A Global Steering Method for Driftless Control-Affine Systems

Yacine Chitour *, Frédéric Jean † and Ruixing Long ‡§

Abstract

In this paper, we extend the globally convergent steering algorithm introduced in [13] for regular nonholonomic systems to general systems with singularities. This extension is based on the explicit construction of a lifted system which is regular. We also propose an exact motion planning method for nilpotent systems, which makes use of sinusoidal control laws and generalizes the algorithm presented in [19] for steering chained-form systems. It gives rise to smooth trajectories, leading to possible dynamical extension.

Contents

*Y. Chitour is with L2S, Université Paris-Sud XI, CNRS and Supélec, Gif-sur-Yvette, 91192, France. yacine.chitour@lss.supelec.fr

†F. Jean is with UMA, ENSTA ParisTech, Paris, 75739, France. fjean@ensta.fr

‡R. Long is with CMAP, Ecole Polytechnique, Palaiseau, 91128, France. long@cmap.polytechnique.fr

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1 Introduction

Nonholonomic systems have been attracting the attention of the scientific community for many years by the theoretical challenges they offer and several important applications they cover. In this paper, we address the motion planning problem for a general class of nonholonomic systems, i.e, for driftless control-affine nonholonomic system given by

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x), \ \ x \in \Omega, \tag{1}$$

where Ω is an open connected subset of $\mathbb{R}^n$, $X_1, \ldots, X_m$ are C^∞ vector fields on Ω and $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$.

This issue has been solved for some specific classes of driftless noholonomic systems by effective techniques among which a Lie bracket method for steering nilpotentizable systems (see [15] and [16]), sinusoidal controls for chained-form systems (see [19]), and trajectory generation method for flat systems (see [8]). However, there exist nonholonomic systems whose kinematic model does not fall into any of the aforementioned categories. For instance, mobile robots with more than one trailer cannot be transformed in chained-form unless each trailer is hinged to the midpoint of the previous wheel axle, an unusual situation in real vehicles. Another similar example is the rolling-body problem. Even the simplest model in this category, the so-called plate-ball system, does not allow any chained-form transformation and is not flat. For general 2-input systems, as long as the dimension of the state space reaches 5, exact nilpotentizability becomes the exception rather than the rule. Techniques about steering general nonholonomic systems have also been proposed by several authors. We can take some examples such as the iterative method of [16], the generic loop method of [20], and the continuation method of [6] and [24]. However, the two first methods require an a priori estimation of some "critical distance" which is in general an unknown parameter in practice. In the third method, proving global existence of solution for the path lifting equation turns out to be a hard issue and it can be achieved only under strong assumptions (see [4, 5, 7]).

This paper takes as starting point the globally convergent algorithm for steering *regular* nonholonomic systems discussed in [13]. Let us recall how this algorithm works: one first solves the motion planning problem for a control system "approximating" system (1) in a suitable sense, one then applies the resulting input $\hat{u}$ to (1) and iterates the procedure from the current point. If we use $\hat{x}(t, x_a, \hat{u})$, $t \in [0, T]$ to denote a trajectory of the "approximate" system starting from x_a, a local version of this algorithm is summarized as follows, where d is an appropriate distance to be defined in the next section.

We note that Algorithm 1 converges *locally* provided that the function AppSteer is *contractive* with respect to the distance d, i.e., for $x_1 \in \Omega$, there exists $\varepsilon_{x_1} > 0$ and $c \in (0, 1)$ such that

$$d(\text{AppSteer}(x, x_1), x_1) \leq c d(x_1, x), \qquad \text{for } x \in \Omega, \text{ and } d(x_1, x) < \varepsilon_{x_1}. \tag{2}$$

Algorithm 1 Local Approximation Steering (LAS)

Require: x_0, x_1, e

$k = 0$;

$x^k = x_0$;

while $d(x^k, x_1) > e$ **do**

Compute $\hat{u}^k$ such that $x_1 = \hat{x}(T, x^k, \hat{u}^k)$;

$x^{k+1} = \text{AppSteer}\ (x^k, x_1) := x(T, x^k, \hat{u}^k)$;

