ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES

UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 46 00. Fax: 01 69 33 46 46
http://www.cmap.polytechnique.fr/

Pricing and hedging gap risk

Peter Tankov

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Peter Tankov
Centre de Mathématiques Appliquées
Ecole Polytechnique
e-mail: peter.tankov@polytechnique.org

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Abstract

We analyze a new class of exotic equity derivatives called gap options or gap risk swaps. These products are designed by major banks to sell off the risk of rapid downside moves, called gaps, in the price of the underlying. We show that to price and manage gap options, jumps must necessarily be included into the model, and present explicit pricing and hedging formulas in the single asset and multi-asset case. The effect of stochastic volatility is also analyzed.

Key words: Gap risk, gap option, exponential Lévy model, quadratic hedging, Lévy copula

1 Introduction

The gap options are a class of exotic equity derivatives offering protection agains rapid downside market moves (gaps). These options have zero delta, allowing to make bets on large downside moves of the underlying without introducing additional sensitivity to small fluctuations, just as volatility derivatives allow to make bets on volatility without going short or long delta. The market for gap options is relatively new, and they are known under many different names: gap options, crash notes, gap notes, daily cliquets, gap risk swaps etc. The gap risk often arises in the context of constant proportion portfolio insurance (CPPI) strategies [9, 17]. The sellers of gap options (who can be seen as the buyers of the protection against gap risk) are typically major banks who want to get off their books the risk associated to CPPI products. The buyers of gap options and the sellers of the protection are usually hedge funds looking for extra returns.

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The pay-off of a gap option is linked to the occurence of a *gap event*, that is, a 1-day downside move of sufficient size in the underlying. The following single-name gap option was commercialized by a big international bank in 2007 under the name of *gap risk swap*:

Example 1 (Single-name gap option).

- The protection seller pays the notional amount N to the protection buyer at inception and receives Libor + spread monthly until maturity or the first occurence of the gap event, whichever comes first, plus the notional at maturity if no gap event occurs.
- The gap event is defined as a downside move of over 10% in the DJ Euro Stoxx 50 index within 1 day (close to close).
- If a gap event occurs between dates $t-1$ and t, the protection seller immediately receives the reduced notional $N(1-10*(0.9-R))^+$, where $R = \frac{S_t}{S_{t-1}}$ is the index performance at gap, after which the product terminates.

The gap options are therefore similar to equity default swaps, with a very important difference, that in EDS, the price change from the inception date of the contract to a given date is monitored, whereas in gap options, only 1-day moves are taken into account.

The pay-off of a multi-name gap option depends of the total number of gap events occuring in a basket of underlyings during a reference period. We are grateful to Zareer Dadachanji from Credit Suisse for the following example.

Example 2 (Multiname gap option).

- As before, the protection seller pays the notional amount N to the protection buyer and receives Libor + spread monthly until maturity. If no gap event occurs, the protection seller receives the full notional amount at the maturity of the contract.
- A gap event is defined as a downside move of over 20% during one business day in any underlying from a basket of 10 names.
- If a gap event occurs, the protection seller receives at maturity a reduced notional amount kN, where the reduction factor k is determined from the number M of gap events using the following table:

M	0	1	2	3	≥ 4
k	1	1	1	0.5	0

The gap options are designed to capture stock jumps, and clearly cannot be priced within a diffusion model with continuous paths, since any such model will largely underestimate the gap risk. For instance, for a stock with a 25% volatility, the probability of having an 10% gap on any one day during one year is 3×10^{-8}, and the probability of a 20% gap is entirely negligible. In this paper we therefore suggest to price and hedge gap options using models based on processes with discontinuous trajectories.

There is ample evidence for crash fears and jump risk premia in quoted European option prices [4, 6, 12, 16] and many authors have argued that jump models allow a precise calibration to short-term European calls and puts and provide an adequate vision of short-term crash risk [1, 3, 8]. Gap options capture exactly the same kind of risk; we will see in section 4 that an approximate hedge of a gap option can be constructed using out of the money puts. It is therefore natural to price and risk manage gap options within a model with jumps, calibrated to market quoted near-expiry Europeans.

The rest of the paper is structured as follows. Section 2 deals with the risk-neutral pricing of single-name gap options, discusses the necessary approximations and provides explicit formulas. The effect of stochastic volatility is also analyzed here. In section 3, we show how gap notes can be approximately hedged with short-dated OTM European options quoted in the market, derive the hedge ratios and illustrate the efficiency of hedging with numerical experiments. Multiname gap options are discussed in section 4.

2 Pricing single asset gap options

Suppose that the time to maturity T of a gap option is subdivided onto N periods of length Δ (e.g. days): $T = N\Delta$. The return of the k-th period will be denoted by $R_k^\Delta = \frac{S_{k\Delta}}{S_{(k-1)\Delta}}$. For the analytic treatment, we formalize the single-asset gap option as follows.

Definition 1 (Gap option)**.** Let α denote the return level which triggers the gap event and k^* be the time of first gap expressed in the units of Δ: $k^* := \inf\{k : R_k^\Delta \leq \alpha\}$. The gap option is an option which pays to its holder the amount $f(R_{k^*}^\Delta)$ at time Δk^*, if $k^* \leq N$ and nothing otherwise.

Supposing that the interest rate is deterministic and equal to r, it is easy to see that the pay-off structure of example 1 can be expressed as a linear combination of pay-offs of definition 1.

We first treat the case where the log-returns are independent and stationary.

Proposition 1. *Let the log-returns $(R_k^\Delta)_{k=1}^N$ be i.i.d. and denote the distribution of $\log R_1^\Delta$ by $p_\Delta(dx)$. Then the price of a gap option as of definition 1 is given by*

$$G_\Delta = e^{-r\Delta} \int_{-\infty}^{\beta} f(e^x) p_\Delta(dx) \frac{1 - e^{-rT}\left(\int_\beta^\infty p_\Delta(dx)\right)^N}{1 - e^{-r\Delta}\int_\beta^\infty p_\Delta(dx)}, \tag{1}$$

with $\beta := \log\alpha < 0$.

