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When and how do cracks propagate?

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Abstract

Crack propagation in an isotropic 2d brittle material is widely viewed as the interplay between two separate criteria. Griffith's cap on the energy release rate along the crack path decides when the crack propagates, while the Principle of Local Symmetry (PSL) decides how, that is, in which direction, that crack propagates. The PSL, which essentially predicts mode I propagation, cannot possibly hold in an anisotropic setting. Further it disagrees with its competitor, the principle of maximal energy release, according to which the direction of propagation should coincide with that of maximal energy release. Also, continuity of the time propagation is always implicitly assumed.

In the spirit of the rapidly growing variational theory of fracture, we revisit crack path in the light of an often used tool in physics, *i.e.,* energetic metastability of the current state among suitable competing crack states. In so doing, we do not need to appeal to either isotropy, or continuity in time. Here, we illustrate the impact of metastability in a 2d setting. In a 2d isotropic setting, it recovers the PSL for smooth crack paths. In the anisotropic case, it gives rise to a new criterion. But, of more immediate concern to the community, it also demonstrates that 2d crack kinking in an isotropic setting is incompatible with continuity in time of the propagation. Consequently, if viewing time continuity as non-negotiable, our work implies that the classical view of crack kinking along a single crack branch is not correct and that a change in crack direction necessarily involves more subtle geometries or evolutions.

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1 Introduction

Two-dimensional classical brittle fracture claims that, thanks to Griffith's energetic criterion [8] referred to henceforth as *Griffith's Criterion*, a priori knowledge of the crack path is all that is needed for an accurate report of the trajectory of the crack tip along that path. The criterion amounts to a balance between the release rate of the potential energy – that is the elastic energy minus the work of the external loads – and the rate of energy spent to create additional crack length. In an isotropic setting that rate is assumed constant and equal to $\mathsf{G_c}$, the bond breaking energy for two atoms in the underlying lattice. So mechanical wisdom has it that knowledge of the how will entail knowledge of the when.

Consensus is breached when addressing how a path is selected by the crack. In particular, competing criteria have been proposed for the kinking of a straight crack. Still in an isotropic setting, a well regarded criterion is the *Principle of Local Symmetry* [7] which states that the crack always propagates in mode I, that is with in-plane tractions that remain perpendicular to the crack in a small neighborhood of the crack tip [2]. A possible alternative is the $\mathsf{G}_{\max}$-Criterion which states that the crack will kink along a direction that maximizes the release rate of its potential energy among all kinking angles. It was shown [3] that the two criteria generically yield different kinking angles. There is in truth scant evidence that would support one criterion over another, and even less so in anti-plane shear because the crack is in mode III, so that the notion of mode I, or mode II propagation is rendered meaningless.

We propose in this Letter to pair Griffith's Criterion with the following least action principle, referred to henceforth as the *Stability Criterion* : At each time, the total energy, *i.e.,* the sum of potential and surface energies, is evaluated along all possible small variations of the crack from its present state and stability of the latter is declared when the total energy is smallest at that state. Similar (meta)-stability principles are common occurrence in solid mechanics, a striking example being finite elasticity. In all such settings, such principles can never be justified through a mere investigation of mechanical balance which produces at best stationarity principles. But, if meta-stability implies stationarity, the converse is generically false in any non-convex setting.

Rather than provide a detailed account of the mathematical intricacies associated with (meta)-stability [4], we propose in this Letter to derive a few striking results about kinking. Those can be summed up as follows:

- A new kinking crietrion criterion is obtained in an anisotropic setting (Proposition 2);
- In an isotropic setting, the principle of local symmetry strictly overestimates the time at which a pre-crack starts extending (Equation (12));
- In an isotropic setting, a crack cannot kink while propagating continuously in time (Proposition 3).