$k = k + 1$;

end while

Assume now that we have a *uniformly contractive* function AppSteer on a compact set $K \subset \Omega$, i.e. there exists $\varepsilon_K > 0$ and $c \in (0, 1)$ such that

$$d(\text{AppSteer}(x, x_1), x_1) \leq c d(x_1, x), \qquad \text{for } x, x_1 \in K, \text{ and } d(x_1, x) < \varepsilon_K. \tag{3}$$

Based on the local algorithm, a global approximate steering algorithm on K can be built along the line of the following idea[1]: Consider a parameterized path $\gamma \subset K$ connecting x_0 to x_1. Then choose a finite sequence of intermediate goals $\{x_0^d = x_0, x_1^d, \ldots, x_n^d = x_1\}$ on γ such that $d(x_{i-1}^d, x_i^d) < c/2$, $i = 0, \ldots, n$. One can prove that the iterated application of a uniformly contractive AppSteer(x^{i-1}, x_i^d) from the current state to the next subgoal (having set $x_i^d = x_1$, for $i \geq n$) yields a sequence x^i converging to x_1.

To turn the above idea into a practically efficient algorithm, three issues must be successfully addressed:

(P-1) Construct a *uniformly* contractive local approximate steering method;

(P-2) The "approximate" control $\hat{u}^k$ must be *exact* for steering the "approximate system" from the current point x^k to the final point x_1. As this computation occurs in each iteration, it must be performed in a reasonable time;

(P-3) Since the knowledge of the "critical distance" ε_K is not available in practice, the algorithm should achieve global convergence without knowing ε_K.

In [13], Issue (P-1) was solved by assuming the control system to be *regular*. As regards Issue (P-2), a general method was proposed in [15] and [16] for computing $\hat{u}$. Then, the authors proposed in [13] a globally convergent motion planning algorithm solving Issue (P-3) and not requiring a priori knowledge on the "critical distance" ε_K. However, two main drawbacks come up along the lines of the previous solution. Firstly, the *regularity* assumption is restrictive since general nonholonomic systems do exhibit singularities. Secondly, $\hat{u}$ computed as above is not suited for practical applications: for instance, a large number of maneuvers is unavoidable as well as the inversion of a nonlinear algebraic system.

In this paper, we first completely solve Issue (P-1), i.e., remove the regularity assumption of [13] and extend it to general driftless control-affine nonholonomic systems. This generalization is based on the construction of a "lifted" control system which generates a free Lie algebra up to certain step. This system contains only regular points and the algorithm introduced in [13] can thus be applied.

[1] A similar idea was proposed in [16].

In a second step, we present an algorithm using sinusoidal inputs for *exact* steering of general nilpotent systems. In particular, the algorithm is applied for controlling the approximate system used in [13], which is nilpotent. Our method generalizes the one proposed in [19] for controlling chained-form systems and we next briefly recall it. After havin put the system under a "canonical" form, one proceeds by controlling component after component by using, for each component, two sinusoids with suitable frequencies. For general systems, we show, in the present paper, that with more frequencies for each component, one can steer an arbitrary component independently on the other components. As a consequence, we are able to construct control laws which give rise to C^1 trajectories. This property will enable us to deal with dynamical extensions.

The paper is organized as follows. In Section 2, we define properly the notion of first order approximation. We then propose in Section 3 a purely polynomial desingularization procedure based on a *lifting* method. In Section 4, we describe in detail the globally convergent steering algorithm given in [13] for regular systems together with a proof of convergence. In Section 5, we present an exact steering method for nilpotent systems using sinusoids. Before the conclusion, we provide in Section 6 a user's guide which summarizes the global motion planning strategy developed in this paper.

2 Steering by Approximation

2.1 First order approximation and approximate steering method

Let Ω be an open connected subset of $\mathbb{R}^n$, and $VF(\Omega)$ the set of C^∞ vector fields on Ω. Let m be an integer smaller than n. Consider m vector fields $X_1, \dots, X_m$ of $VF(\Omega)$, and the associated driftless control-affine nonholonomic system given by

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x), \;\; x \in \Omega, \tag{4}$$

where the input $u(t) = (u_1(t), \dots, u_m(t)) \in \mathbb{R}^m$ is an integrable vector-valued function defined on $[0, T]$ with T a positive real number.

