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Abstract. We give uniqueness theorem and reconstruction algorithm for the nonlinearized problem of finding the dielectric anisotropy f of the medium from non-overdetermined polarization tomography data. We assume that the medium has uniform background parameters and that the anisotropic (dielectric permeability) perturbation is described by symmetric and sufficiently small matrix-function f. On a pure mathematical level this article contributes to the theory of non-abelian Radon transforms and to iterative methods of inverse scattering.

1. Introduction

We consider the system

$$\theta\partial_x\eta = \pi_\theta f(x)\eta, \quad x \in \mathbb{R}^3, \quad \theta \in \mathbb{S}^2, \tag{1.1}$$

where

$$\theta\partial_x = \sum_{j=1}^{3} \theta_j \frac{\partial}{\partial x_j}, \tag{1.2}$$

$$\eta \;\text{ at fixed }\; \theta \;\text{ is a function on }\; \mathbb{R}^3 \;\text{ with values in }\; Z_\theta, \tag{1.3}$$

$$Z_\theta = \{z \in \mathbb{C}^3 : \; z\theta = 0\}, \tag{1.4}$$

$$f \;\text{ is a sufficiently regular function on }\; \mathbb{R}^3 \;\text{ with values in }\; \mathcal{M}_{3,3} \text{ (that is in } 3\times 3 \text{ complex matrices) with sufficient decay at infinity,} \tag{1.5}$$

$$\pi_\theta \;\text{ is the orthogonal projector on }\; Z_\theta. \tag{1.6}$$

In (1.1) the unit vector θ is considered as a spectral parameter.

System (1.1) arises in the electromagnetic polarization tomography and is a system of differential equations for the polarization vector-function η in a medium with zero conductivity, unit magnetic permeability and appropriately small anisotropic perturbation of some uniform dielectric permeability. This anisotropic perturbation of the dielectric permeability tensor corresponds to the matrix-function f of (1.1). In addition, by some physical arguments, f must be skew-Hermition, $f_{ij} = -\bar{f}_{ji}$. For more information on physics of the electromagnetic polarization tomography see [Sh1], [NS], [Sh3] and references therein (and, in particular, [KO] and [A]).

Let

$$\omega \in \mathbb{S}^2, \quad \theta \in \mathbb{S}^1_\omega, \quad \theta^\perp = \omega \times \theta, \tag{1.7}$$

$$\mu_1 = \eta\omega, \quad \mu_2 = \eta\theta^\perp \;\text{ for }\; \eta \in Z_\theta, \tag{1.8}$$

where

$$\mathbb{S}^1_\omega = \{\theta \in \mathbb{S}^2 : \ \theta\omega = 0\}, \tag{1.9}$$

$\times$ denotes vector product, Z_θ is defined by (1.4).

From (1.1)-(1.9) it follows that

$$\begin{aligned} &\theta\partial_x\mu = F(x,\theta,\omega)\mu, \quad x \in \mathbb{R}^3, \quad \theta \in \mathbb{S}^1_\omega, \\ &F(x,\theta,\omega) = \begin{pmatrix} \omega f(x)\omega & \omega f(x)\theta^\perp \\ \theta^\perp f(x)\omega & \theta^\perp f(x)\theta^\perp \end{pmatrix}, \quad \xi f(x)\zeta = \sum_{1\le i,j\le 3} f_{ij}(x)\xi_i\zeta_j, \end{aligned} \tag{1.10}$$

where μ is related with η of (1.1) by (1.8) and is a $\mathbb{C}^2$-valued function on $\mathbb{R}^3$ for fixed ω and θ.

We consider also (1.10) for μ taking its values in $\mathcal{M}_{2,2}$ (that is in 2×2 complex matrices). Let μ^+ denote the solution of (1.10) such that

$$\lim_{s\to-\infty} \mu^+(x+s\theta,\theta,\omega) = Id \ \text{ for } \ x \in \mathbb{R}^3, \tag{1.11}$$

where Id is the 2×2 identity matrix. Let

$$S(x,\theta,\omega) = \lim_{s\to+\infty} \mu^+(x+s\theta,\theta,\omega), \quad (x,\theta) \in \Gamma_\omega, \tag{1.12}$$

where

$$\begin{aligned} &\Gamma_\omega = \{(x,\theta) : \ x \in X_\theta, \ \theta \in \mathbb{S}^1_\omega\}, \quad \omega \in \mathbb{S}^2, \\ &X_\theta = Re\, Z_\theta, \quad \theta \in \mathbb{S}^2, \end{aligned} \tag{1.13}$$

where $\mathbb{S}^1_\omega$ is defined by (1.9), Z_θ is defined by (1.4). In addition, μ^+ and S are well defined due to (1.5).

Note that

$$\Gamma_\omega \subset T\mathbb{S}^2, \quad \omega \in \mathbb{S}^2, \tag{1.14}$$

where

$$T\mathbb{S}^{d-1} = \{(x,\theta) : \ x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \quad x\theta = 0\}. \tag{1.15}$$

In addition, we interpret $T\mathbb{S}^{d-1}$ as the set of all rays in $\mathbb{R}^d$. As a ray γ we understand a straight line with fixed orientation. If $\gamma = (x,\theta) \in T\mathbb{S}^{d-1}$, then $\gamma = \{y \in \mathbb{R}^d : \ y = x + s\theta, \quad s \in \mathbb{R}\}$ (modulo orientation) and θ gives the orientation of γ. Note also that

$$\Gamma_\omega \approx \mathbb{R}^2 \times \mathbb{S}^1, \tag{1.16a}$$

or, more precisely,

$$(x,\theta) \in \Gamma_\omega \Leftrightarrow x = \xi_1\theta^\perp + \xi_2\omega, \quad \xi = (\xi_1,\xi_2) \in \mathbb{R}^2, \quad \theta \in \mathbb{S}^1_\omega \approx \mathbb{S}^1, \tag{1.16b}$$

where $\omega \in \mathbb{S}^2$, $\theta^\perp = \omega \times \theta$. In addition, we consider $(\xi,\theta) \in \mathbb{R}^2 \times \mathbb{S}^1$ as coordinates on Γ_ω.

One can see that S of (1.12) is a matrix-function on

$$\Sigma = \{(\gamma,\omega): \ \gamma \in \Gamma_\omega, \ \ \omega \in \mathbb{S}^2\} = \\ \{(\gamma,\omega): \ \gamma = (x,\theta) \in T\mathbb{S}^2, \ \ \omega \in \mathbb{S}^1_\theta\}, \tag{1.17}$$

where $\mathbb{S}^1_\theta$ is defined as in (1.9). On the other hand, one can show that

$$\begin{aligned} &S(x,\theta,\omega) \ \text{ at fixed } \ \omega \in \mathbb{S}^1_\theta \ \text{ and } \ (x,\theta) \in T\mathbb{S}^2 \\ &\text{uniquely determines } \ S(x,\theta,\cdot) \ \text{ on } \ \mathbb{S}^1_\theta \ \text{ and,} \\ &\text{as a corollary, } \ S \ \text{ can be considered as a matrix} - \text{function on } \ T\mathbb{S}^2. \end{aligned} \tag{1.18}$$

The matrix-function S can be considered as a non-abelian ray transform of f. See [MZ], [V], [Sh2], [N], [FU], [E], [M], [DP], [P] and references therein for some other non-abelian ray transforms.

In the present work we say that S of (1.11)-(1.13) is the polarization ray transform of f.

Using the terminology of the scattering theory one can say also that S is the "scattering" matrix for system (1.10).

The basic problem of the polarization tomography in the framework of the model described by (1.1), (1.10) consists in finding f on $\mathbb{R}^3$ from S on Λ, where Λ is some appropriate subset of Σ of (1.17). It is especially natural to consider this problem for the case when $dim\,\Lambda = 3$.

From results of [NS] it follows that there is a non-uniqueness in this problem if f is not symmetric even if S is given on $\Lambda = \Sigma$. Results of [NS] also imply a local uniqueness theorem (up to a natural obstruction if f is not symmetric) for the case when S is given on Σ (or on $T\mathbb{S}^2$ in the sense (1.18)).

In the present work we consider the following inverse problem for equations (1.1), (1.10).

Problem 1.1. Find symmetric f, $f_{ij} = f_{ji}$, from S on Λ (or from partial information about S on Λ), where

$$\Lambda = \{(\gamma,\omega): \ \gamma \in \Gamma_\omega, \ \ \omega \in \{\omega^1,\ldots,\omega^k\}\}, \tag{1.19}$$

where $\omega^1,\ldots,\omega^k$ are some fixed points of $\mathbb{S}^2$.

