepsilon\int_{\mathbb{R}} g^4\Phi_\varepsilon^2(t)dx+\int_{\mathbb{R}} g^2(N_\varepsilon^2(t)+P_\varepsilon^2(t)+|\partial_x\Phi_\varepsilon(t)|^2)dx+\frac{\alpha}{2}\int_0^t\int_{\mathbb{R}} g^2(|\partial_x N_\varepsilon|^2+|\partial_x P_\varepsilon|^2)dxdt\le C(T)$$

for all $t\in[0,T]$. To obtain a uniform bound of Φ_ε in L^2, we use lemma 2 so we get for $t\in[0,T]$

$$\int_{\mathbb{R}}|\Phi_\varepsilon(t)|^2dx\le C(T).$$

Now we set $V_\varepsilon=(N_\varepsilon,P_\varepsilon)$, $\Theta=(\theta,\theta)$ and $U_\varepsilon=g(\tilde{A}(V_\varepsilon)\partial_x(V_\varepsilon+\Theta)+\tilde{B}(U_\varepsilon)\partial_x(\Phi_\varepsilon+\theta_1))$. We have $g^2\partial_tV_\varepsilon=g'U_\varepsilon+g\partial_xU_\varepsilon$ so $g\partial_tV_\varepsilon=g'g^{-1}U_\varepsilon+\partial_xU_\varepsilon$. Therefore $\|g\partial_tV_\varepsilon\|_{L^2(0,T;H^{-1}(\mathbb{R}))}\le C\|U_\varepsilon\|_{L^2(Q_T)}\le C(T)+C(\|\partial_xV_\varepsilon\|_{L^2_g}+\|\partial_x\Phi_\varepsilon\|_{L^2_g})$. The result follows using (29) and lemma 2. □

5 Passing to the limit

Thanks to (29) and (30) there exists subsequences (not relabeled) of $N_\varepsilon, P_\varepsilon, \Phi_\varepsilon$ and three functions $N,P\in L^2\left(0,T;H^1_g(\mathbb{R})\right)\cap H^1\left(0,T;(H^1_g(\mathbb{R}))'\right)$, $\Phi\in L^\infty\left(0,T;H^1_{1,g}(\mathbb{R})\right)$ such that as $\varepsilon\to 0$

$$\begin{aligned} N_\varepsilon &\rightharpoonup N \text{ weakly in } L^2\left(0,T;H^1_g(\mathbb{R})\right)\cap H^1\left(0,T;(H^1_g(\mathbb{R}))'\right) &(33)\\ P_\varepsilon &\rightharpoonup P \text{ weakly in } L^2\left(0,T;H^1_g(\mathbb{R})\right)\cap H^1\left(0,T;(H^1_g(\mathbb{R}))'\right) &(34)\\ \Phi_\varepsilon &\rightharpoonup \Phi \text{ weakly star in } L^\infty\left(0,T;H^1_{1,g}(\mathbb{R})\right). &(35)\end{aligned}$$

In order to pass to the limit in the nonlinear terms we need some strong convergence result. The compactness of the embedding $H^1_g(\mathbb{R})\hookrightarrow L^2(\mathbb{R})$ and Aubin lemma imply that

$$N_\varepsilon\to N\ ,\ P_\varepsilon\to P \text{ strongly in } L^2\left(0,T;L^2(\mathbb{R})\right) \text{ and a.e. in } Q_T$$

then thanks to lemma 9,

$$N+\theta,P+\theta\ge 0,\ N+P+2\theta\le 1 \text{ a.e. in } Q_T.$$

Moreover since the operators $\tilde{A}$ and $\tilde{B}$ are lipschitz continuous, we get

$$\tilde{A}(N_\varepsilon,P_\varepsilon)\to\tilde{A}\,(N,P)\ ,\ \tilde{B}(N_\varepsilon,P_\varepsilon)\to\tilde{B}\,(N,P) \text{ strongly in } L^2(Q_T).$$

Therefore (N,P,Φ) satisfy in the sense of distributions the equations

$$\left\{\begin{array}{ll} g^2\partial_t(N,P)=\partial_x[g^2(\tilde{A}\,(N,P)\,\partial_x(N+\theta,P+\theta)+\tilde{B}\,(N,P)\,\partial_x(\Phi+\vartheta))] & \text{in } Q_T\\ -\partial_x(g^2\partial_x\Phi)=-g^2(N-P+q(x)) & \text{in } Q_T\end{array}\right.\tag{36}$$

and we have the regularity $N, P \in L^\infty(0,T; L^2_g(\mathbb{R}))$, $g^2\partial_t N, g^2\partial_t P \in L^2\left(0,T; (H^1_g(\mathbb{R}))'\right)$, $g\partial_x^2\Phi \in L^\infty(0,T; L^2(\mathbb{R}))$ which lead to a weak solution of our problem according to definition 1.