Given $x_a \in \Omega$, let $x(s, x_a, u)$, $s \in [0, T]$ be the trajectory of (4) starting from x_a under the action of the input function u. A point $x \in \Omega$ is said to be *accessible* from x_a if there exists an input $u : [0, T] \to \mathbb{R}^m$ and a time $t \in [0, T]$ such that $x = x(t, x_a, u)$. Chow's Theorem states that any two points in Ω are accessible from each other if the elements of the Lie algebra $L(X)$ generated by the vectors fields $X_1, \dots, X_m$ form an n-dimensional vector space at each point (see [1]). As System (4) is driftless, Chow's condition implies controllability in the usual sense (see [21]). Throughout this paper, we assume that System (4) is controllable. Then, the *motion planning problem* is the following: given two points x_0, $x_1 \in \Omega$, find an input u such that $x(T, x_0, u) = x_1$. Before bringing a solution to this problem, we first provide useful definitions. We refer the reader to [1] for more details.

Definition 2.1 (*Length of an input*)**.** The *length of an input* u is defined by

$$l(u) = \int_0^T \sqrt{u_1^2(t) + \cdots + u_m^2(t)} dt,$$

and the *length of a trajectory* $x(\cdot, x_a, u)$ is defined by

$$l(x(\cdot, x_a, u)) := l(u).$$

The appropriate notion of distance associated with the control system (4) and closely related to the notion of accessibility is the *sub-Riemannian distance*, also called *control distance*.

Definition 2.2 (*Sub-Riemannian distance*)**.** The vector fields $X_1, \ldots, X_m$ induce a function d on Ω, defined by

$$d(x_1, x_2) = \inf_u l(x(\cdot, x_1, u)), \tag{5}$$

where the infimum is taken over all the inputs u such that $x(\cdot, x_1, u)$ is defined on $[0, T]$ and $x(T, x_1, u) = x_2$. We will say that the function d is the *sub-Riemannian distance* associated with $X_1, \ldots, X_m$.

Remark 2.1. The function d defined above is a *distance* in the usual sense, i.e., it verifies (i) $d(x_1, x_2) \geq 0$; (ii) $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$; (iii) symmetry: $d(x_1, x_2) = d(x_2, x_1)$; (iv) triangle inequality: $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$.

Remark 2.2. We have that $d(x_1, x_2) < \infty$ if and only if x_1 and x_2 are accessible from each other.

Definition 2.3 (*Nonholonomic derivatives of a function*)**.** The *first-order nonholonomic derivatives* of f are the Lie derivatives $X_i f$ of f along X_i, $i = 1, \ldots, m$. Similarly, $X_i(X_j f)$, $i, j = 1, \ldots, m$ are called the *second-order nonholonomic derivatives* of f, and more generally, $X_{i_1} \cdots X_{i_k} f$, $i_1, \ldots, i_k \in \{1, \ldots, m\}$ are the k^{th}*-order nonholonomic derivatives* of f.

Proposition 2.1 ([1, Proposition 4.10, page 34])**.** *Let s be a non-negative integer. For a smooth function f defined near $x_a \in \Omega$, the following conditions are equivalent:*

(i) *$f(x) = O(d^s(x, x_a))$ for x in a neighborhood of x_a;*

(ii) *All the nonholonomic derivatives of order $\leq s - 1$ of f vanish at x_a.*

Definition 2.4 (*Nonholonomic order of a function*)**.** Let $s \in \mathbb{N}$ and f be a smooth real-valued function defined on Ω. If Condition (i) or (ii) of Proposition 2.1 holds, we say that f is of *order* $\geq s$ at x_a. If f is of order $\geq s$ but not of order $\geq s + 1$ at x_a, we say that f is of *order* s at x_a. The order of f at x_a will be denoted by $\mathrm{ord}_{x_a}(f)$.

Definition 2.5 (*Nonholonomic order of a vector field*)**.** Let $q \in \mathbb{Z}$. A vector field $Y \in VF(\Omega)$ is of *order* $\geq q$ at x_a if, for every non-negative integer s and every smooth function f of order s at x_a, Yf is of order $\geq q + s$ at x_a. If Y is of order $\geq q$ but not $\geq q + 1$, it is of *order* q at x_a. The order of Y at x_a will be denoted by $\mathrm{ord}_{x_a}(Y)$.