One can see that Problem 1.1 is a version of the aforementioned basic problem of the polarization tomography with $dim\,\Lambda = 3$, see definitions (1.13), (1.17), (1.19).

The main results of the present work consist in uniqueness theorem and reconstruction algorithm for nonlinearized Problem 1.1 with sufficiently small f, where only the element S_{11} of $S = (S_{ij})$ on Λ is used as the data and where

$$\begin{aligned} &k = 6, \ \omega^1 = e_1, \ \omega^2 = e_2, \ \omega^3 = e_3, \\ &\omega^4 = (e_1+e_2)/\sqrt{2}, \ \omega^5 = (e_1+e_3)/\sqrt{2}, \ \omega^6 = (e_2+e_3)/\sqrt{2}, \end{aligned} \tag{1.20}$$

where e_1, e_2, e_3 is the basis in $\mathbb{R}^3$. See Sections 2, 3 and, in particular, Theorem 3.1.

One can see that this reconstruction is non-overdetermined: we reconstruct 6 functions f_{ij}, $1 \le i \le j \le 3$, on $\mathbb{R}^3$ from 6 functions $S_{11}(\cdot, \omega)$, $\omega \in \{\omega^1, \ldots, \omega^6\}$, of 3 variables.

Our reconstruction is iterative and its first apprximation more or less coincides with the linearized polarization tomography reconstruction of Section 5.1 of [Sh1]. In addition, we give estimates on the reconstruction error $f - f^n$ for the approximation f^n with number $n \in \mathbb{N}$, see Theorem 3.1 . To our knowledge even $f - f^1$ was not estimated rigorously in the literature.

The main results of the present work are presented in detail in Sections 2 and 3.

Some possible development of the present work and some open questions are mentioned in Section 6.

2. Reconstruction algorithm

Consider the classical ray transform I defined by the formula

$$If(\gamma) = \int\limits_{\mathbb{R}} f(x + s\theta)ds, \quad \gamma = (x, \theta) \in T\mathbb{S}^{d-1}, \tag{2.1}$$

for any complex-valued sufficiently regular function f on $\mathbb{R}^d$ with sufficient decay at infinity, where $T\mathbb{S}^{d-1}$ is defined by (1.15) (and where $d = 2$ or $d = 3$).

We use the following Radon-type inversion formula for I in dimension $d = 2$:

$$\begin{aligned} f(x) &= \frac{1}{4\pi} \int\limits_{\mathbb{S}^1} h'(x\theta^\perp, \theta)d\theta, \quad h'(s, \theta) = \frac{d}{ds} h(s, \theta), \\ h(s, \theta) &= \frac{1}{\pi} p.v. \int\limits_{\mathbb{R}} \frac{g(t, \theta)}{s - t} dt, \end{aligned} \tag{2.2}$$

where $g(s, \theta) = If(s\theta^\perp, \theta)$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$, $\theta^\perp = (-\theta_2, \theta_1)$, $s \in \mathbb{R}$, $d\theta$ is the standard element of arc length on $\mathbb{S}^1$.

We use the following slice by slice reconstruction of f on $\mathbb{R}^3$ from $g = If$ on Γ_ω of (1.13) for fixed $\omega \in \mathbb{S}^2$:

$$g\Big|_{T\mathbb{S}^1(Y)} \overset{(2.2)}{\rightarrow} f\Big|_Y \tag{2.3a}$$

for each two-dimensional plane Y of the form

$$Y = X(\mathbb{S}^1_\omega) + y, \quad y \in X^\perp(\mathbb{S}^1_\omega), \tag{2.3b}$$

where $\mathbb{S}^1_\omega$ is defined by (1.9), $X(\mathbb{S}^1_\omega)$ is the linear span of $\mathbb{S}^1_\omega$ in $\mathbb{R}^3$, $X^\perp(\mathbb{S}^1_\omega)$ is the orthogonal complement of $X(\mathbb{S}^1_\omega)$ in $\mathbb{R}^3$, $T\mathbb{S}^1(Y)$ is the set of all oriented straight lines lying in Y. In addition,

$$\Gamma_\omega = \cup_{y \in X^\perp(\mathbb{S}^1_\omega)} T\mathbb{S}^1(X(\mathbb{S}^1_\omega) + y). \tag{2.4}$$

Consider the three-dimensional transverse ray transformation J defined by the formula (see Section 5.1 of [Sh1]):

$$Jf(\gamma,\omega) = I(\omega f\omega)(\gamma) = \int\limits_{\mathbb{R}} \omega f(x+s\theta)\omega ds, \quad (\gamma,\omega)\in\Sigma, \quad \gamma=(x,\theta), \tag{2.5}$$

for any $\mathcal{M}_{3,3}$-valued sufficiently regular function f on $\mathbb{R}^3$ with sufficient decay at infinity, where $\omega f\omega$ is defined as in (1.10), Σ is defined by (1.17).

We use the following reconstruction of symmetric f on $\mathbb{R}^3$ from Jf on Λ of (1.19) for $\omega^1,\ldots,\omega^k$ given by (1.20):

$$\begin{aligned} f_{jj} &= I^{-1}_{\omega^j} g_{\omega^j}, \quad j=1,2,3,\\ f_{12} &= I^{-1}_{\omega^4} g_{\omega^4} - \frac{1}{2}(f_{11}+f_{22}),\\ f_{13} &= I^{-1}_{\omega^5} g_{\omega^5} - \frac{1}{2}(f_{11}+f_{33}),\\ f_{23} &= I^{-1}_{\omega^6} g_{\omega^6} - \frac{1}{2}(f_{22}+f_{33}), \end{aligned} \tag{2.6}$$

where $g_\omega = Jf\big|_{\Gamma_\omega}$ and I^{-1}_ω denotes the slice by slice reconstruction via inversion formulas (2.2), (2.3) for I from data on Γ_ω.

Now we are ready to present our iterative reconstruction of sufficiently small, symmetric and compactly supported f from the element S_{11} of $S=(S_{ij})$ on Λ, where S is defined by (1.12), Λ is defined by (1.19), (1.20).

Thus, in addition to (1.5), we assume that

$$\begin{aligned} &f \ \text{ is symmetric}, \quad f_{ij}=f_{ji},\\ &f(x)\equiv 0 \ \text{ for } \ |x|\geq r_0,\\ &f \ \text{ is sufficiently small.} \end{aligned} \tag{2.7}$$

Let

$$\Delta^0 = (S_{11}-1)\big|_\Lambda, \tag{2.8}$$

$$f^1 = \chi J^{-1}_\Lambda \Delta^0, \tag{2.9}$$

where J^{-1}_Λ denotes the reconstruction via inversion formulas (2.6) for J from data on Λ, χ denotes the multiplication by smooth χ such that

$$\begin{aligned} \chi(x)&\equiv 1 \ \text{ for } \ |x|\leq r_0,\\ \chi(x)&\equiv 0 \ \text{ for } \ |x|\geq r_1, \end{aligned} \tag{2.10}$$

where r_0 is the number of (2.7), $r_1>r_0$.

In our iterative reconstruction, f^1 is the first approximation to f.

From the approximation f^n with number n the approximation f^{n+1} with number $n+1$ is constructed as follows:

(1) We find the element μ_{11}^{n+} of $\mu^{n+} = (\mu_{ij}^{n+})$ on

$$\mathcal{V} = \{(x,\theta,\omega): \ x \in \mathbb{R}^3, \ \theta \in \mathbb{S}^1_\omega, \ \omega \in \{\omega^1, \ldots, \omega^6\}\}, \tag{2.11}$$

where μ^{n+} satisfies (1.10), (1.11) with f^n in place of f in (1.10);

(2) We find

$$S_{11}^n(x,\theta,\omega) = \lim_{s\to+\infty} \mu_{11}^{n+}(x+s\theta,\theta,\omega), \quad (x,\theta,\omega) \in \Lambda, \tag{2.12}$$

$$\Delta^n = (S_{11} - S_{11}^n)\big|_\Lambda; \tag{2.13}$$

(3) Finally, we find

$$f^{n+1} = \chi(f^n + J_\Lambda^{-1}\Delta^n), \tag{2.14}$$

where J_Λ^{-1} and χ are the same that in (2.9).