Thus, our results essentially settle a longstanding debate between proponents of various kinking criteria in isotropic 2d brittle fracture. The negative answer they provide demonstrate that, modulo the acceptance of metastability, the current vision of crack kinking should be completely overhauled. This stands in sharp contrast to a host of papers that pre-assume the validity of the Principle of Local Symmetry, e.g. [6, 1], or else derive it – and its anisotropic analogue – from a mere energetic stationarity principle involving both inner and outer variations; se e.g. [12, 9]. The arguments presented below can be recast as mathematical propositions in the framework developed in [5].

2 The Stability Criterion

Consider a 2d structure made of a linearly elastic, maybe anisotropic material and subject to a time-dependent load. A crack is propagating through the structure. The evolution is assumed to be quasi-static: at each time t, the structure is in elastic equilibrium with the load at t, and this for any admissible crack state Γ at t. The potential energy is denoted by $\mathcal{P}(t,\Gamma)$, while, following Griffith, the surface energy for the crack state Γ, denoted by $\mathcal{S}(\Gamma)$, is defined as follows. For a homogeneous, isotropic material, it is proportional to the crack length, *i.e.,* $\mathcal{S}(\Gamma) = \mathsf{G_c}\ \mathrm{length}(\Gamma)$, where $\mathsf{G_c}$, the fracture toughness, is a material characteristic. In the anisotropic case, toughness is orientation-dependent and, if $\theta(s)$ is the angle of the tangent to Γ with a set direction at the point with arclength s, the surface energy becomes

$$\mathcal{S}(\Gamma) = \int_0^{\mathrm{length}(\Gamma)} \mathsf{G_c}(\theta(s))\ ds. \tag{1}$$

The total energy at t for a crack state Γ is

$$\mathcal{E}(t,\Gamma) = \mathcal{P}(t,\Gamma) + \mathcal{S}(\Gamma).$$

At t, the actual crack state $\Gamma(t)$ has a tip at the interior point x, whereas

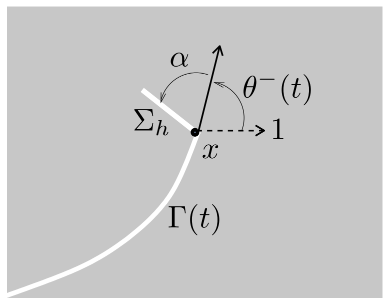

Figure 1: An admissible virtual crack that tests the meta-stability of the actual crack state $\Gamma(t)$ at t

an admissible virtual crack is obtained by adding at the tip x of $\Gamma(t)$ a line segment Σ_h of length h in a direction with angle α with the tangent to $\Gamma(t)$ at x, cf. Fig. 1. Upon computing the various energies associated with those two different crack states, the Stability Criterion demands that the actual total energy be smaller than, or equal to that for the virtual state $\Gamma(t) \cup \Sigma_h$. This should hold for any sufficiently small h and for any kinking direction α. So, for h small enough and for any α,

$$\mathcal{E}(t,\Gamma(t)) \leq \mathcal{E}(t,\Gamma(t) \cup \Sigma_h). \tag{2}$$

As already mentioned in the introduction, meta-(stability) is classical in solid mechanics for many systems with an internal variable [11]. In the setting of fracture that internal variable is the crack set and the definition of (meta)-stability entails a decision on the class of admissible test cracks. In this paper, we only consider finite unions of connected small line segments as admissible variations and will eschew consideration of a larger class of variations [5]. We should emphasize that, because fracture is classically seen as an irreversible process (at least at the structural level), we are only at liberty to increase the crack set and will refrain to test meta-stability with smaller cracks.

The physics underlying meta-stability may be roughly viewed as follows: if an admissible crack path can be found between the actual crack state and a nearby state of smaller energy, then any perturbation of the actual state should dynamically evolve the crack to the lower energy state.