Definition 2.6 (*Nonholonomic first order approximation at x_a*)**.** An m-tuple

$$\widehat{X}^{x_a} := (\widehat{X}_1^{x_a}, \ldots, \widehat{X}_m^{x_a})$$

defined on $B(x_a, \rho_{x_a}) := \{x \in \Omega,\ d(x, x_a) \leq \rho_{x_a}\}$ with $\rho_{x_a} > 0$ is the *nonholonomic first order approximation of $X := (X_1, \ldots, X_m)$ at x_a* if the vector fields $X_i - \widehat{X}_i^{x_a}$, for $i = 1, \ldots, m$, are of order ≥ 0 at x_a. The positive number ρ_{x_a} is called *the approximate radius at x_a*.

Remark 2.3. As a consequence of Definition 2.6, one gets that the nonholonomic order at x_a defined by the vector fields $\widehat{X}_1^{x_a}, \ldots, \widehat{X}_m^{x_a}$ coincides with the one defined by $X_1, \ldots, X_m$.

Definition 2.7 (*Nonholonomic first order approximation on* Ω)**.** The *nonholonomic first order approximation of* X *on* Ω is a mapping $\mathcal{A}$ which associates to every $x_a \in \Omega$ the first order approximation of X at x_a defined on $B(x_a, \rho_{x_a})$, i.e., $\mathcal{A}(x_a) := \widehat{X}^{x_a}$ on $B(x_a, \rho_{x_a})$. The *approximation radius function* of $\mathcal{A}$ is the function $\rho : \Omega \to (0, \infty)$ which associates to every x_a its approximate radius ρ_{x_a}, i.e., $\rho(x_a) := \rho_{x_a}$.

In the sequel, the *nonholonomic first-order approximations* will simply be called *approximations.* Useful properties of approximations are continuity and nilpotency.

Definition 2.8 (*Continuity and nilpotency of an approximation*)**.** Let $\mathcal{A} : x_a \mapsto \widehat{X}$ be an approximation on Ω.

- We say that $\mathcal{A}$ is *continuous* if
  - (i) the mapping $(x_a, x) \mapsto \mathcal{A}(x_a)(x)$ is defined and, for every $x_a \in \Omega$, is continuous on a neighborhood of $(x_a, x_a) \in \Omega \times \Omega$;
  - (ii) the approximation radius function ρ of A is continuous.
- We say that $\mathcal{A}$ is *nilpotent of step* $s \in \mathbb{N}$ if, for every $x_a \in \Omega$, the Lie algebra generated by $\widehat{X}^{x_a}$ is nilpotent of step s.

We also need to define precisely the notion of *steering law for an approximation.*

Definition 2.9 (*Steering law for an approximation*)**.** Let $\mathcal{A} : x_a \mapsto \widehat{X}$ be an approximation on Ω and ρ its approximation radius function. A *steering law* of $\mathcal{A}$ is a mapping which, to every pair $(x, x_a) \in \Omega \times \Omega$ verifying $d(x_a, x) < \rho(x_a)$, associates an integrable input function $\hat{u} : [0, t] \mapsto \mathbb{R}^m$, henceforth called a *steering control*, such that the trajectory $\hat{x}(\cdot, x, \hat{u})$ of the *approximate control system*

$$\dot{x} = \sum_{i=1}^{m} u_i \widehat{X}_i^{x_a}(x), \tag{6}$$

is defined on $[0, T]$ and satisfies $\hat{x}(T, x, \hat{u}) = x_a$. In other words, $\hat{u}(\cdot)$ steers (6) from x to x_a.

Given X, an approximation $\mathcal{A}$ of X, and a steering law for $\mathcal{A}$, we define a *local approximate steering* method for X as follows.

Definition 2.10 (*Local approximate steering*)**.** Let $x_a \in \Omega$. For $x \in B(x_a, \rho(x_a))$, let $\hat{u}(\cdot)$ be one steering control of $\mathcal{A}(x_a)$ between x and x_a. The *local approximate steering (LAS for short)* method associated to $\mathcal{A}$ and its steering law is the function defined on $\Omega \times \Omega$ by

$$\text{AppSteer}(x, x_a) := x(T, x, \hat{u}).$$

Definition 2.11 (*Contractive and uniformly contractive*)**.** A LAS method is *contractive* if, for every $x_a \in \Omega$, there exists $\varepsilon_{x_a} > 0$ such that the following implication holds true:

$$d(x_a, x) < \varepsilon_{x_a} \implies d(x_a, \text{AppSteer}(x, x_a)) \leq d(x_a, x)^{1+\beta},$$

where $\beta > 0$ is independent of x_a. A LAS method is *uniformly contractive* on a compact set $K \subset \Omega$ if it is contractive and if ε_{x_a} is independent of x_a, i.e., there exists $\varepsilon_K > 0$ such that, for every pair $(x_a, x) \in K \times K$, the following implication holds true:

$$d(x_a, x) < \varepsilon_K \implies d(x_a, \text{AppSteer}(x, x_a)) \leq d(x_a, x)^{1+\beta}.$$

