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Quasi-stationary distributions for structured birth and death processes with mutations

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Abstract

We study the probabilistic evolution of a birth and death continuous time measure-valued process with mutations and ecological interactions. The individuals are characterized by (phenotypic) traits that take values in a compact metric space. Each individual can die or generate a new individual. The birth and death rates may depend on the environment through the action of the whole population. The offspring can have the same trait or can mutate to a randomly distributed trait. We assume that the population will be extinct almost surely. Our goal is the study, in this infinite dimensional framework, of quasi-stationary distributions when the process is conditioned on non-extinction. We firstly show in this general setting, the existence of quasi-stationary distributions. This result is based on an abstract theorem proving the existence of finite eigenmeasures for some positive operators. We then consider a population with constant birth and death rates per individual and prove that there exists a unique quasi-stationary distribution with maximal exponential decay rate. The proof of uniqueness is based on an absolute continuity property with respect to a reference measure.

1 Introduction and main results

1.1 Introduction

We consider a general discrete model describing a structured population with a microscopic individual-based and stochastic point of view. The dynamics takes into account all reproduction and death events. Each individual is characterized by an heritable quantitative parameter, usually called *trait*, which can for example be the expression of its genotype or phenotype. During the reproduction process, mutations of the trait can occur, implying some variability in the trait space. Moreover, the individuals can die. In the general model, the individual reproduction and death rates, as well as the mutation distribution, depend on the trait of the individual and on the whole population. In particular, cooperation or competition between individuals in this population are taken into account.

In our model the set of traits $\mathbb{T}$ is a compact metric space with metric d. For convenience we assume diameter$(\mathbb{T}) = 1$. Let $\mathcal{B}(\mathbb{T})$ be the class of Borel sets in $\mathbb{T}$. The structured population is described by a finite point measure on $\mathbb{T}$. Thus, the state space, denoted by $\mathcal{A}$, is the set of all finite point measures which is contained in $\mathcal{M}(\mathbb{T})$, the set of positive measures on $\mathbb{T}$.

A configuration $\eta \in \mathcal{A}$ is described by $(\eta_y : y \in \mathbb{T})$ with $\eta_y \in \mathbb{Z}_+ = \{0, 1, ...\}$, where only a finite subset of elements $y \in \mathbb{T}$ satisfy $\eta_y > 0$. The finite set of present traits (i.e. traits of alive individuals) is denoted by

$$\{\eta\} := \{y \in \mathbb{T} : \eta_y > 0\}$$

and called the support of η. For a function f defined on the trait space $\mathbb{T}$, we will denote the integral of f with respect to η by

$$\langle \eta, f \rangle = \sum_{y \in \{\eta\}} f(y)\eta_y.$$

Let $|\cdot|$ be the cardinal number of a set. We denote by $\#\eta = |\{\eta\}|$ the number of active traits and by $\|\eta\| = \sum_{y \in \{\eta\}} \eta_y$ the total number of individuals in η. The void configuration is denoted by $\eta = 0$, so $\#0 = \|0\| = 0$ and we define $\mathcal{A}^{-0} := \mathcal{A} \setminus \{0\}$ the set of nonempty configurations.

The structured population dynamics is given by an individual-based model, taking into account each (clonal or mutation) birth and death events.

The *clonal birth rate*, the *mutation birth rate* and the *death rate* of an individual with trait y and a population $\eta \in \mathcal{A}$, are denoted respectively by $b_y(\eta)$,

$m_y(\eta)$ and $\lambda_y(\eta)$. The total reproduction rate for an individual with trait $y \in \{\eta\}$ is equal to $b_y(\eta) + m_y(\eta)$. We assume $\lambda_y(0) = b_y(0) = m_y(0) = 0$ for all $y \in \mathbb{T}$, which is natural for population dynamics. In what follows we assume that the functions

$$\lambda_y(\eta),\, b_y(\eta),\, m_y(\eta) : \mathbb{T} \times \mathcal{A}^{-0} \to \mathbb{R}_+ \text{ are continuous and strictly positive.} \quad (1)$$

Let σ be a fixed *non-atomic probability measure* on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$. The *density location* function of the mutations is $g : \mathbb{T} \times \mathbb{T} :\to \mathbb{R}_+$, $(y, z) \to g_y(z)$, where $g_y(\cdot)$ is the probability density of the trait of the new mutated individual born from y. It satisfies

$$\int_\mathbb{T} g_y(z) d\sigma(z) = 1 \text{ for all } y \in \mathbb{T}\,. \quad (2)$$

We assume that the function $g.(\cdot)$ is jointly continuous. To simplify notations we express the mutation part using location kernel $G(\eta, z) : \mathcal{A} \times \mathcal{B}(\mathbb{T}) \to \mathbb{R}^+$ given by

$$\forall\, \eta \in \mathcal{A} \,\forall z \in \mathbb{T}\,,\ G(\eta, z) = \sum_{y \in \{\eta\}} \eta_y m_y(\eta) g_y(z) = \langle \eta, m.(\eta) g.(z) \rangle. \quad (3)$$

Note that the ratio $G(\eta, z) d\sigma(z) / \int_\mathbb{T} G(\eta, z) d\sigma(z)$ is the probability that, given there is a mutation from η, the new trait is located at z. Hypothesis (1) and Lemma 1.4 stated below imply that the function G is continuous on $\mathcal{A} \times \mathbb{T}$.

We define a continuous time pure jump Markov process $Y = (Y_t)$ taking values on $\mathcal{A}$. We denote by $Q : \mathcal{A} \times \mathcal{B}(\mathcal{A}) \to \mathbb{R}_+$, $(\eta, B) \to Q(\eta, B)$, the kernel of measure jump rates given by

$$Q(\eta, B) = \sum_{y \in \{\eta\}, \eta + \delta_y \in B} \eta_y b_y(\eta) + \sum_{y \in \{\eta\}, \eta - \delta_y \in B} \eta_y \lambda_y(\eta) + \int_{\eta + \delta_z \in B} G(\eta, z) d\sigma(z)\,. \quad (4)$$

The total mass $Q(\eta)$ of the kernel at η is always finite and given by

$$Q(\eta) = \sum_{y \in \{\eta\}} (Q(\eta, \eta + \delta_y) + Q(\eta, \eta - \delta_y)) + \int_{\mathbb{T} \backslash \{\eta\}} Q(\eta, \eta + \delta_z) d\sigma(z)\,. \quad (5)$$

Observe that because of (1)

$$\forall k \geq 1\,,\ Q_+(k) = \sup\{Q(\eta) : \|\eta\| \leq k\} < \infty\,. \quad (6)$$

The construction of a process Y with càdlàg trajectories associated with the kernel Q, is the canonical one. Assume that the process starts from $Y_0 = \eta$. Then, after an exponential time of parameter $Q(\eta)$, the process jumps to $\eta + \delta_y$ for $y \in \{\eta\}$ with probability $Q(\eta, \eta + \delta_y)/Q(\eta)$, or to $\eta - \delta_y$ for $y \in \{\eta\}$ with probability $Q(\eta, \eta - \delta_y)/Q(\eta)$, or to a point $\eta + \delta_z$ for $z \in \mathbb{T} \setminus \{\eta\}$ with probability density $Q(\eta, \eta + \delta_z)/Q(\eta)$ with respect to σ. The process restarts independently at the new configuration.

The process Y can have explosions. To avoid this phenomenon and other reasons, throughout the paper we shall assume that

$$B^* = \sup_{\eta \in \mathcal{A}} \sup_{y \in \{\eta\}} \big(b_y(\eta) + m_y(\eta) \big) < \infty. \tag{7}$$

This condition also guarantees the existence of the process $(Y_t : t \geq 0)$ as the unique solution of a stochastic differential equation driven by Poisson point measures. This is done in Section 2 following [11], [5].

Since $Q(0) = 0$, the void configuration is an absorbing state for the process Y. We denote by

$$T_0 = \inf\{t \geq 0 : Y_t = 0\}$$

the extinction time. In what follows, we will assume that the process a.s. extincts when starting from any initial configuration:

$$\forall\, \eta \in \mathcal{A} : \quad \mathbb{P}_\eta(T_0 < \infty) = 1 \,. \tag{8}$$

So, in our setting we assume that competition between individuals, often due to the sharing of limited amount of resources, yields the discrete population to extinction with probability 1. Nevertheless, the extinction time T_0 can be very large compared to the typical life time of individuals, and for some species one can observe fluctuations of the population size for large amounts of time before extinction ([17]). To capture this phenomenon, we work with the notion of quasi-stationary measure, that is the class of probability measures that are invariant under the conditioning to non-extinction. This notion has been extensively studied since the pioneering work of Yaglom for the branching process in [22] and the classification of killed processes introduced by Vere-Jones in [21]. The description of quasi stationary distributions (q.s.d. for short) for finite state Markov chains was done in [6]. For countable Markov chains the infinitesimal description of q.s.d. on countable spaces was studied in [16] and [20] among others, and the more general existence result in the countable case was shown in [10]. For one-dimensional diffusions there is the pioneering work of Mandl [13] further developed in [4], [14], [18] and

for bounded regions one can see [15] among others. For models of population dynamics and demography see [2], [12] and [3].

Let us recall the definition of a quasi-stationary distribution (q.s.d.).

Definition 1.1. *A probability measure ν supported by the set of nonempty configurations $\mathcal{A}^{-0}$ is said to be a q.s.d. if*

$$\forall\, B \in \mathcal{B}(\mathcal{A}^{-0}) \,: \quad \mathbb{P}_\nu(Y_t \in B \,|\, T_0 > t) = \nu(B)\,, \tag{9}$$

where $\mathcal{B}(\mathcal{A}^{-0})$ is the class of Borel sets of $\mathcal{A}^{-0}$ and where as usual we put $\mathbb{P}_\nu = \int_{\mathcal{A}^{-0}} \mathbb{P}_\eta d\nu(\eta)$.

When starting from a q.s.d. ν, the absorption at the state 0 is exponentially distributed (for instance see [10]). Indeed, by the Markov property, the q.s.d. equality $\mathbb{P}_\nu(Y_t \in d\eta, T_0 > t) = \nu(d\eta)\mathbb{P}_\nu(T_0 > t)$ gives

$$
\begin{aligned}
\mathbb{P}_\nu(T_0 > t+s) &= \int_{\mathcal{A}^{-0}} \mathbb{P}_\nu(Y_t \in d\eta, T_0 > t+s) = \mathbb{P}_\nu(T_0 > t) \int_{\mathcal{A}^{-0}} \nu(d\eta)\mathbb{P}_\eta(T_0 > s) \\
&= \mathbb{P}_\nu(T_0 > t)\mathbb{P}_\nu(T_0 > s).
\end{aligned}
$$

Hence there exists $\theta(\nu) \geq 0$, the exponential decay rate (of absorption), such that

$$\forall\, t \geq 0 \,: \quad \mathbb{P}_\nu(T_0 > t) = e^{-\theta(\nu)t}\,. \tag{10}$$

In nontrivial situations as ours, $0 < \mathbb{P}_\nu(T_0 > t) < 1$ (for $t > 0$), then $0 < \theta(\nu) < \infty$.

1.2   The main results

Let us introduce the global quantity

$$\lambda_* = \inf_{\eta \in \mathcal{A}^{-0}} \inf_{y \in \{\eta\}} \lambda_y(\eta)\,. \tag{11}$$

Theorem 1.2. *Under the assumption*

$$B^* < \lambda_* \tag{12}$$

there exists a q.s.d ν, with exponential decay rate

$$\theta(\nu) = -\log\beta \quad with \quad \beta = \frac{\int \mathbb{E}_\eta(\|Y_1\|)\, d\nu(\eta)}{\int \|\eta\|\, d\nu(\eta)} > 0\,.$$

This result is shown in Section 4. It is based on an intermediate abstract theorem proving the existence of finite eigenmeasures for some positive operators (Theorem 4.2).

In Section 5, we will introduce a natural σ-finite measure μ and show that absolute continuity with respect to μ is preserved by the process. We study the Lebesgue decomposition of a q.s.d. with respect to μ.

In Section 6 we will study the uniform case, which is given by

$$\lambda_y(\eta) = \lambda, \ b_y(\eta) = b(1 - \rho), \ m_y(\eta) = b\rho, \tag{13}$$

where λ, b and ρ are positive numbers with $\rho < 1$. The property (12) reads $\lambda > b$. In this case it can be shown that $\beta = e^{-(\lambda-b)}$, so Theorem 1.2 ensures the existence of a q.s.d. with exponential decay rate $\lambda - b$. We will prove that this q.s.d. is the unique one with this decay rate, under the (recurrence) condition

$$\sigma \otimes \sigma\{(y, z) \in \mathbb{T}^2 : g_y(z) = 0\} = 0, \tag{14}$$

and that given the weights of the configuration, the locations of the traits under this q.s.d. are absolutely continuous with respect to σ.

Theorem 1.3. *In the uniform case assume that $\lambda > b$ and (14). Then there is a unique q.s.d. ν on $\mathcal{A}^{-0}$, associated with the exponential decay rate $\theta = \lambda - b$. Moreover ν satisfies the absolutely continuous property,*

$$\nu\left(\vec{\eta} \in \bullet \mid \overline{\eta}\right) << \sigma^{\otimes \#\eta}(\bullet).$$

In this statement, $\vec{\eta}$ denotes the ordered sequence of the elements of the support $\{\eta\}$, (the compact metric space $(\mathbb{T}, d)$ being ordered in a measurable way, see Subsection 2.1), and

$$\overline{\eta} = (\eta_y : y \in \{\eta\}) \tag{15}$$

is the associated sequence of strictly positive weights ordered accordingly.

In all what follows, the set $\mathcal{A}$ will be endowed with the Prohorov metric which makes it a Polish space (complete separable metric space). This metric induces the weak convergence topology for which $\mathcal{A}$ is closed in the finite positive measure set. (See for example [7] Chapter 7 and Appendix).

Let us give a general smoothness result which will be used several times later on.

Lemma 1.4. *Let $F : \mathcal{A} \times \mathbb{T} \to \mathbb{R}$ be a continuous function on $\mathbb{T}$. Then the function $\hat{F}$ defined on $\mathcal{A} \times \mathbb{T}$ by*

$$\hat{F}(\eta, z) = \int_{\mathbb{T}} F(y, \eta, z)\eta(dy),$$

is continuous.