Note that in (2.9), (2.14) we do not assume that Δ^0, Δ^n are in the range of J. However, $J_\Lambda^{-1} g$ is well-defined on the basis of (2.6) for any

$$\begin{aligned} & g = (g_{\omega^1}, \ldots, g_{\omega^6}), \quad \text{where} \\ & g_{\omega^i} \ \text{ is a complex} - \text{valued sufficiently regular} \\ & \text{function on } \ \Gamma_{\omega^i} \ \text{ with sufficient decay at} \\ & \text{infinity (see (1.16)) for each } \ i \in \{1,\ldots,6\}. \end{aligned} \tag{2.15}$$

3. Convergence

We consider

$$\hat{L}^{\infty,\sigma}(\mathbb{R}^3) = \{u: \ \hat{u} \in L^\infty(\mathbb{R}^3), \ \ \|u\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)} < +\infty\}, \ \ \sigma \geq 0, \tag{3.1}$$

where

$$\hat{u}(p) = \left(\frac{1}{2\pi}\right)^3 \int\limits_{\mathbb{R}^3} e^{ipx} u(x) dx, \quad p \in \mathbb{R}^3, \tag{3.2}$$

$$\begin{aligned} & \|u\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)} = \|\hat{u}\|_{L^{\infty,\sigma}(\mathbb{R}^3)}, \\ & \|\hat{u}\|_{L^{\infty,\sigma}(\mathbb{R}^3)} = ess \ \sup_{p\in\mathbb{R}^3} (1+|p|)^\sigma |\hat{u}(p)|. \end{aligned} \tag{3.3}$$

We consider

$$\hat{C}^{\alpha,\sigma}(\mathbb{R}^3) = \{u: \ \hat{u} \in C^{[\alpha]}(\mathbb{R}^3), \ \ \|u\|_{\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)} < +\infty\}, \ \alpha \geq 0, \ \sigma \geq 0, \tag{3.4}$$

where $\hat{u}$ is defined by (3.2), $C^{[\alpha]}$ denotes $[\alpha]$-times continuously differentiable functions, $[\alpha]$ is the integer part of α,

$$\|u\|_{\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)} = \|\hat{u}\|_{C^{\alpha,\sigma}(\mathbb{R}^3)}, \tag{3.5}$$

$$\|\hat{u}\|_{C^{\alpha,\sigma}(\mathbb{R}^3)} = \sup_{|J|\le[\alpha],\ p\in\mathbb{R}^3} (1+|p|)^{\sigma}|\partial^J\hat{u}(p)| \quad \text{for} \quad \alpha=[\alpha], \tag{3.6}$$

$$\begin{aligned} &\|\hat{u}\|_{C^{\alpha,\sigma}(\mathbb{R}^3)} = \max\left(N_1, N_2\right) \quad \text{for} \quad \alpha>[\alpha],\\ &N_1 = \|\hat{u}\|_{C^{[\alpha],\sigma}(\mathbb{R}^3)},\\ &N_2 = \sup_{|J|=[\alpha],\ p\in\mathbb{R}^3,\ p'\in\mathbb{R}^3,\ |p-p'|\le 1} (1+|p|)^{\sigma}\frac{|\partial^J\hat{u}(p')-\partial^J\hat{u}(p)|}{|p-p'|^{\alpha-[\alpha]}}, \end{aligned} \tag{3.7}$$

where

$$\partial^J\hat{u}(p) = \frac{\partial^{|J|}\hat{u}(p)}{\partial p_1^{J_1}\partial p_2^{J_2}\partial p_3^{J_3}}, \quad J=(J_1,J_2,J_3)\in(\mathbb{N}\cup 0)^3, \ |J|=J_1+J_2+J_3. \tag{3.8}$$

In addition, in (3.1)-(3.8) we assume that u, $\hat{u}$ are $\mathcal{M}_{n_1,n_2}$-valued functions, in general, where $\mathcal{M}_{n_1,n_2}$ is the space of $n_1\times n_2$ matrices with complex elements,

$$|M| = \max_{\substack{1\le i\le n_1\\ 1\le j\le n_2}} |M_{ij}| \quad \text{for} \quad M\in\mathcal{M}_{n_1,n_2}. \tag{3.9}$$

In addition, in the present work we always have that $1\le n_1\le 3$, $1\le n_2\le 3$.

Lemma 3.1. *Let $u\in\hat{L}^{\infty,\sigma}(\mathbb{R}^3)$, $v\in\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)$, where $\alpha\ge 0$, $\sigma>3$. In addition, in general, we assume that u is $\mathcal{M}_{n_1,n_2}$- valued and v is $\mathcal{M}_{m_1,m_2}$- valued, where $m_2=n_1$ or/and $n_2=m_1$ (and where $1\le n_1,n_2,\ m_1,m_2\le 3$). Let*

$$\begin{aligned} &\textit{either } w=vu \quad \textit{for} \quad m_2=n_1\\ &\textit{or } w=uv \quad \textit{for} \quad n_2=m_1. \end{aligned} \tag{3.10}$$

Then for each of w of (3.10) *the following estimate holds:*

$$\begin{aligned} &w\in\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)\\ &\|w\|_{\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)} \le \lambda_1(\alpha,\sigma)\|v\|_{\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)}\|u\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)} \end{aligned} \tag{3.11}$$

for some positive $\lambda_1=\lambda_1(\alpha,\sigma)$.

Lemma 3.1 is proved in Section 4.

We consider

$$\hat{L}^{\infty,\sigma}(\Lambda) = \{g:\ \hat{g}\in L^{\infty}(\Lambda),\ \ \|g\|_{\hat{L}^{\infty,\sigma}(\Lambda)}<+\infty\}, \quad \sigma\ge 0, \tag{3.12}$$

where g is complex-valued, Λ is defined by (1.19) with $\omega^1,\dots,\omega^k$ given by (1.20),

$$\begin{aligned}&g=(g_{\omega^1},\dots,g_{\omega^k}),\quad \hat g=(\hat g_{\omega^1},\dots,\hat g_{\omega^k}),\\&g_{\omega^i}=g\big|_{\Gamma_{\omega^i}},\quad \hat g_{\omega^i}=\hat g\big|_{\Gamma_{\omega^i}},\\&\hat g_{\omega^i}(p,\theta)=\Big(\frac{1}{2\pi}\Big)^2\int\limits_{X_\theta}e^{ipx}g_{\omega^i}(x,\theta)dx,\quad (p,\theta)\in\Gamma_{\omega^i},\quad i=1,\dots,k,\end{aligned}\tag{3.13}$$

$$\begin{aligned}&\|g\|_{\hat L^{\infty,\sigma}(\Lambda)}=\|\hat g\|_{L^{\infty,\sigma}(\Lambda)},\\&\|\hat g\|_{L^{\infty,\sigma}(\Lambda)}=\max_{i\in\{1,\dots,k\}}\ ess\sup_{(p,\theta)\in\Gamma_{\omega^i}}(1+|p|)^\sigma|\hat g_{\omega^i}(p,\theta)|,\end{aligned}\tag{3.14}$$

where Γ_ω and X_θ are defined according to (1.13). Actually, in (3.12) we consider $L^\infty(\Lambda)$ as

$$L^\infty(\Lambda)=L^\infty(\Gamma_{\omega^1})\oplus\dots\oplus L^\infty(\Gamma_{\omega^k}).\tag{3.15}$$

We assume that

$$\begin{aligned}&f\ \text{ is a }\ \mathcal{M}_{3,3}-\ \text{valued function on }\ \mathbb{R}^3,\\&f\in\hat L^{\infty,\sigma}(\mathbb{R}^3)\ \text{ for some }\ \sigma>3,\\&f\ \text{ is symmetric},\quad f_{ij}=f_{ji},\\&f(x)\equiv 0\ \text{ for }\ |x|\ge r_0,\end{aligned}\tag{3.16}$$

$$\begin{aligned}&\chi\ \text{ is a nonnegative real}-\text{valued function on }\ \mathbb{R}^3,\\&\chi\in C^m(\mathbb{R}^3)\ \text{ for some }\ m\in\mathbb{N},\ m\ge\sigma,\\&\chi(x)\ge\chi(y)\ \text{ if } |y|\ge|x|,\\&\chi(x)\equiv 1\ \text{ for }\ |x|\le r_0,\\&\chi(x)\equiv 0\ \text{ for }\ |x|\ge r_1,\end{aligned}\tag{3.17}$$

where r_0, r_1 are some fixed real numbers, $r_0<r_1$. Properties (3.16), (3.17) imply, in particular, that

$$\begin{aligned}&\chi f=f,\\&\chi\in\hat C^{\alpha,\sigma}(\mathbb{R}^3)\ \text{ for any }\ \alpha\ge 0.\end{aligned}\tag{3.18}$$

Theorem 3.1. *Let f and χ satisfy* (3.16), (3.17). *Let*

$$\begin{aligned}&\|f\|_{\hat L^{\infty,\sigma}(\mathbb{R}^3)}\le\varepsilon\le\varepsilon_0(\alpha,\sigma,\rho),\\&\|\chi\|_{\hat C^{1+\alpha,\sigma}(\mathbb{R}^3)}\le\rho\end{aligned}\tag{3.19}$$

for some $\alpha\in]0,1[$, $\rho>0$, $\varepsilon_0>0$, where $\varepsilon_0=\varepsilon_0(\alpha,\sigma,\rho)$ is sufficiently small. Let S be the polarization ray transform of f (see Section 1 and, in particular, formulas (1.11), (1.12)).