3 A few hints at how cracks propagate

We proceed to derive a few necessary conditions for propagation in the light of the Stability Criterion. The (virtual) energy release rate $\mathsf{G}(t, \alpha)$ associated with the virtual test crack $\Gamma(t) \cup \Sigma_h$ at t is given by

$$\mathsf{G}(t, \alpha) := \lim_{h \downarrow 0} \frac{1}{h} \left(\mathcal{P}(t, \Gamma(t)) - \mathcal{P}(t, \Gamma(t) \cup \Sigma_h)\right).$$

As such, it depends upon α, t, but also upon $\Gamma(t)$. Similarly, the (virtual) surface energy creation rate is given by

$$\mathsf{G_c}(\alpha + \theta^-(t)) = \lim_{h \downarrow 0} \frac{1}{h} \left(\mathcal{S}(\Gamma(t) \cup \Sigma_h) - \mathcal{S}(\Gamma(t))\right)$$

where $\theta^-(t)$ is the angle of the tangent to the crack $\Gamma(t)$ with the fixed direction 1 at its tip x.

Dividing (2) by h and letting h tend to 0 yields the following property

P1 *The crack path $\Gamma(t)$ must be such that, at each t, all virtual kinks produce an energy release rate which remains smaller than the surface energy creation rate, that is*

$$\mathsf{G}(t, \alpha) \le \mathsf{G_c}(\theta^-(t) + \alpha), \quad \forall \alpha. \tag{3}$$

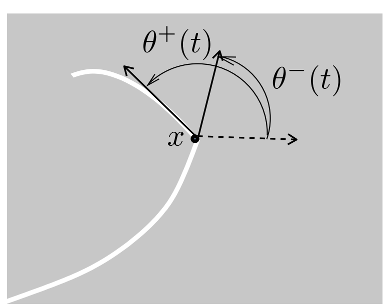

Figure 2: The crack propagates at t from the tip x in the direction $\theta^+(t)$. If $\theta^+(t) \neq \theta^-(t)$, kinking occurs, whereas, if $\theta^+(t) = \theta^-(t)$, there is no kinking.

Assume now that the time t is the actual propagation time, that is the time at which $\Gamma(t)$ is going to extend from x, *continuously as a function of time*, in a direction $\theta^+(t)$ that may coincide, or differ from $\theta^-(t)$, cf. Fig. 2. Griffith's Criterion [1] implies that the (real) energy release rate must be equal to the (real) surface energy creation rate, that is

$$\mathsf{G}(t, \theta^+(t) - \theta^-(t)) = \mathsf{G_c}(\theta^+(t)). \tag{4}$$

The reader is reminded that Griffith's Criterion (4) becomes meaningless if the crack propagation near x ceases to be smooth in time. Indeed, if the crack tip was to jump from x to y at t, it would become impossible to express the balance of energy in terms of rates and one should then generalize energy balance to a setting that accommodates finite increments in crack length [4].

Recalling Stability (3) and Griffith (4), we conclude that

P2 *At each continuous propagation time t, the crack orientation $\theta^+(t)$ must be such that*

$$1 = \frac{\mathsf{G}(t, \theta^+(t) - \theta^-(t))}{\mathsf{G_c}(\theta^+(t))} = \max_{\alpha} \frac{\mathsf{G}(t,\alpha)}{\mathsf{G_c}(\theta^-(t)+\alpha)}.$$

The above result, which applies to any kind of anisotropy, seemingly settles the issue of how and when the crack should propagate. If applied to an isotropic setting for which $\mathsf{G_c}$ is a constant, it does seem to favor the G-max criterion at the expense of the Principle of Local Symmetry. But such is not the case, as will be demonstrated below. To pursue the analysis further, we recall Irwin's formula [10] which relates the energy release upon kinking by an angle α at time t to the stress intensity factors, namely, in our notation,

$$\mathsf{G}(t,\alpha) = \mathcal{C}\left\{ {\mathsf{K}_1}^2(t,\alpha) + {\mathsf{K}_2}^2(t,\alpha) \right\},$$