Proof. Let $\eta, \tilde{\eta} \in \mathcal{A}$ and $z, z' \in \mathbb{T}$. Thus

$$
\begin{aligned}
|\hat{F}(\eta, z) - \hat{F}(\tilde{\eta}, z')| \quad \leq \quad &\langle \eta, |F(., \eta, z) - F(., \eta, z')| \rangle + \langle \eta, |F(., \eta, z') - F(., \tilde{\eta}, z')| \rangle \\
&+ |\langle \eta - \tilde{\eta}, F(., \tilde{\eta}, z') \rangle|.
\end{aligned}
$$

Since $\mathbb{T}$ is a compact set, it is immediate that the two first terms are small if z is close to z' and η close to $\tilde{\eta}$. If $\tilde{\eta}$ is in a small enough neighborhood of η, these two atomic measures have the same weights, and the corresponding traits are close. In particular, $\tilde{\eta}$ belongs to a compact set, and the smallness of the last term follows by the equicontinuity of F on compact sets. $\qquad\square$

2 Poisson construction, martingale and Feller properties

Recall that (1) and (7) are assumed. We now give a pathwise construction of the process Y. As a preliminary result, we introduce an equivalent representation of the finite point measures as a finite sequence of ordered elements.

2.1 Representation of the finite point measures

Since $(\mathbb{T}, d)$ is a compact metric space there exists a countable basis of open sets $(\mathcal{U}_i : i \in \mathbb{N} = \{1, 2, ..\})$, that we fix once for all. The representation

$$\mathcal{R} : \mathbb{T} \to \{0, 1\}^{\mathbb{N}}, \quad z \to \mathcal{R}(z) = (c_i : i \in \mathbb{N}) \text{ with } c_i = \mathbf{1}(z \in \mathcal{U}_i)$$

is an injective measurable mapping, where the set $\{0, 1\}^{\mathbb{N}}$ is endowed with the product σ−field. On $\{0, 1\}^{\mathbb{N}}$ we consider the lexicographical order $\leq_l$ which induces the following order on $\mathbb{T}$: $z \preceq z' \Leftrightarrow \mathcal{R}(z) \leq_l \mathcal{R}(z')$. This order relation is measurable.

The support $\{\eta\}$ of a configuration can be ordered by $\preceq$ and represented by the tuple $\vec{\eta} = (y_1, ..., y_{\#\eta})$ and its discrete structure is $\overline{\eta} = (\overline{\eta}(k) := \eta_{y_k} : k \in \{1, ..., \#\eta\})$. Let us define $S_0(\eta) = 0$ and $S_k(\eta) = \sum_{l=1}^{k} \eta_{y_l}$ for $k \in \{1, ..., \#\eta\}$.

Remark that $S_{\#\eta}(\eta) = \|\eta\|$. It is convenient to add an extra topologically isolated point ∂ to $\mathbb{T}$. Now we can introduce the functions $H^i : \mathcal{A} \mapsto \mathbb{T} \cup \{\partial\}$ by $H^0(\eta) = 0$ for all $\eta \in \mathcal{A}$ and for $i \geq 1$

$$H^i(\eta) = \begin{cases} y_k & \text{if } i \in (S_{k-1}(\eta), S_k(\eta)] \text{ for } k \leq \#\eta \\ \partial & \text{otherwise}. \end{cases}$$

The functions H^i are measurable. We extend the functions b, λ and m to ∂ by putting $b_\partial(\eta) = \lambda_\partial(\eta) = m_\partial(\eta) = 0$ for all $\eta \in \mathcal{A}$.

2.2 Pathwise Poisson construction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space in which there are defined two independent Poisson point measures:

- (ii) $M_1(ds, di, dz, d\theta)$ is a Poisson point measure on $[0, \infty) \times \mathbb{N} \times \mathbb{T} \times \mathbb{R}^+$, with intensity measure $ds \left(\sum_{k \geq 1} \delta_k(di) \right) d\sigma(z) d\theta$ (the birth Poisson measure).

- (i) $M_2(ds, di, d\theta)$ is a Poisson point measures on $[0, \infty) \times \mathbb{N} \times \mathbb{R}^+$, with the same intensity measure $ds \left(\sum_{k \geq 1} \delta_k(di) \right) d\theta$ (the death Poisson measure).

We denote $(\mathcal{F}_t : t \geq 0)$ the canonical filtration generated by these processes.

We define the process $(Y_t : t \geq 0)$ as a $(\mathcal{F}_t : t \geq 0)$-adapted stochastic process such that a.s. and for all $t \geq 0$,

$$Y_t = Y_0 + \int_{[0,t] \times \mathbb{N} \times \mathbb{T} \times \mathbb{R}^+} \mathbf{1}_{\{i \leq \|Y_{s-}\|\}} \left\{ \delta_{H^i(Y_{s-})} \mathbf{1}_{\left\{ \theta \leq b_{H^i(Y_{s-})} g_{H^i(Y_{s-})}(z) \right\}} \right.$$

$$\left. + \delta_z \mathbf{1}_{\left\{ b_{H^i(Y_{s-})} g_{H^i(Y_{s-})}(z) \leq \theta \leq b_{H^i(Y_{s-})} g_{H^i(Y_{s-})}(z) + m_{H^i(Y_{s-})} g_{H^i(Y_{s-})}(z) \right\}} \right\} M_1(ds, di, dz, d\theta)$$

$$- \int_{[0,t] \times \mathbb{N} \times \mathbb{R}^+} \delta_{H^i(Y_{s-})} \mathbf{1}_{\{i \leq \|Y_{s-}\|\}} \mathbf{1}_{\left\{ \theta \leq \lambda_{H^i(Y_{s-})}(Y_{s-}) \right\}} M_2(ds, di, d\theta). \tag{16}$$

The existence of such process is proved in [11], as well as its uniqueness in law. Its jump rates are those given by (4) so this process has the same law as the process introduced in Subsection 1.1. In particular, the law of Y does not depend on the choice of the functions H^i neither on the order defined in Subsection 2.1.

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Proposition 2.1. *For any $\eta \in \mathcal{A}$, any $p \geq 1$ and $t_0 > 0$, there exists two positive constants c_p and b_p such that*

$$\mathbb{E}_\eta(\sup_{t \in [0,t_0]} \|Y_t\|^p) \leq c_p \, e^{b_p t_0} < \infty. \tag{17}$$

Proof. Let us introduce the following hitting times,

$$\forall K \in \mathbb{N} : \quad T_K = \inf\{t \geq 0 : \|Y_t\| \geq K\}. \tag{18}$$

From (16) and neglecting the non-positive term, we easily obtain that for any K,

$$\|Y_{t \wedge T_K}\|^p \leq \|Y_0\|^p + \int_{D_0} ((\|Y_{s-}\| + 1)^p - \|Y_{s-}\|^p) M_1(ds, di, dz, d\theta),$$

where D_0 is the subset of $[0, t \wedge T_K] \times \mathbb{N} \times \mathbb{T} \times \mathbb{R}_+$ which satisfies $i \leq \|Y_{s-}\|$ and $\theta \leq b_{H^i(Y_{s-})} g_{H^i(Y_{s-})}(z) + m_{H^i(Y_{s-})} g_{H^i(Y_{s-})}(z)$.

Then, by taking expectations, from (7) and convexity inequality, we obtain

$$\mathbb{E}_\eta(\sup_{t \leq t_0} \|Y_{t \wedge T_K}\|^p) \leq \|\eta\|^p + B^* p 2^{p-2} \int_0^{t_0} (1 + \mathbb{E}_\eta(\sup_{u \leq s \wedge T_K} \|Y_u\|^p)) ds.$$

Standard arguments (Gronwall's lemma) allow us to control the growth of the r.h.s. with respect to t_0. In particular, we have

$$\mathbb{E}_\eta(\sup_{t \wedge T_K} \|Y_t\|^p) \leq (\|\eta\|^p + p 2^{p-2} B^*) e^{p 2^{p-2} B^* t_0}.$$

With $p = 1$, (17) implies

$$K \, \mathbb{P}_\eta(T_K < t_0) \leq \|\eta\| \, e^{B^* t_0}, \tag{19}$$

and thus $T_K \to +\infty$ a.s. as $K \to +\infty$ and the process is well defined on $\mathbb{R}_+$. Letting K go to infinity leads to the conclusion of the proof. $\qquad \square$

Observe that the process $\|Y\|$ is dominated everywhere by the integer-valued process Z solution of

$$Z_t = \|Y_0\| + \int_{[0,t] \times \mathbb{N} \times \mathbb{T} \times \mathbb{R}^+} \mathbf{1}_{\{i \leq \|Z_{s-}\|\}} \mathbf{1}_{\{\theta \leq B^* \, g_{H^i(Z_{s-})}(z)\}} M_1(ds, di, dz, d\theta), \tag{20}$$

which is a birth process with rate B^*. This means that a.s. $\|Y_t\| \leq Z_t$. We can establish the following stronger result.

Lemma 2.2. *The process $\|Y\|$ is dominated by a birth and death process with birth rate B^* and death rate λ_*. Then if we assume that $\lambda_* > B^*$, the process Y is absorbed exponentially fast.*

Proof. We introduce a coupling on the subset $\mathcal{J}$ of $\mathcal{A} \times \mathbb{N}$ defined by

$$\mathcal{J} = \left\{ (\eta, m) \in \mathcal{A} \times \mathbb{N} \; : \; \|\eta\| \le m \right\}.$$

The coupled process is defined by its infinitesimal generator J, given by the rates

$$
\begin{aligned}
J(\eta, m; \eta + \delta_y, m + 1) &= \eta_y b_y(\eta), \;\; y \in \{\eta\} \,, \\
J(\eta, m; \eta + \delta_z, m + 1) &= G(\eta, z), \;\; z \notin \{\eta\} \,, \\
J(\eta, m; \eta, m + 1) &= mB^* - \sum_{y \in \{\eta\}} \eta_y (b_y(\eta) + m_y(\eta)) \,, \\
J(\eta, m; \eta - \delta_y, m - 1) &= \lambda_* \eta_y \,, \;\; y \in \{\eta\} \,, \\
J(\eta, m; \eta - \delta_y, m) &= \eta_y (\lambda_y(\eta) - \lambda_*) \,, \;\; y \in \{\eta\} \,, \\
J(\eta, m; \eta, m - 1) &= \lambda_* \big(m - \sum_{y \in \{\eta\}} \eta_y \big) \,.
\end{aligned}
$$

It is immediate to check that the coordinates of this process have respectively the law of Y and the law of a birth and death process with birth rate B^* and death rate λ_*. On the other hand when the coupled process starts from $\mathcal{J}$ it remains in $\mathcal{J}$ forever, so the domination follows.

In [20] it is shown that the condition $\lambda_* > B^*$ implies that the birth and death chain is exponentially absorbed. The above domination implies that so does $\|Y\|$.

$\square$

It is useful to prove at this stage the following result on hitting times that only requires the property $B^* < \infty$.

Lemma 2.3. *For any $t \ge 0$ and any $\eta \in \mathcal{A}^{-0}$, there is a number $c = c(t, \|\eta\|) \in (0, 1)$ such that,*

$$\forall K > 0 : \quad \mathbb{P}_\eta\big(T_K \le t\big) \le c^{-1} e^{-cK} \,. \tag{21}$$

Proof. The proof follows immediately from the domination of $\|Y\|$ by the birth process Z introduced in (20). Indeed, assume $\|Y_0\| < K$ and denote by T_M^Z the smallest time such that $Z_t \ge M$. Then $T_K \le T_K^Z$ a.s.. Therefore, for any $t \ge 0$, and any $\eta \in \mathcal{A}^{-0}$

$$\mathbb{P}_\eta\big(T_K \le t\big) \le \mathbb{P}_{\|\eta\|}\big(T_K^Z \le t\big) \,.$$

For a pure birth process (see for example [9]) we have

$$\mathbb{P}_{\|\eta\|}\left(T_K^Z \leq t\right) \leq \mathbb{P}_{\|\eta\|}\left(Z_t \geq K\right) = \sum_{m=K}^{\infty} \binom{m-1}{\|\eta\|-1} e^{-B^* \|\eta\| t}\left(1 - e^{-B^* t}\right)^{m-\|\eta\|}.$$

The result follows at once from this estimate. $\qquad\square$

2.3 Martingale properties

The process Y is Markovian and we describe its infinitesimal generator, in a weak form, using related martingales. The main hypotheses here are the boundedness of the total birth rate per individual (see (7)) and the following bound for the death individual rate: there exist $p \geq 1$ and $c > 0$ such that

$$\sup_{y \in \mathbb{T}} \lambda_y(\eta) \leq c \|\eta\|^p. \tag{22}$$

We define the weak generator of Y. Given $f : \mathcal{A} \to \mathbb{R}$, a measurable and locally bounded function with $f(0) = 0$, we define Lf as

$$\begin{aligned}
Lf(\eta) &= \sum_{y \in \{\eta\}} \eta_y b_y(\eta)\left(f(\eta + \delta_y) - f(\eta)\right) \tag{23}\\
&\quad + \sum_{y \in \{\eta\}} \eta_y m_y(\eta) \int_{\mathbb{T}} \left(f(\eta + \delta_z) - f(\eta)\right) g_y(z) d\sigma(z)\\
&\quad + \sum_{y \in \{\eta\}} \eta_y \lambda_y(\eta)(f(\eta - \delta_y) - f(\eta)).
\end{aligned}$$

Proposition 2.4. *Let $f : \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}$ be a measurable function such that for any $\rho \in \mathcal{A}$ the marginal function $f(\bullet, \rho)$ is C^1. We assume $f(\bullet, 0) = 0$ and we take $Y_0 = \eta$.*

(i) If f and $\partial_s f$ are bounded on $[0, t_0] \times \mathcal{A}$, for any $t_0 \geq 0$, then

$$\mathscr{M}_t^f =: f(t, Y_t) - f(0, \eta) - \int_0^t (\partial_s f(s, Y_s) + Lf(Y_s))ds \tag{24}$$

is a càdlàg $(\mathcal{F}_t : t \geq 0)$-martingale.