Let Λ *be defined by* (1.19), (1.20). *Then the element* S_{11} *of* $S = (S_{ij})$ *on* Λ *uniquely determines* f *by the iterative reconstruction algorithm of Section 2; in addition,*

$$\begin{aligned} &f^n \to f \quad in \quad \hat{L}^{\infty,\sigma}(\mathbb{R}^3) \quad as \quad n \to +\infty, \\ &\|f - f^n\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)} \le a(\alpha,\sigma,\rho)(b(\alpha,\sigma,\rho))^{n-1}\varepsilon^{n+1}, \end{aligned} \tag{3.20}$$

for some a *and* b, *where* f^n, $n \in \mathbb{N}$, *are defined by* (2.9), (2.14).

Theorem 3.1 follows from Lemma 3.1, properties (3.18), and Propositions 3.1, 3.2, 3.3.

Proposition 3.1. *Let*

$$g \in \hat{L}^{\infty,\sigma}(\Lambda) \quad for\ some \quad \sigma > 3, \tag{3.21}$$

where g *is complex-valued (and* Λ *is defined by* (2.19), (2.20)). *Let* J_Λ^{-1} *be defined as in* (2.9). *Then*

$$\begin{aligned} &J_\Lambda^{-1} g \in \hat{L}^{\infty,\sigma}(\mathbb{R}^3), \\ &\|J_\Lambda^{-1} g\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)} \le \lambda_2 \|g\|_{\hat{L}^{\infty,\sigma}(\Lambda)} \end{aligned} \tag{3.22}$$

for some positive λ_2. *In addition, if* $g = Jf$, *where* f *satisfies* (3.16), *then* g *satisfies* (3.21) *and*

$$J_\Lambda^{-1} g = f. \tag{3.23}$$

Proposition 3.1 is proved in Section 4.

Proposition 3.2. *Let the assumptions of Theorem 3.1 be fulfilled. Let* $\delta^0 Jf$ *on* Λ *and* $\delta^1 f$ *on* $\mathbb{R}^3$ *be defined by the formulas*

$$\Delta^0 = Jf + \delta^0 Jf \quad on \quad \Lambda, \tag{3.24}$$

$$J_\Lambda^{-1}\Delta^0 = f + \delta^1 f \quad on \quad \mathbb{R}^3, \tag{3.25}$$

where Δ^0, $J_\Lambda^{-1}\Delta^0$ *are defined as in* (2.8), (2.9). *Then*

$$\begin{aligned} &\delta^0 Jf \in \hat{L}^{\infty,\sigma}(\Lambda), \\ &\|\delta^0 Jf\|_{\hat{L}^{\infty,\sigma}(\Lambda)} \le \lambda_3(\alpha,\sigma,\rho)\varepsilon^2, \end{aligned} \tag{3.26}$$

$$\begin{aligned} &\delta^1 f \in \hat{L}^{\infty,\sigma}(\mathbb{R}^3), \\ &\|\delta^1 f\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)} \le \lambda_2\lambda_3(\alpha,\sigma,\rho)\varepsilon^2, \end{aligned} \tag{3.27}$$

for some positive $\lambda_3 = \lambda_3(\alpha,\sigma,\rho)$.

Proposition 3.2 is proved in Section 4.

Proposition 3.3. *Let*

$$f^n = \chi(f + \delta^n f), \tag{3.28}$$

where f, χ *satisfy* (3.16), (3.17),

$$\begin{aligned}
&\delta^n f \in \hat{L}^{\infty,\sigma}(\mathbb{R}^3),\\
&\|f\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)} \le \varepsilon \le \varepsilon_1(\alpha,\sigma,\rho),\\
&\|f+\delta^n f\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)} \le \varepsilon \le \varepsilon_1(\alpha,\sigma,\rho),\\
&\|\chi\|_{\hat{C}^{1+\alpha,\sigma}(\mathbb{R}^3)} \le \rho
\end{aligned} \tag{3.29}$$

for some $\alpha \in]0,1[$, $\rho > 0$, $\varepsilon_1 > 0$, *where* $\varepsilon_1 = \varepsilon_1(\alpha,\sigma,\rho)$ *is sufficiently small. Let* f^{n+1} *be constructed from* S_{11} *(for* f*) and* f^n *as described in Section 2. Then*

$$f^{n+1} = \chi(f + \delta^{n+1} f), \tag{3.30}$$

where

$$\begin{aligned}
&\delta^{n+1} f \in \hat{L}^{\infty,\sigma}(\mathbb{R}^3),\\
&\|\delta^{n+1} f\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)} \le \lambda_4(\alpha,\sigma,\rho)\varepsilon\|\delta^n f\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)}
\end{aligned} \tag{3.31}$$

for some positive $\lambda_4 = \lambda_4(\alpha,\sigma,\rho)$.

Proposition 3.3 is proved in Section 5.

To obtain Theorem 3.1 we assume that ε_0 of (3.19) and ε_1 of (3.29) are so small that

$$\begin{aligned}
&\varepsilon_0 + \lambda_2\lambda_3\varepsilon_0^2 \le \varepsilon_1, \quad \lambda_4\varepsilon_1 < 1,\\
&\lambda_4(1+\lambda_2\lambda_3\varepsilon_0) \le b, \quad b\varepsilon_0 < 1,
\end{aligned} \tag{3.32}$$

for some $b = b(\alpha,\sigma,\rho)$, where λ_2, λ_3, λ_4, ε_1 are the constants of Propositions 3.1, 3.2, 3.3. Under these assumptions, Theorem 3.1 follows directly from Propositions 3.2, 3.3 and Lemma 3.1. In addition, we use Proposition 3.3 with ε given as $\varepsilon + \lambda_2\lambda_3\varepsilon^2$ in terms of ε of Theorem 3.1.

4. Proofs of Lemma 3.1 and Propositions 3.1 and 3.2

Let

$$L^{\infty,\sigma}(\mathbb{R}^3) = \{\hat{u} \in L^\infty(\mathbb{R}^3) : \ \|\hat{u}\|_{L^{\infty,\sigma}(\mathbb{R}^3)} < +\infty\}, \ \ \sigma \ge 0, \tag{4.1}$$

where $\|\cdot\|_{L^{\infty,\sigma}(\mathbb{R}^3)}$ is defined as in (3.3),

$$C^{\alpha,\sigma}(\mathbb{R}^3) = \{\hat{u} \in C^{[\alpha]}(\mathbb{R}^3) : \ \|\hat{u}\|_{C^{\alpha,\sigma}(\mathbb{R}^3)} < +\infty\}, \ \ \alpha \ge 0, \ \sigma \ge 0, \tag{4.2}$$

where $C^{[\alpha]}$ and $\|\cdot\|_{C^{\alpha,\sigma}(\mathbb{R}^3)}$ are defined as in (3.4) and (3.6), (3.7),

$$L^{\infty,\sigma}(\Lambda) = \{\hat{g} \in L^\infty(\Lambda) : \ \|\hat{g}\|_{L^{\infty,\sigma}(\Lambda)} < +\infty\}, \ \ \sigma \ge 0, \tag{4.3}$$

where $\|\cdot\|_{L^{\infty,\sigma}(\Lambda)}$ is defined as in (3.14). In addition, we assume that $\hat{u}$ of (4.1), (4.2) is matrix-valued (of some fixed size), in general, and $\hat{g}$ of (4.3) is complex-valued.

Proof of Lemma 3.1. We use, in particular, that

$$\widehat{u_1 u_2} = \hat{u}_1 * \hat{u}_2,$$
$$\hat{u}_1 * \hat{u}_2(p) = \int_{\mathbb{R}^3} \hat{u}_1(p-p')\hat{u}_2(p')dp', \quad p \in \mathbb{R}^3, \tag{4.4}$$

where u_1, u_2 are test functions on $\mathbb{R}^3$, $\hat{u}$ is defined by (3.2). In addition, u_1, u_2 are matrix-valued, in general, where the matrix product is defined in the standard way.

