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Time Consistent Dynamic Limit Order Books Calibrated on Options

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Abstract

In an incomplete financial market, the axiomatic of Time Consistent Pricing Procedure (TCPP), recently introduced, is used to assign to any financial asset a dynamic limit order book, taking into account both the dynamics of basic assets and the limit order books for options.
Kreps-Yan fundamental theorem is extended to that context. A characterization of TCPP calibrated on options is given in terms of their dual representation. In case of perfectly liquid options, these options can be used as the basic assets to hedge dynamically. A generic family of TCPP calibrated on option prices is constructed, from càdlàg BMO martingales.
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1 Introduction

The problem of dynamic pricing is the problem of extending a function that gives the prices of marketed financial instruments to a larger class of financial instruments. The usual way of dynamic pricing in financial mathematics is to start with a (No Free Lunch) dynamic model for the stock prices and to use the theory of portfolios constructed from these basic assets to price the other financial instruments. The first step along these lines was made by Black Scholes and Merton. In a complete market the dynamic price of any financial instrument X is then equal to the dynamic price of the replicating portfolio. As pointed out by Avellaneda and Paras [1] and [2], the market prices of options give important informations on the volatility. Therefore the prices constructed from the theory associated with options have to be compatible with their observed bid and ask prices. If the constructed prices do not lie within the interval defined by the observed bid and ask prices, this means that the choosen dynamics for the basic assets induce an arbitrage in the financial market. For example the Black Scholes model with constant volatility is not compatible with the call and put options prices. This is the volatility smile effect.

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In a Brownian setting, the notion of implied volatility has been introduced, inverting the Black Scholes formula for the options prices. Then a wide litterature has been developped trying to modelize the implied volatility. Necessary conditions for the resulting model to be arbitrage free have been given by Schönbucher [20]. However there exists no dynamic model for implied volatility leading to arbitrage free prices. Other approaches have been developped in order to produce, in a arbitrage free way, dynamic prices consistant with observed prices for options. The local volatility model introduced by Dupire [12] is an arbitrage free dynamic model of one stock, in a Brownian setting, assuming a particular shape for the volatility. It assumes also that one observes in the market the prices of call options on this asset for all strikes and all maturity dates and furthermore that the corresponding function is very smooth (in particular of C^2 class in the strike). This leads to a non robust model. In addition this model is a complete model for one stock which is not compatible with some observations in the market. Other approaches have been introduced recently in order to price dynamically in a arbitrage free way, taking into account the observed prices for options. Jacod and Protter [14] and Schweizer and Wissel [21] assume that only options with one fixed payoff function but all maturities are traded. Schweizer and Wissel [22] consider also the case where call options with one fixed maturity but all strikes are traded. In both cases the dynamics of the stock and of the options are modelized simultaneously in a arbitrage free way. However in real financial markets options of various kind with various strikes and various maturities are traded. Only a finite number of options are traded and not a continuum.
Furthermore the options are not all perfectly liquid. At some fixed instant only a limit order book is observed for some options and not a price. For n large enough the ask price of nX is larger than n times the ask price of X. This implies that the market is incomplete. When the model for the stock prices is not complete, there are several equivalent local martingale measures for the stock prices, or equivalently financial assets are not perfectly replicated by portfolios in the basic assets. Thus a natural way of assigning a dynamic ask price to a financial asset X defined at time T (for example an option of maturity date T), using the theory of portfolios, is to consider portfolios in the basic assets dominating at time T this asset X. This leads to the super-replication price, originally studied by El Karoui and Quenez [13], which is the minimal price of portfolios in the basic assets dominating X. The super-replication price is sublinear. The dynamic super-replication price is equal to $\mathrm{esssup}_{Q\in\mathcal{Q}}E_Q(X|\mathcal{F}_t)$, where $\mathcal{Q}$ is the set of all equivalent local martingale measures for the stock prices. However for many models this super-replication price is too high. It doesn't lie within the interval defined from the observed bid and ask prices associated with the option. Notice also that in case of linear or sublinear ask prices, the ask price associated with nX ($n \geq 0$) for any financial asset X is equal to n times the ask price associated with X. Linear or sublinear prices don't take into account the liquidity risk. On the contrary the observation of limit order books leads to the conclusion that for any traded asset Y the ask price of nY is a convex, not sublinear, function of n.

The context of the present paper is that of an incomplete and illiquid market. We construct a dynamic pricing theory taking into account both the dynamics of basic assets and the limit order books of options on these assets. This is done making use of the theory of No Free Lunch TCPP introduced in [5]. We consider a reference family composed of two kinds of assets: the basic assets $(S^k)_{0 \leq k \leq d+1}$ for which the dynamic process is assumed to be known , and the assets $(Y^l)_{1 \leq l \leq d})$ (for examples options) which are only revealed at their maturity date (the stopping time τ_l) and for which one observes a limit order book at time 0. One of the basic asset S^0 is assumed to be strictly positive and is taken as numéraire.
The first question we address is the question of non existence of arbitrage for the reference family $((S^k)_{0 \leq k \leq d+1}, (Y^l)_{1 \leq l \leq d})$ and the observed limit order books associated with the assets Y^l. We extend to that context the notion of No Free Lunch, replacing the usual notion of dynamic strategy with respect to the basic assets (S^k) by the sum of a dynamic strategy with respect to the basic assets (S^k) and of a static strategy with respect to the options Y^l. We prove the following generalization of Kreps-Yan Theorem: there is No Free Lunch with respect to the reference family if and only there is an equivalent local martingale measure Q for the process $(S^k)_k$ such that, for every l, and any $n \geq 0$, $C_{bid}(nY^l) \leq E_Q(nY^l) \leq C_{ask}(nY^l)$. The conditional expectation with respect to Q provides then a linear pricing procedure calibrated on the reference family. However as mentioned above, in order to take into account the liquidity risk, we do not want to restrict to linear nor sublinear pricing procedures. The theory of No Free Lunch TCPP takes into account the liquidity risk and allows for the construction of dynamic limit order books associated with any financial instrument. This construction is done in a arbitrage free way and consistently in time.
The second main result is the characterization of TCPP calibrated on the reference family $((S^k)_{0 \leq k \leq d}, (Y^l)_{1 \leq l \leq p})$ and the limit order books in terms of their dual representation. We also study the supply curves for such TCPP.
The third result concerns the study of the hedge in the case where both the basic assets S^k and the options Y^l are assumed to be very liquid. In that case we prove that the options can be used to hedge dynamically as well as the basic assets.
The last important result of the paper is the generic construction of a family of convex No Free Lunch TCPP calibrated on option prices. We prove the existence of a non sublinear TCPP calibrated on option prices belonging to the class first introduced in [3], making use of the theory of right continuous BMO martingales, as soon as the reference family satisfies the robust No Free Lunch condition. This construction is made in a very general setting of locally bounded stochastic processes, for which jumps are allowed.

2 First Fundamental Theorem

2.1 The economic model

We work with a filtered probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\in I\!R^+}, P)$ throughout this paper. The filtration $(\mathcal{F}_t)_{t\in I\!R^+}$ satisfies the usual assumptions of right continuity and completeness and $\mathcal{F}_0$ is assumed to be the σ-algebra generated by the P null sets of $\mathcal{F}_\infty$. We assume that the time horizon is infinite, which is the most general case. Indeed if the time horizon is finite equal to T we define $\mathcal{F}_s = \mathcal{F}_T$ for every $s \geq T$.

The usual way of dynamic pricing is to start with some reference assets for which the dynamics is assumed to be known and to construct a dynamic pricing procedure extending the dynamics of these reference assets. In order to use more information from the market, we want to take also into account the limit order books associated with some options.

Therefore the reference family will be composed of two kinds of assets: As usual, we consider that there are some basic assets $(S^k)_{0\leq k\leq d}$ for which we have a good idea of the evolution of their dynamics and we want to take into account all these dynamics. From a newer point of view there are also assets $(Y^l)_{1\leq l\leq p}$ as options of various maturity dates on one or several of the basic assets on which there are a lot of transactions, so that it is meaningful to take into account the corresponding limit order books observed in the market. Notice that even if one knows the dynamics of the underlying assets, one doesn't know, in an incomplete market, the dynamics of options. The option is revealed at time τ_l which is the maturity date of the option. The value at time τ_l of this option is therefore modeled by a $\mathcal{F}_{\tau_l}$ measurable function Y^l. We assume that in the market at time 0, a limit order book is observed for each of the options $(Y^l)_{1\leq l\leq p}$.

We assume that S^0 is always positive, and we can take it as numéraire. So from now on, $(S^0)_t = 1 \ \forall \ t \in I\!R^+$, $S_t = (S^k_t)_{1\leq k\leq d}$ models the discounted price process of d risky assets, and Y^l the discounted prices of options (at time τ_l). S is assumed to be a locally bounded stochastic process with a.s. càdlàg trajectories.

For any stopping time τ, denote $\mathcal{F}_\tau$ the σ-algebra defined by
$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty | \forall \ t \in I\!R^+ \ A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$. Denote $L^\infty(\Omega, \mathcal{F}_\tau, P)$ the Banach algebra of essentially bounded real valued $\mathcal{F}_\tau$ measurable functions. We will always identify an essentially bounded $\mathcal{F}_\tau$ measurable function with its class in $L^\infty(\Omega, \mathcal{F}_\tau, P)$.

The aim of this section is to define a notion of no arbitrage extending the usual one and to prove in this new context a first fundamental theorem generalizing the Kreps Yan theorem. Before that we want to point out some properties of the limit order books.

2.2 Limit order books

Let Y^l be a traded financial asset. One assumes that at time t_0 one observes a limit order book associated with the asset Y^l. The limit order book takes into account only the non executed orders at time t_0. One assumes that all the non executed orders on the asset Y^l are written in the following tabular.

Bid	
quantity	limit
M_1	C^1_{bid}
M_2	C^2_{bid}
...	...
M_p	C^p_{bid}

Ask	
limit	quantity
C^1_{ask}	N_1
C^2_{ask}	N_2
...	...
C^q_{ask}	N_q

with

$$C^p_{bid} < ... < C^1_{bid} < C^1_{ask} < ... < C^q_{ask} \tag{1}$$

If there is also a transaction at time t_0 on the asset Y^l, we denote C^0 the price of the transaction and $N_0 = M_0$ the number of shares exchanged at time t_0 (if there is no transaction on Y^l at time t_0, $N_0 = M_0 = 0$). Necessarily,

$$C^1_{bid} \leq C^0 \leq C^1_{ask} \tag{2}$$

Taking into account the limit order book, one can canonically associate to any positive integer $n \leq \sum_{0\leq i\leq q} N_i = N^l$ the ask price $C_{ask}(nY^l)$ defined as follows: Let $j \leq q$ be such that $\sum_{0\leq i\leq j-1} N_i \leq n < \sum_{0\leq i\leq j} N_i$. Define

$$C_{ask}(nY) = \sum_{0\leq i\leq j-1} N_i C^i_{ask} + (n - \sum_{0\leq i\leq j-1} N_i)C^j_{ask}$$

if $0 \leq n \leq N_0$, $C_{ask}(nY) = nC^0$. The bid price associated with nY for $n \leq \sum_{0\leq i\leq p} M_i = M^l$ is defined in a similar way.
Notice that it is easy to verify from the definition of $C_{ask}(nY)$ and the relations (1) and (2) that the map $n \in I\!N \to C_{ask}(nY)$ is convex and the map $n \in I\!N \to C_{bid}(nY)$ is concave. In particular $n \in I\!N \to \frac{C_{ask}(nY)}{n}$ is increasing and $n \in I\!N \to \frac{C_{bid}(nY)}{n}$ is decreasing. Also for any n, m $\frac{C_{bid}(mY)}{m} \leq \frac{C_{ask}(nY)}{n}$.