(ii) Moreover, if there exists a finite p such that for any $t_0 \geq 0$ we have

$$\sup_{0 \leq t \leq t_0} |f(t, \eta)| + |\partial_t f(t, \eta)| \leq C(t_0)(1 + \|\eta\|^p),$$

for some finite $C(t_0)$, then $\mathscr{M}^f$ is a martingale.

(iii) If the functions $f, \partial_s f$ are assumed to be continuous, or more generally locally bounded, then $\mathscr{M}^f$ is a local martingale and for any $T_N = \inf\{t > 0 : \|Y_t\| \geq N\}$ the process $(\mathscr{M}^f_{T_N \wedge t} : t \geq 0)$ is a martingale.

Proof. Let us prove the first part of the Proposition. For all $t \geq 0$

$$f(t, Y_t) - f(0, \eta) - \int_0^t \partial_s f(s, Y_s) ds = \sum_{s \leq t} (f(s, Y_{s-} + (Y_s - Y_{s-})) - f(s, Y_{s-}))$$

holds $\mathbb{P}_\eta$ almost surely. A simple computation shows that

$$f(t, Y_t) - f(0, \eta) - \int_0^t \partial_s f(s, Y_s) ds =$$

$$\int_{[0,t] \times \mathbb{N} \times \mathbb{T} \times \mathbb{R}^+} \mathbf{1}_{\{i \leq \|Y_{s-}\|\}} \left\{ \left(f(s, Y_{s-} + \delta_{H^i(Y_{s-})}) - f(s, Y_{s-}) \right) \mathbf{1}_{\{\theta \leq b_{H^i(Y_{s-})}(Y_{s-}) g_{H^i(Y_{s-})}(z)\}} \right.$$

$$\left. + (f(s, Y_{s-} + \delta_z) - f(s, Y_{s-})) \mathbf{1}_{\{\theta \leq m_{H^i(Y_{s-})}(Y_{s-}) g_{H^i(Y_{s-})}(z)\}} \right\} M_1(ds, di, dz, d\theta)$$

$$+ \int_{[0,t] \times \mathbb{N} \times \mathbb{R}^+} \left(f(s, Y_{s-} - \delta_{H^i(Y_{s-})}) - f(s, Y_{s-}) \right) \mathbf{1}_{\{i \leq \|Y_{s-}\|, \theta \leq \lambda_{H^i(Y_{s-})}(Y_{s-})\}} M_2(ds, di, d\theta),$$

where both integrals belong to $L^1(\mathbb{P}_\eta)$. Compensating each Poisson measure, using Fubini's Theorem, and the fact that $\int_\mathbb{T} g_y(z) d\sigma(z) = 1$, we obtain

$$f(t, Y_t) - f(0, \eta) - \int_0^t (\partial_s f(s, Y_s) + L f(Y_s)) ds$$

is a martingale. The rest of the Proposition is proved by localization arguments, justified by the result and the proof of Proposition 2.1. $\qquad\square$

2.4 Feller property of the semi-group

Let $\mathcal{N} = (\mathcal{N}_t : t \geq 0)$ be the number of jumps for the process Y. We shall prove by induction the following result.

Lemma 2.5. *Assume that $f : \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}$ is a bounded continuous function. Then for all $m \geq 0$*

$$(t, \eta) \to \mathbb{E}_\eta(f(t, Y_t), \mathcal{N}_t = m)$$

is a continuous function.

Proof. We notice that continuity and uniform continuity on every $\mathcal{A}_k$, $k \geq 1$ are equivalent because these sets are compact. Also we have that $|f|$ is bounded on $[0, t_0] \times (\bigcup_{k=1}^n \mathcal{A}_k)$ for any t_0, n and we denote by $\|f\|_{t_0,n}$ its supremum on this set. Denote by $\ell = \|\eta\|$ and $n = m + \ell + 1$.

We first prove the continuity on time. For this purpose, we assume that $0 \leq u \leq t \leq t_0$ where we assume that t, u are close and t_0 is fixed. From

$$f(t, Y_t) = f(t, Y_u)\mathbf{1}_{\mathcal{N}_t = \mathcal{N}_u} + f(t, Y_t)\mathbf{1}_{\mathcal{N}_t \neq \mathcal{N}_u}$$

we find (recall the notation (6)),

$$|\mathbb{E}_\eta(f(t, Y_t), \mathcal{N}_t = m) - \mathbb{E}_\eta(f(u, Y_u), \mathcal{N}_u = m)|$$
$$\leq \sup_{\|\xi\| \leq \ell + m} |f(t, \xi) - f(u, \xi)| + \|f\|_{t_0,n} Q_+(m + \ell)\left((t - u) + o(t - u)\right).$$

Since the set $\{\xi \,\big|\, \|\xi\| \leq \ell + m\}$ is compact, it follows from the uniform continuity of f on compact sets that the first term on the r.h.s. is small if $|t - u|$ is small. Hence the result follows.

So in what follows we consider that $t = u$ and we prove continuity on η. We will do it by induction on m. In the case $m = 0$ we have $\mathbb{E}_\eta(f(t, Y_t), \mathcal{N}_t = 0) = f(t, \eta)e^{-Q(\eta)t}$ which is clearly continuous on η. Now we prove the induction step, so we assume that the statement holds for m and all continuous functions f. We have

$$\mathbb{E}_\eta(f(t, Y_t), \mathcal{N}_t = m+1) = \int_0^t (A_1(\eta, m, t-s) + A_2(\eta, m, t-s) + A_3(\eta, m, t-s))e^{-Q(\eta)s}\, ds,$$

where

$$A_1(\eta, m, t-s) = \sum_{y \in \eta} \eta_y b_y(\eta)\mathbb{E}_{\eta+\delta_y}(f(t-s, Y_{t-s}), \mathcal{N}_{t-s} = m)$$

$$A_2(\eta, m, t-s) = \sum_{y \in \eta} \eta_y \lambda_y(\eta)\mathbb{E}_{\eta-\delta_y}(f(t-s, Y_{t-s}), \mathcal{N}_{t-s} = m)$$

$$A_3(\eta, m, t-s) = \int_{\mathbb{T}} \mathbb{E}_{\eta+\delta_z}(f(t-s, Y_{t-s}), \mathcal{N}_{t-s} = m)G(\eta, z)\sigma(dz).$$

Using Lemma 2.5, condition (1) and Lemma 1.4, it is immediate that the functions A_1, A_2 and A_3 are continuous in (t, η). We conclude by the Dominated Convergence Theorem since f is bounded. $\qquad\square$

Proposition 2.6. *Let* $f : \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}$ *be a bounded continuous function. Then*

$$(t, \eta) \to \mathbb{E}_\eta(f(t, Y_t))$$

is a continuous bounded function.

Proof. Using Lemma 2.2 (1) and the proof of Lemma 2.3, we obtain that for each $\eta \in \mathcal{A}$, $t > 0$, there exists $a = a(t, \|\eta\|) > 0$ such that for any positive integer K,

$$
\begin{aligned}
\mathbb{P}_\eta\big(T_K^{\mathcal{N}} \leq t\big) &= \mathbb{P}_\eta\big(\mathcal{N} \geq M\big) \leq \mathbb{P}_\eta\big(Z \geq K + \|\eta\|\big) \\
&= \mathbb{P}_{\|\eta\|}\big(T_K^Z \leq t\big) \leq a^{-1} e^{-aK} .
\end{aligned}
\tag{25}
$$

Assume that η' is closed to η, and consider u, t close and smaller than t_0 fixed. Then

$$
|\mathbb{E}_\eta(f(t, Y_t)) - \mathbb{E}_{\eta'}(f(u, Y_u))| \leq 2\|f\| \mathbb{P}_{\|\eta\|}(Z_{t_0} \geq M + \|\eta\|) + \\
\sum_{m=0}^{K} |\mathbb{E}_\eta(f(t, Y_t), \mathcal{N}_t = m) - \mathbb{E}_{\eta'}(f(u, Y_u), \mathcal{N}_u = m)|.
$$

The result follows by taking a large K, and by applying the bound (25) and Lemma 2.5. $\qquad \square$

3 Quasi-stationary distributions

3.1 The process killed at 0.

Let us recall that the state 0 is absorbing for the population process Y. We have moreover assumed in (8) that the population goes almost surely to extinction, that is $\mathbb{P}(T_0 < \infty) = 1$. This is in particular true if $\lambda_* > B^*$. Our aim is the study of existence and possibly uniqueness of a q.s.d. ν, which is a probability measure on $\mathcal{A}^{-0}$ satisfying $\mathbb{P}_\nu(Y_t \in B \,|\, T_0 > t) = \nu(B)$. Let us now give some preliminary results for quasi-stationary distributions (q.s.d.).

Since by condition (12), the process Y is almost surely but not immediately absorbed, and since starting from a q.s.d. ν, the absorption time is exponentially distributed (see (10)), then its exponential decay rate satisfies $0 < \theta(\nu) < \infty$. Since 0 is absorbing it holds $\mathbb{P}_\nu(Y_t \in B) = \mathbb{P}_\nu(Y_t \in B, T_0 > t)$ for $B \in \mathcal{B}(\mathcal{A}^{-0})$. So, the q.s.d. equation can be written as,

$$\forall B \in \mathcal{B}(\mathcal{A}^{-0}), \quad \nu(B) = e^{\theta(\nu)t} \mathbb{P}_\nu(Y_t \in B). \tag{26}$$

From the above relations we deduce that for all $\theta < \theta(\nu)$, $\mathbb{E}_\nu(e^{\theta T_0}) < \infty$. So, for all $\theta < \theta(\nu)$, $\nu-$a.e. in η it holds: $\mathbb{E}_\eta(e^{\theta T_0}) < \infty$. Then, a necessary condition for the existence of a q.s.d. is exponential absorption at 0, that is

$$\exists \eta \in \mathcal{A}^{-0}, \quad \exists \theta > 0, \quad \mathbb{E}_\eta(e^{\theta T_0}) < \infty. \tag{27}$$

Let $(P_t : t \geq 0)$ be the semigroup of the process before killing at 0, acting on the set $C_b(\mathcal{A}^{-0})$ of real continuous bounded functions defined on $\mathcal{A}^{-0}$:

$$\forall \eta \in \mathcal{A}^{-0}, \ \forall f \in C_b(\mathcal{A}^{-0}) : \quad (P_t f)(\eta) = \mathbb{E}_\eta(f(Y_t), T_0 > t).$$

Let us observe that for any continuous and bounded function $h : \mathcal{A} \to \mathbb{R}$ and for any $\eta \in \mathcal{A}^{-0}$, we have

$$\mathbb{E}_\eta(h(Y_t)) = \mathbb{E}_\eta(h(Y_t), T_0 > t) + h(0)\mathbb{P}_\eta\big(T_0 \leq t\big). \tag{28}$$

In particular, if $h(0) = 0$, we get $\mathbb{E}_\eta(h(Y_t)) = \mathbb{E}_\eta(h(Y_t), T_0 > t)$.

We denote by $P_t^\dagger$ the action of the semigroup on $\mathcal{M}(\mathcal{A}^{-0})$, defined for any positive measurable function f and any $v \in \mathcal{M}(\mathcal{A}^{-0})$ by

$$P_t^\dagger v(f) = v(P_t f).$$

From relation (26) we get that a probability measure ν is a q.s.d. if and only if there exists $\theta > 0$ such that for all $t \geq 0$

$$\nu(P_t f) = e^{-\theta t}\nu(f),$$

holds for all positive measurable function f, or equivalently for all $f \in C_b(\mathcal{A}^{-0})$. Then ν is a q.s.d. with exponential decay rate θ if and only if it verifies

$$\forall t \geq 0 : \quad P_t^\dagger \nu = e^{-\theta t}\nu. \tag{29}$$

3.2 Some properties of q.s.d.

Let us show that the existence of a q.s.d. will be proved if for a fixed strictly positive time, the eigenmeasure equation (29) is satisfied. In what follows we denote by $\mathscr{P}(\mathcal{A}^{-0})$ the set of probability measures on $\mathcal{A}^{-0}$.

Lemma 3.1. *Let $\tilde\nu \in \mathscr{P}(\mathcal{A}^{-0})$ and $\beta > 0$ such that $P_1^\dagger \tilde\nu = \beta\tilde\nu$. Then $\beta < 1$ and there exists $\tilde\nu$ a q.s.d. with exponential decay rate $\theta := -\log\beta > 0$.*

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Proof. From $\beta = \tilde{\nu} P_1(\mathcal{A}^{-0}) = \mathbb{P}_{\tilde{\nu}}(T_0 > 1) < 1$, we get $\beta < 1$, so $\theta := -\log \beta > 0$. We must show that there exists $\nu \in \mathscr{P}(\mathcal{A}^{-0})$ such that $P_t^\dagger \nu = e^{-\theta t} \nu$ for all $t \geq 0$. Consider,

$$\nu = \int_0^1 e^{\theta s} P_s^\dagger \tilde{\nu} \, ds \, .$$

For $t \in (0, 1)$ we have

$$
\begin{aligned}
P_t^\dagger \nu &= \int_0^1 e^{\theta s} P_{t+s}^\dagger \tilde{\nu} \, ds = \int_0^{1-t} e^{\theta s} P_{t+s}^\dagger \tilde{\nu} \, ds + \int_{1-t}^1 e^{\theta s} P_{t+s}^\dagger \tilde{\nu} \, ds \\
&= \int_t^1 e^{\theta(u-t)} P_u^\dagger \tilde{\nu} \, du + \int_1^{1+t} e^{\theta(u-t)} P_u^\dagger \tilde{\nu} \, du \\
&= e^{-\theta t} \int_t^1 e^{\theta u} P_u^\dagger \tilde{\nu} \, du + e^{-\theta t} \int_0^t e^{\theta u} e^\theta P_u^\dagger P_1^\dagger \tilde{\nu} \, du = e^{-\theta t} \nu \, .
\end{aligned}
$$

For $t \geq 1$ we write $t = n + r$ with $0 \leq r < 1$ and $n \in \mathbb{N}$. We have

$$P_t^\dagger \nu = P_r^\dagger P_n^\dagger \nu = \beta^n P_r^\dagger \nu = e^{-n\theta} e^{-r\theta} \nu = e^{-\theta t} \nu \, .$$

$\square$

Note that

$$\theta(\nu) = \lim_{t \to 0^+} \frac{1}{t} \left(1 - \mathbb{P}_\nu(T_0 > t) \right) = \lim_{t \to 0^+} \frac{\mathbb{P}_\nu(T_0 \leq t)}{t} \, . \tag{30}$$

In the next result we give an explicit expression for the exponential decay rate associated to a q.s.d. We will use the identification between $y \in \mathbb{T}$ and the singleton configuration that gives unit weight to the trait y.