Remark 2.4. We will show that if $\widehat{X}$ is an approximation of X at x_a, the corresponding AppSteer function is contractive in a neighborhood of x_a. By the Fixed Point Theorem, one gets local convergence of Algorithm 1 (LAS). However, in order to obtain a globally convergent algorithm from LAS, one needs AppSteer to be *uniformly* contractive. In other words, the mapping $\mathcal{A}$ needs to be *continuous* in the sense of Definition 2.8.

2.2 Privileged coordinates and distance estimation

A special class of coordinates, called *privileged coordinates* and defined below, turns out to be a useful tool to compute the order of functions and vector fields, and to estimate the sub-Riemannian distance d.

Recall first that the *length* of elements in $L(X)$ is defined by induction as

$$\begin{aligned} \Delta(X_i) &:= 1, \text{ for } i = 1, \dots, m; & (7)\\ \Delta([X_I, X_J]) &:= \Delta(X_I) + \Delta(X_J), \text{ with } X_I, X_J \in L(X). & (8) \end{aligned}$$

We will use $L^s(X)$ to denote the Lie sub-algebra of elements of length not greater than $s \in \mathbb{N}$. Take $x \in \Omega$ and let $L^s(x)$ be the vector space generated by the values at x of elements belonging to $L^s(X)$. The controllability of System (4) guarantees that there exists a smallest integer $r := r(x)$ such that $\dim L^r(x) = n$. This integer is called the *degree of nonholonomy* at x.

Definition 2.12 (*Growth vector*)**.** For $x_a \in \Omega$, let $n_s(x_a) := \dim L^s(x_a)$, $s = 1, \dots, r$. The sequence

$$(n_1(x_a), \dots, n_r(x_a))$$

is the *growth vector* of X at x_a.

Definition 2.13 (*Regular and singular points*)**.** A point $x_a \in \Omega$ is said to be *regular* if the growth vector remains constant in a neighborhood of $x_a$and, otherwise, x_a is said to be *singular*.

Note that regular points form an open and dense set in Ω.

Definition 2.14 (*Weight*)**.** For $x_a \in \Omega$ and $j = 1, \dots, n$, let $w_j := w_j(x_a) \in \mathbb{N}$ be defined by setting $w_j := s$ if $n_{s-1} < j \le n_s$, with $n_s := n_s(x_a)$ and $n_0 := 0$. The integers w_j, for $j = 1, \dots, n$ are called the *weight at* x_a.

Remark 2.5. The meaning of Definition 2.14 is best understood in term of a basis of the tangent space of Ω at x_a denoted by $T_{x_a}\Omega$. Choose first some vector fields $W_1, \dots, W_{n_1}$ in $L^1(X)$ such that $W_1(x_a), \dots, W_{n_1}(x_a)$ form a basis of $L^1(x_a)$. Choose then other vectors fields $W_{n_1+1}, \dots, W_{n_2}$ in $L^2(X)$ such that $W_1(x_a), \dots, W_{n_2}(x_a)$ form a basis of $L^2(x_a)$. For every $s \in \mathbb{N}$, choose $W_{n_{s-1}+1}, \dots, W_{n_s}$ in $L^s(X)$ such that $W_1(x_a), \dots, W_{n_s}(x_a)$ form a basis of $L^s(x_a)$. We obtain in this way a sequence of vector fields $W_1, \dots, W_n$ such that

$$\begin{cases} W_1(x_a), \dots, W_n(x_a) \text{ is a basis of } T_{x_a}\Omega, \\ W_i \in L^{w_i}, i = 1, \dots, n. \end{cases} \tag{9}$$

A sequence of vector fields verifying Eq. (9) is called an *adapted frame at* x_a. The word “adapted" means “adapted to the flag $L^1(x_a) \subset L^2(x_a) \subset \cdots \subset L^r(x_a) = T_{x_a}\Omega$”, since the values at x_a of an adapted frame contain a basis $W_1(x_a), \dots, W_{n_s}(x_a)$ of every subspace $L^s(x_a)$ of the flag. The values of $W_1, \dots, W_n$ at x_b near x_a form also a basis of $T_{x_b}\Omega$. However, this basis is not adapted to the flag $L^1(x_b) \subset L^2(x_b) \subset \cdots \subset L^r(x_b) = T_{x_b}\Omega$ if x_a is singular.