Remark 2. *The result of this work remains valid in the case where* $k_n \neq k_p$ *and assumption (H2) replaced by one of the following assumptions*

$$k_n \geq k_w, \quad k_p \geq k_w, \quad k_n + k_p < 4k_w$$

$$k_n \leq k_w, \quad k_p \leq k_w, \quad k_n > \max(\frac{4k_w - k_p}{5}, 4k_w - 5k_p)$$

$$k_n \geq k_w, \quad k_p \leq k_w, \quad k_p > \max(k_n - 2k_w, \frac{2k_w + k_n}{5})$$

$$k_n \leq k_w, \quad k_p \geq k_w, \quad k_n > \max(k_p - 2k_w, \frac{2k_w + k_p}{5}).$$

Indeed in these cases, one has

$$4(k_n - (k_n - k_\omega)n)(k_p - (k_p - k_\omega)p) > ((k_n - k_\omega)p + (k_p - k_\omega)n)^2$$

so that the diffusion matrix $A(n,p)$ *is positive definite if* $0 \leq n\,, p\,,\, n+p \leq 1$.

6 Appendix: Proof of lemma 5

Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we distinguish two cases

1/ If $u^+ + v^+ \leq 1$ we have

$$A^+(u,v)\xi.\,\xi = \left(k - k'u^+\right)\xi_1^2 + \left(k - k'v^+\right)\xi_2^2 - k'(u^+ + v^+)\xi_1\xi_2.$$

We use the elementary inequality

$$ax^2 + by^2 + cxy \geq \min\left(\frac{4ab - c^2}{8b}, \frac{b\,(4ab - c^2)}{4ab + c^2}\right)\left(x^2 + y^2\right) \quad \text{for all } a, b > 0 \tag{37}$$

to obtain

$$A^+(u,v)\xi.\,\xi \geq \min\left(\frac{F_1(u,v)}{8\,(k - k'v^+)}, \frac{(k - k'v^+)\,F_1(u,v)}{G_1(u,v)}\right)\left(\xi_1^2 + \xi_2^2\right)$$

where

$$\begin{cases} F_1(u,v) = 4\,(k - k'u^+)\,(k - k'v^+) - k'^2(u^+ + v^+)^2 \\ G_1(u,v) = 4\,(k - k'u^+)\,(k - k'v^+) + k'^2(u^+ + v^+)^2. \end{cases}$$

We are proving that $F_1(u,v) \geq \alpha_0 > 0$, for this we consider two cases. In the case $k' \geq 0$, this inequality is equivalent to say that $4k^2 - k'^2 - 4kk' > 0$ which is satisfied thanks to

(H2). Similarly, if $k' < 0$ therefore **(H2)** implies that $k'^2 < 4k^2 - 4kk'$ then $k'^2 (u^+ + v^+) < 4k^2 (u^+ + v^+) - 4kk' (u^+ + v^+)$ which gives

$$k'^2 \left(u^+ + v^+\right)^2 < 4k^2 + 4k'^2 u^+ v^+ - 4kk' \left(u^+ + v^+\right)$$

2/ If $u^+ + v^+ \geq 1$ we have

$$A^+(u,v)\xi.\xi = \frac{1}{u^+ + v^+}\left(\left(kv^+ + k_w u^+\right)\xi_1^2 + \left(ku^+ + k_w v^+\right)\xi_2^2 - (k - k_w)(u^+ + v^+)\xi_1\xi_2\right).$$

Once again applying the elementary inequality (37), we find

$$A^+(u,v)\xi.\xi \geq \frac{1}{u^+ + v^+}\min\left(\frac{F_2(u,v)}{8\,(ku^+ + k_w v^+)}, \frac{(ku^+ + k_w v^+)\,F_2(u,v)}{G_2(u,v)}\right)\left(\xi_1^2 + \xi_2^2\right)$$

where

$$\begin{cases} F_2(u,v) = 4\,(kv^+ + k_w u^+)\,(ku^+ + k_w v^+) - (k - k_w)^2\,(u^+ + v^+)^2 \\ G_2(u,v) = 4\,(kv^+ + k_w u^+)\,(ku^+ + k_w v^+) + (k - k_w)^2\,(u^+ + v^+)^2. \end{cases}$$

We see that $F_2(u,v) = u^+v^+[4(k^2 + k_w^2) - ((u^+)^2 + (v^+)^2)(k^2 + k_w^2 - 6kk_w)] \geq -((u^+)^2 + (v^+)^2)(k^2 + k_w^2 - 6kk_w) \geq C((u^+)^2 + (v^+)^2)$

that is

$$\frac{F_2(u,v)}{(u^+ + v^+)\,(ku^+ + k_w v^+)} \geq \alpha \text{ and } \frac{(ku^+ + k_w v^+)\,F_2(u,v)}{(u^+ + v^+)\,G_2(u,v)} \geq \alpha \text{ if } u^+ + v^+ \geq 1$$

then matrix A^+ is definite positive.

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