Proof.

$$
\begin{aligned}
G_\Delta &= E\left[e^{-\Delta k^* r} f(R^\Delta_{k^*}) 1_{k^* \le N}\right] \\
&= \sum_{n=1}^N \mathbb{P}[k^* = n] E[f(R^\Delta_n)|k^* = n] e^{-\Delta n r} \\
&= \sum_{n=1}^N \mathbb{P}[R^\Delta_n \le \alpha] E[f(R^\Delta_n)|R^\Delta_n \le \alpha] e^{-\Delta n r} \prod_{l=1}^{n-1} \mathbb{P}[R^\Delta_l > \alpha] \\
&= e^{-r\Delta} \int_{-\infty}^{\beta} f(e^x) p_\Delta(dx) \frac{1 - e^{-rT}\left(\int_\beta^\infty p_\Delta(dx)\right)^N}{1 - e^{-r\Delta}\int_\beta^\infty p_\Delta(dx)}.
\end{aligned}
$$

□

Numerical evaluation of prices Formula (1) allows to compute gap option prices by Fourier inversion. For this, we need to be able to evaluate the cumulative distribution function $F_\Delta(x) := \int_{-\infty}^x p_\Delta(d\xi)$ and the integral

$$\int_{-\infty}^{\beta} f(e^x) p_\Delta(dx). \tag{2}$$

Let ϕ_Δ be the characteristic function of p_Δ, and suppose that p_Δ satisfies $\int |x| p_\Delta(dx) < \infty$ and $\int_{\mathbb{R}} \frac{|\phi_\Delta(u)|}{1+|u|} du < \infty$. Let F' be the CDF and ϕ' the characteristic function of a Gaussian random variable with zero mean and standard deviation $\sigma' > 0$. Then by Lemma 1 in [9],

$$F_\Delta(x) = F'(x) + \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux} \frac{\phi'(u) - \phi_\Delta(u)}{iu} du. \tag{3}$$

The Gaussian random variable is only needed to obtain an integrable expression in the right hand side and can be replaced by any other well-behaved random variable.

The integral (2) is nothing but the price of a European option with payoff function f and maturity Δ. For arbitrary f it can be evaluated using the Fourier transform method proposed by Lewis [15]. However, in practice, the pay-off of a gap option is either a put option or a put spread. Therefore, for most practical purposes it is sufficient to compute this integral for $f(x) = (K - x)^+$, in which case a simpler method can be used. From [7, chapter 11], the price of such a put option with log forward moneyness $k = \log(K/S) - r\Delta$ is given by

$$P_\Delta(k) = P^{BS}_\Delta(k) + \frac{S_0}{2\pi}\int_{\mathbb{R}} e^{-ivk} \tilde{\zeta}_\Delta(v) dv, \tag{4}$$

where

$$\tilde{\zeta}_\Delta(v) = \frac{\phi_\Delta(v-i) - \phi^\sigma_\Delta(v-i)}{iv(1+iv)},$$

$\phi_\Delta^\sigma(v) = \exp\left(-\frac{\sigma^2 T}{2}(v^2 + iv)\right)$ and $P_\Delta^{BS}(k)$ is the price of a put option with log-moneyness k and time to maturity Δ in the Black-Scholes model with volatility $\sigma > 0$. Once again, the auxiliary Black-Scholes price is needed to regularize $\tilde{\zeta}$ and the exact value of σ is not very important.

Equations (3) and (4) can be used to compute the exact price of a gap option. In practice, the corresponding integrals will be truncated to a finite interval $[-L, L]$. Since Δ is small, the characteristic function $\phi_\Delta(u)$ decays slowly at infinity, which means that L must be sufficiently big (typically $L \sim 10^2$), and the computation of the integrals will be costly. On the other hand, precisely the fact that Δ is small allows, in exponential Lévy models, to construct an accurate approximation of the gap option price.

Approximate pricing formula In this section, we suppose that $S_t = S_0 e^{X_t}$, where X is a Lévy process. This means that p_Δ as defined above is the distribution of X_t.

Since $r\Delta \sim 10^{-4}$ and the probability of having a gap on a given day $\int_{-\infty}^{\beta} p_\Delta(dx)$ is also extremely small, with very high precision,

$$G_\Delta \approx \int_{-\infty}^{\beta} f(e^x) p_\Delta(dx) \frac{1 - e^{-rT - N\int_{-\infty}^{\beta} p_\Delta(dx)}}{r\Delta + \int_{-\infty}^{\beta} p_\Delta(dx)}. \tag{5}$$

Our second approximation is less trivial. From [18], we know that for all Lévy processes and under very mild hypotheses on the function f, we have

$$\int_{-\infty}^{\beta} g(x) p_\Delta(dx) \sim \Delta \int_{-\infty}^{\beta} g(x)\nu(dx),$$

as $\Delta \to 0$, where ν is the Lévy measure of X. Consequently, when Δ is nonzero but small, we can replace the integrals with respect to the density with the integrals with respect to the Lévy measure in formula (5), obtaining an approximate but explicit expression for the gap option price:

$$G_\Delta \approx G_0 = \lim_{\Delta \to 0} G_\Delta = \int_{-\infty}^{\beta} f(e^x)\nu(dx) \frac{1 - e^{-rT - T\int_{-\infty}^{\beta} \nu(dx)}}{r + \int_{-\infty}^{\beta} \nu(dx)}. \tag{6}$$

This approximation is obtained by making the time interval at which returns are monitored (a priori, one day), go to zero. It is similar to the now standard approximation used to replicate variance swaps:

$$\sum_{i=1}^{T/\Delta} (X_{i\Delta} - X_{(i-1)\Delta})^2 \approx \lim_{\Delta \to 0} \sum_{i=1}^{T/\Delta} (X_{i\Delta} - X_{(i-1)\Delta})^2 = \int_0^T \sigma_t^2 dt.$$

We now illustrate how this approximation works on a parametric example.