Definition 2.15 (*Privileged coordinates at x_a*)**.** A *system of privileged coordinates* at $x_a \in \Omega$ is a system of local coordinates $(z_1, \dots, z_n)$ centered at x_a (the image of x_a is 0) such that $\mathrm{ord}_{x_a}(z_j) = w_j$, for $j = 1, \dots, n$.

Remark 2.6. For every system of local coordinates $(y_1, \dots, y_n)$ centered at x_a, we have, up to a re-ordering, $\mathrm{ord}_{x_a}(y_j) \leq w_j$ or without re-ordering, $\sum_{j=1}^n \mathrm{ord}_{x_a}(y_j) = \sum_{j=1}^n w_j$.

The order at $x_a \in \Omega$ of functions and vector fields expressed in a system of privileged coordinates $(z_1, \dots, z_n)$ centered at x_a can be evaluated algebraically as follows:

- The order of the monomial $z_1^{\alpha_1} \dots z_n^{\alpha_n}$ is equal to its *weighted degree*

$$w(\alpha) := w_1\alpha_1 + \dots + w_n\alpha_n;$$

- The order of a function $f(z)$ at $z = 0$ is the least weighted degree of the monomials occurring in the Taylor expansion of f at 0;
- The order of the monomial vector field $z_1^{\alpha_1} \dots z_n^{\alpha_n}\partial_{z_j}$ is equal to its *weighted degree* $w(\alpha) - w_j$, where one assigns the weight $-w_j$ to ∂_{z_j} at 0;
- The order of a vector field $h(z) = \sum_{j=1}^n h_j(z)\partial_{z_j}$ at $z = 0$ is the least weighted degree of the monomials occurring in the Taylor expansion of h at 0.

Definition 2.16 (*Continuously varying system of privileged coordinates on Ω*)**.** A *continuously varying system of privileged coordinates on* Ω is a mapping Φ, with values in $\mathbb{R}^n$, defined and continuous on a neighborhood of $(x_a, x_a) \in \Omega \times \Omega$ such that the partial mapping $z := \Phi(x_a, \cdot)$ is a system of privileged coordinates at x_a. In this case, there exists a continuous function $\bar{\rho} : \Omega \to (0, +\infty)$ such that the coordinates $\Phi(x_a, \cdot)$ are defined on $B(x_a, \bar{\rho}(x_a))$. We call $\bar{\rho}$ an *injectivity radius function* of Φ.

Definition 2.17 (*Pseudonorm*)**.** Given the system of privileged coordinates $(z_1, \dots, z_n)$ centered at x_a, the function

$$\|z\|_{x_a} := |z_1|^{1/w_1} + \dots + |z_n|^{1/w_n},$$

where $w_1, \dots, w_n$ are weights at x_a, is called a *pseudonorm* at x_a.

Privileged coordinates provide estimates of the sub-Riemannian distance d, according to the following result.

Theorem 2.2 (Ball-Box Theorem [1])**.** *Consider $(X_1, \dots, X_m) \in VF(\Omega)^m$, a point $x_a \in \Omega$, and a system of privileged coordinates z at x_a. There exist positive constants $C_d(x_a)$ and $\varepsilon_d(x_a)$ such that, for every $x \in \Omega$ with $d(x_a, x) < \varepsilon_d(x_a)$, one has*

$$\frac{1}{C_d(x_a)} \|z(x)\|_{x_a} \leq d(x_a, x) \leq C_d(x_a) \|z(x)\|_{x_a}. \tag{10}$$

If Ω contains only regular points and if Φ is a continuously varying system of privileged coordinates on Ω, then there exist continuous positive functions $C_d(\cdot)$ and $\varepsilon_d(\cdot)$ on Ω such that inequality (10) holds true with $z = \Phi(x_a, \cdot)$ at all (x, x_a) satisfying $d(x, x_a) < \varepsilon_d(x_a)$.