2.3 Fundamental Theorem

The first step is to define the notion of admissible simple strategy in this new setting. The investor can use two kinds of assets. The basic assets S^k for which the dynamics are assumed to be known, therefore an investor can trade dynamically using the S^k. He can also invest in the assets Y^l but these assets are only known at their maturity date τ_l and not at any intermediate date, therefore we restrict to static investments on Y^l between the dates 0 and τ_l. From the observation of the limit order book associated with Y^l at time 0 we associate, as in the previous Section 2.2, to any $n \leq N^l$ an ask price $C_{ask}(nY^l)$ and to any $n \leq M^l$ a bid price $C_{bid}(nY^l)$.

Definition 2.1 *An admissible simple strategy with respect to the reference assets* $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ *is the sum of a dynamic simple strategy* H *with respect to the process* (S^k) *and of a static strategy with respect to the random variables* (Y^l).
$H = \sum_{i=1}^n h_i \mathcal{X}_{]\sigma_{i-1},\sigma_i]}$, *where* $0 = \sigma_0 \leq \sigma_1 \leq ... \leq \sigma_n$ *are finite stopping times and* h_i *are essentially bounded* $\mathbb{R}^d$ *valued* $\mathcal{F}_{\sigma_{i-1}}$ *measurable functions and the stopped process* $(S^k)^{\sigma_n}$ *is uniformly bounded.*

Define now the convex set of contingent claims available at zero or negative price, using admissible simple strategies, taking into account the fact that for the random variables Y^l, one observes a limit order book. In all the following the limit order book observed for Y^l will be denoted $(C_{bid}(mY^l), C_{ask}(nY^l))$. This means:
$(C_{bid}(mY^l)_{0\leq m\leq M^l}, C_{ask}(nY^l)_{0\leq n\leq N^l})$.

Definition 2.2 *The convex set of portfolios available at zero cost is:*

$$K = \{\sum_{i=1}^n \sum_{k=1}^d (h_i^k)(S_{\sigma_i}^k - S_{\sigma_{i-1}}^k) + \sum_{l=1}^p (\gamma^l - \beta^l)Y^l + (\gamma^0 - \beta^0) \ ; (h^k)_i \in L^\infty(\mathcal{F}_{\sigma_{i-1}}),$$

$$\beta^l, \gamma^l \in \mathbb{N} \ \beta^l \leq N^l, \ \gamma^l \leq M^l \mid \sum_{l=1}^p (C_{ask}(\gamma^l Y^l) - C_{bid}(\beta^l Y^l) + (\gamma^0 - \beta^0) \leq 0\}.$$

Notice that adding the condition: for any l, either γ^l or β^l is equal to 0 in the definition of K would not change the set K. An element of K is the sum of a static portfolio in the options Y^l corresponding to γ^l long position in Y^l and β^l short position in Y^l and of a dynamic portfolio in the assets S^k available at price 0. The convexity of K follows from the convexity (resp concavity) of the map $\gamma \to C_{ask}(\gamma Y^l)$ (resp $\beta \to C_{bid}(\beta Y^l)$). Denote $\tilde{K}$ the set of portfolios dominated by an element of K, $\tilde{K} = K - L_+^\infty$.
In this setting, we say that there is No Arbitrage if there is no non trivial non negative attainable claim ot zero cost, i.e. $K \cap L_+^\infty(\Omega, \mathcal{F}, P) = \{0\}$. Notice that this condition is equivalent to $C \cap L_+^\infty(\Omega, \mathcal{F}, P) = \{0\}$, where C is the cone generated by $\tilde{K}$.
We prove now a theorem generalizing the Kreps Yan theorem to that context. As in the usual setting, the notion of No Arbitrage is not sufficient, we have to pass to the notion of No Free Lunch.

Definition 2.3 *The reference family* $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq d})$ *satisfies the No Free Lunch condition with respect to the limit order books* $C_{bid}(mY^l), C_{ask}(nY^l)$ *if the closure* $\overline{C}$ *of* C *with respect to the weak* topology of* $L^\infty(\Omega, \mathcal{F}, P)$ *satisfies* $\overline{C} \cap L_+^\infty(\Omega, \mathcal{F}, P) = \{0\}$.

First fundamental theorem generalizing Kreps-Yan theorem:

Theorem 2.4 *The following conditions are equivalent:*

i) The reference family $((S^k)_{0\le k\le d}, (Y^l)_{1\le l\le d})$ satisfies the No Free Lunch condition with respect to the limit order books $C_{bid}(mY^l), C_{ask}(nY^l)$.

ii) There is an equivalent local martingale measure R for $(S^k)_{0\le k\le d}$ such that for any $l \in 1,...,p$, for $m \le M^l$ $C_{bid}(mY^l) \le E_R(mY^l)$ and for $n \le N^l$ $E_R(nY^l) \le C_{ask}(nY^l)$.

We will give a more complete version of this Theorem in Section 3.4 after having discussed the notion of TCPP calibrated on option prices. The proof is in Appendix A.1.

3 TCPP calibrated on options

3.1 TCPP calibrated on a reference family

Recall briefly the definition of TCPP (Time Consistent Dynamic Pricing Procedure) , that we have introduced in [5] in order to assign to any financial product a dynamic limit order book in a financial market with transaction costs and liquidity risk. Other definitions close to the following one can be found in Peng [17], with deterministic times instead of stopping times and in the restrictive context of a Brownian filtration, in Cheridito et al [6] in a discrete time setting, and in Klöppel and Schweizer [15] with just one deterministic time.

Definition 3.1 *Let $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, P)$ be a filtered probability space. A TCPP $(\Pi_{\sigma,\tau})_{0\le\sigma\le\tau}$ (where $\sigma \le \tau$ are stopping times) is a family of maps*

$$\Pi_{\sigma,\tau} : L^\infty(\mathcal{F}_\tau) \to L^\infty(\mathcal{F}_\sigma)$$

satisfying the properties of monotonicity, translation invariance, convexity, normalization, continuity from below and time consistency.
For any $X \in L^\infty(\mathcal{F}_\tau)$, the dynamic ask (resp. bid) price process of X is $(\Pi_{\sigma,\tau}(X))_\sigma$ (resp. $(-\Pi_{\sigma,\tau}(-X))_\sigma$).
A TCPP is called sublinear if furthermore $\forall\lambda > 0$ $\forall X \in L^\infty(\mathcal{F}_\tau)$, $\Pi_{\sigma,\tau}(\lambda X) = \lambda\Pi_{\sigma,\tau}(X)$.

Recall that for any $X \in L^\infty(\mathcal{F}_\tau)$, $-\Pi_{\sigma,\tau}(-X) \le \Pi_{\sigma,\tau}(X)$ Notice that it follows from time consistency and normalization that for any $\nu \le \sigma \le \tau$, $\Pi_{\nu,\sigma}$ is the restriction of $\Pi_{\nu,\tau}$ to $L^\infty(\mathcal{F}_\sigma)$. Remark that a TCPP assigns to any financial instrument $X \in L^\infty(\mathcal{F}_\infty)$ not only a dynamic bid and ask prices, but also a dynamic limit order book $(-\Pi_{\sigma,\infty}(-nX), \Pi_{\sigma,\infty}(nX))_\sigma$, satisfying at any time, the properties observed for limit order books in real financial markets (Section 2.2).
A TCPP is equal, up to a minus sign, to a normalized time consistent dynamic risk measure.
In this paper we restrict our attention to No Free Lunch TCPP. For the definition and the general study of No Free Lunch TCPP we refer to [5]. Recall in

particular that any No Free Lunch TCPP has a dual representation in terms of equivalent probability measures of finite penalty:

$$\forall\, X \in L^\infty(\mathcal{F}_\tau),\;\; \Pi_{\sigma,\tau}(X) = \mathrm{esssup}_{\mathrm{Q}\in\mathcal{M}^{1,e}(\mathrm{P})}(\mathrm{E_Q(X}|\mathcal{F}_\sigma) - \alpha^{\mathrm{m}}_{\sigma,\tau}(\mathrm{Q})) \tag{3}$$

where

$$\mathcal{M}^{1,e}(P) = \{Q \sim P \text{ and } \;\alpha^m_{0,\infty}(Q) < \;\infty\} \tag{4}$$

Recall also that from [5], the No Free Lunch property implies that the set $\mathcal{M}^0$ of probability measures equivalent with P with zero minimal penalty is non empty.

$$\mathcal{M}^0 = \{R \sim P,\;\; \alpha^m_{0,\infty}(R) = 0\} \tag{5}$$

Recall also that any probability measure $Q \in \mathcal{M}^{1,e}(P)$ satisfies the cocycle condition (cf [5]):

$$\forall \nu \leq \sigma \leq \tau \;\; \alpha^m_{\nu,\tau}(Q) = \alpha^m_{\nu,\sigma}(Q) + E_Q(\alpha^m_{\sigma,\tau}(Q)) \tag{6}$$

We have proved in [5], that for any $R \in \mathcal{M}^0$, the ask price (resp bid price) process associated with any $X \in L^\infty(\mathcal{F}_\infty)$ is then a R-supermartingale (resp R-submartingale) admitting a càdlàg modification.

Remark 3.2 *For any $R \in \mathcal{M}^0$, for any stopping times $\sigma \leq \tau$, $\alpha^m_{\sigma,\tau}(R) = 0$.*

This is an easy consequence of the non negativity and of the cocycle condition satisfied by the minimal penalty (equation 6).

Remark 3.3 *Assume now that X belongs to $L^0(\Omega, \mathcal{F}_\infty, P)$, is no more essentially bounded but is such that $X^- \in L^\infty(\Omega, \mathcal{F}_\infty, P)$. X is the increasing limit of a sequence $(X_n)_{n\in\mathbb{N}}$ of elements in $L^\infty(\Omega, \mathcal{F}_\infty, P)$. And therefore using the continuity from below of the TCPP, for any σ, $\Pi_{\sigma,\infty}(X)$ is defined as the increasing limit of $\Pi_{\sigma,\infty}(X_n)$.*

We give now the definition of calibration of a TCPP on a reference family, definition extending the notion first introduced in [5].