Lemma 3.2. *If $\nu \in \mathscr{P}(\mathcal{A}^{-0})$ is a q.s.d. then its exponential decay rate $\theta(\nu)$ satisfies*

$$\theta(\nu) = \int_{\eta \in \mathcal{A}_1} Q(\eta, 0) \, \nu(d\eta) = \int_{\mathbb{T}} Q(y, 0) d\nu(y) = \int_{\mathbb{T}} \lambda_y(y) d\nu(y) \, . \tag{31}$$

Proof. Since 0 is absorbing we get that for all fixed $\eta \in \mathcal{A}^{-0}$ the absorption probability $\mathbb{P}_\eta(T_0 \leq t)$ is increasing in time t. Let us denote

$$a_2(t) = \sup\{\mathbb{P}_\eta(T_0 \leq t) : \|\eta\| = 2\} \, .$$

Obviously we have $a_2(s) \leq a_2(t)$ when $0 \leq s \leq t$. We claim that

$$\sup\{\mathbb{P}_\eta(T_0 \leq t) : \|\eta\| \geq 2\} \leq a_2(t) \tag{32}$$

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Indeed let $\widehat{T}_2 = \inf\{t \geq 0 : \|Y_t\| = 2\}$. Since the process will be a.s. extinct for all $\eta \in \mathcal{A}$ with $\|\eta\| > 2$, we have $\mathbb{P}_\eta(\widehat{T}_2 < \infty) = 1$. From the Markov property and the monotonicity in time of $a_2(t)$ we get that for all $\eta \in \mathcal{A}$ with $\|\eta\| > 2$,

$$
\begin{aligned}
\mathbb{P}_\eta(T_0 \leq t) &= \sum_{\xi:\|\xi\|=2} \int_0^t \mathbb{P}_\eta(\widehat{T}_2 = ds, Y_{\widehat{T}_2} = \xi) \mathbb{P}_\xi(T_0 \leq t - s) \\
&\leq a_2(t) \sum_{\xi:\|\xi\|=2} \int_0^t \mathbb{P}_\eta(\widehat{T}_2 = ds, Y_{\widehat{T}_2} = \xi) = a_2(t)\,.
\end{aligned}
$$

Now let us show that $a_2(t) = o(t)$, that is $\lim\limits_{t \to 0^+} a_2(t)/t = 0$.

Let $\eta \in \mathcal{A}_2$ be a fixed initial configuration, thus $\|\eta\| = 2$. We denote by $A_\downarrow$ the subset of trajectories such that the function $(\|Y_t\| : t \leq T_0)$ is decreasing, that is at all the jumps of the trajectory, an individual dies. Remark that, in the complement set $A_\downarrow^c$ of $A_\downarrow$, either at the first or at the second jump of the trajectory, the number of individuals increases. Therefore, from (32), the Markov property and the monotonicity in time of $\mathbb{P}_\eta(T_0 \leq t, A_\downarrow)$, we get that

$$
\sup\{\mathbb{P}_\eta(T_0 \leq t, A_\downarrow^c) : \|\eta\| = 2\} \leq \sup\{\mathbb{P}_\eta(T_0 \leq t, A_\downarrow) : \|\eta\| = 2\}\,.
$$

Let us now denote by τ the time of the first jump of the process Y, and y_1, y_2 are the locations of the points in η (they can be equal). We have

$$
\mathbb{P}_\eta(T_0 \leq t, A_\downarrow) \leq \mathbb{P}_\eta(T_0 \leq t, A_\downarrow, Y_\tau = y_1) + \mathbb{P}_\eta(T_0 \leq t, A_\downarrow, Y_\tau = y_2)\,,
$$

and

$$
\mathbb{P}_\eta(T_0 \leq t, A_\downarrow, Y_\tau = y_i) = \mathbb{P}(\mathbf{e}_\eta + \mathbf{e}_{y_i} \leq t, \text{both events are deaths}) = \int_0^t f_i(s)ds\,,
$$

where $\mathbf{e}_\eta$ and $\mathbf{e}_{y_i}$ are two independent random variables exponentially distributed with parameters $Q(\eta)$ and $Q(y_i)$ respectively. Moreover, conditionally to the fact that the two jump events occur before time t, the probability to obtain two death events is $\frac{\lambda_{y_2}(\eta)}{Q(\eta)} \times \frac{\lambda_{y_1}(y_1)}{Q(y_1)}$. We have for $i = 1$ (a similar computation holds for $i = 2$),

$$
f_1(s) = \int_0^s \lambda_{y_2}(\eta)e^{-Q(\eta)u}\lambda_{y_1}(y_1)e^{-Q(y_1)(s-u)}du \leq Q(y_1)(1 - e^{-Q(\eta)s})\,.
$$

By using the bounds in (6) we find,

$$
\sup\{\mathbb{P}_\eta(T_0 \leq t, A_\downarrow) : \|\eta\| = 2\} \leq Q_+(1) \int_0^t (1 - e^{-Q_+(2)s})ds = Q_+(1)\,o(t)\,.
$$

So $a_2(t) = o(t)$ holds and from (30) we obtain,

$$\theta(\nu) = \lim_{t \to 0^+} \frac{1}{t} \int_{\mathbb{T}} \mathbb{P}_y(T_0 \leq t) d\nu(y) \, .$$

Similar arguments as those just developed allow to get $\mathbb{P}_y(T_0 \leq t) = \lambda_y(y)(1 - e^{-Q(y)t}) + k \, a_2(t)$, where k is a positive constant. Then the result follows. $\quad\square$

4 Proof of the existence of q.s.d.

In this section we give a proof of Theorem 1.2. The proof is based upon a more general result, Theorem 4.2, which shows that for a class of positive linear operators defined in some Banach spaces, whose elements are real functions with domain in a Polish space, there exist finite eigenmeasures. We show Theorem 1.2 in Subsection 4.2. For this purpose we construct the appropriate Banach spaces and the operator, in order that the eigenmeasure given by Theorem 4.2 is a q.s.d. of the original problem.

4.1 An abstract result

In this paragraph, $(\mathcal{X}, d)$ is a Polish metric space. We will denote by $C_b(\mathcal{X})$ the set of bounded continuous functions on $\mathcal{X}$. This set becomes Banach space when equipped with the supremum norm.

Let S be a bounded positive linear operator on $C_b(\mathcal{X})$. We will also make the following hypothesis.

Hypothesis $\mathscr{H}$: There exists a continuous function φ_2 on $(\mathcal{X}, d)$ such that

$\quad \mathscr{H}_1 \quad \varphi_2 \geq 1.$

$\quad \mathscr{H}_2 \quad$ For any $u \geq 0$, the set $\varphi_2^{-1}([0, u])$ is compact.

It follows from $\mathscr{H}_2$ that if $(\mathcal{X}, d)$ is not compact, there is a sequence $(x_j : j \in \mathbb{N})$ in $\mathcal{X}$ such that $\lim_{j \to \infty} \varphi_2(x_j) = \infty$.

Before stating the main result of this section we state and prove a lemma which will be useful later on.

Lemma 4.1. *Let v be a continuous nonnegative linear form on $C_b(\mathcal{X})$. Assume there is a positive number K such that for any function $\psi \in C_b(\mathcal{X})$ satisfying $0 \leq \psi \leq \varphi_2$, we have*

$$v(\psi) \leq K \, .$$

Then there exists a positive measure ν on $\mathcal{X}$ such that for any function $f \in C_b(\mathcal{X})$

$$v(f) = \int f \, d\nu \, .$$

Proof. Let $C_0(\mathcal{X})$ be the set of continuous functions vanishing at infinity. Let ϖ be a real continuous non-increasing nonnegative function on $\mathbb{R}^+$. Assume that $\varpi = 1$ on the interval $[0, 1]$ and $\varpi(2) = 0$ (hence $\varpi = 0$ on $[2, \infty)$). For any integer m, let v_m be the continuous positive linear form defined on $C_0(\mathcal{X})$ by

$$v_m(f) = v\big(\varpi(\varphi_2/m) \, f\big) \, .$$

This linear form has support in the set $\varphi_2^{-1}([0, 2m])$ in the sense that it vanishes on those functions which vanish on this set. Note also that $\varphi_2^{-1}([0, 2m])$ is compact by hypothesis $\mathscr{H}_2$. Therefore it can be identified with a nonnegative measure ν_m on $\mathcal{X}$, namely for any $f \in C_b(\mathcal{X})$ we have

$$v_m(f) = \int f \, d\nu_m \, .$$

We now prove that this sequence of measures is tight. Let $u > 0$ and define the set

$$K_u = \varphi_2^{-1}([0, u]).$$

Again, by hypothesis $\mathscr{H}_2$, for any $u > 0$ this is a compact set. We now observe that $\mathbf{1}_{K_u^c} \leq 1 - \varpi(2\,\varphi_2/u)$. Therefore,

$$\nu_m\big(K_u^c\big) \leq \nu_m\big(1 - \varpi(2\,\varphi_2/u)\big) = v_m\big(1 - \varpi(2\,\varphi_2/u)\big)$$

$$= v\big(\varpi(\varphi_2/m)\,\big(1 - \varpi(2\,\varphi_2/u)\big)\big) \, ,$$

We now use the fact that the function $\varphi_2\varpi(\varphi_2/m)\,\big(1 - \varpi(2\,\varphi_2/u)\big)$ is in $C_b(\mathcal{X})$ and satisfies

$$\frac{u}{2}\varpi(\varphi_2/m)\,\big(1 - \varpi(2\,\varphi_2/u)\big) \leq \varpi(\varphi_2/m)\,\big(1 - \varpi(2\,\varphi_2/u)\big)\varphi_2 \leq \varphi_2$$

to obtain from the hypothesis of the lemma that

$$v\big(\varpi(\varphi_2/m)\,\big(1 - \varpi(2\,\varphi_2/u)\big)\big) \leq \frac{2}{u}v\big(\varpi(\varphi_2/m)\,\big(1 - \varpi(2\,\varphi_2/u)\big)\varphi_2\big) \leq \frac{2K}{u} \, .$$

In other words, for any $u > 0$ we have for any integer m

$$\nu_m\big(K_u^c\big) \leq \frac{2K}{u} \, .$$

The sequence of measures ν_m is therefore tight, and we denote by ν an accumulation point which is a nonnegative measure on $\mathcal{X}$. We now prove that for any $f \in C_b(\mathcal{X})$ we have $\nu(f) = v(f)$. For this purpose, we write

$$v(f) = v\big(\varpi(\varphi_2/m)\, f\big) + v\big((1 - \varpi(\varphi_2/m))\, f\big) \;.$$

We now use the inequality

$$\varphi_2 \geq (1 - \varpi(\varphi_2/m))\varphi_2 \geq m(1 - \varpi(\varphi_2/m)) \;,$$

to conclude using the hypothesis of the lemma (since $(1 - \varpi(\varphi_2/m))\varphi_2 \in C_b(\mathcal{X})$) that

$$\Big|v\big((1 - \varpi(\varphi_2/m))\, f\big)\Big| \leq v\big((1 - \varpi(\varphi_2/m))\,|f|\big) \leq \|f\|\, v\big(1 - \varpi(\varphi_2/m)\big) \leq \frac{K}{m} \;.$$

In other words, we have for any $f \in C_b(\mathcal{X})$

$$\big|v(f) - \nu_m(f)\big| \leq \frac{K}{m} \;.$$

From the tightness bound, we have for any $f \in C_b(\mathcal{X})$

$$\lim_{m \to \infty} \nu_m(f) = \nu(f) \;,$$

see for example [1], and therefore $\nu(f) = v(f)$ which completes the proof of the lemma. $\qquad\square$

We now state the general result.

Theorem 4.2. *Assume hypotheses $\mathscr{H}_1$ and $\mathscr{H}_2$, and assume also that there exist three constants $c_1 > \gamma > 0$ and $D > 0$ such that*

$$S(1) \geq c_1$$

and for any $\psi \in C_b(\mathcal{X})$ with $0 \leq \psi \leq \varphi_2$

$$S\psi \leq \gamma\varphi_2 + D \;.$$

Then there is a probability measure ν on $\mathcal{X}$ such that $\nu \circ S = \beta\nu$, with $\beta = \nu(S(1)) > 0$.

Proof. In the dual space $C_b(\mathcal{X})^*$, we define for any real $K > 0$ the convex set $\mathscr{K}_K$ given by

$$\mathscr{K}_K = \left\{ v \in C_b(\mathcal{X})^* : v \geq 0,\ v(1) = 1,\ \sup_{\psi \in C_b(\mathcal{X}),\, 0 \leq \psi \leq \varphi_2} v(\psi) \leq K \right\} \;,$$

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Note that by Lemma 4.1, the elements of $\mathscr{K}_K$ are positive measures.

We observe that for any K large enough the set $\mathscr{K}_K$ is non empty. It suffices to consider a Dirac measure δ_x on a point $x \in \mathcal{X}$ and to take $K \geq \varphi_2(x)$. Since for any $K \geq 0$, $\mathscr{K}_K$ is an intersection of weak* closed subsets, it is closed in the weak* topology.