where $\mathcal{C}$ is a constant that only depends upon the elasticity of the material and upon the type of setting (plane strain, or plane stress) for the specific problem under consideration. The coefficients $\mathsf{K}_1(t,\alpha)$ and $\mathsf{K}_2(t,\alpha)$ are the coefficients of the singularity at the crack tip after kinking. We also recall [3] that $\mathsf{K}_1(t,\alpha)$ and $\mathsf{K}_2(t,\alpha)$ are related to their pre-kinking analogues $\mathsf{K}_1(t,0)$ and $\mathsf{K}_2(t,0)$ through

$$\mathsf{K}_i(t,\alpha) = \mathsf{F}_{ij}(\alpha)\mathsf{K}_j(t,0),$$

[1]Introduced here independently of the Stability Criterion, Griffith's Criterion may be deduced from that Stability Criterion, together with a general principle of energy conservation [4].

where the coefficients $\mathsf{F}_{ij}(\alpha)$ are universal constants that only depend upon the kinking angle α. An analytic expression for the matrix $F(\alpha)$ is lacking at present, although asymptotic expansions around $\alpha = 0$ have been derived as well as numerical plots for all values of α[3]. In particular, it is known [3] that

$$\begin{aligned} \mathsf{F}'_{11}(0) = 0,\ \mathsf{F}'_{12}(0) = -3/2, \\ \mathsf{F}'_{21}(0) = 1/2,\ \mathsf{F}'_{22}(0) = 0, \end{aligned} \tag{5}$$

and also that

$$\mathsf{F}^2_{12}(\alpha) + \mathsf{F}^2_{22}(\alpha) = 1 + (3/2 - 8/\pi^2)\alpha^2 + O(\alpha^4). \tag{6}$$

Consider at first a crack that propagates continuously in both space and time in a homogeneous, isotropic material. At the propagation time t, $\alpha = 0$ and thus, according to Property **P2**, $\alpha = 0$ maximizes $\mathsf{G}(t, \alpha)$ among all α's. Because of Irwin's formula, this means in particular that

$$\mathsf{K}_1(t, 0)\mathsf{K}_1{}'(t, 0) + \mathsf{K}_2(t, 0)\mathsf{K}_2{}'(t, 0) = 0.$$

In view of (5), this is not possible unless $\mathsf{K}_1(t, 0)\mathsf{K}_2(t, 0) = 0$, while, if $\mathsf{K}_1(t, 0) = 0$, we should have, by maximality, that $\mathsf{F}^2_{12}(\alpha) + \mathsf{F}^2_{22}(\alpha) \leq 1$, for all α's, which is not the case in view of (6). Thus, we must have $\mathsf{K}_2(t, 0) = 0$. In other words, the Principle of Local Symmetry holds in such a setting and we conclude that

P3 *Assuming the validity of the Griffith and Stability Criteria, a crack cannot propagate continuously in space and time in a homogeneous, isotropic material unless it propagates in mode I.*

In this setting, the Principle of Local Symmetry derives from the Stability Criterion. But note that the Stability Criterion has a much longer reach than the Principle of Local Symmetry. It could in principle be applied in the anisotropic case and serve to derive an anisotropic equivalent of the Principle of Local Symmetry, although, in all fairness, this amounts to little more than wishful thinking in the absence of a more appropriate knowledge of the matrix $\mathsf{F}(\alpha)$.

Consider now a crack that propagates (in a homogeneous, isotropic material) continuously in time, while kinking in space at the point x at time t. Since, after kinking, propagation resumes continuously in both space and time, Property **P3** implies Mode I propagation after kinking, and, by passing to the limit in time down to time t, we conclude, thanks to the continuity of the stress intensity factors as a function of the kinked crack length, that $\mathsf{K}_2(t, [\![\theta]\!]) = 0$. But, in view of **P2**, $[\![\theta]\!]$ must also maximize $\mathsf{G}(t, \alpha)$ among all α's. In other words, the kinking angle must satisfy both the $\mathsf{G}_{\max}$-Criterion and the $\mathsf{K}_2 = 0$-Criterion. The numerical plots for the $\mathsf{F}_{ij}(\alpha)$'s strongly indicate that only $[\![\theta]\!] = 0$ can satisfy both and that the corresponding loading mode must be Mode I.