Definition 3.4 *A TCPP $(\Pi_{\sigma,\tau})_{0\leq\sigma\leq\tau}$ is calibrated on the reference family $((S^k)_{0\leq k,\leq d}$, $(Y^l)_{1\leq l\leq p})$ and the limit order books $(C_{bid}(mY^l), C_{ask}(nY^l))_{1\leq l\leq p}$ if*

- *it extends the dynamics of the process $(S^k)_{0\leq k\leq d}$, i.e. for any finite stopping time τ such that the stopped process $(S^k)^\tau$ is uniformly bounded,*

$$\forall\, n \in \mathbb{Z} \;\; \forall\, 0 \leq \sigma \leq \tau \quad \Pi_{\sigma,\tau}(nS^k_\tau) = nS^k_\sigma \tag{7}$$

- *it is compatible with the limit order books of Y^l: $\forall 1 \leq l \leq p$,*

$$\begin{aligned} \forall m \leq M^l \;\; C_{bid}(mY^l) &\leq -\Pi_{0,\tau_l}(-mY^l) \\ \forall n \leq N^l \;\; \Pi_{0,\tau_l}(nY^l) &\leq C_{ask}(nY^l) \end{aligned} \tag{8}$$

In the condition (7) we consider only integer multiples of the process $(S^k)_{0\le k\le d}$ because there are the only one that can be traded (however considering the definition with real numbers instead of integers would not affect the results).
The preceding notion of calibration assumes that the assets $(S^k)_{0\le k\le d}$ are perfectly liquid. This is of course not completely realistic. If we want to take into account the existence of a limit order book associated with the S^k, we have to weaken the preceding condition. This is the subject of the next subsection.

3.2 Weak calibration for a TCPP

In this section we introduce a weaker notion of calibration on the reference family, taking into account the fact that the financial assets $(S^k)_{0\le k\le d}$ are not perfectly liquid. We want to construct a dynamic for the limit order books, taking into account both the limit order books observed for the process (S^k) at time 0, $(C_{bid}(nS^k),\ C_{ask}(nS^k))$, and the dynamics of S^k. Thus we introduce the following definition of weak calibration:

Definition 3.5 *A TCPP $(\Pi_{\sigma,\tau})_{0\le\sigma\le\tau}$ is weakly calibrated on the reference family $((S^k)_{0\le k\le d},(Y^l)_{1\le l\le p})$ and the observed limit order books $(C_{bid}(mS^k),C_{ask}(nS^k))_{1\le k\le d}$, $(C_{bid}(mY^l),C_{ask}(nY^l))_{1\le l\le p}$ if it satisfies the following conditions:*
1. Weak extension of the process $(S^k)_{0\le k\le d}$: For any finite stopping time τ such that the stopped process $(S^k)^\tau$ is uniformly bounded, $\forall\ 0\le k\le d\ \forall\ 0\le\sigma\le\tau$

- *i)* $\forall n\in I\!N\ -\Pi_{\sigma,\tau}(-nS^k_\tau)\le nS^k_\sigma\le\Pi_{\sigma,\tau}(nS^k_\tau)$
- *ii)* $-\Pi_{\sigma,\tau}(-S^k_\tau)=\Pi_{\sigma,\tau}(S^k_\tau)=S^k_\sigma$
- *iii) compatibility with the limit order books of $(S^k)_{0\le k\le d}$,*

$$\forall n\le M(k)\ C_{bid}(nS^k)\le-\Pi_{0,\tau}(-nS^k_\tau)$$

$$\forall n\le N(k)\Pi_{0,\tau}(nS^k_\tau)\le C_{ask}(nS^k)$$

2. Compatibility with the limit order books of $(Y^l)_{1\le l\le p}$: equation (8) of Definition 3.4.

Remark 3.6 *For a sublinear TCPP there is no difference between calibration and weak calibration.*

As proved in the next lemma, if there is a transaction on each S^k at time zero, the condition 1.*ii)* is a consequence of the assumptions 1.*i) and iii)*.

Lemma 3.7 *Assume that there is at time zero a transaction on each S^k, i.e. $C_{bid}(S^k)=C_{ask}(S^k)$. Assume that the pricing procedure satisfies the conditions 1.i) and iii) of Definition 3.5, then it also satisfies condition 1.ii).*

Lemma 3.7 is a consequence of the following general lemma which will be also useful in the study of the hedge (Section 6).

Lemma 3.8 *Let $(\Pi_{\sigma,\tau})_{\sigma\leq\tau}$ be a No Free Lunch TCPP. Let τ be a stopping time. Assume that for some $X \in L^\infty(\Omega, \mathcal{F}_\tau, P)$ there is $\nu \leq \tau$ and $A \in \mathcal{F}_\nu$ such that:*
$-\Pi_{\nu,\tau}(-X)1_A = \Pi_{\nu,\tau}(X)1_A$.
Then for all $\nu \leq \sigma \leq \tau$, $-\Pi_{\sigma,\tau}(-X)1_A = \Pi_{\sigma,\tau}(X)1_A$.

Proof: For any $\nu \leq \sigma \leq \tau$,

$$-\Pi_{\sigma,\tau}(-X)1_A \leq \Pi_{\sigma,\tau}(X)1_A \tag{9}$$

As the TCPP has No Free Lunch, there is a probability measure $R \sim P$ with zero minimal penalty. From equations (3) and (9) and time consistency it follows that

$$\begin{aligned} -\Pi_{\nu,\tau}(-X)1_A &\leq E_R(-\Pi_{\sigma,\tau}(-X)1_A|\mathcal{F}_\nu) \\ &\leq E_R(\Pi_{\sigma,\tau}(X)1_A|\mathcal{F}_\nu) \leq \Pi_{\nu,\tau}(X)1_A \end{aligned} \tag{10}$$

By hypothesis $-\Pi_{\nu,\tau}(-X)1_A = \Pi_{\nu,\tau}(X)1_A$. Thus any inequality in expression (10) is in fact an equality. As $R \sim P$, it thus follows from (9) that

$$-\Pi_{\sigma,\tau}(-X)1_A = \Pi_{\sigma,\tau}(X)1_A$$

$\square$

Proof of Lemma 3.7. The equality $C_{bid}(S^k) = C_{ask}(S^k)$ and the hypotheses 1.i and *iii)* of Definition 3.5 implie that $-\Pi_{0,\tau}(-S^k_\tau) = \Pi_{0,\tau}(S^k_\tau) = S^k_0$. We apply Lemma 3.8 with $X = S^k_\tau$ $A = \Omega$ and $\nu = 0$. It follows that $-\Pi_{\sigma,\tau}(-S^k_\tau) = \Pi_{\sigma,\tau}(S^k_\tau)$. From condition 1.$i$, it is also equal to S^k_σ. Thus ii) is proved. $\square$

3.3 Characterization of the calibration

The following theorem characterizes the calibration and weak calibration conditions for a No Free Lunch TCPP.

Theorem 3.9 *1. A No Free Lunch TCPP $(\Pi_{\sigma,\tau})_{0\leq\sigma\leq\tau}$ is weakly calibrated on the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the observed limit order books $(C_{bid}(mS^k),\ C_{ask}(nS^k))_{1\leq k\leq d}$, $(C_{bid}(mY^l), C_{ask}(nY^l)_{1\leq l\leq p})$ if and only if:*
- Local martingale property:
Any probability measure R equivalent with P with zero minimal penalty (i.e. $R \in \mathcal{M}^0$) is an equivalent local martingale measure for the process $(S^k)_{0\leq k\leq d}$.
- Threshold condition: for any $R \sim P$,

$$\alpha^m_{0,\tau}(R) \geq \sup_{\tau_l\leq\tau}\Big(\sup_{m\leq M^l} (C_{bid}(mY^l) - E_R(mY^l)),\ \sup_{n\leq N^l}((E_R(nY^l) - C_{ask}(nY^l)\Big)\Big) \tag{11}$$

$$\alpha^m_{0,\tau}(R) \geq \sup_{1\leq k\leq d} |S^k_0 - E_R(S^k_\tau)| \tag{12}$$

$$\alpha^m_{0,\tau}(R) \geq \sup_{m\leq M(k)} (C_{bid}(mS^k) - E_R(mS^k_\tau)),\ \sup_{n\leq N(k)} (E_R(nS^k_\tau) - C_{ask}(nS^k)) \tag{13}$$

2. A No Free Lunch TCPP $(\Pi_{\sigma,\tau})_{0\leq\sigma\leq\tau}$ is calibrated on the reference family $((S^k)_{0\leq k\leq d},\ (Y^l)_{1\leq l\leq p})$ and the limit order books $(C_{bid}(mY^l), C_{ask}(nY^l))_{1\leq l\leq p}$ if and only if any probability measure $R\in\mathcal{M}^{1,e}(P)$ (i.e. $R\sim P$ of finite penalty) is a local martingale measure for the process $(S^k)_{0\leq k\leq p}$, and the threshold condition (11) is satisfied.

Remark 3.10 *: The fundamental difference between the calibration and the weak calibration for a No Free Lunch TCPP in terms of their dual representation, is the following:*
in case of calibration, any probability measure in the dual representation is a local martingale measure for the process $(S^k_t)_{1\leq k\leq d}$ while in case of weak calibration, this is only the case for the probability measures with zero penalty.

Proof of Theorem 3.9
Proof of *1.*
- Assume first that the No Free Lunch TCPP is weakly calibrated on the reference family. Let τ be a stopping time such that the stopped process $(S^k)^\tau_{0\leq k\leq d}$ is uniformly bounded. Let $0\leq\sigma\leq\tau$. $\Pi_{\sigma,\tau}(S^k_\tau) = -\Pi_{\sigma,\tau}(-S^k_\tau) = S^k_\sigma$. Let $R\in\mathcal{M}^0$, $\alpha^m_{\sigma,\tau}(R)=0$. From the dual representation, equation (3), it follows that

$$S^k_\sigma = -\Pi_{\sigma,\tau}(-S^k_\tau)\leq E_R(S^k_\tau|\mathcal{F}_\sigma)\leq\Pi_{\sigma,\tau}(S^k_\tau) = S^k_\sigma$$

Thus any R equivalent with P with zero minimal penalty is a local martingale measure for $(S^k)_{0\leq k\leq d}$. The threshold condition follows from the expression of the minimal penalty $\alpha^m_{0,\tau}(Q) = \sup_{Z\in L^\infty(\mathcal{F}_\tau)}(E_Q(Z)-\Pi_{0,\tau}(Z))$
- Conversely, assume that the No Free Lunch TCPP satisfies the local martingale property and the threshold condition. We have to prove that the pricing procedure satisfies the conditions of definition 3.5.
Let τ be a stopping time such that the stopped process $(S^k)^\tau_{0\leq k\leq d}$ is uniformly bounded. Let $R\in\mathcal{M}^0$. From the dual representation of $\Pi_{\sigma,\tau}$, equation (3), as R is a local martingale measure for $(S^k)_{0\leq k\leq d}$, it follows then that

$$\forall\sigma\leq\tau\ \ \forall n\in I\!N\quad -\Pi_{\sigma,\tau}(-nS^k_\tau)\leq nS^k_\sigma\leq\Pi_{\sigma,\tau}(nS^k_\tau) \tag{14}$$