Corollary 2.3. *Let K be a compact subset of Ω. Assume that K contains only regular points and there exists a continuously varying system of privileged coordinates Φ on K. Then the* AppSteer *function in the LAS method is uniformly contractive on K. Moreover there exist positive constants C_K and ε_K such that, for every pair $(x_a, x) \in K \times K$ verifying $d(x_a, x) < \varepsilon_K$, one has*

$$\frac{1}{C_K}\|z(x) - z(x_a)\|_{x_a} \leq d(x_a, x) \leq C_K\|z(x) - z(x_a)\|_{x_a}, \tag{11}$$

where $z := \Phi(x_a, \cdot)$.

Privileged coordinates also allow one to measure the error obtained when X is replaced by an approximation $\widehat{X}$.

Proposition 2.4 ([1, Prop. 7.29])**.** *Consider a point $x_a \in \Omega$, a system of privileged coordinates z at x_a, and an approximation $\widehat{X}$ of X at x_a. Then, there exist positive constants $C_e(x_a)$ and $\varepsilon_e(x_a)$ such that, for every $x \in \Omega$ with $d(x_a, x) < \varepsilon_e(x_a)$ and every integrable input function $u(\cdot)$ with $\ell(u) < \varepsilon_e(x_a)$, one has*

$$\|z(x(T, x, u)) - z(\hat{x}(T, x, u))\|_{x_a} \leq C_e(x_a) \max\left(\|z(x)\|_{x_a}, \ell(u)\right)\, \ell(u)^{1/r}, \tag{12}$$

where r is the degree of nonholonomy at x_a, $x(\cdot, x, u)$ and $\hat{x}(\cdot, x, u)$ are the trajectories of $\dot{x} = \sum_{i=1}^m u_i X_i(x)$, and $\dot{x} = \sum_{i=1}^m u_i \widehat{X}_i(x)$ respectively.

If Ω contains only regular points, Φ is a continuously varying system of privileged coordinates on Ω, and $\mathcal{A}$ a continuous approximation of X on Ω, then there exist continuous positive functions $C_e(\cdot)$ and $\varepsilon_e(\cdot)$ such that inequality (12) holds true, with $z = \Phi(x_a, \cdot)$ and $\widehat{X} = \mathcal{A}(x_a)$, for every pair $(x, x_a) \in \Omega \times \Omega$ with $d(x, x_a) < \varepsilon_e(x_a)$ and every integrable input function $u(\cdot)$ with $\ell(u) < \varepsilon_e(x_a)$.

Corollary 2.5. *Let K be a compact subset of Ω. Assume that K contains only regular points and there exists a continuously varying system of privileged coordinates Φ on K, and $\mathcal{A}$ a continuous approximation of X on K. Then, up to reducing ε_K occurring in* Corollary 2.3, *for every pair $(x_a, x) \in K \times K$ verifying $d(x_a, x) < \varepsilon_K$, one has*

$$d(\text{AppSteer}(x, x_a), x_a) \leq \frac{1}{2} d(x, x_a), \tag{13}$$

$$\|z(\text{AppSteer}(x, x_a))\|_{x_a} \leq \frac{1}{2}\|z(x)\|_{x_a}. \tag{14}$$

Remark 2.7. Since the growth vector and the weights do not remain constant in any open neighborhood of a singular point, privileged coordinates z cannot vary continuously in any open neighborhood of that singular point . Therefore, around a singular point, the distance estimations Eqs. (11) and (14) based on privileged coordinates do not hold true uniformly. In particular, if (x_{a_n}) is a sequence of regular points converging to a singular point x_a (this is possible since regular points are dense in Ω), the sequences $\varepsilon_d(x_{a_n})$ and $\varepsilon_e(x_{a_n})$ tend to zero whereas $\varepsilon_d(x_a)$ and $\varepsilon_d(x_a)$ are not equal to zero.

Remark 2.8. A similar discontinuity issue of course occurs for the approximate system. Indeed, if x_a is a singular point, the growth vector and the weights of the associated privileged coordinates at x_a change around x_a, implying a change of the truncation order in the Taylor expansion of the vector fields. Therefore, the approximate vector fields cannot vary continuously when approaching a singular point.