Example 3 (Gap option pricing in Kou's model). In this example we suppose that the stock price follows the exponential Lévy model [14] where the driving

Lévy process has a non-zero Gaussian component and a Lévy density of the form

$$\nu(x) = \frac{\lambda(1-p)}{\eta_+} e^{-x/\eta_+} 1_{x>0} + \frac{\lambda p}{\eta_-} e^{-|x|/\eta_-} 1_{x<0}. \tag{7}$$

Here, λ is the total intensity of positive and negative jumps, p is the probability that a given jump is negative and η_- and η_+ are characteristic lengths of respectively negative and positive jumps. In this case, for most common choices of f, the integrals in (6) can be computed explicitly:

$$\int_{-\infty}^{\beta} \nu(dx) = \lambda p e^{\beta/\eta_-}$$

and if we set $f(x) = (K-x)^+$ with $\log K \leq \beta$ then

$$\int_{-\infty}^{\beta} f(e^x)\nu(dx) = \frac{\lambda p \eta_-}{1+\eta_-} K^{1+1/\eta_-}.$$

The model parameter estimation is a tricky issue here: it is next to impossible to estimate the probability of a 10% gap from historical data, since the historical data simply does not contain negative daily returns of this size: for example, during the 6-year period from 2002 to 2008, the strongest negative return was -7%. The fact that 10% gap options do have positive prices can be explained by a peso effect: even though 10% negative return has never occured yet, the market participants believe that it has a positive probability of occurence in the future. The same effect explains prices of short maturity OTM puts [4]. This suggests to extract the information about the probability of sharp downside moves from short maturity OTM put prices by calibrating an exponential Lévy model to market option quotes, and use it to price gap options.

European options on the DJ Euro Stoxx 50 index are quoted on the Eurex exchange. Figure 1 shows the implied volatilities corresponding to the market option prices (observed on July 7, 2008) and the implied volatilities in the Kou model calibrated to these prices. The calibration was carried out by least squares with several starting points chosen at random to avoid falling into a local minimum. The calibrated parameter values are $\sigma = 0.23$, $\lambda = 7.04$, $p = 0.985$, $\eta_+ = 0.0765$ and $\eta_- = 0.0414$. Since the upward-sloping part of the smile is very small, the parameters of the positive jumps cannot be calibrated in a reliable manner but they are irrelevant for gap option pricing anyway. The gap option price is most affected by the intensity $p\lambda$ and the characteristic size η_- of negative jumps, and these are calibrated quite precisely from the negative-sloping part of the smile. The calibrated parameter values correspond to approximately one negative jump greater than 10% in absolute values every two years.

The calibrated parameter values were used to price the single-asset gap option of example 1 (with duration 1 year). With the exact formula (1) we obtained a price of 15.1% (this is interpreted as the percentage of the notional that the

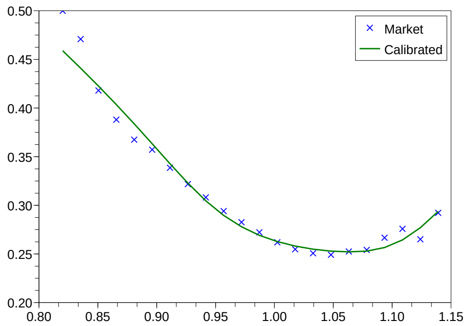

Figure 1: Observed and calibrated implied volatilities of 10 day options on the DJ Euro Stoxx 50 index, as a function of moneyness K/S_0.

protection buyer must pay to the protection seller in exchange in the beginning), and the approximate formula (6) gives 14.3%. Since the gap is a tail event and its probability cannot be estimated with high precision anyway, we conclude that the approximate formula provides sufficient accuracy in this context, and can be used to price and risk manage this product.

A modified gap option For a better understanding of the risks of a gap option, it is convenient to interpret the pricing formula (6) as an exact price of a modified gap option rather than the true price of the original option. From now on, we define the single-asset gap option as follows.

Definition 2 (Modified gap option)**.** Let $\tau = \inf\{t : \Delta X_t \leq \beta\}$ be the time of the first jump of X smaller than β. The gap option as a product which pays to its holder the amount $f\left(\frac{S_\tau}{S_{\tau-}}\right) = f(e^{\Delta X_\tau})$ if $\tau \leq T$ and zero otherwise.

The price of this product is given by

$$G = E^Q[e^{-r\tau} f(e^{\Delta X_\tau}) 1_{\tau \leq T}]$$

which is easily seen to be equal to G_0:

Proposition 2. *Suppose that the underlying follows an exponential Lévy model: $S_t = S_0 e^{X_t}$, where X is a Lévy process with Lévy measure ν. Then the price of the gap option as of definition 2, or, equivalently, the approximate price of the gap option as of definition 1 is given by*

$$G = \int_{-\infty}^{\beta} f(e^x)\nu(dx) \frac{1 - e^{-rT - T\int_{-\infty}^{\beta} \nu(dx)}}{r + \int_{-\infty}^{\beta} \nu(dx)}$$

with $\beta := \log \alpha$.

The gap option then arises as a pure jump risk product, which is only sensitive to negative jumps larger than β in absolute value, but not to small fluctuations of the underlying. In particular, it has zero delta. This new definition of gap option pay-off allows us to develop a number of extensions.

Stochastic interest rates Formula (6) is easily generalized to the case where the short interest rate r_t is a stochastic process. In this case the price of a gap option is given by

$$G = E^Q[e^{-\int_0^\tau r_s ds} f(e^{\Delta X_\tau}) 1_{\tau \leq T}].$$

Suppose that the process $(r_t)_{t\geq 0}$ is independent from the *jump part* of X. Then, conditioning the expectation on $(r_t)_{t\geq 0}$, we obtain

$$G = \int_{-\infty}^{\beta} f(e^x)\nu(dx) \int_0^T e^{-\lambda^* t} B(t) dt,$$

where $\lambda^* := \nu((-\infty, \beta])$ is the intensity of gap events and $B(t)$ is the price of a zero-coupon with maturity t (observed from the yield curve).