Thus condition 1. *i)* of Definition 3.5 is satisfied. From the threshold condition, for $n\leq N(k)$, for any $Q\in\mathcal{M}^{1,e}(P)$, $E_Q(nS^k_\tau)-\alpha^m_{0,\tau}(Q)\leq C_{ask}(nS^k)$. So applying the equation of representation (3) to $\Pi_{0,\tau}$, we get

$$\Pi_{0,\tau}(nS^k_\tau)\leq C_{ask}(nS^k)\ \ \forall n\leq N(k)$$

The inequality $C_{bid}(mS^k)\leq -\Pi_{0,\tau}(-mS^k_\tau)\ \forall m\leq M(k)$ is proved in the same way. Thus conditions 1.*iii)* of Definition 3.5 is satisfied.
The proof of condition 2. of Definition 3.5 is analogous.
We prove now that the condition 1. *ii)* is satisfied. From equation (14), $-\Pi_{0,\tau}(-S^k_\tau)\leq S^k_0\leq\Pi_{0,\tau}(S^k_\tau)$. The converse inequality is a consequence of the threshold condition (inequation (12)) and of the equation of representation

(3) applied to $\Pi_{0,\tau}$. Thus $S_0^k = \Pi_{0,\tau}(S_\tau^k) = -\Pi_{0,\tau}(-S_\tau^k)$. Applying now Lemma 3.8 , we get the equality

$$\Pi_{\sigma,\tau}(S_\tau^k) = -\Pi_{\sigma,\tau}(-S_\tau^k)$$

Using (14) it is also equal to S_σ^k thus condition 1. *ii)* of Definition 3.5 is satified. This proves *1* .
For the proof of *2* we refer to [5]. □

3.4 Extended Version of Kreps Yan First Fundamental Theorem

We state now an extended version of Theorem 2.4 of Section 2.3.

Theorem 3.11 *The following conditions are equivalent:*

i) *The reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ satisfies the No Free Lunch condition with respect to the limit order books $(C_{bid}(mY^l), C_{ask}(nY^l))$.*

ii) *There is an equivalent local martingale measure R for $(S^k)_{0\leq k\leq d}$ such that for any $l \in \{1, ..., p\}$, for $m \leq M^l$ $C_{bid}(mY^l) \leq E_R(mY^l)$ and for $n \leq N^l$ $E_R(nY^l) \leq C_{ask}(nY^l)$.*

iii) *There is a No Free Lunch TCPP calibrated on the reference family $(S^k_{0\leq k\leq d}, Y^l_{1\leq l\leq p})$ and the limit order books $(C_{bid}(mY^l), C_{ask}(nY^l))$.*

iv) *There is a No Free Lunch TCPP weakly calibrated on the reference family $(S^k_{0\leq k\leq d}, Y^l_{1\leq l\leq p})$ and the limit order books $(C_{bid}(mS^k), C_{ask}(nS^k))$, $(C_{bid}(mY^l), C_{ask}(nY^l))$.*

The proof of this extended version of the First Fundamental Theorem is given in Appendix A.1. A key tool in this proof is the existence of an equivalent probability measure with zero minimal penalty. The aim of the proof is the same as that of the proof of Kreps Yan Theorem given in [10].
Theorem 3.11 shows also that the notion of No Free Lunch TCPP is well adapted to the questions related to No Arbitrage.

4 Properties of the Supply Curve

Let $(\Pi_{\sigma,\tau})_{\sigma\leq\tau}$ be a No Free Lunch TCPP. Let X be an essentially bounded non negative financial asset. For $x \in I\!R^{+}*$,(resp $x \in I\!R^{-}*$) denote $X(t, x, \omega)$ the ask price (resp bid price) at time t per share for an order of size x, which means that $X(t, x, \omega) = \frac{\Pi_{t,\infty}(xX)(\omega)}{x}$. In the following proposition we list the properties of the supply curve.

Proposition 4.1 *1. For any x, $(t, \omega) \to X(t, x, \omega)$ is a càdlàg stochastic process.*

2. *There is an equivalent probability measure R such that for any $x \geq 0$, the process $X(t,x,.)$ is a R-supermartingale and for any $x \leq 0$ the process $X(t,x,.)$ is a R-submartingale.*

3. *For any τ, $x \in I\!R^* \to X(\tau,x,.) \in L^\infty(\Omega,\mathcal{F}_\tau,P)$ is non decreasing. $\forall\tau$, P a.s., $x \to X(\tau,x,\omega)$ is continuous, admits a right and a left derivative at any point. It is twice derivable almost surely.*

4. *limit in zero: For any τ, $x \to X(\tau,x,.)$ has a right (resp. a left) limit in 0 in $L^\infty(\Omega,\mathcal{F}_\tau,P)$ denoted $X^+(\tau,0,.)$ (resp.$X^-(\tau,0,.)$*

$$X^+(\tau,0,.) = \mathrm{esssup}_{\mathrm{Q}\in\mathcal{M}^0}(\mathrm{E_Q}(\mathrm{X}|\mathcal{F}_\tau)$$
$$X^-(\tau,0,.) = \mathrm{essinf}_{\mathrm{Q}\in\mathcal{M}^0}(\mathrm{E_Q}(\mathrm{X}|\mathcal{F}_\tau) \quad (15)$$

where $\mathcal{M}^0$ is the set of probability measures with zero minimal penalty in the dual representation of the TCPP (equations (3) and (5)).

5. *Asymptotic limit: For any $\tau \in I\!R^+$, $X(\tau,x,.)$ has a limit as $x \to +\infty$ (resp $x \to -\infty$) denoted $X^\infty(\tau,.)$ (resp $X^{-\infty}(\tau,.)$). $X^\infty(\tau,.)$ and $X^{-\infty}(\tau,.)$ are càdlàg processes.*

$$X^\infty(\tau,.) = \mathrm{esssup}_{\mathrm{Q}\in\mathcal{M}^{1,e}(\mathrm{P})}(\mathrm{E_Q}(\mathrm{X}|\mathcal{F}_\tau) \quad (16)$$
$$X^{-\infty}(\tau,.) = \mathrm{essinf}_{\mathrm{Q}\in\mathcal{M}^{1,e}(\mathrm{P})}(\mathrm{E_Q}(\mathrm{X}|\mathcal{F}_\tau) \quad (17)$$

with the notations of (3) and (4).

Proof. As the TCPP has No Free Lunch, it follows from [5] that $\mathcal{M}_0$ is non empty. let $R \in \mathcal{M}_0$, *1* and *2* follow then from [4] Corollary 1 of Theorem 3.
3. follows from the convexity of $\Pi_{\tau,\infty}$ and normalization (i.e. $\Pi_{\tau,\infty}(0)=0$)
4. Let $X \in L^\infty(\Omega,\mathcal{F}_\infty,P)$, $\forall x \in I\!R^+*$, $\frac{\Pi_{\tau,\infty}(xX)}{x} \geq \mathrm{esssup}_{\mathrm{Q}\in\mathcal{M}^0}(\mathrm{E_Q}(\mathrm{X}|\mathcal{F}_\tau)$, so

$$X^+(\tau,0,.) \geq \mathrm{esssup}_{\mathrm{Q}\in\mathcal{M}^0}(\mathrm{E_Q}(\mathrm{X}|\mathcal{F}_\tau) \quad (18)$$

$$E_P(X^+(\tau,0,.) \leq \inf_{x\in I\!R^+*}(\sup_{Q\in\mathcal{M}^{1,e}(P)}(E_P(E_Q(X|\mathcal{F}_\tau) - \frac{E_P(\alpha^m_{\tau,\infty}(Q)}{x})) \quad (19)$$

If $E_P(\alpha^m_{\tau,\infty}(Q)) \neq 0$, $\frac{E_P(\alpha^m_{\tau,\infty}(Q))}{x} \to \infty$ as $x \to 0$. Therefore we can restrict in (19) to probability measures $Q \sim P$ such that $\alpha^m_{\tau,\infty}(Q) = 0$ P *a.s.*. Choose $R \in \mathcal{M}^0$. denote $\tilde{Q}$ the probability measure of Radon Nykodim derivative

$$\frac{d\tilde{Q}}{dP} = E(\frac{dR}{dP}|\mathcal{F}_\sigma)\frac{\frac{dQ}{dP}}{E(\frac{dQ}{dP}|\mathcal{F}_\sigma)}$$

for all X, $E_Q(X|\mathcal{F}_\tau) = E_{\tilde{Q}}(X|\mathcal{F}_\tau)$ and $\tilde{Q} \in \mathcal{M}^0$. Thus
$E_P(X^+(\tau,0,.)) \leq \inf_{x\in I\!R^+*}(\sup_{\tilde{Q}\in\mathcal{M}^0} E_{\tilde{Q}}(X))$. *4.* follows then from (18).
5. The increasing limit of $X(t,x)$ as $x \to \infty$, defines a sublinear No Free Lunch TCPP. Denote it $\Pi^\infty_{\sigma,\tau}$. From the dual representation of Π, and the

non negativity of the minimal penalty, it follows that for any $X \geq 0$, for any $x \in I\!R^+$, $\frac{\Pi_{\sigma,\tau}(xX)}{x} \leq \text{esssup}_{Q\in\mathcal{M}^{1,e}(P)}(E_Q(X|\mathcal{F}_\sigma)$. Thus $(\Pi^\infty)_{\sigma,\tau}(X) \leq \text{esssup}_{Q\in\mathcal{M}^{1,e}(P)}(E_Q(X|\mathcal{F}_\sigma)$. For any $Q \in \mathcal{M}^{1,e}(P)$, $\alpha_{0,\infty}(Q) < \infty$ $\alpha^\infty_{0,\infty}(Q) = \sup_{Y\in L^\infty(\Omega,\mathcal{F}_\infty,P)}(E_Q(Y)-\Pi^\infty_{0,\infty}(Y))$ From the inequality $\Pi_{0,\infty}(Y)) \leq \Pi^\infty_{0,\infty}(Y))$, it follows that $\alpha^\infty_{0,\infty}(Q) < \infty$. Π^∞ is sublinear so $\alpha^\infty_{0,\infty}(Q) = 0$. Thus $(\Pi^\infty)_{\sigma,\tau}(X) \geq \text{esssup}_{Q\in\mathcal{M}^{1,e}(P)}(E_Q(X|\mathcal{F}_\sigma)$ and *5.* is proved. □

In the particular case where the TCPP is calibrated on the reference family we get the following corollary.

Corollary 4.2 *Assume that the TCPP is calibrated on the reference family* $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ *and the limit order book* $(C_{bid}(mY^l), C_{ask}(nY^l))_{1\leq l\leq p}$.