We now introduce the non-linear operator T having domain $\mathscr{K}_K$ and defined by

$$T(v) = \frac{v \circ S}{v\big(S(1)\big)} \ .$$

Note that since $S(1) > c_1 > 0$, we have $v\big(S(1)\big) \geq c_1 v(1)$ and this operator T is well defined on $\mathscr{K}_K$. We have obviously $T(v)(1) = 1$. We now prove that T maps $\mathscr{K}_K$ into itself. Let $\psi \in C_b(\mathcal{X})$ with $0 \leq \psi \leq \varphi_2$. Since

$$S\psi \leq \gamma \varphi_2 + D$$

and obviously

$$0 \leq S\psi \leq \gamma \frac{\|S\psi\|}{\gamma}$$

we get

$$0 \leq S\psi \leq \gamma\big(\varphi_2 \wedge (\|S\psi\|/\gamma)\big) + D \ .$$

Therefore since the function $\psi' = \varphi_2 \wedge (\|S\psi\|/\gamma)$ satisfies $\psi' \in C_b(\mathcal{X})$ and $0 \leq \psi' \leq \varphi_2$. We conclude that for $v \in \mathscr{K}_K$

$$T(v)\big(\psi\big) \leq \frac{\gamma v(\psi') + D}{c_1} \ .$$

From the bound $v(\psi') \leq K$ we get

$$T(v)\big(\psi\big) \leq \frac{\gamma v(\psi') + D}{c_1} \leq \frac{\gamma}{c_1} K + \frac{D}{c_1} \leq K$$

if $K > D/(c_1 - \gamma)$. Therefore, for any K large enough, the set $\mathscr{K}_K$ is non empty and mapped into itself by T.

It is easy to show that T is continuous on $\mathscr{K}_K$ in the weak* topology. This follows at once from the continuity of the operator S. We can now apply Tychonov's fixed point theorem (see [19] or [8]) to deduce that T has a fixed point. This implies that there is a point $\nu \in \mathscr{K}_K$ such that $\nu \circ S = v(S(1))\nu$. This concludes the proof of the Theorem. $\qquad\square$

4.2 Construction of the function φ_2 and the proof of Theorem 1.2.

We assume that the hypotheses of Theorem 1.2 hold. In our application we have a semi-group P_t acting on $C_b(\mathcal{A}^{-0})$. We will use Lemma 3.1 to construct a q.s.d. This lemma is proved using Theorem 4.2 applied to $S = P_1$:

$$Sf(\eta) = P_1 f(\eta) = \mathbb{E}_\eta\big(f(Y_1)\,, T_0 > 1\big)\,, \quad \eta \in \mathcal{A}^{-0}\,.$$

Here the Polish metric space $(\mathcal{X}, d)$ of the previous paragraph will be $(\mathcal{A}^{-0}, d_P)$, and so the function φ_2 will have domain in the set of nonempty configurations.

We recall the elementary formula valid for any continuous and bounded function f on $\mathcal{A}$ and any $t \geq 0$

$$\mathbb{E}_\eta\big(f(Y_t)\big) = \mathbb{E}_\eta\big(f(Y_t)\,, T_0 > t\big) + f(0)\mathbb{P}_\eta\big(T_0 \leq t\big)\,.$$

We start with the following bounds.

Lemma 4.3. *Let* $\overline{\lambda}_1 = \sup\limits_{\eta\,:\,\|\eta\|=1}\ \sup\limits_{y \in \eta} \lambda_y(\eta) < \infty$, *then*

$$-\overline{\lambda}_1 \leq L\mathbf{1} \leq 0\,.$$

and for all $t \geq 0$,

$$e^{-\overline{\lambda}_1 t} \leq P_t\mathbf{1} \leq \mathbf{1}\,.$$

Proof. The proof follows at once from Lemma 2.4 and a computation of $L\mathbf{1}_{\mathcal{A}^{-0}}$ (see formula (23)). $\qquad\square$

Lemma 4.4. *Consider for any* $a > 0$ *the function* $\varphi_2^a(\eta) = e^{a\|\eta\|}\mathbf{1}_{\mathcal{A}^{-0}}(\eta)$. *Then*

$$L\varphi_2^a(\eta) \leq \big(B^*\,(e^a - 1) + \lambda_*\,(e^{-a} - 1)\big)\,\|\eta\|\,\varphi_2^a(\eta).$$

Proof. We compute $L\varphi_2^a(\eta)$ using (23). For $\eta \in \mathcal{A}^{-0}$ we have

$$
\begin{aligned}
L\varphi_2^a(\eta) \ &=\ \sum_{y \in \{\eta\}} \eta_y\big(b_y(\eta) + m_y(\eta)\big)\,(e^a - 1)\,e^{a\|\eta\|} \\
&+ \sum_{y \in \{\eta\}} \eta_y\lambda_y(\eta)\,(e^{-a} - 1)\,e^{a\|\eta\|} - \lambda_y(\eta)\mathbf{1}_{\|\eta\|=1} \qquad (33) \\
&\leq\ B^*\,\|\eta\|\,\varphi_2^a(\eta)\,(e^a - 1) + \lambda_*\,\|\eta\|\,\varphi_2^a(\eta)\,(e^{-a} - 1)\,.
\end{aligned}
$$

$\qquad\square$

To define the function φ_2, we will need the two following results.

Lemma 4.5. *The differential equation*

$$\frac{da}{dt} = \lambda_* \left(1 - e^{-a}\right) + B^* \left(1 - e^a\right) \tag{34}$$

has two fixed points $a = 0$ and $a = \log(\lambda_*/B^*)$. The trajectory of any initial condition $a_0 \in (0, \log(\lambda_*/B^*))$ is increasing in time and converges to $\log(\lambda_*/B^*)$.

Proof. Left to the reader. $\square$

Lemma 4.6. *Assume (12). Let $a(t)$ be the solution of (34) with initial condition $a_0 \in (0, \log(\lambda_*/B^*))$. Then*

$$\sup_{t \in \mathbb{R}_+} \mathbb{E}_\eta(e^{-\lambda_* t} e^{a(t)\|Y_t\|}, T_0 > t) \le e^{a_0\|\eta\|} .$$

Proof. We introduce the function

$$f(t, \eta) = e^{-\lambda_* t} e^{a(t)\|\eta\|} \mathbf{1}_{\mathcal{A}^{-0}}(\eta) ,$$

and for any integer N we denote by f^N the function

$$f^N(t, \eta) = f(t, \eta) \mathbf{1}_{\|\eta\| \le N} .$$

Note that $f^N(t, \eta)$ is continuous with compact support $\{\eta : \|\eta\| \le N\}$. Using Proposition 2.4 *(iii)* we get

$$f^N\left(t, Y_{t \wedge T_M}\right) = f^N\left(0, Y_0\right) + \int_0^{t \wedge T_M} \left(\partial_s f^N(s, Y_s) + L f^N(s, Y_s)\right) ds + \mathcal{M}_{t \wedge T_M}^{f^N} ,$$

where $\mathcal{M}^{f^N}$ is a martingale. Then we obtain

$$\mathbb{E}_\eta\left(f^N\left(t, Y_{t \wedge T_M}\right)\right) = f^N\left(0, Y_0\right) + \mathbb{E}_\eta\left(\int_0^{t \wedge T_M} \left(\partial_s f^N(s, Y_s) + L f^N(s, Y_s)\right) ds\right) .$$

Observe that if $N > M$ and $s \le T_M$ we have $f^N(s, Y_s) = f(s, Y_s)$. Let N tend to infinity to get

$$\mathbb{E}_\eta\left(f\left(t, Y_{t \wedge T_M}\right)\right) = f\left(0, Y_0\right) + \mathbb{E}_\eta\left(\int_0^{t \wedge T_M} \left(\partial_s f(s, Y_s) + L f(s, Y_s)\right) ds\right) .$$

Using Lemma 4.4 we have

$$\partial_s f(s, Y_s) + L f(s, Y_s) =$$
$$e^{-\lambda_* s} \left(\left(\lambda_* \left(1 - e^{-a(s)}\right) + B^* \left(1 - e^{a(s)}\right)\right) \|Y_s\| - \lambda_*\right) \varphi_2^{a(s)}(Y_s) + L \varphi_2^{a(s)}(Y_s) \le 0 .$$

Therefore
$$\mathbb{E}_\eta\left(f\left(t, Y_{t \wedge T_M}\right)\right) \le f\left(0, Y_0\right) .$$
Letting M tend to infinity and by using the Monotone Convergence Theorem we obtain,
$$\mathbb{E}_\eta\left(f\left(t, Y_t\right)\right) \le f\left(0, Y_0\right) .$$
The result follows from the definition of f $\qquad\square$

We take as function φ_2 the function
$$\varphi_2 = \varphi_2^{a(1)},$$
for a solution of (34) with initial condition $a_0 \in (0, \log(\lambda_*/B^*))$. The operator S is given by $S = P_1$, and hence is positive and maps continuously $C_b(\mathcal{A}^{-0})$ into itself.

We must now show that $S = P_1$, and φ_2 satisfy the hypothesis of Theorem 4.2.

Lemma 4.7.

   (i) The hypotheses $\mathscr{H}_1$ and $\mathscr{H}_2$ are satisfied.

   (ii) $S(1) > c_1 > 0$, with $c_1 = e^{-\bar{\lambda}_1}$.

   (iii) For any $\gamma > 0$, there is a constant $D = D(\gamma) > 0$ such that for any $\psi \in C_b(\mathcal{X})$ with $0 \le \psi \le \varphi_2$
$$S\psi \le \gamma \varphi_2 + D .$$

Proof. The hypotheses $\mathscr{H}_1$ and $\mathscr{H}_2$ are easy to check using the Feller property of P_1 (see Proposition 2.6).

(ii) follows at once from Lemma 4.3. We now prove (iii).
Let $\psi \in C_b(\mathcal{X})$ with $0 \le \psi \le \varphi_2$. We have from Lemma 4.6
$$P_1\psi(\eta) = \mathbb{E}_\eta\left(\psi(Y_1), T_0 > 1\right) \le \mathbb{E}_\eta(\varphi_2(Y_1), T_0 > 1) \le e^{\lambda_*}\, e^{a_0 \|\eta\|} .$$
Since $a(1) > a_0$ by Lemma 4.5, for any $\gamma > 0$ there is an integer m_γ such that for any $m \ge m_\gamma$ we have
$$e^{\lambda_*}\, e^{a_0\, m} \le \gamma e^{a(1)m} .$$
Therefore, for any η we have
$$P_1\psi(\eta) \le e^{\lambda_*}\, e^{a_0\|\eta\|} \le \gamma e^{a(1)\,\|\eta\|} + e^{\lambda_*}\, e^{a_0\, m_\gamma} .$$
In other words, we have proved (iii) with the constant $D = e^{\lambda_*}\, e^{a_0\, m_\gamma}$. $\quad\square$

Theorem 1.2 follows immediately from the previous Lemma and Theorem 4.2.

5 The process and absolute continuity

In this section we introduce a natural $\sigma-$finite measure μ. We will show that the process Y preserves the absolutely continuity with respect to μ and that when the process starts from any point measure after any positive time the absolutely continuous part of the marginal distribution does not vanish.

5.1 The measures

We will denote by $\widehat{\mathbb{T}^k}$ the set of all $k-$tuples in $\mathbb{T}$ ordered by $\preceq$ defined in Subsection 2.1. So, for $\eta \in \mathcal{A}$, its ordered support $\vec{\eta} = (y_1, ..., y_{\#\eta})$ belongs to $\widehat{\mathbb{T}^{\#\eta}}$. The discrete structure $\overline{\eta}$ is an element in $\mathbb{N}^{\#\eta}$ and the set of all discrete structures is denoted by
$$\Sigma(\mathbb{N}) = \bigcup_{n \in \mathbb{Z}_+} \mathbb{N}^n \,.$$

Here $\mathbb{N}^0$ contains a unique element denoted by 0 and it is the discrete structure of the void configuration $\eta = 0$. A generic element of $\Sigma(\mathbb{N})$ will be denoted by $\vec{q}$. Moreover for each $\vec{q} \in \Sigma(\mathbb{N})$ we put $\#\vec{q} = k$ if $\vec{q} \in \mathbb{N}^k$. We put

$$\mathcal{A}_{\vec{q}} = \{\eta \in \mathcal{A} : \overline{\eta} = \vec{q}\} \ \text{ for } \vec{q} \in \Sigma(\mathbb{N}) \,,$$

and for $B \subseteq \mathcal{A}$,

$$B_{\vec{q}} = \{\eta \in B : \overline{\eta} = \vec{q}\} \ \text{ for } \vec{q} \in \Sigma(\mathbb{N}) \,.$$

In the sequel for $\vec{q} \in \mathbb{N}^k$ and $C \subseteq \widehat{\mathbb{T}^k}$ we denote

$$\{\vec{q}\} \times C := \{\eta \in \mathcal{A} : \overline{\eta} = \vec{q}, \vec{\eta} \in C\} \,. \tag{35}$$

We denote by $\mathscr{M}_f(\mathcal{A})$ the set of measures on $(\mathcal{A}, \mathcal{B}(\mathcal{A}))$ that give finite weight to all sets $\mathcal{A}_k$. By $\mathscr{M}_f(\Sigma(\mathbb{N}))$ we mean the set of measures on $\Sigma(\mathbb{N})$ giving finite weight to all the subsets $\mathbb{N}^k$, and $\mathscr{M}_f(\mathbb{N})$) denotes the measures on $\mathbb{N}$ giving finite weight to all its points. Every measure $v \in \mathscr{M}_f(\mathcal{A})$ defines a measure $\overline{v} \in \mathscr{M}_f(\Sigma(\mathbb{N}))$ by

$$\overline{v}(\vec{q}) = v(\mathcal{A}_{\vec{q}}) \,. \tag{36}$$

Also v defines a set of conditional measures $v_{\vec{q}} \in \mathscr{M}(\widehat{\mathbb{T}^{\#\vec{q}}})$ by

$$v_{\vec{q}}(\bullet) = \begin{cases} 0 & \text{if } \overline{v}(\vec{q}) = 0, \\ v(\eta \in \mathcal{A} : \overline{\eta} = \vec{q}, \ \vec{\eta} \in \bullet)/\overline{v}(\vec{q}) & \text{otherwise} \,. \end{cases}$$

Then $v_{\vec{q}} \in \mathscr{P}(\widehat{\mathbb{T}^{\#\vec{q}}})$ is a probability measure if $\bar{v}(\vec{q}) > 0$. In the case that $v \in \mathscr{P}(\mathcal{A})$ we have

$$v_{\vec{q}}(\bullet) = v\left(\vec{\eta} \in \bullet \,|\, \bar{\eta} = \vec{q}\right). \tag{37}$$