By the assumptions of Lemma 3.1 we have that

$$\hat{u} \in L^{\infty,\sigma}(\mathbb{R}^3), \ \hat{v} \in C^{\alpha,\sigma}(\mathbb{R}^3). \tag{4.5}$$

Formulas (3.11) follow from (4.4), (4.5), where we use, in particular, that

$$\int_{\mathbb{R}^3} \frac{dp'}{(1+|p-p'|)^\sigma (1+|p'|)^\sigma} \le \frac{c_1(\sigma)}{(1+|p|)^\sigma} \ \text{ for } \ \sigma > 3, \tag{4.6}$$

for some positive $c_1 = c_1(\sigma)$.

Lemma 3.1 is proved.

Proof of Proposition 3.1. We use that $I_\omega^{-1} g_\omega$ of (2.3), (2.6) can be defined also as

$$\begin{aligned} &I_\omega^{-1} g_\omega = u_\omega, \quad \text{where} \\ &(4\pi)^{-1}(\hat{g}_\omega(p,\theta) + \hat{g}_\omega(p,-\theta)) = \hat{u}_\omega(p), \quad p \in X_\theta, \quad \theta \in \mathbb{S}^1_\omega, \end{aligned} \tag{4.7}$$

where u_ω and $\hat{u}_\omega$ are related by (3.2), g_ω and $\hat{g}_\omega$ are related as in (3.13). Indeed, (4.7) means that

$$\begin{aligned} &(2\pi)^{-3} \int_{X_\theta} e^{ipx} g_\omega^{sym}(x,\theta)dx = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ipx} u_\omega(x)dx, \\ &p \in X_\theta \backslash \{x = s\omega, s \in \mathbb{R}\}, \quad \theta \in \mathbb{S}^1_\omega, \quad \omega \in \mathbb{S}^2, \end{aligned} \tag{4.8}$$

where

$$g_\omega^{sym}(x,\theta) = \frac{1}{2}(g_\omega(x,\theta) + g(x,-\theta)), \quad (x,\theta) \in \Gamma_\omega. \tag{4.9}$$

Representing p in (4.8) as

$$p = p_1\theta^\perp + p_2\omega \tag{4.10a}$$

and representing x in the left hand side of (4.8) as

$$x = \xi_1\theta^\perp + \xi_2\omega \tag{4.10b}$$

and integrating (4.8) with $e^{-ip_2\xi_2'}$, one can see that u_ω of (4.7) is the function such that

$$\begin{aligned} &(2\pi)^{-2} \int_{\mathbb{R}^2} e^{i\xi_1 p_1} g_\omega^{sym}(\xi_1\theta^\perp + \xi_2\omega, \theta)\delta(\xi_2 - \xi_2')d\xi_1 d\xi_2 = \\ &(2\pi)^{-2} \int_{\mathbb{R}^3} e^{ip_1\theta^\perp x} u_\omega(x)\delta(\omega x - \xi_2')dx, \quad p_1 \in \mathbb{R}\backslash 0, \quad \xi_2' \in \mathbb{R}. \end{aligned} \tag{4.11}$$

It remains to note that it is actually well-known that the determination of $u_\omega\big|_Y$ from $g_\omega\big|_{TS^1(Y)}$ via (4.9), (4.11) is equivalent to such a determination on the basis of (2.3), where $y = \xi_2'\omega$.

Formulas (4.7) imply that

$$\begin{aligned} &\hat u_\omega \in L^{\infty,\sigma}(\mathbb{R}^3),\\ &\|\hat u_\omega\|_{L^{\infty,\sigma}(\mathbb{R}^3)} \le (4\pi)^{-1}\|\hat g_\omega\|_{L^{\infty,\sigma}(\Gamma_\omega)}\\ &\text{if}\ \ \hat g_\omega \in L^{\infty,\sigma}(\Gamma_\omega), \end{aligned} \tag{4.12}$$

where $L^{\infty,\sigma}(\Gamma_\omega)$ is considered as $L^{\infty,\sigma}(\Lambda)$ (see (4.3)) for the case when Λ is reduced to the single Γ_ω.

Proposition 3.1 follows from (2.6) and (4.12).

Proof of Proposition 3.2. We consider I_θ, D_θ defined by

$$I_\theta u(x) = \int\limits_{\mathbb{R}} u(x+s\theta)ds,\quad x\in X_\theta,\quad \theta\in\mathbb{S}^2, \tag{4.13}$$

$$D_\theta u(x) = \int\limits_0^{+\infty} u(x+s\theta)ds,\quad x\in\mathbb{R}^3,\quad \theta\in\mathbb{S}^2, \tag{4.14}$$

where u is a matrix-valued test function on $\mathbb{R}^3$, X_θ is defined in (1.13).

We use that

$$\mu^+(\cdot,\theta,\omega) = Id + D_{-\theta}(F(\cdot,\theta,\omega)\mu^+(\cdot,\theta,\omega))\ \text{ on }\ \mathbb{R}^3, \tag{4.15}$$

$$S(\cdot,\theta,\omega) = Id + I_\theta(F(\cdot,\theta,\omega)\mu^+(\cdot,\theta,\omega))\ \text{ on }\ X_\theta, \tag{4.16}$$

where μ^+, F, S are defined in Section 1 (see (1.10), (1.11), (1.12)), $\theta\in\mathbb{S}^1_\omega$, $\omega\in\mathbb{S}^2$. In addition, (4.15) is an integral equation for μ^+, (4.16) is a formula for S.

Lemma 4.1. *Let $u\in\hat C^{0,\sigma}(\mathbb{R}^3)$ for some $\sigma>3$. Then*

$$\begin{aligned} &I_\theta u \in \hat C^{0,\sigma}(X_\theta),\\ &\|I_\theta u\|_{\hat C^{0,\sigma}(X_\theta)} \le 2\pi\|u\|_{\hat C^{0,\sigma}(\mathbb{R}^3)},\quad \theta\in\mathbb{S}^2, \end{aligned} \tag{4.17}$$

where

$$\hat C^{0,\sigma}(X_\theta) = \{g:\ \ \hat g\in C(X_\theta),\ \ \|g\|_{\hat C^{0,\sigma}(X_\theta)} < +\infty\}, \tag{4.18}$$

$$\hat g(p) = \Big(\frac{1}{2\pi}\Big)^2 \int\limits_{X_\theta} e^{ipx} g(x)dx,\quad p\in X_\theta, \tag{4.19}$$

$$\begin{aligned} &\|g\|_{\hat C^{0,\sigma}(X_\theta)} = \|\hat g\|_{C^{0,\sigma}(X_\theta)},\\ &\|\hat g\|_{C^{0,\sigma}(X_\theta)} = \sup_{p\in X_\theta}(1+|p|)^\sigma|\hat g(p)|. \end{aligned} \tag{4.20}$$

Lemma 4.1 follows from the formula

$$(2\pi)^{-1}\hat{g} = \hat{u}\big|_{X_\theta} \ \text{ for } \ g = I_\theta u \tag{4.21}$$

and definitions (3.2)-(3.6), (4.18)-(4.20).

Lemma 4.2. *Let*

$$v(x,\theta,\omega) = \begin{pmatrix} \omega u(x)\omega & \omega u(x)\theta^\perp \\ \theta^\perp u(x)\omega & \theta^\perp u(x)\theta^\perp \end{pmatrix}, \tag{4.22}$$

where u is a $\mathcal{M}_{3,3}$- valued function on $\mathbb{R}^3$, $x \in \mathbb{R}^3$, ω, θ, $\theta^\perp$ are vectors of (1.7). *Then*

$$\|v(\cdot,\theta,\omega)\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)} \le c_2\|u\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)} \ \ \textit{for} \ \ u \in \hat{L}^{\infty,\sigma}(\mathbb{R}^3), \ \sigma \ge 0, \tag{4.23}$$

$$\|v(\cdot,\theta,\omega)\|_{\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)} \le c_2\|u\|_{\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)} \ \ \textit{for} \ \ u \in \hat{C}^{\alpha,\sigma}(\mathbb{R}^3), \ \alpha \ge 0, \ \sigma \ge 0, \tag{4.24}$$

where c_2 is some positive constant.

Lemma 4.2 follows from definitions (4.22), (3.1)-(3.8).