1. *For any k,* $S^k(t,x,\omega) = S^k(t,\omega)$

2. *Assume that there is a transaction at time* 0 *on the option* Y^l*, then the process* $Y^l(t,x,\omega)$ *has a limit as* x *tends to* 0*,* $\forall t,\quad (Y^l)^+(t,0,.) = (Y^l)^-(t,0,.) = \Pi_{t,\tau_l}(Y^l)$

3. *For any financial instrument* X*, the asymptotic limit* $X^\infty(t,.)$*is less or equal to the surreplication price (with respect to the basic assets* $((S^k)_{0\leq k\leq d})$*, i.e.*

$$X^\infty(t,.) \leq \text{esssup}_{Q\in\mathcal{M}(S)}E_Q(X|\mathcal{F}_t)(\omega)$$

where $\mathcal{M}(S)$ *denotes the set of all equivalent local martingale measures for the process* $S = (S^k)$*. There is equality in the above equation if and only if the set of probability measures* $\mathcal{M}^{1,e}(P)$ *in the dual representation of the TCPP is equal to* $\mathcal{M}(S)$ *(with the notations of (3) and (4)).*

Proof. *1* follows from definition of calibration.
2 As there is a transaction at time 0 on Y^l, $C_{bid}(Y^l) = c_{ask}(Y^l) = \Pi_{0,\infty}(Y^l) = -\Pi_{0,\infty}(-Y^l)$. Thus,from Lemma 3.8, for any $t \geq 0$, $\Pi_{t,\infty}(Y^l) = -\Pi_{t,\infty}(-Y^l)$ $\forall x \in]0,1[,\quad -\Pi_{t,\infty}(-Y^l) \leq \frac{\Pi_{t,\infty}(-xY^l)}{-x} \leq \frac{\Pi_{t,\infty}(xY^l)}{x} \leq \Pi_{t,\infty}(Y^l)$ Therefore $(Y^l)^+(t,0,\omega) = \Pi_{t,\infty}(Y^l) = \Pi_{t,\tau_l}(Y^l)$
3 follows from Theorem 3.9 and Proposition 4.1.

Remark 4.3 *In this paper we work in a very general framework of illiquid market represented by a general filtered probability space. From a very simple axiomatic for TCPP we have proved properties (Proposition (4.1)) satisfied by the supply curve associated with any financial product. We can compare these properties with the properties which were assumed in [7] for one asset. Notice first that our model is an infinite dimensional model. The supply curve is defined for any asset i.e; any essentially bounded random variable. We list the main differences for the supply curve:*
-We have proved that the sample paths are càdlàg (and therefore allow for jumps) whereas in [7] the sample paths were assumed to be continuous.
We do not have in general a limit as x *tends to* 0 *for* $S(t,x,.)$ *but only a right limit and a left limit.*
A strong hypothesis of smoothness is made in [7]: $S(t,x,\omega)$ *is assumed to be of* C^2 *class in* x*. We get here that it is twice derivable almost surely in* x*.*

5 Robustness for TCPP calibrated on option prices

In this section we study the robustness of TCPP calibrated on the reference family $((S^k)_{1\leq k\leq d}, (Y^l)_{1\leq l\leq d})$ and the limit order books $((C_{bid}(mY^l))_{m\leq M^l}$, $(C_{ask}(nY^l))_{n\leq N^l}$.

Proposition 5.1 *The maximal bid-ask interval associated to No Free Lunch TCPP calibrated on the reference family* $((S^k)_{1\leq k\leq d}, (Y^l)_{1\leq l\leq d})$ *and the limit order books* $((C_{bid}(mY^l))_{m\leq M^l}, (C_{ask}(nY^l))_{n\leq N^l}$ *is given, for any financial asset* X*, by*

$$[m_X, M_X] = [\inf_{Q\in\mathcal{M}_e}(E_Q(X) + \beta(Q), \sup_{Q\in\mathcal{M}_e}(E_Q(X) - \beta(Q)]$$

with

$$\beta(Q) = \sup_l[\sup_{m\leq M^l}((C_{bid}(mY^l) - E_Q(mY^l), \sup_{n\leq N^l} E_Q(nY^l)) - C_{ask}(nY^l)] \quad (20)$$

where $\mathcal{M}_e$ *is the set of equivalent local martingale measures for* $(S^k)_{1\leq k\leq d}$

Proof. This results from Theorem 3.9.
It follows from this Proposition that a little move for $C_{bid}(nY^l)$ and $C_{ask}(nY^l)$ induces for any X a small change in the maximal bid-ask spread associated with X. Indeed, denote $\beta'(Q)$ the minimal penalty associated to No Free lunch TCPP calibrated on the limit order books $((C'_{bid}(mY^l))_{m\leq M^l}$, $(C'_{ask}(nY^l))_{n\leq N^l}$. let ϵ such that $\epsilon \geq |(C_{bid}(mY^l) - (C_{bid}(mY^l)|$ for $m \leq M^l$ and $\epsilon \geq |(C_{ask}(nY^l) - (C'_{ask}(nY^l)|$ for $n \leq N^l$. From equation (20), it follows that $|\beta(Q) - \beta'(Q)| \leq \epsilon$ and thus $|m_X - m'_X| \leq \epsilon$, $|M_X - M'_X| \leq \epsilon$ for any X.

6 A hedging result for TCPP calibrated on liquid options

The aim of this Section is to study No Free Lunch TCPP calibrated on perfectly liquid options Y^l, and to prove a hedging result.
We assume that for any of the reference options $(Y^l)_{1\leq l\leq d}$, $C_{bid}(nY^l) = C_{ask}(nY^l)$. Denote $C^l = C_{bid}(Y^l) = C_{ask}(Y^l)$. From the convexity of $n \to C_{ask}(nY^l)$, (and concavity of $n \to C_{bid}(nY^l)$ it follows that $nC^l = C_{ask}(nY^l) = C_{bid}(nY^l)$. In that case we simply say that the TCPP is calibrated on the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the observed prices $(C^l)_{1\leq l\leq p}$. Let $(\Pi_{\sigma,\tau})_{\sigma\leq\tau}$ be such a No Free Lunch TCPP. From equation (3) and Theorem 3.9 , it follows that there is a set $\mathcal{Q}$ of equivalent local martingale measures for $(S^k)_{0\leq k\leq d}$, with $\alpha_{0,\infty}(Q) < \infty$, such that

$$\forall\sigma \leq \tau \ \ \Pi_{\sigma,\tau}(X) = \mathrm{esssup}_{Q\in\mathcal{Q}}(\mathrm{E_Q(X|}\mathcal{F}_\sigma) - \alpha^{\mathrm{m}}_{\sigma,\tau}(\mathrm{Q})) \quad (21)$$

We say that the No Free Lunch TCPP is represented by the set $\mathcal{Q}$.
The following lemma is a key result for the study of the hedge: the process $Z^l_t = \Pi_{t,\infty}(Y^l)$ is a martingale for any Q in $\mathcal{Q}$.

Lemma 6.1 *Let $(\Pi_{\sigma,\tau})_{\sigma\leq\tau}$ be a No Free Lunch TCPP calibrated on the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the prices $(C^l)_{1\leq l\leq p}$. Then*

$$\forall 1 \leq l \leq p,\ \ \forall\sigma,\ \ \forall n \in I\!N,\ \ \Pi_{\sigma,\infty}(nY^l) = -\Pi_{\sigma,\infty}(-nY^l)$$

The process $Z^l_t = \Pi_{t,\infty}(Y^l)$ is a martingale with respect to any probability measure in $\mathcal{Q}$.

Proof. For all $n \in I\!N$, $-\Pi_{0,\infty}(-nY^l) = \Pi_{0,\infty}(nY^l) = nC^l$. From Lemma 3.8 applied with $\nu = 0$, it follows that for any σ,

$$-\Pi_{\sigma,\infty}(-nY^l) = \Pi_{\sigma,\infty}(nY^l) \tag{22}$$

The convexity of $\Pi_{\sigma,\infty}$ implies that

$$-\Pi_{\sigma,\infty}(-nY^l) \leq -n\Pi_{\sigma,\infty}(-Y^l) \leq n\Pi_{\sigma,\infty}(Y^l) \leq \Pi_{\sigma,\infty}(nY^l)$$

From (22), it follows that any inequality in the above relation is in fact an equality. Thus $n\Pi_{\sigma,\infty}(Y^l) = \Pi_{\sigma,\infty}(nY^l)\ \forall n \in Z\!\!Z$. From equation (21), it follows that $\forall Q \in \mathcal{Q},\ \ \forall n \in I\!N$, $\alpha^m_{\sigma,\infty}(Q) \geq n|\Pi_{\sigma,\infty}(Y^l) - E_Q(Y^l|\mathcal{F}_\sigma)|$ a.s. From the cocycle equation (6), $\forall Q \in \mathcal{Q}\ \ E_Q(\alpha^m_{\sigma,\infty}(Q)) < \infty$.
Then $Z^l_\sigma = \Pi_{\sigma,\infty}(Y^l) = E_Q(Y^l|\mathcal{F}_\sigma) = E_Q(Z^l_\tau|\mathcal{F}_\sigma) a.s.,\ \ \forall\tau \geq \sigma$. □

Theorem 6.2 *Let $(\Pi_{\sigma,\tau})_{\sigma\leq\tau}$ be a No Free Lunch TCPP calibrated on the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the observed prices $(C^l)_{1\leq l\leq p}$. Then $\forall X \in L^\infty(\Omega, \mathcal{F}, P)$,*

$$\Pi_{0,\infty}(X) \leq \inf\{x|\ there\ is\ h \in \mathcal{C}\ with\ x + h = X\} \tag{23}$$

where $\mathcal{C} = (K_0 - L^0_+) \cap L^\infty$ and

$$\begin{aligned} K_0 = \{&(H.S)_\infty + (K.Z)_\infty \mid H, K\ admissible\ and \\ &(H.S)_\infty = \lim_{t\to\infty}(H.S)_t\ exists\ a.s.\ idem\ for\ K.Z\} \end{aligned}$$

This gives a better superhedge result than the usual one.

Proof. Denote $\mathcal{M}^e(S^k, Z^l)$ the set of equivalent local martingale measures for the process (S^k, Y^l). From Lemma 6.1, and part *2.* of Theorem 3.9, any probability measure in $\mathcal{Q}$ is an equivalent local martingale measure for (S^k, Z^l), i.e. $\mathcal{Q} \subset \mathcal{M}^e(S^k, Z^l)$. Therefore from equality (21), we get $\Pi_{0,\infty}(X) \leq \sup_{Q\in\mathcal{M}^e(S^k,Z^l)} E_Q(X)$. One can now apply the superhedge result of Delbaen and Schachermayer [9] (Theorem 9.5.8 in [10]). This proves (23). □
Economic interpretation of this result: When the options Y^l are perfectly liquid, a TCPP $(\Pi)_{\sigma,\tau}$ calibrated on the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the prices $(C^l)_{1\leq l\leq p}$ is a TCPP extending the dynamics of the processes $((S^k_t)_{0\leq k\leq p}, Z^l_t = E_Q(Y^l|\mathcal{F}_t)_{1\leq l\leq d})$ where Q is any equivalent probability measure involved in the representation of $(\Pi)_{\sigma,\tau}$ (equation (21)). The options Y^l can be used to hedge dynamically as well as the assets S^k.