Stochastic volatility Empirical evidence suggests that independence of increments is not a property observed in historical return time series: stylized facts such as volatility clustering show that the amplitude of returns is positively correlated over time. This and other deviations from the case of IID returns can be accounted for introducing a "stochastic volatility" model for the underlying asset. It is well known that the stochastic volatility process with continuous paths

$$\frac{dS_t}{S_t} = \sigma_t dW_t$$

has the same law as a time-changed Geometric Brownian motion

$$S_t = e^{-\frac{v_t}{2} + W_{v_t}} = \mathcal{E}(W)_{v_t}, \quad \text{where} \quad v_t = \int_0^t \sigma_s^2 ds,$$

where the time change is given by the integrated volatility process v_t, provided that volatility is independent from the Brownian motion W governing the stock price.

In the same spirit, Carr et al. [5] have proposed to construct "stochastic volatility" models with jumps by time-changing an exponential Lévy model for the discounted stock price:

$$S_t^* = \mathcal{E}(L)_{v_t}, \qquad v_t = \int_0^t \sigma_s^2 ds$$

where L is a Lévy process and σ_t is a positive process. The stochastic volatility thus appears as a random time change governing the intensity of jumps, and can be seen as reflecting an intrinsic market time scale ("business time"). The volatility process most commonly used in the literature (and by practitioners) is the process

$$d\sigma_t^2 = k(\theta - \sigma_t^2)dt + \delta\sigma_t dW. \tag{8}$$

introduced in [11] which has the merit of being positive, stationary and analytically tractable. Other specifications such as positive Lévy-driven Ornstein-Uhlenbeck processes [2] can also be used. The Brownian motion W driving the volatility is assumed to be independent from the Lévy process L. For the specification (8), the Laplace transform of the integrated variance v is known in explicit form [11]:

$$\mathcal{L}(\sigma,t,u) := E[e^{-uv_t}|\sigma_0 = \sigma] = \frac{\exp\left(\frac{k^2\theta t}{\delta^2}\right)}{\left(\cosh\frac{\gamma t}{2} + \frac{k}{\gamma}\sinh\frac{\gamma t}{2}\right)^{\frac{2k\theta}{\delta^2}}} \exp\left(-\frac{2\sigma_0^2 u}{k + \gamma\coth\frac{\gamma t}{2}}\right)$$

with $\gamma := \sqrt{k^2 + 2\delta^2 u}$. In this approach, the stochastic volatility modifies the intensity of jumps, but not the distribution of jump sizes. The price of a gap option (definition 2) can be computed by first conditioning the expectation on the trajectory of the stochastic volatility. Since the formula (6) is exponential in T, we still get an explicit expression:

$$G_{\sigma_0} = \frac{1 - \mathcal{L}(\sigma_0, T, r + \int_{-\infty}^{\beta}\nu(dx))}{r + \int_{-\infty}^{\beta}\nu(dx)} \int_{-\infty}^{\beta} f(e^x)\nu(dx). \tag{9}$$

A few properties of gap option prices can be deduced from this formula directly.

- The price of gap risk protection is increasing in volatility σ_0: greater volatility makes time run faster and increases the frequency of gap events.
- Since the formula (6) is concave in T, taking into account the stochastic nature of volatility will reduce the price of a gap option compared to the constant volatility case.

Figure 2 shows the gap option price as function of the initial volatility level σ_0 with other parameters $\theta = 1$, $k = 2$ and $\delta = 2$. Since the volatility here is representing the intensity of the time change, the case $\sigma_0 = 1$ corresponds to the situation where the stochastic time runs, on average, at the same speed as the original time. As expected, the gap option price in this case is slightly smaller than in the constant volatility case.

3 Hedging gap options with short-dated European options

As remarked above, the (modified) gap option is a zero-delta product, which means that the associated risk cannot be delta-hedged and more generally, it

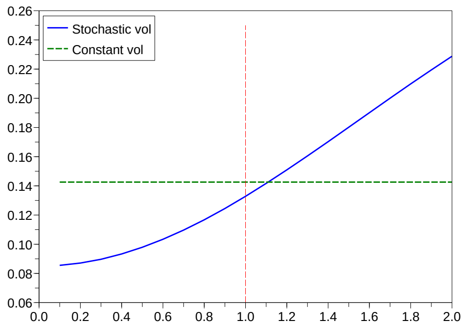

Figure 2: Solid line: the gap option price as function of the initial volatility level σ_0. Dashed line: gap option price with constant volatility corresponding to $\sigma_0 = 1$.

is hopeless to try to hedge it with the underlying. Moreover, the gap options are designed to offset jump risk, and the markets with jumps are typically incomplete [7]. Therefore, one can only try to approximately hedge a gap option, for example, in the sense of L^2 approximation, and even then one would need to find a suitable hedging product, which is sensitive to extreme downside moves of the underlying and has little sensitivity to the small everyday movements. A natural example of such product is an out of the money put option. As shown in figure 1, the strikes of market-traded 10 day puts can be as far as 15% out of the money. Since a 15% donwside move in 10 days is highly unlikely in a diffusion model, we conclude that these put options offer protection against jumps, that is, against the same kind of risk as the gap option. The gap option itself is nothing but a strip of 1-day puts, and if such options were traded, this would enable us to construct a perfect hedge. However, options with maturity below 1 week are not liquidly traded, so we will instead construct an approximate hedge using options maturing in 1-2 weeks.

Our aim is now to compute the optimal quadratic hedge ratio for hedging a gap option with an OTM put, that is, the hedge ratio minimizing the expected squared hedging error. Following [10], we suppose that this expected squared error is computed under the *martingale* probability. We start by expressing the martingale dynamics of a gap option price.