Lemma 4.3. *Let $u \in \hat{C}^{\alpha,\sigma}(\mathbb{R}^3)$ for some $\alpha \in]0,1[$, $\sigma > 3$. Let $v \in \hat{C}^{1+\alpha,\sigma}(\mathbb{R}^3)$. Then*

$$\begin{aligned} &vD_{-\theta}u \in \hat{C}^{\alpha,\sigma}(\mathbb{R}^3), \\ &\|vD_{-\theta}u\|_{\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)} \le c_3(\alpha,\sigma)\|v\|_{\hat{C}^{1+\alpha,\sigma}(\mathbb{R}^3)}\|u\|_{\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)}, \quad \theta \in \mathbb{S}^2, \end{aligned} \tag{4.25}$$

for some positive $c_3 = c_3(\alpha,\sigma)$.

In Lemma 4.3, u, v are matrix-valued, in general, where the matrix product is defined in the standard way.

Proof of Lemma 4.3. We use that

$$\begin{aligned} &D_{-\theta}u(x) = G_\theta^+ * u(x) = \int\limits_{\mathbb{R}^3} G_\theta^+(x-y)u(y)dy, \\ &G_\theta^+(x) = \delta(\omega x)\delta(\theta^\perp x)h(\theta x), \end{aligned} \tag{4.26}$$

where ω, θ, $\theta^\perp$ are related as in (1.7), δ is the Dirac function,

$$\begin{aligned} &h(s) = 1 \ \text{ for } \ s > 0, \\ &h(s) = 0 \ \text{ for } \ s \le 0. \end{aligned} \tag{4.27}$$

Further in this proof we assume for simplicity that

$$\theta = e_1, \quad \theta^\perp = e_2, \quad \omega = e_3, \tag{4.28}$$

where e_1, e_2, e_3 is the basis in $\mathbb{R}^3$.

We use that

$$\widehat{u_1 * u_2} = (2\pi)^3 \hat{u}_1 \hat{u}_2, \tag{4.29}$$

where u_1, u_2 are the same that in (4.4).

We use also that, under assumptions (4.28),

$$\begin{gathered}\hat{G}^+_\theta(p) = (2\pi)^{-3} \int\limits_{\mathbb{R}^3} e^{ipx}\delta(x_3)\delta(x_2)h(x_1)dx = \\ (2\pi)^{-2}\frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{ip_1x_1}h(x_1)dx_1 = (2\pi)^{-2}\frac{-1}{2\pi i}\frac{1}{p_1+i0} = (2\pi)^{-3}\frac{i}{p_1+i0}, \quad p \in \mathbb{R}^3.\end{gathered} \tag{4.30}$$

Due to (4.26)-(4.30), we have that

$$\widehat{D_\theta u}(p) = \frac{i\hat{u}(p)}{p_1+i0}, \quad p \in \mathbb{R}^3. \tag{4.31}$$

Due to (4.4), (4.31), we have that

$$\widehat{vD_\theta u}(p) = \int\limits_{\mathbb{R}^3} \hat{v}(p-p')\frac{i\hat{u}(p')}{p'_1+i0}dp', \quad p \in \mathbb{R}^3. \tag{4.32}$$

To prove Lemma 4.3, it is sufficient to prove that

$$|\widehat{vD_\theta u}(p)| \le \frac{c_{3,1}(\alpha,\sigma)\|\hat{v}\|_{C^{\alpha,\sigma}(\mathbb{R}^3)}\|\hat{u}\|_{C^{\alpha,\sigma}(\mathbb{R}^3)}}{(1+|p|)^\sigma}, \tag{4.33a}$$

$$\left|\frac{\partial \widehat{vD_\theta u}(p)}{\partial p_j}\right| \le \frac{c_{3,2}(\alpha,\sigma)\|\hat{v}\|_{C^{1+\alpha,\sigma}(\mathbb{R}^3)}\|\hat{u}\|_{C^{\alpha,\sigma}(\mathbb{R}^3)}}{(1+|p|)^\sigma}, \tag{4.33b}$$

for $p \in \mathbb{R}^3$, $j = 1, 2, 3$ and some positive $c_{3,1}$, $c_{3,2}$.

Proceeding from (4.32) we have that

$$\widehat{vD_\theta u}(p) = \Big(\int\limits_{|p'_1|<1} + \int\limits_{|p'_1|\ge 1}\Big)\hat{v}(p-p')\frac{i\hat{u}(p')}{p'_1+i0}dp' = A(p) + B(p), \tag{4.34}$$

$$A(p) = \pi \int\limits_{p'_1=0} \hat{v}(p-p')\hat{u}(p')dp' + p.v. \int\limits_{|p'_1|<1} \hat{v}(p-p')\frac{i\hat{u}(p')}{p'_1}dp' = A_1(p) + A_2(p), \tag{4.35}$$

$$\begin{gathered}|A_1(p)| \le \pi \int\limits_{p'_1=0} \frac{\|\hat{v}\|_{C^{0,\sigma}(\mathbb{R}^3)}\|\hat{u}\|_{C^{0,\sigma}(\mathbb{R}^3)}}{(1+|p-p'|)^\sigma(1+|p'|)^\sigma}dp' \le \\ c_{3,1,1}(\sigma)\|\hat{v}\|_{C^{0,\sigma}(\mathbb{R}^3)}\|\hat{u}\|_{C^{0,\sigma}(\mathbb{R}^3)}(1+|p|)^{-\sigma},\end{gathered} \tag{4.36}$$

$$|A_2(p)| \le Const(\sigma) \int\limits_{|p_1'|<1} \left(\int\limits_{p_1'=0} \frac{\|\hat v\|_{C^{\alpha,\sigma}(\mathbb{R}^3)} \|\hat u\|_{C^{\alpha,\sigma}(\mathbb{R}^3)}}{(1+|p-p'|)^\sigma (1+|p'|)^\sigma} dp' \right) \frac{|p_1'|^\alpha}{|p_1'|} dp_1' \le$$
$$c_{3,1,2}(\alpha,\sigma) \|\hat v\|_{C^{\alpha,\sigma}(\mathbb{R}^3)} \|\hat u\|_{C^{\alpha,\sigma}(\mathbb{R}^3)} (1+|p|)^{-\sigma}, \tag{4.37}$$

$$|B(p)| \le \int\limits_{|p_1'|\ge 1} \frac{\|\hat v\|_{C^{0,\sigma}(\mathbb{R}^3)} \|\hat u\|_{C^{0,\sigma}(\mathbb{R}^3)}}{(1+|p-p'|)^\sigma (1+|p'|)^\sigma} dp' \le$$
$$c_1(\sigma) \|\hat v\|_{C^{0,\sigma}(\mathbb{R}^3)} \|\hat u\|_{C^{0,\sigma}(\mathbb{R}^3)} (1+|p|)^{-\sigma}, \tag{4.38}$$

where $p \in \mathbb{R}^3$, $c_{3,1,1}$, $c_{3,1,2}$ are some positive constants, c_1 is the constant of (4.6).

Estimate (4.33a) follows from (4.34)-(4.38). Estimate (4.33b) follows from (4.34)-(4.38) with $\hat v(p)$ replaced by $\partial \hat v(p)/\partial p_j$.

Lemma 4.3 is proved.

We continue the proof of Proposition 3.2.

Using (4.15), the property that

$$F(\cdot,\theta,\omega) = \chi F(\cdot,\theta,\omega), \tag{4.39}$$

where χ is the function of (3.16), (3.17), and Lemmas 3.1, 4.3 we obtain that

$$\mu^+(\cdot,\theta,\omega) = Id + \sum_{j=1}^{+\infty} D_{-\theta} w_j(\cdot,\theta,\omega),$$
$$w_j(\cdot,\theta,\omega) = \underbrace{F(\cdot,\theta,\omega) D_{-\theta} \ldots F(\cdot,\theta,\omega) D_{-\theta}}_{j-1} F(\cdot,\theta,\omega), \tag{4.40}$$

$$F(\cdot,\theta,\omega)\mu^+(\cdot,\theta,\omega) = F(\cdot,\theta,\omega) + \sum_{j=1}^{+\infty} w_{j+1}(\cdot,\theta,\omega), \tag{4.41}$$

where

$$\|w_j(\cdot,\theta,\omega)\|_{\hat C^{\alpha,\sigma}(\mathbb{R}^3)} \le \left(c_3(\alpha,\sigma)\|F(\cdot,\theta,\omega)\|_{\hat C^{1+\alpha,\sigma}(\mathbb{R}^3)}\right)^{j-1} \|F(\cdot,\theta,\omega)\|_{\hat C^{\alpha,\sigma}(\mathbb{R}^3)} \le$$
$$q_1 (q_2)^{j-1} \varepsilon^j, \tag{4.42}$$

where

$$q_1 = \lambda_1(\alpha,\sigma) c_2 \|\chi\|_{\hat C^{\alpha,\sigma}(\mathbb{R}^3)},$$
$$q_2 = \lambda_1(1+\alpha,\sigma) c_2 c_3(\alpha,\sigma) \|\chi\|_{\hat C^{1+\alpha,\sigma}(\mathbb{R}^3)}, \tag{4.43}$$

and λ_1, c_2, c_3, ε are the numbers of Lemmas 3.1, 4.2, 4.3 and Theorem 3.1.