Definition 6.3 *The TCPP Π^0 is said maximal among the TCPP calibrated on the reference family (S^k, Y^l) and the prices (C^l) if for any TCPP Π calibrated on (S^k, Y^l) and the prices (C^l), for any $Y \geq 0$, $\Pi(Y) \leq \Pi^0(Y)$.*

From Theorem 6.2, we deduce the following result:

Corollary 6.4 *Let Π^0 be a maximal TCPP calibrated on (S^k, Y^l) and the prices (C^l) then Π^0 is sublinear and represented by the set of all equivalent local martingal measures for the process $((S^k, Z^l))$ $\mathcal{M}^e(S^k, Z^l)$. The inequality (23) becomes an equality, leading to a perfect hedge result.*

7 TCPP in a stochastic volatility model

In this Section we assume that the price process S of a primitive asset expressed in terms of the numéraire S^0 satisfies a stochastic volatility model. We assume that for any of the Y^l one observes a limit order book. The notations for the limit order book are those of Subsection 2.2.
Assuming that the reference family satisfies the No Free Lunch condition, there is (Theorem 2.4) an equivalent local martingale measure R for the process S such that for any l, $C^l_{bid} \leq E_R(Y^l) \leq C^l_{ask}$. We take this probability measure R as the new reference probability. Therefore we assume that the price process S of the primitive asset expressed in terms of the numéraire S^0 is given by

$$\begin{cases} \dfrac{dS_t}{S_t} &= \sigma_t(\sqrt{1-\rho^2}dW^1_t + \rho dW^2_t) \\ d\sigma_t &= \alpha(t, S_t, \sigma_t)dt + \gamma(t, S_t, \sigma_t)dW^2_t \end{cases} \tag{23}$$

where W^1 and W^2 are two independent Brownian motions and $\rho \in]-1, 1[$.

Proposition 7.1 *Any TCPP calibrated on the reference family $(S, (Y^l)_{1\leq l\leq p})$ can be written*

$$\Pi_{\sigma,\tau}(X) = \mathrm{esssup}_{\{\nu | \int_0^\infty \nu_s^2 \mathrm{ds} < \infty\}}(\mathrm{E}_{\mathrm{Q}_\nu}(\mathrm{X}|\mathcal{F}_\sigma) - \alpha_{\sigma,\tau}(\mathrm{Q}_\nu))$$

with

$$\frac{dQ_\nu}{dR} = exp(-\int_0^\infty \frac{\rho\nu_s}{\sqrt{1-\rho^2}}dW^1_s + \int_0^\infty \nu_s dW^2_s - \frac{1}{2}\int_0^\infty \frac{\nu_s^2}{1-\rho^2}ds) \tag{24}$$

Furthermore for any such TCPP for any $X \in L^\infty(\Omega, \mathcal{F}_t, P)$,

$$\Pi_{0,t}(X) \leq \sup_\nu (E_{Q_\nu}(X) - \alpha^m_{0,t}(Q_\nu))$$

with

$$\alpha^m_{0,\tau}(Q_\nu) = \sup(0, \sup_{\tau_l \leq \tau}(\sup_{n \leq M^l} C_{bid}(nY^l) - E_{Q_\nu}(nY^l), \sup_{n \leq N^l} E_{Q_\nu}(nY^l) - C_{ask}(nY^l)) \tag{25}$$

Proof. From Theorem 3.9, any TCPP extending the dynamics of S has a representation in terms of equivalent local martingale measures for S. Any probability measure equivalent with R is characterized by its Radon Nikodym derivative

$$\frac{dQ}{dR} = exp(\int_0^\infty \lambda_s dW_s^1 - \frac{1}{2}\int_0^\infty \lambda_s^2 ds + \int_0^\infty \nu_s dW_s^2 - \frac{1}{2}\int_0^\infty \nu_s^2 ds) \tag{26}$$

Let $W_t^1(Q) = W_t^1 - \int_0^\infty \lambda_s ds$ and $W_t^2(Q) = W_t^2 - \int_0^\infty \nu_s ds$. From Girsanov's Theorem, $(W_t^1(Q), W_t^2(Q))$ is a two dimensional Brownian motion under the probability measure Q and the dynamics of S can be written

$$\begin{cases} \frac{dS_t}{S_t} &= \sigma_t[(\lambda_t\sqrt{1-\rho^2} + \rho\nu_t)dt + (\sqrt{1-\rho^2}dW_t^1(Q) + \rho dW_t^2(Q))] \\ d\sigma_t &= (\alpha(t, S_t, \sigma_t) + \nu_t\gamma(t, S_t, \sigma_t))dt + \gamma(t, S_t, \sigma_t))dW_t^2(Q) \end{cases} \tag{26}$$

Therefore Q is a local martingale measure for S if and only if $\lambda_t\sqrt{1-\rho^2} + \rho\nu_t = 0 \quad a.s.$. Let Q_ν be the equivalent local martingale measure of Radon Nikodym derivative given by the formula (26) with $\lambda_t = -\frac{\rho\nu_t}{\sqrt{1-\rho^2}}$. Then Q_ν satisfies equation (24). From Theorem 3.9, the minimal penalty has to satisfy the threshold condition (11), leading then to the result.

8 TCPP calibrated on options from BMO martingales

8.1 General construction of TCPP calibrated on a reference family

In Section 3, we have characterized No Free Lunch TCPP calibrated on a reference family in terms of their dual representation. The next step is to construct such No Free Lunch TCPP in a very general setting where the processes $(S^k)_{0\leq k\leq d}$ can allow for jumps. Assume that the reference family satisfies the No Free Lunch condition. Denote $\mathcal{M}$ the set of equivalent local martingale measure for $(S^k)_{0\leq k\leq d}$ and

$$\mathcal{M}_1 = \{R \in \mathcal{M} \,|\forall l \in \{1, ..., p\} \;\; C_{bid}(nY^l) \leq E_R(nY^l) \leq C_{ask}(nY^l)\}$$

Let $R \in \mathcal{M}_1$. The conditional expectation with respect to R provides thus a linear TCPP calibrated on the reference family. However as soon as one calibrates on options which are not perfectly liquid, one has to construct non linear, and even more, non sublinear TCPP in order to take care of the non liquidity. One has introduced in [3] a general method to construct convex TCPP starting with a stable set of equivalent probability measures and defining on it a penalty. In general the penalty constructed is not the minimal one. The following result gives sufficient conditions on the set of probability measures and on the penalty for the construction of TCPP calibrated on the reference family. For the definition of stability of a set of probability measures we refer to [8] and [4]. For the definition of locality for the penalty α we refer to [4] definition 5. The cocycle condition is given by the equation (6) with α instead of α^m.

Proposition 8.1 *Let $\mathcal{M}$ be a stable set of probability measures all equivalent to P. Let α be a non negative penalty function on $\mathcal{M}$. Assume that there is $Q \in \mathcal{M}$ such that $\alpha_{0,\infty}(Q) = 0$. Assume that the penalty function α is local and satisfies the cocycle condition.*
Consider the No Free Lunch TCPP $(\Pi_{\sigma,\tau})_{\sigma\leq\tau}$ defined by

$$\forall\, X \in L^\infty(\mathcal{F}_\tau) \;\; \Pi_{\sigma,\tau}(X) = \mathrm{esssup}_{\mathrm{Q}\in\mathcal{M}}(\mathrm{E_Q(X|}\mathcal{F}_\sigma) - \alpha_{\sigma,\tau}(\mathrm{Q})) \tag{27}$$

1. *The TCPP is calibrated on the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the limit order books $(C_{bid}(mY^l))_{m\leq M^l}, (C_{ask}(nY^l)_{n\leq N^l})_{1\leq l\leq p})$ if it satisfies the two following conditions:*
   i) Local martingale property: Any element of $\mathcal{M}$ is an equivalent local martingale measure for $(S^k)_{0\leq k\leq d}$
   ii) threshold condition for the penalty: for any $R \in \mathcal{M}$,

$$\alpha_{0,\tau}(R) \geq \sup_{\tau_l\leq\tau}(\sup_{m\leq M^l} C_{bid}(mY^l) - E_R(mY^l), \sup_{n\leq N^l} E_R(nY^l) - C_{ask}(nY^l)) \tag{28}$$

2. *It is weakly calibrated on the reference family if*
   i') any $R \in \mathcal{M}$ with zero penalty is an equivalent local martingale measure for S^k.
   ii') threshold condition: for any $R \in \mathcal{M}$, inequality (28) is satisfied as well as (12) and (13) with $\alpha_{0,\tau}$ instead of $\alpha^m_{0,\tau}$.

Proof. From Theorem 4.4 of [3] and its extended version Proposition 3 of [4], formula (27) defines a time consistent dynamic pricing procedure. $\alpha_{0,\infty}(Q) = 0$, thus the minimal penalty $\alpha^m_{0,\infty}(Q)$ is also equal to 0. Therefore the TCPP has No Free Lunch.
Notice that the part of the proof of Theorem 3.9 starting with "conversely" does not use the specific expression of the minimal penalty. It applies to any penalty. This proves *1* and *2*. □
In order to construct a No Free Lunch TCPP calibrated on the reference family,, we start with the construction of a stable set of equivalent local martingale measures for S^k. This is the easy part. Then we have to prove the existence of penalties satisfying the threshold condition inequality (28), this is the difficult part. One has to prove that the bound is satisfied uniformly for any $R \in \mathcal{M}$. The examples of TCPP that we construct here belong all to the new class that we first introduced in [3], using right continuous BMO martingales. For the theory of right continuous BMO martingales we refer to Doléans-Dade and Meyer [11].

8.2 A generic family of convex TCPP calibrated on option prices

Assume that the locally bounded process $(S^k)_{0\leq k\leq d}$ satisfies the usual No Free Lunch condition. Let Q_0 be a local martingale measure for (S^k). For simplicity one assumes that $(S^k)_{0\leq k\leq d}$ is a square integrable martingale with respect to Q_0.

Proposition 8.2 *Let $M^1, ..., M^j$ be strongly orthogonal square integrable right continuous martingales in $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t}, P)$. Assume that each M^i is furthermore strongly orthogonal to the martingale $(S^k)_{1\leq k\leq d}$. Let $(\Phi_i)_{1\leq i\leq j}$ be a non negative predictable processes such that the stochastic integral $\Phi.M^i$ is a BMO martingale of BMO norm m^i. Any martingale in*

$$\mathcal{M} = \{ \sum_{1\leq i\leq j} H_i.M^i, \quad H_i \ \textit{predictable} \ |H_i| \leq \Phi_i \ \textit{a.s.}\}$$

is BMO with BMO norm bounded by $(\sum_{1\leq i\leq j}(m^i)^2)^{\frac{1}{2}} = m$.
If $m < \frac{1}{16}$, $\mathcal{Q}(\mathcal{M}) = \{Q_M \ ; \ \frac{dQ_M}{dP} = \mathcal{E}(M) \mid M \in \mathcal{M}\}$ is a stable set of probability measures which are all equivalent martingale measures for $(S^k)_{1\leq k\leq d}$.
When the M^i are continuous the preceding result is true without any restriction on m.