Conversely a probability measure $v \in \mathscr{P}(\mathcal{A})$ is given by a probability measure $\bar{v} \in \mathscr{P}(\Sigma(\mathbb{N}))$ and the family of conditional measures $(v_{\bar{\eta}} \in \mathscr{P}(\mathcal{A}_{\bar{\eta}}))$ so that

$$v(B) = \sum_{\vec{q} \in \Sigma(\mathbb{N})} \bar{v}(\vec{q})\, v_{\vec{q}}(B_{\vec{q}}), \;\; B \in \mathcal{B}(\mathcal{A}^{-0}).$$

In this sense

$$dv(\eta) = \bar{v}(\bar{\eta})dv_{\bar{\eta}}(\eta). \tag{38}$$

Let $\varphi : \mathcal{A} \to \mathbb{R}$ be a function. Observe that its restriction to $\mathcal{A}_{\vec{q}}$ can be identified with a function $\varphi\big|_{\mathcal{A}_{\vec{q}}}$ with domain in $\widehat{\mathbb{T}^{\#\vec{q}}}$ by the formula $\varphi\big|_{\mathcal{A}_{\vec{q}}}(\vec{\eta}) = \varphi(\eta)$. Let $\varphi : \mathcal{A} \to \mathbb{R}$ be a $v-$integrable function, we have

$$\int_{\mathcal{A}} \varphi(\eta)\, dv(\eta) = \sum_{\vec{q} \in \Sigma(\mathbb{N})} \bar{v}(\vec{q}) \int_{\widehat{\mathbb{T}^{\#\vec{q}}}} \varphi\big|_{\mathcal{A}_{\vec{q}}}(\vec{y})\, dv_{\vec{q}}(\vec{y}).$$

Now we define the measure μ by,

$$\mu(\mathbb{N}^0 \times \mathbb{T}^0) = \mu(\{0\}) = 1 \text{ and } \mu\big|_{\mathbb{N}^k \times \widehat{\mathbb{T}^k}} = \ell^k \times \widehat{\sigma^k} \text{ for } k \geq 1, \tag{39}$$

where ℓ^k is the point measure on $\mathbb{N}^k$ that gives a unit mass to every point, and $\widehat{\sigma^k}$ is the restriction to $\widehat{\mathbb{T}^k}$ of product measure $\sigma^{\otimes k}$. Note that $v \in \mathscr{M}_f(\mathcal{A})$ satisfies

$$v \;<<\; \mu \;\Leftrightarrow\; \left(\forall \vec{q} \in \Sigma(\mathbb{N}) : v_{\vec{q}} << \widehat{\sigma^{\#\vec{q}}}\right).$$

Hence, if $v \in \mathscr{P}(\mathcal{A})$ is such that $v << \mu$, then v is of the form

$$v\left(\eta \in \mathcal{A} : \bar{\eta} = \vec{q}, \vec{\eta} \in d\,\vec{y}\right) = \bar{v}(\vec{q})\, \varphi_{\vec{q}}(\vec{y})\, d\widehat{\sigma^{\#\vec{q}}}(\vec{y}). \tag{40}$$

where $\bar{v} \in \mathscr{P}(\Sigma(\mathbb{N}))$ and for each fixed $\vec{q} \in \Sigma(\mathbb{N})$, $\varphi_{\vec{q}}(\bullet)$ is a density function in $\widehat{\mathbb{T}^{\#\vec{q}}}$ with respect to $\widehat{\sigma^{\#\vec{q}}}$.

5.2 Absolutely continuity is preserved

Proposition 5.1. *The process Y preserves the absolutely continuity with respect to μ, that is*

$$\forall\, v \in \mathscr{P}(\mathcal{A}),\; v << \mu \;\Rightarrow\; \forall t > 0\; \mathbb{P}_v(Y_t \in \bullet) << \mu(\bullet). \tag{41}$$

Proof. Let us define the jump time sequence,

$$\tau_0 = 0 \text{ and } \tau_n = \inf\{t > \tau_{n-1} : Y_t \neq Y_{\tau_{n-1}}\} \text{ for } n \geq 1.$$

In particular $\tau = \tau_1$ is the time of the first jump. Remark that the sequence τ_n tends a.s. to infinity, as it can be deduced from (19). When we need to emphasize the dependence on the initial condition $Y_0 = \eta$ we will denote τ_n^η and τ^η instead of τ_n and τ, respectively. We have

$$\mathbb{P}_v(Y_t \in \bullet) = \sum_{n \geq 0} \mathbb{P}_v(Y_t \in \bullet, \tau_n \leq t < \tau_{n+1}).$$

Since the sum of absolutely continuous measures is also absolutely continuous it suffices to prove that

$$\mathbb{P}_v(Y_t \in \bullet, \tau_n \leq t < \tau_{n+1}) << \mu \text{ for all } n \geq 0. \tag{42}$$

First, let us show the case $n = 0$. For $B \in \mathcal{B}(\mathcal{A})$ we have that the expression

$$\mathbb{P}_v(Y_t \in B, t < \tau) = \mathbb{P}_v(Y_0 \in B, t < \tau) = \int_B v(d\eta)\mathbb{P}_\eta(t < \tau) \tag{43}$$

vanishes if $v(B) = 0$, so also when $\mu(B) = 0$.

Before considering the case $n \geq 1$ in (42) let us prove the relation

$$v << \mu \Rightarrow \mathbb{P}_v(Y_\tau \in \bullet) << \mu. \tag{44}$$

It suffices to fix $\vec{q} \in \Sigma(\mathbb{N})$ and to show that the measure $\mathbb{P}_v(Y_\tau \in \bullet)$ restricted to the class of sets $(\{\vec{q}\} \times C : C \in \mathcal{B}(\widehat{\mathbb{T}^{\#\vec{q}}}))$ is absolutely continuous with respect to $\widehat{\sigma^{\#\vec{q}}}$ (see 35). Let $k = \#\vec{q}$. For $k = 0$ the claim holds because $\mu(\{0\}) = 1$. Let $k \geq 1$. Recall that the probability measure v has the form stated in (40), so $\varphi_{\overline{\eta}} = dv_{\overline{\eta}}/d\widehat{\sigma^{\#\eta}}$ is the density function on the space $\widehat{\mathbb{T}^{\#\eta}}$. Below, for $\eta \in \mathcal{A}$ we denote

$$\eta^{-y} = \eta - \delta_y, \ y \in \{\eta\}; \quad \eta^{+z} = \eta + \delta_z, \ \forall z \in \mathbb{T};$$
$$\overline{\eta}^{-y} := \overline{\eta^{-y}}; \quad \overline{\eta}^{+z} := \overline{\eta^{+z}}; \quad \vec{\eta}^{-y} := \overrightarrow{\eta^{-y}}; \quad \vec{\eta}^{+z} := \overrightarrow{\eta^{+z}}. \tag{45}$$

We denote $v_k = \widehat{\sigma^k}(\widehat{\mathbb{T}^k})$ (for $k = 0$ we set $v_0 = 1$). Let $k \in \mathbb{N}$, $\vec{q} \in \mathbb{N}^k$ and

$C \in \mathcal{B}(\widehat{\mathbb{T}^k})$ be fixed. We have

$$\mathbb{P}_v(Y_\tau \in \{\vec{q}\} \times C) = \int_{\mathcal{A}} dv(\eta') \int_{\{\vec{q}\} \times C} \frac{1}{Q(\eta')} Q(\eta', d\eta)$$

$$= \int_C \mathbf{1}(\overline{\eta} = \vec{q}) \left(\sum_{y \in \{\eta\}\,:\,\eta_y > 1} \frac{(\eta_y - 1) b_y(\eta^{-y})}{Q(\eta^{-y})} \overline{v}(\vec{q}^{-y}) \varphi_{\overline{\eta}^{-y}}(\vec{\eta}^{-y}) \right) d\widehat{\sigma^k}(\vec{\eta})$$

$$+ \int_C \mathbf{1}(\overline{\eta} = \vec{q}) \left(\sum_{y \in \{\eta\}} \frac{(\eta_y - 1) \lambda_y(\eta^{+y})}{Q(\eta^{+y})} \overline{v}(\eta^{+y}) \varphi_{\overline{\eta}^{+y}}(\vec{\eta}^{+y}) \right) d\widehat{\sigma^k}(\vec{\eta})$$

$$+ \frac{\mathrm{v}_{k+1}}{\mathrm{v}_k} \int_C \mathbf{1}(\overline{\eta} = \vec{q}) \left(\int_{z \in \mathbb{T} \backslash \{\eta\}} \frac{\lambda_z(\eta^{+z})}{Q(\eta^{+z})} \overline{v}(\overline{\eta}^{+z}) \varphi_{\overline{\eta}^{+z}}(\vec{\eta}^{+z}) d\sigma(z) \right) d\widehat{\sigma^k}(\vec{\eta})$$

$$+ \frac{\mathrm{v}_{k-1}}{\mathrm{v}_k} \int_C \mathbf{1}(\overline{\eta} = \vec{q}) \left(\sum_{y \in \{\eta\}\,:\,\eta_y > 1} \overline{v}(\overline{\eta}^{-y}) \varphi_{\overline{\eta}^{-y}}(\vec{\eta}^{-y}) \sum_{y' \in \{\eta\} \backslash \{y\}} \frac{\eta_{y'} m_{y'}(\eta^{-y}) g_{y'}(y)}{Q(\eta^{-y})} \right) d\widehat{\sigma^k}(\vec{\eta}) \,,$$

where in the last two terms we have used the following relations

$$d\sigma(z) d\widehat{\sigma^k}(\vec{\eta}) = \frac{\mathrm{v}_k}{\mathrm{v}_{k+1}} d\widehat{\sigma^{k+1}}(\vec{\eta}^{+z}) \,, \quad z \notin \{\eta\}$$

and

$$d\widehat{\sigma^k}(\vec{\eta}) = \frac{\mathrm{v}_k}{\mathrm{v}_{k-1}} d\sigma(y') \, d\widehat{\sigma^{k-1}}(\vec{\eta}^{-y}) \,, \quad y \in \{\eta\}.$$

Hence, the relation (44) is proved. An inductive argument gives

$$\mathbb{P}_v(Y_{\tau_n} \in \bullet) << \mu \quad \text{for every } n \geq 1 \,. \tag{46}$$

Now, let us show that (42) holds for $n \geq 1$. Denote by F_n the distribution of τ_n. By the strong Markov property and Fubini theorem we get

$$\mathbb{P}_v(Y_t \in \bullet, \tau_n \leq t < \tau_{n+1}) = \int_0^t \mathbb{E}_v \left(\mathbb{P}_{Y_{\tau_n}}(Y_{t-s} \in \bullet, \tau \geq t - s) \right) dF_n(s) \,.$$

which, by using relations (43) and (46), is absolutely continuous with respect to μ. $\qquad \square$

5.3   Evolution after the first mutation

We want to study the absolute continuity with respect to μ of the law of Y_t initially distributed according to a general measure v. To this aim we

will introduce the first mutation time. Note that a mutant individual has a different trait from those of its parent, so the time of first mutation is

$$\chi = \inf\{t \geq 0 : \{Y_t\} \not\subseteq \{Y_0\}\}. \tag{47}$$

When χ is finite we have $\{Y_\chi\} \neq \emptyset$, so $(\chi < \infty) \Rightarrow (\chi < T_0)$.

Now, let us consider the first time where the traits of the initial configuration disappear,

$$\kappa = \inf\{t \geq 0 : \{Y_t\} \cap \{Y_0\} = \emptyset\},$$

and for a fixed η, the first time where the traits of η disappear, $\kappa^\eta = \inf\{t \geq 0 : \{Y_t\} \cap \{\eta\} = \emptyset\}$. When $\{\eta\} \cap \{Y_0\} = \emptyset$, then $\kappa^\eta=0$.

We have $\kappa \neq \chi$ except when $\kappa = \chi = \infty$. Obviously $\kappa \leq T_0$. Moreover

$$(\chi > \kappa) \Leftrightarrow (\infty = \chi > \kappa) \Leftrightarrow (\chi > \kappa = T_0) \quad \text{and} \quad (\kappa < T_0) \Leftrightarrow (\chi < \kappa < T_0). \tag{48}$$

Also note that $(\kappa < \chi) \cap (\kappa \leq t) \subseteq (\kappa = T_0 \leq t)$. Since $\mathbb{P}_\eta(Y_t \in \bullet, T_0 \leq t) = \delta_0(\bullet)$ is concentrated at $\eta = 0$, then

$$\mathbb{P}_\eta(Y_t \in \bullet, \kappa < \chi, \kappa \leq t) = \delta_0(\bullet).$$

The unique nontrivial cases are the following two ones.

Proposition 5.2. *Let $\eta \in \mathcal{A}^{-0}$ and $t \geq 0$, we have:*

(i) $\mathbb{P}_\eta(Y_t \in \bullet, \chi < \kappa \leq t < T_0)$ is absolutely continuous with respect to μ and it is concentrated in $\mathcal{A}^{-0}$;

(ii) $\mathbb{P}_\eta(Y_t \in \bullet, t < \kappa)$ is singular with respect to μ.

Proof. Let us show (i). From the Markov property we have,

$$\mathbb{P}_\eta(Y_t \in \bullet, \chi < \kappa \leq t) = \sum_{\xi: \emptyset \neq \{\xi\} \subseteq \{\eta\}} \mathbb{P}_\eta(\chi < \kappa \leq t, Y_{\chi^-} = \xi, Y_t \in \bullet) \tag{49}$$

$$= \sum_{\xi: \emptyset \neq \{\xi\} \subseteq \{\eta\}} \sum_{y \in \{\xi\}} \frac{\xi_y m_y(\xi)}{Q(\xi)} \int_0^t \mathbb{P}_\eta(\chi \in ds, Y_{s^-} = \xi) \times$$

$$\int_{\mathbb{T}\backslash\{\xi\}} g_y(z) \mathbb{P}_{\xi+z}(Y_{t-s} \in \bullet, \kappa^\xi \leq t-s) d\sigma(z).$$

Hence, it is sufficient to show that for every $u > 0$, $\eta \in \mathcal{A}^{-0}$ and $y \in \{\eta\}$, it holds

$$\int_{\mathbb{T}\backslash\{\eta\}} \mathbb{P}_{\eta+z}(Y_u \in \bullet, \kappa^\eta \leq u) \, g_y(z) \, d\sigma(z) \ll \mu(\bullet).$$

By using $\int\limits_{\mathbb{T}\backslash\{\eta\}} \mathbb{P}_{\eta+z}\left(\{Y_u\} \cap \{\eta\} \neq \emptyset, \kappa^\eta \leq u\right) g_y(z) d\sigma(z) = 0$, and since the measure σ is non-atomic, a similar proof to the one showing Proposition 5.1 works and proves the result. Indeed, for each $t > 0$, the singular part with respect to μ of $\mathbb{P}_\eta(Y_t \in \cdot)$ is a measure on the set of atomic measures with support contained in $\{\eta\}$ (corresponding to death or clonal events from individuals initially alive).