Due to (4.41)-(4.43), we have that

$$\|F(\cdot,\theta,\omega)\mu^+(\cdot,\theta,\omega) - F(\cdot,\theta,\omega)\|_{\hat C^{\alpha,\sigma}(\mathbb{R}^3)} \le$$
$$q_1 \varepsilon \sum_{j=1}^{+\infty} (q_2\varepsilon)^j = \frac{q_1 q_2 \varepsilon^2}{1 - q_2\varepsilon}, \tag{4.44}$$

under the condition that $q_2\varepsilon < 1$.

Due to (4.16), (4.44) and Lemma 4.1, we have that

$$\|S(\cdot,\theta,\omega) - Id - I_\theta F(\cdot,\theta,\omega)\|_{\hat{C}^{0,\sigma}(X_\theta)} \le \frac{2\pi q_1 q_2 \varepsilon^2}{1 - q_2\varepsilon} \tag{4.45}$$

and, in particular,

$$\|S_{11}(\cdot,\theta,\omega) - 1 - I_\theta \omega f \omega\|_{\hat{C}^{0,\sigma}(X_\theta)} \le \frac{2\pi q_1 q_2 \varepsilon^2}{1 - q_2\varepsilon}. \tag{4.46}$$

To obtain (3.26) we use (4.46), (4.43) and the following estimate

$$\|\chi\|_{\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)} \le c_4(\alpha,\sigma)\|\chi\|_{\hat{C}^{1+\alpha,\sigma}(\mathbb{R}^3)}, \tag{4.47}$$

where c_4 is an appropriate positive constant. In addition, it is assumed that ε_0 of Theorem 3.1 is so small that

$$q_2\varepsilon_0 \le 1/2. \tag{4.48}$$

As a result, proceeding from (4.46) we obtain (3.24), (3.26).

Finally, (3.27) follows from (3.24)-(3.26) and Proposition 3.1.

Proposition 3.2 is proved.

5. Proof of Proposition 3.3

We will use that in the construction of f^{n+1} from f^n and S_{11} the steps given by (2.13), (2.14) can be rewritten as

$$f^{n+1} = \chi J_\Lambda^{-1}(S_{11} - 1 - T_{11}^n), \tag{5.1}$$

where T_{11}^n is the element of $T^n = (T_{ij}^n)$, where

$$T^n(x,\theta,\omega) = \lim_{s\to+\infty} U^n(x+s\theta,\theta,\omega), \quad (x,\theta,\omega)\in\Lambda, \tag{5.2}$$

$$\begin{aligned} &\theta\partial_x U^n(x,\theta,\omega) = F^n(x,\theta,\omega)(\mu^{n+}(x,\theta,\omega) - Id),\\ &\lim_{s\to-\infty} U^n(x+s\theta,\theta,\omega) = 0, \quad (x,\theta,\omega)\in\mathcal{V}, \end{aligned} \tag{5.3}$$

where F^n and μ^{n+} are defined as F and μ^+ of (1.10), (1.11) with f^n in place of f, $\mathcal{V}$ is defined by (2.11).

Indeed, (5.2), (5.3) and the definition of μ^{n+} imply that

$$U^n(x,\theta,\omega) = \mu^{n+}(x,\theta,\omega) - Id - D_{-\theta}F^n(x,\theta,\omega), \quad (x,\theta,\omega)\in\mathcal{V}, \tag{5.4}$$

$$T_{11}^n(x,\theta,\omega) = S_{11}^n(x,\theta,\omega) - 1 - I_\theta\omega f^n(x)\omega, \quad (x,\theta,\omega)\in\Lambda, \tag{5.5}$$

where I_θ and D_θ are defined by (4.13), (4.14). Using (5.5), (2.5) one can see that (5.1) is equivalent to

$$f^{n+1} = \chi J_\Lambda^{-1}(S_{11} - S_{11}^n + Jf^n). \tag{5.6}$$

Using that $J_\Lambda^{-1}Jf^n = f^n$ (this is completely similar to (3.23)), one can see that f^{n+1} of (5.1), (5.6) coincides with f^{n+1} of (2.14).

Thus, it is sufficient to prove Proposition 3.3, where (2.14) is written as (5.1).

In this proof, in addition to T^n and U^n, we consider also T and U (defined in a completely similar way with T^n, U^n):

$$T(x,\theta,\omega) = \lim_{s\to+\infty} U(x+s\theta,\theta,\omega), \quad (x,\theta,\omega)\in\Lambda, \tag{5.7}$$

$$\begin{aligned} &\theta\partial_x U(x,\theta,\omega) = F(x,\theta,\omega)(\mu^+(x,\theta,\omega) - Id),\\ &\lim_{s\to-\infty} U(x+s\theta,\theta,\omega) = 0, \quad (x,\theta,\omega)\in\mathcal{V}, \end{aligned} \tag{5.8}$$

In addition, (in a completely similar way with (5.4), (5.5)) we have that

$$U(x,\theta,\omega) = \mu^+(x,\theta,\omega) - Id - D_{-\theta}F(x,\theta,\omega), \quad (x,\theta,\omega)\in\mathcal{V}, \tag{5.9}$$

$$T_{11}(x,\theta,\omega) = S_{11}(x,\theta,\omega) - 1 - I_\theta\omega f^n(x)\omega, \quad (x,\theta,\omega)\in\Lambda. \tag{5.10}$$

One can see that

$$\begin{aligned} &f^{n+1} \overset{(5.1)}{=} \chi J_\Lambda^{-1}(S_{11} - 1 - T_{11}) + \chi J_\Lambda^{-1}(T_{11} - T_{11}^n) \overset{(2.5),(5.10)}{=}\\ &\chi J_\Lambda^{-1}(Jf) + \chi J_\Lambda^{-1}(T_{11} - T_{11}^n) \overset{(3.23)}{=} \chi(f + \delta^{n+1}f), \end{aligned} \tag{5.11}$$

where

$$\delta^{n+1}f = J_\Lambda^{-1}(T_{11} - T_{11}^n). \tag{5.12}$$

Thus, to complete the proof of Proposition 3.3 it is sufficient to obtain an appropriate estimate on $T_{11} - T_{11}^n$ (estimate (5.31) given below).

Let

$$\delta^n T_{11} = T_{11}^n - T_{11}, \tag{5.13}$$

$$\delta^n F = F^n - F, \tag{5.14}$$

$$\delta^n\mu^+ = \mu^{n+} - \mu^+. \tag{5.15}$$

Using the definitions of T^n and T one can see that $\delta^n T_{11}$ is the element of $\delta^n T = (\delta^n T_{ij})$, where

$$\delta^n T = I_\theta(F^n(\cdot,\theta,\omega)(\mu^{n+}(\cdot,\theta,\omega) - Id) - F(\cdot,\theta,\omega)(\mu^+(\cdot,\theta,\omega) - Id)), \tag{5.16}$$

where I_θ is defined by (4.13). In addition, using (5.14), (5.15) formula (5.16) can be rewritten as

$$\delta^n T = I_\theta(\delta^n F(\cdot,\theta,\omega)(\mu^+(\cdot,\theta,\omega) - Id) + F^n(\cdot,\theta,\omega)\delta^n\mu^+(\cdot,\theta,\omega)). \tag{5.17}$$

The following estimates hold:

$$\begin{aligned} &\|\delta^n F(\cdot,\theta,\omega)(\mu^+(\cdot,\theta,\omega) - Id)\|_{\hat{C}^{\alpha,\sigma}(\mathbb{R}^3)} \le\\ &\lambda_1(\alpha,\sigma)c_2\Big(\frac{q_1q_3\varepsilon}{1-q_2\varepsilon}\Big)\|\delta^n f\|_{\hat{L}^{\infty,\sigma}(\mathbb{R}^3)}, \end{aligned} \tag{5.18}$$

$$\|F^n(\cdot,\theta,\omega)\delta^n\mu^+(\cdot,\theta,\omega)\|_{\hat C^{\alpha,\sigma}(\mathbb{R}^3)} \leq \frac{q_1q_2\varepsilon\|\delta^n f\|_{\hat L^{\infty,\sigma}(\mathbb{R}^3)}}{(1-q_2\varepsilon)^2}, \tag{5.19}$$

under the condition that $q_2\varepsilon < 1$, where q_1, q_2 are given by (4.43),

$$q_3 = c_3(\alpha,\sigma)\|\chi\|_{\hat C^{1+\alpha,\sigma}(\mathbb{R}^3)}, \tag{5.20}$$

ε is the number of (3.29), λ_1, c_2, c_3 are the numbers of Lemmas 3.1, 4.2, 4.3, $\theta \in \mathbb{S}^1_\omega$, $\omega \in \mathbb{S}^2$.