Proof. From Lemma 4.11 of [3] $\mathcal{Q}(\mathcal{M})$ is a stable set of probability measures equivalent to P. From the results on strongly orthogonal martingales [18], Chapter IV Section 3, it follows that for any $M \in \mathcal{M}$, $\mathcal{E}(M)$ is strongly orthogonal to S^k for any k and Q_M is an equivalent martingale measure for $(S^k)_{1\leq k\leq d}$.$\square$

To construct TCPP extending the dynamics of reference assets S^k, one can take any stable subset of the set of equivalent local martingale measures for S^k,for example the set $\mathcal{Q}(\mathcal{M})$ of Proposition 8.2, this defines a TCPP calibrated on the reference family. As soon as one adds options in the reference family, the threshold condition (28) has to be satisfied. Notice that the set $\mathcal{M}_1$ introduced at the begining of Section 8.1 is not stable in general. Our next goal is to construct a universal example of penalties which provides a convex TCPP calibrated on options. The technics used are those of right continuous BMO martingales and the proof applies to any model S^k, processes with jumps as well as processes in a Brownian filtration.
In [19] Schachermeyer introduced a notion of Robust No Arbitrage meaning that there is No Arbitrage with respect to a smaller bid ask spread. In the same way we define here the notion of Robust No Free Lunch.

Definition 8.3 *The reference family $((S^k)_{1\leq k\leq d}, (Y^l)_{1\leq l\leq d})$ satisfies the Robust No Free Lunch Condition if there is $\epsilon > 0$ such that is satisfies the No Free Lunch Condition when one replaces every $C_{bid}(Y^l)$ (resp. $C_{ask}(Y^l)$) by $C_{bid}(Y^l) + \epsilon$ (resp. $C_{ask}(Y^l) - \epsilon$) for any l such that $C_{bid}(Y^l) \neq C_{ask}(Y^l)$.*

One assumes in what follows, for simplicity, that for any l, $C^l_{bid} \neq C^l_{ask}$.
Denote $\mathcal{P}$ the predictable σ-algebra on $I\!R^+ \times \Omega$, and $\mathcal{B}(I\!R^j)$ the Borel σ-algebra on $I\!R^j$.

Theorem 8.4 *Assume that the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq d})$ satisfies the robust No Free Lunch condition with respect to the limit order books $C_{bid}(mY^l)_{m\leq M^l}$, $C_{ask}(nY^l)_{n\leq N^l}$. Denote Q_0 an equivalent local martingale measure for S^k and $\epsilon > 0$ such that for any l, $C^l_{bid} + \epsilon < E_{Q_0}(Y^l) < C^l_{ask} - \epsilon$. Let $\mathcal{M}$ be as in Proposition 8.2. Assume that $m < \frac{1}{16}$. Let $\mathcal{Q}(\mathcal{M})$ be the corresponding set of equivalent probability measures $(Q_M)_{M\in\mathcal{M}}$ of Radon Nikodym derivative $\frac{dQ_M}{dQ_0} = \mathcal{E}(M)$. let $(b_i)_{1\leq i\leq j}$, $b_i : I\!R^+ \times \Omega \times I\!R^j \to I\!R^+$ be non negative measurable maps with respect to the σ-algebra $\mathcal{P} \times \mathcal{B}(I\!R^j)$ such that $b_i(s,\omega,0,...,0) = 0$. Assume that there is a constant $B > 0$ such that $b_i(s,\omega,x_1,...,x_j) \geq Bx_i^2$ for all i. Denote $b_i(s,H_1,...,H_j)$ the predictable process defined as $b_i(s,H_1,...,H_j)(\omega) = b_i(s,\omega,H_{1,s}(\omega),...,H_{j,s}(\omega))$. For $M \in \mathcal{M}$ and stopping times $0 \leq \sigma \leq \tau$ let*

$$\alpha_{\sigma,\tau}(Q_M) = E_{Q_M}(\sum_{1\leq i\leq j} \int_\sigma^\tau b_i(s,H_1,...,H_j)d[M^i,M^i]_s|\mathcal{F}_\sigma) \tag{29}$$

Then

$$\Pi_{\sigma,\tau}(X) = \mathrm{esssup}_{\mathrm{Q_M}\in\mathcal{Q}(\mathcal{M})}(\mathrm{E_{Q_M}}(\mathrm{X}|\mathcal{F}_\sigma) - \alpha_{\sigma,\tau}(\mathrm{Q_M})) \tag{30}$$

defines a TCPP. Furthermore for B large enough, The TCPP is calibrated on the reference family. Notice that the minimal acceptable B depends only on m, ϵ, $\max(M^l,N^l)$ and $\max||Y^l||_\infty$ and not on the dynamics of S^k nor on the set $\mathcal{M}$.

The proof is given in the Appendix.

Remark 8.5 *We get a similar result for weak calibration.*

9 Conclusion

The motivation of this paper was to study and construct dynamic pricing procedures assigning to any financial instrument a dynamic limit order book in a arbitrage free way, extending the dynamics of given basic assets and compatible with the observed limit order books for reference options. This is done by making use of the theory of No Free Lunch TCPP introduced in [5]. We have defined two notions of calibration for a Dynamic Pricing Procedure with respect to a reference family $((S^k)_{0\leq k\leq d+1}, (Y^l)_{1\leq l\leq d})$ composed of two kinds of assets: the basic assets $(S^k)_{0\leq k\leq d+1}$ for which the dynamic process is assumed to be known , and the assets $(Y^l)_{1\leq l\leq d})$ (for example options) which are only revealed at their maturity date (the stopping time τ_l) and for which one observes a limit order book at time 0. One of the basic asset S^0 is assumed to be strictly positive and is taken as numéraire. The first notion of calibration, simply called

calibration, assumes that the basic assets $(S^k)_{0\leq k\leq d}$ are perfectly liquid. A TCPP is said to be calibrated on the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the limit order books $C_{bid}(mY^l)_{m\leq M^l}$, $C_{ask}(nY^l)_{n\leq N^l}$ if it extends the dynamics of the basic assets $(S^k)_{0\leq k\leq d}$ and it is compatible with the observed limit order books for the options $(Y^l)_{1\leq l\leq p}$. The second notion called weak calibration takes into account the limit order books associated with the basic assets. We have characterized TCPP calibrated or weakly calibrated on the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the limit order books in terms of their dual representation. In case of calibration, any probability measure in the dual representation of the TCPP has to be an equivalent local martingale measure for the process $(S^k)_{0\leq k\leq d}$ while in case of weak calibration this is only the case for probability measures with zero penalty. In both cases there is a threshold condition on the penalty.
We have extended to that context the notion of No Free Lunch, replacing the usual notion of dynamic strategy with respect to the basic assets $(S^k)_k$ by the sum of a dynamic strategy with respect to the basic assets S^k and of a static strategy with respect to the options Y^l. We have proved the following generalization of Kreps-Yan Theorem: there is No Free Lunch with respect to the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the limit order books $C_{bid}(mY^l)_{m\leq M^l}$, $C_{ask}(nY^l)_{n\leq N^l}$ if and only if there is an equivalent local martingale measure Q for the process $(S^k)_{0\leq k\leq d}$ such that, for every l, and any $n\geq 0$, $C_{bid}(nY^l)\leq E_Q(nY^l)\leq C_{ask}(nY^l)$. Furthermore, the No Free Lunch condition is also equivalent to the existence of a No Free Lunch TCPP calibrated on the reference family.
We have illustrated our results with two examples: The first one is the case of TCPP calibrated on very liquid options $C_{bid}(nY) = C_{ask}(nY^l) = nC^l \;\; \forall n \in \mathbb{N}$. We have proved in that case that the process $(Z^l)_t = E_Q(Y^l|\mathcal{F}_t)$ is independent on the probability measure Q involved in the dual representation of the TCPP. Therefore the options can be used to hedge (dynamically) as well as the basic assets. The second example is that of a stochastic volatility model.
We have also used the powerful technique of right continuous BMO martingales in order to prove the existence of convex (not sublinear) No Free Lunch TCPP calibrated on the reference family. We have produced a generic construction of a convex No Free Lunch TCPP calibrated on the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ and the limit order books $C_{bid}(mY^l)_{m\leq M^l}$, $C_{ask}(nY^l)_{n\leq N^l}$ as soon as this reference family satisfies the Robust No Free Lunch condition. Such a No Free Lunch TCPP is constructed inside the new family first introduced in [3]. This construction is made in a very general setting of locally bounded stochastic processes for which jumps are allowed.
The advantage of dynamic pricing making use of TCPP is that it not only takes into account the liquidity risk and the properties of the limit order books, but also it induces more robustness in the prices. A small variation in the values of the limit order books of the options on which the TCPP is calibrated induces only a small modification of the constructed TCPP.

A Appendix

A.1 Proof of the extended First Fundamental Theorem

We prove directly the extended version formulated in Theorem 3.11, Section 3.4. It gives also a proof of Theorem 2.4 of Section 2.3.

Proof. We begin with the easiest implications.

- *ii*) implies *iii*): We define the TCPP Π as follows: for any stopping times $\sigma \leq \tau$, $\Pi_{\sigma,\tau}(X) = E_R(X|\mathcal{F}_\sigma)$. As $C_{bid}(mY^l) \leq E_R(mY^l)$ for $m \leq M^l$, and $E_R(nY^l) \leq C_{ask}(nY^l)$ for any $n \leq N^l$, it follows that Π is calibrated on the reference family. It has No Free Lunch as R is equivalent to P and the penalty associated with R is equal to 0.

- *iii*) implies *iv*) is trivial.

- *iv*) implies *i*):

-Let Π be a No Free Lunch TCPP weakly calibrated on the reference family. Consider its dual representation, equation (3). As the TCPP Π has No Free Lunch, the set $\mathcal{M}^0$ of equivalent probability measures with zero minimal penalty (equation(5)) is non empty. Define

$$(\Pi^0)_{0,\infty}(X) = \sup_{Q \in \mathcal{M}^0} E_Q(X)$$

The sets $\overline{C}$ and $\tilde{K}$ are those defined in Section 2.3. Prove that $\forall X \in \overline{C}$, $(\Pi^0)_{0,\infty}(X) \leq 0$. Let $Z \in \tilde{K}$
$Z = \sum_{i=1}^n \sum_{k=1}^d (h^k)_i (S^k_{\sigma_i} - S^k_{\sigma_{i-1}}) + \sum_{l=1}^p (\gamma^l - \beta^l) Y^l + (\gamma^0 - \beta^0) - g$
for some $g \in L^\infty_+(\Omega, \mathcal{F}, P)$. From Theorem 3.9, any Q in $\mathcal{M}^0$ is an equivalent local martingale measure for the process S^k. Thus $E_Q(Z) = \sum_{l=1}^p (\gamma^l - \beta^l) E_Q(Y^l) + (\gamma^0 - \beta^0) - E_Q(g)$. As $\Pi_{\sigma,\tau}$ is weakly calibrated on the reference family, the minimal penalty satisfies equation (11) of Theorem 3.9. For $Q \in \mathcal{M}^0$, $\alpha^m_{0,\infty}(Q) = 0$ so

$$E_Q(Z) \leq \sum_{l=1}^p (C_{ask}(\gamma^l Y^l) - C_{bid}(\beta^l Y^l)) + (\gamma^0 - \beta^0) \ \leq 0.$$

As $\overline{C}$ is the weak* closure of the cone generated by $\tilde{K}$, it follows that for every $X \in \overline{C}$, and Q in $\mathcal{M}^0$, $E_Q(X) \leq 0$. And then $(\Pi^0)_{0,\infty}(X) \leq 0$.