In this section, we suppose that the underlying price follows an exponential Lévy model $S_t = S_0 e^{X_t}$, and we denote by J the jump measure of X: $J([s,t] \times A) := \#\{r \in [s,t] : \Delta X_r \in A\}$. For details on jump measures of Lévy processes see [7, chapter 3]. The compensated version of J will be denoted by $\tilde{J}$: $\tilde{J}([s,t] \times A) := J([s,t] \times A) - (t-s)\nu(A)$. Moreover, to simplify the notation, we assume zero interest rate (in this section only). We use definition 2, and denote by G_t the price of a gap option evaluated at time t. The terminal value of a gap option can then be expressed as an integral with respect to J:

$$G_T = \int_0^T \int_{\mathbb{R}} f(e^x) 1_{t\leq\tau} J(dt \times dx).$$

Taking the conditional expectation under the risk-neutral probability, we can compute the price of a gap option at any time t:

$$G_t = E^Q[G_T|\mathcal{F}_t] = 1_{\tau\leq t} f(e^{\Delta X_\tau}) + 1_{\tau>t} \int_{-\infty}^{\beta} f(e^x)\nu(dx) \frac{1-e^{-\lambda^*(T-t)}}{\lambda^*}. \quad (10)$$

The interpretation of this formula is very simple: if, at time t, the gap event has already occured, then the price of a gap option is constant and equal to its pay-off; otherwise, it is given by the formula (6) applied to the remainder of the interval.

Formula (10) can be alternatively rewritten as a stochastic integral with respect to $\tilde{J}$:

$$G_t = G_0 + \int_0^t \int_{-\infty}^{\beta} 1_{s\leq\tau} e^{-\lambda^*(T-s)} f(e^x) \tilde{J}(ds \times dx).$$

Let $P(t,S)$ denote the price of a European put option evaluated at time t:

$$P(t,S) = E^Q[(K - S_T)^+ | S_t = S].$$

Via Itô's formula, we can express $P(t,S_t)$ as a stochastic integral as well:

$$\begin{aligned} P_t \equiv P(t,S_t) = P(0,S_0) &+ \int_0^t \sigma S_u \frac{\partial P(u,S_u)}{\partial S} dW_u \\ &+ \int_0^t \int_{\mathbb{R}} \{P(u, S_{u-}e^z) - P(u,S_{u-})\} \tilde{J}(du \times dz). \end{aligned}$$

A self-financing portfolio containing ϕ_t units of the put option and the risk-free asset has value V_t given by

$$V_t = c + \int_0^t \phi_s dP_s,$$

where c is the initial cost of the portfolio. The following result is then directly deduced from proposition 4 in [10].

Proposition 3. *The hedging strategy $(\hat{c}, \hat{\phi})$ minimizing the risk-neutral L^2 hedging error*

$$E^Q\left[\left(c + \int_0^T \phi_t dP_t - G_T\right)^2\right]$$

is given by

$$\hat{c} = E^Q[G_T] = G_0. \quad (11)$$

$$\hat{\phi}_t = 1_{t\leq\tau} \frac{\int_{-\infty}^{\beta} \nu(dz) f(e^z) e^{-\lambda^*(T-t)} \{P(t,S_te^z) - P(t,S_t)\}}{\sigma^2 S_t^2 \left(\frac{\partial P}{\partial S}\right)^2 + \int_{\mathbb{R}} \nu(dz) \{P(t,S_te^z) - P(t,S_t)\}^2}. \quad (12)$$

Note that $\hat{\phi}_t$ is nothing but the local regression coefficient of G_t on P_t. The cost of the hedging strategy, $\hat{c}$ coincides with the price of the gap option.

The strategy $\hat{\phi}_t$ is optimal but is does not allow perfect hedging (there is always a residual risk) and it is not feasible, because it requires continuous rebalancing of an option portfolio. In practice, due to relatively low liquidity of the option market, the portfolio will be rebalanced rather seldom, say, once a week or once every two weeks, as the hedging options arriving to maturity are replaced with more long-dated ones.

To test the efficiency of out of the money puts for hedging gap options, we simulate the L^2 hedging error (variance of the terminal $P\&L$) over one rebalancing period (one week or two weeks) using two feasible hedging strategies:

A The trader buys $\hat{\phi}_0$ options in the beginning of the period and keeps the number of the options constant until the end of the period.

B The trader buys $\hat{\phi}_0$ options in the beginning of the period and keeps the number of the options constant until the end of the period *unless a gap event occurs*, in which case the options are sold immediately.

To interpret the results, we also compute the L^2 error without hedging (strategy C) and for the case of continuous rebalancing (strategy D).

Table 1 reports the L^2 errors for the gap option of example 1 (with the notional value $N = 1$), computed over 10^6 scenarios simulated in Kou's model with the parameters calibrated to market option prices and given on page 6. For comparaison, the L^2 error of 10^{-4} correspond to the standard deviation of the hedging portfolio from the terminal gap option pay-off equal to 1% of the notional amount. We see that the strategy where the hedge ratio is constant up to a gap event and zero afterwards achieves a 4-fold reduction in the L^2 error compared to no hedging at all, if 1-week options are used. With 2-week options, the reduction factor is only 2.2. For every strategy, the L^2 error of hedging over a period of 2 weeks is greater than twice the error of hedging over 1 week: it is always better to use 1-week options than 2-week ones.

As seen from figure 3, hedging modifies considerably the shape of the distribution of the terminal P&L, reducing, in particular, the probability of extreme negative pay-offs. Without hedging, the distribution of the terminal pay-off of the gap option has an important peak at -1, corresponding to the maximum possible pay-off (the graphs are drawn from the point of view of the gap option seller). In the presence of hedging, this peak is absent and the distribution is concentrated around zero. If 1-week options are used for hedging, the Value at Risk of the portfolio for the horizon of 1 week and at the level of 0.1% is equal to 0.85 without hedging and only to 0.23 in the presence of hedging: this means that the hedging will allow to reduce the regulatory capital by a factor of four. If 2-week options are used, the 2-week VaR at the level of 0.1% is 0.38 with hedging, and without hedging it is equal to 0.99 (the probability of having a gap event within 2 weeks is slightly greater than 0.1%). We conclude that hedging gap options with OTM puts is feasible, but one should use the shortest

Period	Strategy A (constant hedge)	Strategy B (constant until gap then zero)	Strategy C (no hedging)	Strategy D (continuous rebalancing)
1 week	8.6×10^{-4}	5.6×10^{-4}	2.2×10^{-3}	2.5×10^{-4}
2 weeks	2.9×10^{-3}	2.0×10^{-3}	4.3×10^{-3}	7.6×10^{-4}

Table 1: L^2 errors for hedging a gap option with 1 week and 2-week European put options.