Specifically, the following is easily derived

P4 *In a homogeneous, isotropic elastic material, and, provided that*

$$F_{21}(\alpha)F'_{12}(\alpha) \neq F_{22}(\alpha)F'_{11}(\alpha), \forall\alpha, \tag{7}$$

and also that

$$(F_{11}(\alpha))^2 + (F_{21}(\alpha))^2 < 1,\ \alpha \neq 0, \tag{8}$$

then kinking never occurs with a propagation which is continuous in time.

As already stated, only numerical evidence presently corroborates the validity of (7), (8), for want of explicit analytical expressions for the matrix $\mathsf{F}(\alpha)$.

The argument that led to the negative Property **P4** is quite different from the classical argument put forth by the proponents of the Principle of Local Symmetry. Indeed their starting assumption is not only that

$$[\![\theta]\!] = 0 \quad \Longleftrightarrow \quad \mathsf{K}_2(t, 0) = 0,$$

but also implicitly that *propagation occurs continuously in time.* Then, they argue rightfully that, if the crack kinks by an angle $[\![\theta]\!]$ at a given point, and since kinking essentially pre-assumes the local spatial smoothness of the crack path after the kink, then $\mathsf{K}_2(t', 0) = 0$ for t' near and slightly larger than t. Then continuity of the stress intensity factors implies that $\mathsf{K}_2(t, [\![\theta]\!]) = 0$, hence the Principle of Local Symmetry $\mathsf{K}_2 = 0$. In short, the classical argument goes as follows:

$$\text{Symmetry} + \text{Continuity in time} \Rightarrow \mathsf{K}_2 = 0.$$

For us, the only principle is the Stability Criterion and we obtain that

$$\text{Stability} + \text{Continuity in time} \Rightarrow \mathsf{K}_2 = 0 + \mathsf{G}_{\max}.$$

The continuity in time of the propagation plays an essential role: the crack tip has to pass at some time through every nearby point y of the kinking point x on the kinked trajectory and equilibrium has to be satisfied at that time (or those times if it lingers at a given point). Hence the validity of the limit process $y \to x$. This is precisely what permits one to assert the validity of the condition $\mathsf{K}_2(t, 0) = 0$ at x. Otherwise, we would only have that condition at some point y, maybe close to x, where the crack jumps to at time t, and $\mathsf{K}_2(t, 0)$ would only be 0 at y, which yields no information whatsoever on $[\![\theta]\!]$ at x.

The knowledge that

$$\mathsf{K}_2 = 0 + \mathsf{G}_{\max} \Rightarrow \text{No kinking},$$

leads to the rather arbitrary rejection of the $\mathsf{G}_{\max}$-Criterion. However, adoption of the Stability criterion leads either to a rejection of the continuity in time of the evolution during kinking, or else to the rejection of the Stability Criterion. The reader is certainly at liberty to reject the latter, but, in doing so, she will have to reflect upon the arbitrariness of accepting at faith value identical meta-stability criteria in many other fields of physics and mechanics. Note also that the criterion also leads to the universally acknowledged Mode I propagation in the case of a spatially smooth crack path as shown above.

4 Revisiting the when

The negative Property **P4** renders obsolete the $\mathsf{K}_2 = 0$ versus $\mathsf{G}_{\max}$ conflict and points to a rethinking of kinking in global terms, that is without resorting to the local notion of energy release rates, which become meaningless if a crack jumps at a given time. If resolutely opposed to the notion of a crack jump, one has to abandon the simple picture of a crack trajectory as a piecewise smooth curve in 2d.