- Assume now that $X \in \overline{C} \cap L^\infty_+(\Omega, \mathcal{F}, P)$. $X \geq 0$. If $X \neq 0$ in L^∞, there is $\alpha \in \mathbb{R}^*_+$ and A with $P(A) > 0$ such that $X \geq \alpha 1_A$. Let $Q \in \mathcal{M}^0$. Q is equivalent with P, so $(\Pi^0)_{0,\infty}(X) \geq \alpha Q(A) > 0$. Thus we get a contradiction. So $\overline{C} \cap L^\infty_+(\Omega, \mathcal{F}, P) = \{0\}$.

- *i*) implies *ii*): The proof follows that of Theorem 5.2.2. of [10]. Let $f \in L^\infty_+(\Omega, \mathcal{F}, P)$. As $\overline{C}$ is closed for the weak * topology, and $\{f\}$ is compact, from Hahn Banach Theorem, there is $g \in L^1$, $g \neq 0$, such that

$$sup_{Z \in \overline{C}} E(gZ) < E(fg)$$

As $\overline{C}$ is a cone, $sup_{Z \in \overline{C}} E(gZ) = 0$ and $0 < E(fg)$. We have $-L^\infty_+ \subset \overline{C}$ so $g \geq 0$. The exhaustion argument of the proof of Theorem 5.2.2. of [10] applies without

any change. So we get g_0 strictly positive P a. s. such that $sup_{Z\in\overline{C}}E(g_0Z) = 0$. Denote Q the probability measure whose Radon Nikodym derivative is $\frac{g_0}{E(g_0)}$. $\{(H.S)_\infty\}$ where H is an admissible simple strategy is a vector space contained in $\overline{C}$. The linear form E_Q is non positive on this vector space. So it has to be identically equal to 0 on it.
It follows then from Lemma 5.1.3. of [10] that S is a local martingale under Q. Let $l \in \{1...p\}$ and $m \leq M^l$. As $-mY^l + C_{bid}(mY^l) \in K$ ($\beta^l = m\,\gamma^0 = C_{bid}(mY^l)$) it follows that $C_{bid}(mY^l) - E_Q(mY^l) \leq 0$. In the same way $E_Q(nY^l) \leq C_{ask}(nY^l)$ for $n \leq N^l$ (as $nY^l - C_{ask}(nY^l) \in K$). □

A.2 Proof of Theorem 8.4

Before starting the proof of the theorem we prove two lemmas.

Lemma A.1 *Assume that M is a Q_0-martingale of BMO norm less than m, $m < \frac{1}{16}$. For any stopping time T,*

$$|1 - \mathcal{E}(M)_T| \leq |M_T| + [M,M]_T)exp(|M|_T + [M,M]_T) \tag{A-1}$$

Proof. Recall ([18]) that

$$\mathcal{E}(M)_T = exp(M_T - \frac{1}{2}([M,M]^c)_T)\Pi_{s\leq T}(1+\Delta M_s)e^{-\Delta M_s}$$

As $m < 1$, each term of the product is positive and less than 1, therefore

$$\begin{aligned}\mathcal{E}(M)_T - 1 &\leq exp(M_T - \frac{1}{2}([M,M]^c)_T) - 1 \\ &\leq exp(|M_T|) - 1 \leq |M_T|exp(|M_T|)\end{aligned} \tag{A-2}$$

Apply the inequality $\frac{1+x}{e^x} \geq e^{-x^2}$, for $|x| < \frac{1}{16}$, (cf [11])with $x = \Delta M_s$

$$\mathcal{E}(M)_T \geq exp(-|M_T| - [M,M]_T) \tag{A-3}$$

Therefore

$$1 - \mathcal{E}(M)_T \leq (|M_T| + [M,M]_T) \tag{A-4}$$

Lemma A.1 follows from the equations (A-2) and (A-4).

Lemma A.2 *Let $m < \frac{1}{16}$. There is a constant K and an integer $r > 0$ depending only on m such that for any Q_0-martingale M of BMO norm less than m, for any stopping time T,*

$$E_{Q_0}(|1 - \mathcal{E}(M)_T|) \leq K((E_{Q_0}([M,M]_T))^{\frac{1}{r}} \tag{A-5}$$

Proof. Choose q a positive integer $q > \frac{1}{1-16m}$. let $p \in I\!R^+$ such that $\frac{1}{p}+\frac{1}{q}=1$, then $16pm < 1$. In all the following E means E_{Q_0}. From Lemma A.1 and Hölder inequality, it follows that

$$E(|\mathcal{E}(M_T)-1|) \leq \{E(|M|_T^q)^{\frac{1}{q}} + E([M,M]_T^q)^{\frac{1}{q}}\}\{E(exp(p|M|_T + p[M,M]_T))\}^{\frac{1}{p}} \tag{A-6}$$

Applying the Cauchy Schwartz inequality,

$$E(exp(p|M|_T + p[M,M]_T)) \leq \{E(exp(2p|M|_T))\}^{\frac{1}{2}}\{E(exp(2p[M,M]_T))\}^{\frac{1}{2}} \tag{A-7}$$

Applying John Nirenberg inequality and Lemma 1 of [11], it follows from (A-7) that

$$E(exp(p|M|_T + p[M,M]_T)) \leq (\frac{1}{(1-16pm)})^{\frac{1}{2}}(\frac{1}{(1-2pm^2)})^{\frac{1}{2}} \tag{A-8}$$

On the other hand, again from Cauchy Schwartz inequality,

$$E((|M|_T)^q)^{\frac{1}{q}} \leq \{E([M,M]_T)\}^{\frac{1}{2q}}\{E(|M|_T^{2q-2})\}^{\frac{1}{2q}} \tag{A-9}$$

$$E(([M,M]_T)^q)^{\frac{1}{q}} \leq \{E([M,M]_T\}^{\frac{1}{2q}})\{E([M,M]_T^{2q-1})\}^{\frac{1}{2q}} \tag{A-10}$$

From Burkholder Davis Gundy inequality, (Theorem 30 of [16], there is a constant c such that

$$E(|M|_T^{2q-2}) \leq cE([M,M]_T^{q-1})$$

and for every integer n, it follows from the proof of Lemma 1 of [11] that

$$E([M,M]_T^n) \leq m^{2n}n! \tag{A-11}$$

This proves the lemma with $r = 2q$.

Lemma A.3 *There is a constant K_1 depending only on m such that for any Q_0-martingale M of BMO norm less than m, for any stopping time T,*

$$E([M,M]_T) \leq K_1(E(\mathcal{E}(M)_T[M,M]_T))^{\frac{1}{2}} \tag{A-12}$$

Proof. From Cauchy Schwartz inequality, and then Hölder inequality with p such that $16pm < 1$ $q \in I\!N^*$, and $\frac{1}{p}+\frac{1}{q}=1$, as in the preceding proof,

$$E([M,M]_T) \leq \{E(\mathcal{E}(M)_T[M,M]_T)\}^{\frac{1}{2}}\{E([M,M]_T^q\}^{\frac{1}{2q}}\{E(\mathcal{E}(M)_T^{-p})\}^{\frac{1}{2p}} \tag{A-13}$$

From equation (A-11) $E(([M,M]_T)^q) \leq m^{2q}q!$ From inequalities (A-3) and (A-8),

$$E((\mathcal{E}(M)_T)^{-p}) \leq (\frac{1}{(1-16pm)}\frac{1}{(1-2pm^2)})^{\frac{1}{2}}$$

This proves equation (A-12).

Proof of Theorem 8.4. From proposition 5 of [4], we already know that equation (30) with (29) defines a TCPP. $\alpha_{\sigma,\tau}(Q_0) = 0$ for every $\sigma \leq \tau$, thus the TCPP has No Free Lunch. Furthermore, from Proposition 8.2, for any M, Q_M is an equivalent martingale measure for every S^k. So from proposition 8.1, we just have to find a condition on B such that the threshold condition (28) is satisfied for any stopping time T, and any probability measure Q_M in $\mathcal{Q}(\mathcal{M})$. By hypothesis on b_i, for any M, for any stopping time T $\alpha_{0,T}(Q_M) \geq BE_{Q_M}([M,M]_T)$. Thus it is enough to verify that

$$BE_{Q_0}(\mathcal{E}(M)_T[M,M]_T) \geq \sup_{\{l \mid \tau_l \leq T\}} (\sup_{m \leq M^l}((C_{bid}(mY^l) - E_{Q_M}(mY^l)), \sup_{n \leq N^l}(E_{Q_M}(nY^l) - C_{ask}(nY^l)) \quad \text{(A-14)}$$

Choose $0 < \epsilon \leq inf_{\{1 \leq l \leq p\}}(C_{ask}(Y^l) - E_{Q_0}(Y^l), E_{Q_0}(Y^l) - C_{bid}(Y^l))$, so for any $m \leq M^l$, $E_{Q_0}(mY^l) - C_{bid}(mY^l) \leq m\epsilon$. (idem for $C_{ask}(nY^l)$. Thus to satisfy (A-14), it is sufficient that for any $n \leq \sup(M^l, N^l)$

$$BE(\mathcal{E}(M)_T[M,M]_T) + n\epsilon \geq \sup_{\{l \mid \tau_l \leq T\}} n|(E_{Q_0}(Y^l) - E_{Q_M}(Y^l)| \quad \text{(A-15)}$$

Notice that $(E_{Q_0}(Y^l) - E_{Q_M}(Y^l) = E_{Q_0}((1 - \mathcal{E}(M)_T)Y^l)$.

From Lemma A.2, and Lemma A.3, there is $\tilde{K}$ depending only on m and ϵ such that for any l,

$$|E(\mathcal{E}(M_T) - 1)Y^l)| \leq \tilde{K}||Y^l||_\infty (E(\mathcal{E}(M)_T[M,M]_T))^{\frac{1}{2r}} \quad \text{(A-16)}$$

There is a constant $B_0 > 0$ such that for any $x > 0$,

$$\tilde{K}x^{\frac{1}{2r}}(\max(||Y^l||_\infty) \leq B_0 x + \epsilon$$

Then $B \geq \max(M^l, N^l)B_0$ satisfies the required conditions. □

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