Let us show (ii). Let $\{\eta\} \subset \mathbb{T}$ be the finite set of initial traits and put $k = \#\eta$. Consider the Borel set $B = \{\xi \in \mathcal{A}^{-0} : \{\xi\} \cap \{\eta\} \neq \emptyset\}$ and define $B_{l,n} = \{\xi \in \mathcal{A}^{-0} : \#\xi = n, |\{\xi\} \cap \{\eta\}| = l\}$ for $n \in \mathbb{N}$, $l = 1, ..., n \wedge k$. We have $B = \bigcup\limits_{n\in\mathbb{N},\, l\in\{1,...,n\wedge k\}} B_{l,n}$. Since σ is non-atomic we have $\mu(B_{l,n}) = 0$ for all $l \in \{1, ...n \wedge k\}$. On the other hand, from the definition of κ we have $\mathbb{P}_\eta(Y_t \in B, t < \kappa) = 1$, and the result follows. $\qquad\square$

Let $v \in \mathscr{P}(\mathcal{A})$. We denote by v^t the distribution of Y_t when the distribution of Y_0 is v, that is

$$v^t(B) = \mathbb{P}_v(Y_t \in B), \quad B \in \mathcal{B}(\mathcal{A}), \ t \geq 0. \tag{50}$$

We denote by $v = v^{\mathrm{ac}} + v^{\mathrm{si}}$ the Lebesgue decomposition of v into its absolutely continuous part $v^{\mathrm{ac}} << \mu$ and its singular part v^{si} with respect to μ. For v^t this decomposition is written as $v^t = v^{t,\mathrm{ac}} + v^{t,\mathrm{si}}$. As usual, δ_η is the Dirac measure at $\eta \in \mathcal{A}$, so δ_η^t denotes the measure $\delta_\eta^t(\bullet) = \mathbb{P}_\eta(Y_t \in \bullet)$. We will denote by $\mathrm{supp}(v)$ the closed support of a measure v.

Proposition 5.3. *The process Y verifies:*

(i) *For all $t > 0$ and all $\eta \in \mathcal{A}^{-0}$ we have $\delta_\eta^{t,ac}(\mathcal{A}^{-0}) > 0$;*

(ii) *For all $t > 0$ and all $v \in \mathscr{P}(\mathcal{A}^{-0})$ it holds $v^{t,ac} \geq \int_{\mathcal{A}^{-0}} \delta_\eta^{t,ac} v(d\eta) > 0$;*

(iii) *Assume condition (14). Then for all $\eta \in \mathcal{A}^{-0}$ with $\{\eta\} \subseteq Supp(\sigma)$ and for all $\epsilon > 0$, the following relation holds,*

$$\forall\, t > 0, \quad \delta_\eta^{t,ac}\left(B(\eta,\epsilon)\right) > 0,$$

where $B(\eta, \epsilon) = \{\eta' \in \mathcal{A} : \|\eta' - \eta\| < \epsilon\}$.

Proof. It suffices to show (i) and (iii). Let us show the first part. Fix $t \geq 0$ and $\eta \in \mathcal{A}^{-0}$. We claim that $\mathbb{P}_\eta(\chi < \kappa \leq t < T_0) > 0$. In fact, it suffices to consider the event where a mutation occurs at the first jump and after it

all the initial traits disappear before t and these changes are the unique ones before t. This event has strictly positive probability, so the claim is proved. Proposition 5.2 (i) gives $\mathbb{P}_\eta(Y_t \in \bullet, \chi < \kappa \leq t < T_0) << \mu$ and we deduce,

$$\delta_\eta^{t,\mathrm{ac}}(\mathcal{A}^{-0}) \geq \mathbb{P}_\eta(\chi < \kappa \leq t < T_0) > 0\,.$$

Then (i) holds.

The proof of (iii) is entirely similar to the proof of (i) but we need some previous remarks. For every $y \in \mathbb{T}$ we have $\int_\mathbb{T} g_y(z)d\sigma(z) = 1$, and so, $\sigma(\{z \in \mathrm{Supp}(\sigma) : g_y(z) > 0\}) > 0$. On the other hand from condition (14) the set

$$D = \{y \in \mathrm{Supp}(\sigma) : \sigma(\{z \in \mathrm{Supp}(\sigma) : g_y(z) > 0\}) = 1\}$$

verifies $\sigma(D) = 1$. In particular $\sigma(z \in D : g_y(z) > 0) > 0$ is satisfied for all $y \in \mathbb{T}$. Now, let $\{\eta\} = \{y_i : i = 1, ..., k\}$ and consider the following event: a mutation occurs at the first jump to a trait $y' \in D$, afterwords successive mutations to the traits in $B(y_i, \epsilon) \cap D$ take place, then for each trait y_i there are $q_i - 1$ clonal births, and finally all the initial traits and y' disappear. This history occurs before t and assume that these changes are the unique ones that happen before t.
From condition (14) and the definition of D this event has strictly positive probability and the claim is proved. $\qquad\square$

5.4   Decomposition of q.s.d.

Let us study the Lebesgue decomposition of a q.s.d. with respect to μ.

Proposition 5.4. *Let ν be a q.s.d. on $\mathcal{A}^{-0}$. Then,*

(i) $\nu^{ac} \neq 0$;

(ii) Assume Condition (14). Then $\{\eta \in \mathcal{A}^{-0} : \{\eta\} \subseteq \mathrm{Supp}(\sigma)\} \subseteq \mathrm{Supp}(\nu^{ac})$;

(iii) If $\nu^{si} \neq 0$, the probability measure $\nu^{*\,si} := \nu^{si}/\nu^{si}(\mathcal{A}^{-0})$ satisfies

$$\mathbb{P}_{\nu^{*\,si}}(Y_t \in B) = e^{-\theta(\nu)t}\nu^{*\,si}(B) \; \forall\, B \in \mathcal{B}(B^{si})\,, t \geq 0\,, \tag{51}$$

where $B^{si} \in \mathcal{B}(\mathcal{A}^{-0})$ is a measurable set such that $\mu(B^{si}) = 0$ and $\nu^{si}(B^{si}) = \nu^{si}(\mathcal{A}^{-0})$.

Proof. We first note that the existence of the set B^{si} is ensured by the Radon-Nikodym decomposition theorem. Set $H := \mathcal{A}^{-0} \setminus B^{\text{si}}$. Let us show that

$$\forall\, t > 0,\ \forall\, \eta \in \mathcal{A}^{-0} :\ \delta_\eta^t(H) > 0. \tag{52}$$

Since $\mu(B^{\text{si}}) = 0$ and $\delta_\eta^{t,\text{ac}} << \mu$ we have $\delta_\eta^{t,\text{ac}}(B^{\text{si}}) = 0$. Then $\delta_\eta^{t,\text{ac}}(H) = \delta_\eta^{t,\text{ac}}(\mathcal{A}^{-0})$. By Proposition 5.3, $\delta_\eta^{t,\text{ac}}(\mathcal{A}^{-0}) > 0$ for all $t > 0$ and all $\eta \in \mathcal{A}^{-0}$. So

$$\delta_\eta^t(H) \geq \delta_\eta^{t,\text{ac}}(H) = \delta_\eta^{t,\text{ac}}(\mathcal{A}^{-0}) > 0,$$

and the assertion (52) holds.

Now we prove part (i). We can assume $\nu^{\text{si}} \neq 0$, if not the result is trivial. From (52) we get,

$$\nu^t(H) = \int_{\mathcal{A}^{-0}} \delta_\eta^t(H)\nu(d\eta) > 0.$$

On the other hand, from relation (26) we obtain $\nu(H) = e^{\theta(\nu)t}\nu^t(H) > 0$. Since $\nu^{\text{si}}(H) = 0$, we necessarily have $\nu^{\text{ac}}(H) = \nu(H) > 0$, so (i) holds. Now, from Proposition 5.3 (iii) a similar proof as above shows (ii).

Let us show (iii). Let $\nu^{*\,\text{ac}} := \nu^{\text{ac}}/\nu^{\text{ac}}(\mathcal{A}^{-0})$. For every $B \subseteq B^{\text{si}}$, $B \in \mathcal{B}(\mathcal{A}^{-0})$, we have

$$\nu(B) = e^{\theta(\nu)t}\left(\nu^{\text{ac}}(\mathcal{A}^{-0})\mathbb{P}_{\nu^{*\,\text{ac}}}(Y_t \in B) + \nu^{\text{si}}(\mathcal{A}^{-0})\mathbb{P}_{\nu^{*\,\text{si}}}(Y_t \in B)\right). \tag{53}$$

By Proposition 5.1, Y preserves μ, so $\mathbb{P}_{\nu^{*\,\text{ac}}}(Y_t \in \bullet) << \mu$. Since $\mu(B^{\text{si}}) = 0$ we get $\mathbb{P}_{\nu^{*\,\text{ac}}}(Y_t \in B^{\text{si}}) = 0$. By evaluating (53) at $t = 0$ and since $B \in \mathcal{B}(B^{\text{si}})$ we find $\nu^{*\,\text{si}}(B) = \nu(B)/\nu(B^{\text{si}})$. By putting all these elements together we obtain relation (51). $\qquad\square$

6 The uniform case

6.1 The model

In this section, we assume that the individual jump rates satisfy,

$$\lambda_y(\eta) = \lambda\,,\ b_y(\eta) = b(1 - \rho)\,,\ m_y(\eta) = b\,\rho\,,\quad \forall y \in \{\eta\}\,, \tag{54}$$

λ, b and ρ are positive numbers with $\rho < 1$. Recall that $g : \mathbb{T} \times \mathbb{T} \to \mathbb{R}_+$ is a jointly continuous nonnegative function satisfying $\int_{\mathbb{T}} g_y(c)d\sigma(c) = 1$ for all $y \in \mathbb{T}$ and the condition (14).

We observe that in this case the process of the total number of individuals $\|Y\| = (\|Y_t\| : t \geq 0)$ is a Markov process and that $Y_t = 0 \Leftrightarrow \|Y_t\| = 0$, which means that the time of absorption at 0 of the processes Y and $\|Y\|$ is the same (note that even if the 0's have a different meaning, they are identified).

Now, in [20] it is shown that there exists a q.s.d. for the process $\|Y\|$ killed at 0 if and only if $\lambda > b$. In addition, the extremal exponential decay rate of $\|Y\|$, defined by $\sup\{\theta(\nu); \ \nu \ q.s.d.\}$, is equal to $\lambda - b$ and there exists a unique (extremal) q.s.d. $\zeta^{\mathbf{e}}$ for $\|Y\|$ with this exponential decay rate $\lambda - b$, given by

$$\zeta^{\mathbf{e}}(k) = \left(\frac{b}{\lambda}\right)^{k-1} \left(1 - \frac{b}{\lambda}\right), \quad k \geq 1. \tag{55}$$

When ν is a q.s.d. for Y with exponential decay rate $\theta(\nu)$ then the probability vector $\zeta = (\zeta(k) : k \in \mathbb{N})$ given by

$$\zeta(k) = \nu(\mathcal{A}_k), \quad k \in \mathbb{N}, \tag{56}$$

is a q.s.d. with exponential decay rate $\theta = \theta(\nu)$, associated with the linear birth and death process $\|Y\|$. Hence a necessary condition for the existence of q.s.d. for the process Y is $\lambda > b$. We also deduce that all quasi-stationary probability measures $\tilde{\nu}$ of Y with exponential decay rate $\lambda - b$ are such that $\tilde{\nu}(\mathcal{A}_k) = \zeta^{\mathbf{e}}(k)$, so by (55) we get

$$\tilde{\nu}(\varphi_1) < \infty, \quad \text{where } \varphi_1(\eta) = \|\eta\|.$$

Now, we know from Theorem 1.2 that there exists a q.s.d. ν with exponential decay rate $\theta(\nu) = \nu(P_1(1))$. Moreover, it is immediate to show that φ_1 satisfies $L\varphi_1 = -(\lambda - b)\varphi_1$. Then, from Proposition 2.4 we get $P_1\varphi_1 = e^{-(\lambda-b)}\varphi_1$. Hence, if ν is a q.s.d. provided by Theorem 1.2 its exponential decay rate should be

$$\theta = \lambda - b \ .$$

Let us now consider the semi-group R_t given by,

$$R_t(\varphi)(\eta) = e^{\theta t} P_t(\varphi)(\eta) = e^{\theta t} \mathbb{E}_\eta\big(\varphi(Y_t)\mathbf{1}_{T_0 > t}\big) \ , \quad t \geq 0 \ .$$

The function φ_1 satisfies $R_t\varphi_1 = \varphi_1 \ \nu-$a.e. for all q.s.d. ν with the exponential decay rate θ.

Proposition 6.1. *Every q.s.d. ν with exponential decay rate $\theta = \lambda - b$ is absolutely continuous with respect to μ.*

Proof. Let ν be a q.s.d. which is not absolutely continuous. Then we can write the Lebesgue decomposition

$$\nu = f\mu + \xi \ \text{ that is } \nu(B) = \int_B f d\mu + \xi(B), \ B \in \mathcal{B}(E),$$

where f is a nonnegative $\mu-$integrable function and ξ is a singular measure with respect to μ.