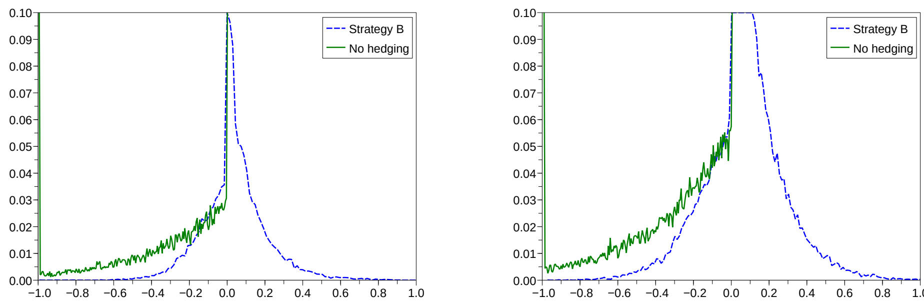

Figure 3: The histograms of the P&L distribution with and without hedging (from the point of view of the gap option seller). The peaks at -1 (maximum pay-off of a gap option) and at 0 (no gap event) were truncated at 0.1. Left: 1-week horizon; right: 2-weeks horizon.

available maturity: while 1-week puts give satisfactory results, hedging with 2-week optons appears problematic.

4 Multi-asset gap options

As explained in the introduction, a multiname (basket) gap option is a product where one monitors the total number of gap events in a basket of underlyings over the lifetime of the option $[0, T]$. A gap event is defined as a negative return of size less than α between consequtive closing prices (close-to-close) in any of the underlyings of the basket. The pay-off of the product at date T is determined by the total number of gap events in the basket over the reference period. To compute the price of a multiname gap option, we suppose that M underlying assets $S^1, \ldots, S^M$ follow an M-dimensional exponential Lévy model, that is, $S_t^i = S_0^i e^{X_t^i}$ for $i = 1, \ldots, M$, where $(X^1, \ldots, X^M)$ is an M-dimensional Lévy process with Lévy measure ν. In this section we will make the same simplifying hypothesis as in section 2 (definition 2), that is, we define a gap event as a negative jump smaller than a given value β in any of the assets, rather than a negative daily return. From now on, we define a multiname gap option as follows.

Definition 3. For a given $\beta < 0$, let

$$N_t = \sum_{i=1}^{M} \#\{(s,i) : s \le t, 1 \le i \le M \text{ and } \Delta X_s^i \le \beta\} \tag{13}$$

be the process counting the total number of gap events in the basket before time t. The multiname gap option is a product which pays to its holder the amount $f(N_T)$ at time T.

The pay-off function f for a typical multiname gap option is given in example 2. Notice that the single-name gap option stops at the first gap event, whereas in the multiname case the gap events are counted up to the maturity of the product.

The biggest difficulty in the multidimensional case, is that now we have to model simultaneous jumps in the prices of different underlyings. The multidimensional Lévy measures can be conveniently described using their tail integrals. The tail integral U describes the intensity of *simultaneous jumps in all components* smaller than the components of a given vector. Given an M-dimensional Lévy measure ν, we define the *tail integral* of ν by

$$U(z_1,\dots,z_M) = \nu(\{x \in \mathbb{R}^M : x_1 \le z_1,\dots,x_M \le z_M\}), \quad z_1,\dots,z_M < 0. \tag{14}$$

The tail integral can also be defined for positive z (see [13]), but we do not introduce this here since we are only interested in jumps smaller than a given negative value.

To describe the intensity of simultaneous jumps of a subset of the components of X, we define the *marginal* tail integral: for $m \le M$ and $1 \le i_1 < \dots < i_m \le M$, the $(i_1,\dots,i_m)$-marginal tail integral of ν is defined by

$$U_{i_1,\dots,i_m}(z_1,\dots,z_m) = \nu(\{x \in \mathbb{R}^M : x_{i_1} \le z_1,\dots,x_{i_m} \le z_m\}), \quad z_1,\dots,z_m < 0. \tag{15}$$

The process N counting the total number of gap events in the basket is clearly a piecewise constant increasing integer-valued process which moves only by jumps of integer size. The jump sizes can vary from 1 (in case of a gap event affecting a single component) to M (simultaneous gap event in all components). The following lemma describes the structure of this process via the tail integrals of ν.

Lemma 1. *The process N counting the total number of gap events is a Lévy process with integer jump sizes $1,\dots,M$ occuring with intensities $\lambda_1,\dots,\lambda_M$ given by*

$$\lambda_m = \sum_{k=m}^{M} (-1)^{k-m} \sum_{1 \le i_1 < \dots < i_k \le M} C_m^k U_{i_1,\dots,i_k}(\beta,\dots,\beta), \quad 1 \le m \le M, \tag{16}$$

where C_m^k denotes the binomial coefficient and the second sum is taken over all possible sets of k integer indices satisfying $1 \le i_1 < \dots < i_k \le M$.

Proof. Since X is a process with stationary and independent increments, it follows from formula (13) that N has stationary and independent increments as well. A jump of size m in N occurs if and only if exactly m components of X jump by an amount smaller or equal to β. Therefore,

$$\lambda_m = \sum_{1\le i_1<\dots<i_m\le M} \nu\left(\{x_i \le \beta\ \forall i \in \{i_1,\dots,i_m\}; x_i > \beta\ \forall i \notin \{i_1,\dots,i_m\}\}\right) \tag{17}$$

The expression under the sum sign can be written as

$$\begin{aligned}
&\nu\left(\{x_i \le \beta\ \forall i \in \{i_1,\dots,i_m\}; x_i > \beta\ \forall i \notin \{i_1,\dots,i_m\}\}\right)\\
&= \nu\left(\{x_i \le \beta\ \forall i \in \{i_1,\dots,i_m\}\}\right)\\
&+ \sum_{p=1}^{M-m} \sum_{\substack{1\le j_1<\dots<j_p\le M\\ \{j_1,\dots,j_p\}\cap\{i_1,\dots,i_m\}=\emptyset}} (-1)^p \nu\left(\{x_i \le \beta\ \forall i \in \{i_1,\dots,i_m\}\cup\{j_1,\dots,j_p\}\}\right)\\
&= U_{i_1,\dots,i_m}(\beta,\dots,\beta) + \sum_{p=1}^{M-m} \sum_{\substack{1\le j_1<\dots<j_p\le M\\ \{j_1,\dots,j_p\}\cap\{i_1,\dots,i_m\}=\emptyset}} (-1)^p U_{i_1,\dots,i_m,j_1,\dots,j_p}(\beta,\dots,\beta)
\end{aligned}$$