Proof of (5.18). Using (3.28), (4.39) and the definiions of F, F^n, $\delta^n F$ we have that

$$\delta^n F(x,\theta,\omega) = \chi(x)\begin{pmatrix} \omega\delta^n f(x)\omega & \omega\delta^n f(x)\theta^\perp \\ \theta^\perp\delta^n f(x)\omega & \theta^\perp\delta^n f(x)\theta^\perp \end{pmatrix}, \tag{5.21}$$

$x \in \mathbb{R}^3$, $\theta \in \mathbb{S}^1_\omega$, $\omega \in \mathbb{S}^2$.

Using (4.40), (4.42) and Lemma 4.3 we obtain that

$$\|\chi(\mu^+(\cdot,\theta,\omega) - Id)\|_{\hat C^{\alpha,\sigma}(\mathbb{R}^3)} \leq q_1q_3\varepsilon\sum_{j=1}^{+\infty}(q_2\varepsilon)^{j-1} = \frac{q_1q_3\varepsilon}{1-q_2\varepsilon} \tag{5.22}$$

under the condition that $q_2\varepsilon < 1$.

Estimate (5.18) follows from (5.21), (5.22) and Lemmas 3.1 and 4.2.

Proof of (5.19). Using (4.40) for μ^+ and for μ^{n+} we have that

$$\begin{aligned} &F^n(\cdot,\theta,\omega)\delta^n\mu^+(\cdot,\theta,\omega) = F^n(\cdot,\theta,\omega)\sum_{j=1}^{+\infty} D_{-\theta}\delta^n w_j(\cdot,\theta,\omega), \\ &\delta^n w_j(\cdot,\theta,\omega) = w_j^n(\cdot,\theta,\omega) - w_j(\cdot,\theta,\omega), \end{aligned} \tag{5.23}$$

where w_j^n is defined as w_j of (4.40), but with F^n in place of F. In addition, one can see that

$$\begin{aligned} &\delta^n w_1(\cdot,\theta,\omega) = \delta^n F(\cdot,\theta,\omega), \\ &\delta^n w_{j+1}(\cdot,\theta,\omega) = \delta^n F(\cdot,\theta,\omega)D_{-\theta}w_j^n(\cdot,\theta,\omega) + F(\cdot,\theta,\omega)D_{-\theta}\delta^n w_j(\cdot,\theta,\omega),\ j \in \mathbb{N}. \end{aligned} \tag{5.24}$$

In addition, using (5.21), (5.24), Lemma 4.3, estimate (4.42) for w_j^n and Lemma 3.1 we have that

$$\begin{aligned} &\|\delta^n w_{j+1}(\cdot,\theta,\omega)\|_{\hat C^{\alpha,\sigma}(\mathbb{R}^3)} \leq \\ &c_3(\alpha,\sigma)\|\delta^n F(\cdot,\theta,\omega)\|_{\hat C^{1+\alpha,\sigma}(\mathbb{R}^3)}q_1q_2^{j-1}\varepsilon^j + \\ &c_3(\alpha,\sigma)\|F(\cdot,\theta,\omega)\|_{\hat C^{1+\alpha,\sigma}(\mathbb{R}^3)}\|\delta^n w_j(\cdot,\theta,\omega)\|_{\hat C^{\alpha,\sigma}(\mathbb{R}^3)} \leq \\ &q_1q_2^j\varepsilon^j\|\delta^n f\|_{\hat L^{\infty,\sigma}(\mathbb{R}^3)} + q_2\varepsilon\|\delta^n w_j(\cdot,\theta,\omega)\|_{\hat C^{\alpha,\sigma}(\mathbb{R}^3)}, \\ &\|\delta^n w_1(\cdot,\theta,\omega)\|_{\hat C^{\alpha,\sigma}(\mathbb{R}^3)} \leq q_1\|\delta^n f\|_{\hat L^{\infty,\sigma}(\mathbb{R}^3)}. \end{aligned} \tag{5.25}$$

Using (5.25) we obtain that

$$\begin{aligned} &\|\delta^n w_j(\cdot,\theta,\omega)\|_{\hat C^{\alpha,\sigma}(\mathbb{R}^3)} \le q_1\|\delta^n f\|_{\hat L^{\infty,\sigma}(\mathbb{R}^3)}Z_j,\\ &Z_{j+1} = (q_2\varepsilon)^j + q_2\varepsilon Z_j,\quad Z_1 = 1. \end{aligned} \tag{5.26}$$

In addition, one can see that

$$Z_j = j(q_2\varepsilon)^{j-1},\quad j\in\mathbb{N}. \tag{5.27}$$

Estimate (5.19) follows from (5.23), (5.26), (5.27), Lemmas 3.1 and 4.3 and the formula

$$\sum_{j=1}^{+\infty} jr^{j-1} = \frac{1}{(1-r)^2},\quad 0<r<1. \tag{5.28}$$

Using (5.12), (5.18), (5.19) and Lemma 4.1 we obtain that

$$\begin{aligned} &\|\delta^n T(\cdot,\theta,\omega)\|_{\hat C^{0,\sigma}(X_\theta)} \le\\ &(\frac{\lambda_1 c_2 q_1 q_3}{1-q_2\varepsilon} + \frac{q_1q_2}{(1-q_2\varepsilon)^2})\varepsilon\|\delta^n f\|_{\hat L^{\infty,\sigma}(\mathbb{R}^3)},\quad \theta\in\mathbb{S}^1_\omega,\quad \omega\in\mathbb{S}^2. \end{aligned} \tag{5.29}$$

Proceeding from (5.29) and assuming that ε_1 of (3.29) is so small that

$$q_2\varepsilon_1 \le 1/2 \tag{5.30}$$

we obtain that

$$\|\delta^n T\|_{\hat L^{\infty,\sigma}(\Lambda)} \le (2\lambda_1 c_2 q_1 q_3 + 4q_1q_2)\varepsilon\|\delta^n f\|_{\hat L^{\infty,\sigma}(\mathbb{R}^3)}, \tag{5.31}$$

where q_1, q_2, q_3, λ_1, c_2 are the same that in (5.18), (5.19).

Finally, (3.31) follows from (5.12), (5.31), (4.47) and Proposition 3.1.

Proposition 3.3 is proved.

6. Final remarks

Remark 6.1. In a subsequent paper we plan to generalize the iterative approach of the present work to the case of the polarization tomography with limited phase measurements, see, for example, [HL], [Sh3] for more information on this problem. Actually, in the framework of the model described by (1.1) the polarization tomography with limited phase measurements is reduced to the inverse problem for (1.10) with $f-(1/2)tr\,(\pi_\theta f\pi_\theta)Id$ in place of f. In this inverse problem we do not plan to restrict the "scattering" matrix S to its element S_{11} only (in contrast with results of the present work).

Remark 6.2. It remains unclear whether a version of the Riemann-Hilbert problem method of [MZ], [N] can be used for solving Problem 1.1, instead of the iterative approach of the present work. The reason is that the dependence of $F(x,\theta,\omega)$ on the spectral parameter θ (and, more precisely, the quadratic dependence on θ of the element

$F_{22}(x,\theta,\omega) = \theta^{\perp} f(x)\theta^{\perp})$ in (1.10) is not appropriate, in general, for direct applications of the Riemann-Hilbert problem method of [MZ], [N].

Remark 6.3. On the other hand (with respect to Remark 1.2), if f is skew-symmetric, $f_{ij} = -f_{ji}$, then $F_{22} \equiv 0$ and the dependence of $F(x,\theta,\omega)$ on θ is appropriate for direct applications of aforementioned Riemann-Hilbert problem method to the inverse problem for equations (1.1), (1.10). We remind that some results on the polarization tomography with skew-symmetric f, including examples of transparent f, were given in [NS]. However, the Riemann-Hilbert problem method was not yet used in these studies.

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