From now on we denote by $R_t^\dagger$ the dual action of R_t on the set of measures defined by $(R_t^\dagger v)(\varphi) = v(R_t\varphi)$ for every measure $v \in \mathscr{M}_f(\mathcal{A})$ and any positive measurable function φ. Since ν is a q.s.d. it is invariant by the adjoint semigroup $R_t^\dagger$, that is $R_t^\dagger\nu = \nu$, then

$$f\mu + \xi = \nu = R_t^\dagger(\nu) = R_t^\dagger(f\mu) + R_t^\dagger(\xi) \ .$$

On the other hand, it follows from Proposition 5.1 that $R_t^\dagger(f\mu) \ll \mu$. Therefore

$$R_t^\dagger(f\mu) \leq f\mu \ .$$

Since φ_1 is ν integrable it must also be $f d\mu$ integrable. From the relation $R_t(\varphi_1) = \varphi_1$ we get,

$$\int \varphi_1 f d\mu = \int \varphi_1 dR_t^\dagger(f\mu),$$

and since φ_1 is strictly positive, we conclude $R_t^\dagger(f\mu) = f\mu$. This implies $R_t^\dagger(\xi) = \xi$. However by Proposition 5.3 (ii) (with $v = \xi$), $R_t^\dagger(\xi)$ cannot be completely singular with respect to μ unless ξ vanishes. This concludes the proof of the proposition. $\qquad\square$

Let us now turn to the study of uniqueness.

Lemma 6.2. *Assume condition (14): $\sigma \otimes \sigma(\{g = 0\}) = 0$. Then the Borel set $\mathcal{A}_{(1,1)} = \{\eta \in \mathcal{A} : \vec{q} = (1,1)\}$ satisfies $\mu(\mathcal{A}_{(1,1)}) > 0$. Moreover, for any q.s.d. ν with exponential decay rate $\theta = \lambda - b$ and for any Borel set B with $\mu(\mathcal{A}_{(1,1)} \cap B) > 0$, we have $\nu(\mathcal{A}_{(1,1)} \cap B) > 0$.*

Proof. Let us consider a q.s.d. ν with exponential decay rate $\theta = \lambda - b$. Lemma 3.2 implies that the restriction $\nu_{(1)}$ of ν to $\mathcal{A}_1 = \{\eta \in \mathcal{A} : \|\eta\| = 1\}$, does not vanish. On the other hand by the previous result, it is absolutely continuous with respect to σ. Then,

$$d\nu_{(1)} = f_1 d\sigma$$

for some nonnegative function f_1 that does not vanish on a set of σ positive measure.

For any function f in $C_b(\mathcal{A})$ such that $f(0) = 0$ and such that is supported in a compact set, that is $f(\eta) = 0$ for all $\|\eta\|$ large enough, it follows from $\nu(P_t f) = \exp(-\theta t)\,\nu(f)$ that

$$\nu(Lf\mathbf{1}_{\mathcal{A}^{-0}}) = -\theta\,\nu(f)\,.$$

Since this is true for any such function, we get (with $\overline{\eta} = (\eta_y : y \in \{\eta\})$ as defined in (15), and notation (36) and (45)),

$$
\begin{aligned}
-\theta d\nu_{\overline{\eta}}(\vec{\eta}) \;=\;& b(1-\rho) \sum_{y:\eta_y>1} \big(\eta_y - 1\big)\overline{\nu}(\overline{\eta}^{-y})d\nu_{\overline{\eta}^{-y}}(\vec{\eta}) \\
& +\lambda \sum_{y\in\{\eta\}} \big(\eta_y+1\big)\overline{\nu}(\overline{\eta}^{+y})d\nu_{\overline{\eta}^{+y}}(\vec{\eta}) + \lambda\overline{\nu}(\overline{\eta}^{+z})\int_{z\in\mathbb{T}\backslash\{\vec{y}\}} d\nu_{\overline{\eta}^{+z}}(\vec{\eta}^{+z}) \\
& + b\,\rho \sum_{y:\eta_y=1} \overline{\nu}(\overline{\eta}^{-y}) \sum_{y'\in\{\eta\}\backslash\{y\}} \eta_{y'}\,g_{y'}(y)\,d\nu_{\overline{\eta}^{-y}}(\vec{\eta}^{-y})\,d\sigma(y) \\
& -(\lambda+b)\left(\sum_{y\in\{\eta\}} \eta_y\right)\overline{\nu}(\overline{\eta})\,d\nu_{\overline{\eta}}(\vec{\eta})\,.
\end{aligned}
\tag{57}
$$

It follows from equation (57) applied to the measure ν and solving for $\nu_{\vec{q}}$ that for some constant $C > 0$ we have

$$d\nu_{(1,1)}\big(y_1, y_2\big) \geq C\,\big(f_1(y_1)g_{y_1}(y_2) + f_1(y_2)g_{y_2}(y_1)\big)\,d\sigma(y_1)\,d\sigma(y_2)\,.$$

Using this lower bound in the equation for $\nu_{(1)}$, we get for some constant $C' > 0$

$$f_1(y) \geq C' \int f_1(u)\,g_u(y)\,d\sigma(u)\,.$$

Therefore for some constant $C'' > 0$ we have the estimate

$$
\begin{aligned}
d\nu_{(1,1)}\big(y_1, y_2\big) \geq\;& C''\,d\sigma(y_1)\,d\sigma(y_2) \times \\
& \left(g_{y_1}(y_2)\int f_1(u)g_u(y_1)\,d\sigma(u) + g_{y_2}(y_1)\int f_1(u)g_u(y_2)d\sigma(u)\right)\,.
\end{aligned}
$$

Let B be a Borel set with $\mu(\mathcal{A}_{(1,1)} \cap B) > 0$. By the identification between $\mathcal{A}_{(1,1)}$ and $(1,1) \times \widehat{\mathbb{T}^2}$ it can be assumed that $B \subseteq \widehat{\mathbb{T}^2}$. We get from Fubini's Theorem

$$\nu(\mathcal{A}_{(1,1)} \cap B) \geq C'' \int_{\widehat{\mathbb{T}^3}} \mathbf{1}_B(y_1, y_2)g_{y_1}(y_2)\,f_1(u)\,g_u(y_1)\,d\sigma(u)\,d\sigma(y_1)\,d\sigma(y_2)\,.$$

Therefore, if $\nu(\mathcal{A}_{(1,1)} \cap B) = 0$, we must have

$$\mathbf{1}_B(y_1, y_2)\, g_{y_1}(y_2)\, f_1(u)\, g_u(y_1) = 0 \quad \sigma \times \sigma \times \sigma - \text{a.e.},$$

which implies from the hypothesis (14) on g

$$\mathbf{1}_B(y_1, y_2)\, f_1(u) = 0 \quad \sigma \times \sigma \times \sigma - \text{a.e.}.$$

However this implies $f_1 = 0$ $\sigma-$a.e., which is a contradiction. $\qquad\square$

Proposition 6.3. *There is a unique q.s.d. associated with the exponential decay rate $\theta = \lambda - b$.*

Proof. Let ν and ν' be two different q.s.d. with the exponential decay rate θ. We can write the Lebesgue decomposition

$$\nu' = f\nu + \xi$$

with f a nonnegative measurable function and ξ a singular measure with respect to ν. Assume $\xi \neq 0$. Applying $R_t^\dagger$ we get

$$\nu' = f\nu + \xi = R_t^\dagger(f\nu) + R_t^\dagger(\xi).$$

If f is bounded, since ν is a q.s.d., we have $R_t^\dagger(f\nu) \ll \nu$. In the general case, the same result holds by approximating f by an increasing sequence of nonnegative functions. Therefore, we must have

$$R_t^\dagger(f\nu) \leq f\nu.$$

Integrating the function φ_1 as before, we conclude that $R_t^\dagger(f\nu) = f\nu$, and therefore $R_t^\dagger(\xi) = \xi$.

Then, we have two q.s.d. ν and ξ with exponential decay rate $\theta = \lambda - b$, which are mutually singular. We claim that this is excluded by Lemma 6.2. Indeed let B be a measurable subset such that $\xi(B) = \nu(B^c) = 0$. Then $\nu(B \cap \mathcal{A}_{(11)}) = \nu(\mathcal{A}_{(11)}) > 0$. Since $\nu << \mu$ we get $\mu(B) \geq \mu(B \cap \mathcal{A}_{(11)}) > 0$. From Proposition 6.2 we deduce $\xi(B \cap \mathcal{A}_{(11)}) > 0$ which is a contradiction. Namely $\xi = 0$ and we conclude that $\nu' = f\nu$. Let us now show that $f \equiv 1$, which will yield $\nu' = \nu$ and so will conclude the uniqueness result.

Recall the following notation on the ordered lattice of measures $\mathscr{M}_f(\mathcal{A}^{-0})$: $|v| = v^+ + (-v)^+$ with $v^+ = \max(v, 0)$. Since the linear operator $R_t^\dagger$ is positive it holds $|R_t^\dagger(v)| \leq R_t^\dagger|v|$, that is for all positive and measurable functions φ,

it holds $\int \varphi \, d|R_t^\dagger(v)| \leq \int \varphi \, dR_t^\dagger|v|$. Moreover, when there exists a couple of sets A_1, A_2 such that $R_t^\dagger(v)(A_1) > 0 > R_t^\dagger(v)(A_2)$ and $R_t^\dagger|v|(A_1) > 0$, $R_t^\dagger|v|(A_2) > 0$, this inequality becomes strict and we put $|R_t^\dagger(v)| < R_t^\dagger|v|$. This means that

$$\forall \, \varphi > 0 \,, \quad \int \varphi \, dR_t^\dagger|v| < \infty \quad \text{where} \quad \int \varphi \, d|R_t^\dagger(v)| < \int \varphi \, dR_t^\dagger|v| \,.$$

Assume that $\nu(f \neq 1) > 0$, which implies $\mu(f \neq 1) > 0$. Thus, the sets $A_1 = \{f < 1\}$ and $A_2 = \{f > 1\}$ fulfill the requirements for the signed measure $v = \nu - \nu'$. Then, by using that ν and ν' are $R_t^\dagger$ invariant, we find

$$\left| \nu - \nu' \right| = \left| R_t^\dagger(\nu) - R_t^\dagger(\nu') \right| < R_t^\dagger\left(\left| \nu - \nu' \right| \right) \,.$$

Since φ_1 is R_t invariant and $\int \varphi_1 \, d|\nu - \nu'| < \infty$, we find

$$\int \varphi_1 \, d\left| \nu - \nu' \right| < \int \varphi_1 \, dR_t^\dagger\left(\left| \nu - \nu' \right| \right) = \int \varphi_1 \, d\left| \nu - \nu' \right| \,,$$

which is a contradiction since $\varphi_1 \geq 1$. The result is shown. $\qquad\square$

The proof of Theorem 1.3 is now complete, it follows from Propositions 6.1 and 6.3.

Acknowledgments

References

[1] BILLINGSLEY, P. (1968). *Convergence of probability measures*, Wiley and Sons, New York.

[2] CATTIAUX P.; COLLET, P.; LAMBERT, A.; MARTINEZ S.; MÉLÉARD S.; SAN MARTÍN, J. (2009) Quasi-stationary distributions and diffusions models in population dynamics. To appear in Ann. Probab.

[3] CATTIAUX P.; MÉLÉARD S. (2009). Competitive or weak cooperatie stochastic lotka-Volterra systems conditioned on non-extinction. To appear in J. Math. Biology.

[4] COLLET, P.; MARTÍNEZ, S.; SAN MARTÍN, J. (1995). Asymptotic laws for onedimensional diffusions conditioned to nonabsorption. Ann. Probab. 23, 1300–1314.

[5] CHAMPAGNAT, N.; FERRIÈRE R.; S. MÉLÉARD S. (2006). Unifying evolutionary dynamics: from individual stochastic processes to macroscopic models. Theoret. Pop. Biology 69, 297–321.

[6] DARROCH, J.N.; SENETA, E. (1965). On quasi-stationary distributions in absorbing discrete-time finite Markov chains. J. Appl. Prob. 2, 88–100.

[7] DALEY, D.J.; VERE-JONES, D. (1988). *An Introduction to the theory of point processes. Springer Series in Statistics.* Springer-Verlag, New York.

[8] DUNFORD, N.; SCHWARTZ, J. (1958). *Linear Operators Vol. I.* Interscience.

[9] FELLER, W. (1968) *An Introduction to probability Theory and Its Applications* John Wiley and Sons, New York

[10] FERRARI, P. A.; KESTEN, H.; MARTINEZ S.; PICCO, P. (1995). Existence of quasi-stationary distributions. A renewal dynamical approach. Ann. Probab. 23, No. 2, 501–521.

[11] FOURNIER, N.; MÉLÉARD, S. (2004). A microscopic probabilistic description of a locally regulated population and macroscopic approximations. Ann. Appl. Probab. 14, No. 4, 1880–1919.

[12] GOSSELIN, F. (2001), Asymptotic behavior os absorbing Markov chains conditional on nonabsorption for applications in conservation biology. Ann. Appl. Probab. 11, No. 1, 261–284.

[13] MANDL, P. (1961), Spectral theory of semi-groups connected with diffusion processes and its applications. Czech. Math. J. 11, 558-569.

[14] S. MARTÍNEZ; SAN MARTÍN, J. (2004), Classification of killed one-dimensional diffusions. Annals of Probability 32, No. 1,530-552.

[15] PINSKY R. (1985), On the convergence of diffusion processes conditioned to remain in bounded region for large time to limiting positive recurrent diffusion processes. Ann. Probab. 13, 363-378.

[16] Nair, M.G.; Pollett, P. (1993). On the relationship between $\mu-$invariant measures and quasi-stationary distributions for continuous-time Markov chains. Adv. Appl. Probab. 25, no. 1, 82–102.

[17] Renault, O.: Ferrière, R.; Porter, J.. The quasi-stationary route to extinction. Private communication.

[18] Steinsaltz, D,; Evans S. (2007), Quasistationary distributions for one-dimensional diffusions with killing. Trans. Amer. Math. Soc. 359, no. 3, 1285–1324.

[19] Tychonov, A.(1935) Ein Fixpunktsatz. Math. Annalen 117, 767–776.

[20] Van Doorn, E. (1991). Quasi-stationary distributions and convergence to quasi-stationary of birth-death processes. Adv. in Appl. Probab. 23, No 4, 683–700.

[21] Vere-Jones, D. (1962). Geometric ergodicity in denumerable Markov chains. Quart. J. Math. Oxford (2) 13, 7-28.

[22] Yaglom, A.M. (1947). Certain limit theorems of the theory of branching processes (in russian). Dokl. Akad. Nauk. SSSR 56, 795-798.

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