Combining this equation with (17) and gathering the terms with identical tail integrals, one obtains (16). □

The process N can equivalently be represented as

$$N_t = \sum_{m=1}^{M} m N_t^{(m)},$$

where $N^{(1)},\dots,N^{(M)}$ are independent Poisson processes with intensities $\lambda_1,\dots,\lambda_M$. Since these processes are independent, the expectation of any functional of N_T (the price of a gap option) can be computed as

$$E[f(N_T)] = e^{-\lambda T} \sum_{n_1,\dots,n_M=0}^{\infty} f\left(\sum_{k=1}^{M} k n_k\right) \prod_{i=1}^{M} \frac{(\lambda_i T)^{n_i}}{n_i!}, \tag{18}$$

where $\lambda := \sum_{i=1}^{M} \lambda_i$. In practice, after a certain number of gap events, the gap option has zero pay-off and the sum in (18) reduces to a finite number of terms. In example 2, $f(n) \equiv 0$ for $n \ge 4$ and

$$E[f(N_T)] = e^{-\lambda T}\Big\{1 + \lambda_1 T + \frac{(\lambda_1 T)^2}{2} + \lambda_2 T \tag{19}$$

$$+ \frac{(\lambda_1 T)^3}{12} + \frac{\lambda_1\lambda_2 T^2}{2} + \frac{\lambda_3 T}{2}\Big\}. \tag{20}$$

The price of the protection (premium over the risk-free rate received by the protection seller) is given by the discounted expectation of $1 - f(N_T)$, that is,

$$e^{-rT} E[1 - f(N_T)]. \tag{21}$$

To make computations with the formula (18), one needs to evaluate the tail integral of ν and all its marginal tail integrals. These objects are determined both by the individual gap intensities of each component and by the dependence among the components of the multidimensional process. For modeling purposes, the dependence structure can be separated from the behavior of individual components via the notion of Lévy copula [7, 13], which is parallel to the notion of copula but defined at the level of jumps of a Lévy process. More precisely we will use the *positive Lévy copulas* which describe the one-sided (in this case, only downward) jumps of a Lévy process, as opposed to general Lévy copulas which are useful when both upward and downward jumps are of interest.

Positive Lévy copulas Let $\overline{\mathbb{R}} := (-\infty, \infty]$ denote the extended real line, and for $a, b \in \overline{\mathbb{R}}^d$ let us write $a \leq b$ if $a_k \leq b_k$, $k = 1, \ldots, d$. In this case, $(a, b]$ denotes the interval

$$(a, b] := (a_1, b_1] \times \cdots \times (a_d, b_d].$$

For a function F mapping a subset $D \subset \overline{\mathbb{R}}^d$ into $\overline{\mathbb{R}}$, the *F-volume* of $(a, b]$ is defined by

$$V_F((a, b]) := \sum_{u \in \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}} (-1)^{N(u)} F(u),$$

where $N(u) := \#\{k : u_k = a_k\}$. In particular, $V_F((a, b]) = F(b) - F(a)$ for $d = 1$ and $V_F((a, b]) = F(b_1, b_2) + F(a_1, a_2) - F(a_1, b_2) - F(b_1, a_2)$ for $d = 2$. If $F(u) = \prod_{i=1}^d u_i$, the F-volume of any interval is equal to its Lebesgue measure.

A function $F : D \to \overline{\mathbb{R}}$ is called *d-increasing* if $V_F((a, b]) \geq 0$ for all $a, b \in D$ such that $a \leq b$. The distribution function of a random vector is one example of a d-increasing function.

A function $F : [0, \infty]^d \to [0, \infty]$ is called a *positive Lévy copula* if it satisfies the following conditions:

1. $F(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \ldots, d\}$,
2. F is d-increasing,
3. $F_i(u) = u$ for any $i \in \{1, \ldots, d\}$, $u \in \mathbb{R}$, where F_i is the one-dimensional margin of F, obtained from F by replacing all arguments of F except the i-th one with ∞:

$$F_i(u) = F(u_1, \ldots, u_d)_{u_i = u, u_j = \infty \, \forall j \neq i}.$$

The positive Lévy copula has the same properties as ordinary copula but is defined on a different domain ($[0, \infty]^d$ instead of $[0, 1]^d$). Higher-dimensional margins of a positive Lévy copula are defined similarly:

$$F_{i_1, \ldots, i_m}(u_1, \ldots, u_m) = F(v_1, \ldots, v_d)_{v_{i_k} = u_k, k = 1, \ldots, m; v_j = \infty, j \notin \{i_1, \ldots, i_m\}}.$$

The Lévy copula links the tail integral to one-dimensional margins; the following result is a direct corollary of Theorem 3.6 in [13].

Proposition 4.

- *Let $X = (X^1, \ldots, X^d)$ be a $\mathbb{R}^d$-valued Lévy process, and let the (one-sided) tail integrals and marginal tail integrals of X be defined by* (14) *and* (15)*. Then there exists a positive Lévy copula F such that the tail integrals of X satisfy*

$$U_{i_1,\ldots,i_m}(x_1, \ldots, x_m) = F_{i_1,\ldots,i_m}(U_{i_1}(x_1), \ldots, U_{i_m}(x_m)) \tag{22}$$

  for any nonempty index set $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, d\}$ and any $(x_1, \ldots, x_m) \in (-\infty, 0)^m$.