Consider a homogeneous, isotropic structure with an initial crack Γ under proportional loading (the load increases proportionally to time while its spatial variation remains fixed). In that case, the potential energy depends quadratically upon t. Also assume that Γ is not in pure Mode I to start with. Call t_i the time at which the crack starts extending, and apply the Stability Criterion with test cracks of the form $\Gamma \cup \Sigma_h$ where Σ_h is an add-crack of length h added at the tip x of Γ. If Σ_h is, as before, a line-segment, then **P1** implies that

$$t_i^2 \leq \frac{\mathsf{G_c}}{\max_\alpha \mathsf{G}(1,\alpha)}. \tag{9}$$

The proponents of the Principle of Local Symmetry usually infer, with the additional help of Griffith's Criterion, the propagation time T_i given by

$$T_i^2 = \frac{\mathsf{G_c}}{\mathsf{G}(1,[\![\theta]\!])}, \tag{10}$$

where $[\![\theta]\!]$ is the kinking angle predicted by the K_2=0 -Criterion , *i.e.,* such that $\mathsf{K}_2(1,[\![\theta]\!]) = 0$. But, since we have assumed that the initial crack is not in pure Mode I, so that, because of the incompatibility between the $\mathsf{G}_{\max}$-Criterion and the K_2=0 -Criterion, $\mathsf{G}(1,[\![\theta]\!]) < \max_\alpha \mathsf{G}(1,\alpha)$, then

$$T_i > t_i.$$

Thus, T_i given by (10) is too large and the crack should have already propagated at the kinking time!

Is the propagation time then equal to t_i? The answer depends on the type of "kink" one allows. For line-segment only add-cracks, the answer is positive, and then we are left with a jump as only possible outcome of a kink, as demonstrated above. More complex geometries of Σ_h produce a different result. Assume for example that Σ_h is a union of two line segments of respective length ηh and $(1-\eta)h$, and respective orientation α_1 and α_2; see Fig. 3. Consider a unit load, and define the energy release rate for this

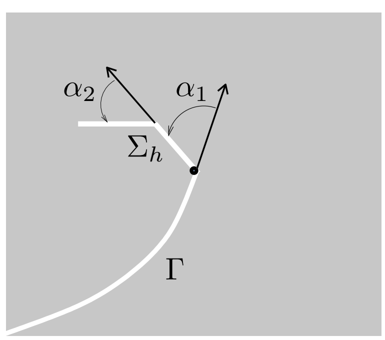

Figure 3: Testing stability with a kinked add-crack

type of add-crack – a kinked add-crack – to be

$$\mathcal{G}(\eta, \alpha_1, \alpha_2) := \lim_{h \downarrow 0} \frac{1}{h} \left(\mathcal{P}(1, \Gamma) - \mathcal{P}(1, \Gamma \cup \Sigma_h) \right). \tag{11}$$

Stability implies a new bound for t_i, namely,

$$t_i^2 \le \frac{\mathsf{G_c}}{\max\limits_{(\eta,\alpha_1,\alpha_2)} \mathcal{G}(\eta, \alpha_1, \alpha_2)}, \tag{12}$$

which is lower than, or equal to that in (9) since taking $\alpha_2 = 0$ would have us recover (9). It can be shown [5] that it is actually strictly lower than the bound given by (9). The proof uses once more the incompatibility between the $\mathsf{G}_{\max}$ and $\mathsf{K}_2\!=\!0$ criteria.

This also demonstrates [5] that the energy release rate due to a line-segment add-crack cannot equal $\mathsf{G_c}$ at the time of kinking.

All of this contributes to our opinion that the actual propagation time is strictly less than t_i. Of course, the perspicacious reader will object that the kinked add-crack is not realistic because the geometric shape of Σ_h changes with h. Maybe so, but this has the arguable merit to demonstrate that the questions of when and how a crack propagates should be simultaneously investigated and that Griffith's Criterion is not sufficient for such a task. We portend that the Stability Criterion is a good candidate for filling in the conceptual gap. In any case, kinking remains a mystery.

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