- *Let F be an M-dimensional positive Lévy copula and $U_i, i = 1, \ldots, d$ tail integrals of real-valued Lévy processes. Then there exists a $\mathbb{R}^d$-valued Lévy process X whose components have tail integrals $U_i, i = 1, \ldots, d$ and whose marginal tail integrals satisfy equation* (22) *for any nonempty index set $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, d\}$ and any $(x_1, \ldots, x_m) \in (-\infty, 0)^m$.*

In terms of the Lévy copula F of X and its marginal tail integrals, formula (16) can be rewritten as

$$\lambda_m = \sum_{k=m}^{M} (-1)^{k-m} \sum_{1 \le i_1 < \cdots < i_k \le M} C_m^k F_{i_1,\ldots,i_k}(U_{i_1}(\beta), \ldots, U_{i_k}(\beta))$$

To compute the intensitites λ_i and price the gap option, it is therefore sufficient to know the individual gap intensities $U_i(\beta)$ (M real numbers), which can be estimated from 1-dimensional gap option prices or from the prices of short-term put options as in section (2), and the Lévy copula F. This Lévy copula will typically be chosen in some suitable parametric family. One convenient choice is the Clayton family of (positive) Lévy copulas defined by

$$F^\theta(u_1, \ldots, u_M) = \left(u_1^{-\theta} + \cdots + u_M^{-\theta}\right)^{-1/\theta}.$$

The dependence structure in the Clayton family is determined by a single parameter $\theta > 0$. The limit $\theta \to +\infty$ corresponds to complete dependence (all components jump together) and $\theta \to 0$ produces independent components. The Clayton family has the nice property of being margin-stable: if X has Clayton Lévy copula then all lower-dimensional margins also have Clayton Lévy copula:

$$F^\theta_{i_1,\ldots,i_m}(u_1, \ldots, u_m) = \left(u_1^{-\theta} + \cdots + u_m^{-\theta}\right)^{-1/\theta}.$$

For the Clayton Lévy copula, equation (16) simplifies to

$$\lambda_m = \sum_{k=m}^{M} (-1)^{k-m} \sum_{1 \le i_1 < \cdots < i_k \le M} C_m^k (U_{i_1}(\beta)^{-\theta} + \cdots + U_{i_k}(\beta)^{-\theta})^{-1/\theta}.$$

This formula can be used directly for baskets of reasonable size (say, less than 20 names). For very large baskets, one can make the simplifying assumption that all individual stocks have the same gap intensity: $U_k(\beta) = U_1(\beta)$ for all k. In this case, formula (16) reduces to the following simple result:

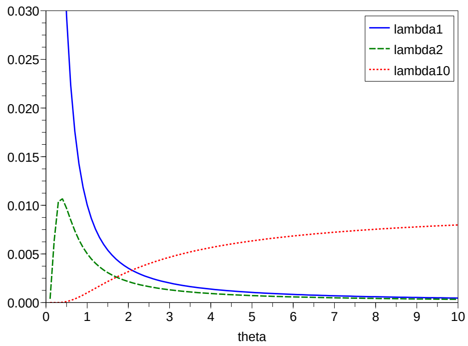

Figure 4: The intensities λ_i of different jump sizes of the gap counting process as a function of θ for $M = 10$ names and a single-name loss probability of 1%.

Proposition 5. *Suppose that the prices of M underlyings follow an M-dimensional exponential Lévy model with Lévy measure ν. If the individual components of the basket are identically distributed and the dependence structure is described by the Clayton Lévy copula with parameter θ, the price of a basket gap option as of definition 13 is given by*

$$E[f(N_T)] = e^{-\lambda T} \sum_{n_1,\ldots,n_M=0}^{\infty} f\left(\sum_{k=1}^{M} k n_k\right) \prod_{i=1}^{M} \frac{(\lambda_i T)^{n_i}}{n_i!},$$

where

$$\lambda_m = U_1(\beta) C_m^M \sum_{j=0}^{M-m} \frac{(-1)^j C_j^{M-m}}{(m+j)^{1/\theta}} \tag{23}$$

Figure 4 shows the behavior of the intensities λ_1, λ_2 and λ_{10} as a funtction of the dependence parameter θ in a basket of 10 names, with a single-name gap probability of 1%. Note that formula (23) implies

$$\lim_{\theta\to\infty} \lambda_m = \begin{cases} 0, & m < M \\ U_1(\beta), & m = M. \end{cases}$$

$$\lim_{\theta\to 0} \lambda_m = \begin{cases} 0, & m > 1 \\ MU_1(\beta), & m = 1. \end{cases},$$

in agreement with the behavior observed in Figure 4.

Figure 5 shows the price of the multiname gap option of example 2 computed using the formula (21). The price achieves a maximum for a finite nonzero value of θ. This happens because for this particular payoff structure, the protection seller does not loose money if only 1 or 2 gap events occur during the lifetime of

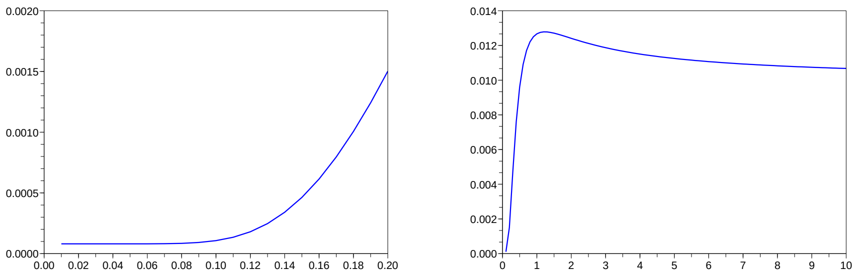

Figure 5: Expected loss of a multi-name gap option in the Credit Suisse example as a function of θ. The single-name loss probability is 1%.

the product, and only start to pay after 3 or more gap events. The probability of having 3 or more gap events is very low with independent components.

While the single-name gap intensity can be approximated from the prices of out of the money puts, the dependence parameter θ is difficult to extract from market data. Moreover, the choice of Lévy copula is far from being trivial and different choices may give different prices for the gap option. Formulas (23) and (18) can therefore only be seen as an crude approximation, which allows to convert, using graphs like figure 5, the trader's views of the probability of simultaneous gap events into a dependence parameter and then into an estimate of the gap option price.

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