3 Desingularization by Lifting

Since general nonholonomic systems exhibit singular points, the estimations (11) and (14) cannot hold uniformly on Ω (see Remark 2.7). Therefore, global convergence of the motion planning algorithm presented in Section 4.2 is not guaranteed for general nonholonomic systems. In this section, we present a purely algebraic procedure of desingularization for general nonholonomic systems. Assume that the vector fields $X = \{X_1, \ldots, X_m\} \subset VF(\Omega)$ are given in a certain system of coordinates $x = (x_1, \ldots, x_n)$ and the maximum degree of nonholonomy of X is equal to r.

The strategy consists in " *lifting* " the vector fields X to some extended ones $\xi = \{\xi_1, \ldots, \xi_m\}$ defined on some extended domain $\widetilde{\Omega} := \Omega \times \mathbb{R}^{\widetilde{n}}$, with $\widetilde{n} \in \mathbb{N}$ to be defined later, so that:

(i) for $i = 1, \ldots, m$, ξ_i has the following form in coordinates $\widetilde{x} := (x, y)$,

$$\xi_i(x, y) := X_i(x) + \sum_{j=1}^{\widetilde{n}} b_j(x, y)\partial_{y_j},$$

where y is a system of coordinates in $\mathbb{R}^{\widetilde{n}}$ and b_j, for $j = 1, \ldots, \widetilde{n}$, are smooth functions;

(ii) the vector fields $\{\xi_1, \ldots, \xi_m\}$ generate a free Lie algebra up to step r.

Point (i) guarantees that, if we consider the canonical projector π from $\widetilde{\Omega}$ to Ω defined by $\pi(\widetilde{x}) = x$ with $\widetilde{x} = (x, y) \in \widetilde{\Omega}$, one has

$$\pi_*\xi_i(\widetilde{x}) = X_i(\pi(\widetilde{x})),$$

where $\pi_*\xi_i$ is the *push-forward* of ξ_i by π, defined by $\pi_*\xi(\widetilde{x}) := D\pi_{\widetilde{x}}\xi_i(\widetilde{x})$, with $D\pi_{\widetilde{x}}$ denoting the value of the differential of π at $\widetilde{x}$. In other words, for $\widetilde{x} \in \widetilde{\Omega}$ and $x = \pi(\widetilde{x})$, one obtains $X_1(x), \ldots, X_m(x)$ by *projecting* $\xi_1(\widetilde{x}), \ldots, \xi_m(\widetilde{x})$ on the tangent space of Ω at x. Thus, the projection by π of trajectories of the following control system

$$\dot{\widetilde{x}} = \sum_{i=1}^{m} u_i\xi_i(\widetilde{x}), \text{ with } \widetilde{x} \in \widetilde{\Omega}, \tag{15}$$

gives rise to trajectories of (4). Therefore, in order to steer System (4) from p to q with $(p, q) \in \Omega \times \Omega$, it suffices to steer System (15) from $\widetilde{p} := (p, 0)$ to $\widetilde{q} := (q, 0)$.

Point (ii) guarantees that System (15) is *regular*. Indeed, since ξ generates a free Lie algebra up to step r, the growth vector is constant at every point $\widetilde{x} \in \widetilde{\Omega}$. Moreover, we will construct during the lifting process a *continuous varying system of privileged coordinates* for ξ such that the nonholonomic first order approximation of (15) is in a "canonical" form which can be exactly controlled by sinusoids (see Section 5). Therefore, the algorithm presented in Section 4.2 can be applied to the motion planning of System (15), and it is globally convergent by Theorem 4.1 and Proposition 4.2.

We start this section by presenting some general facts on free Lie algebras, namely the *P. Hall basis* in Subsection 3.1, and the *canonical form* of a nilpotent Lie algebra of step r in Subsection 3.2. We then give a *desingularization procedure* in Subsection 3.3. The proofs of the results stated in Subsection 3.3 will be gathered in Subsection 3.4.

3.1 P. Hall basis on a free Lie algebra and evaluation map

In this subsection, we present some general facts on free Lie algebras. The reader is referred to [3] for more details. Consider $\mathcal{I} := \{1, \ldots, m\}$, and the free Lie algebra $\mathcal{L}(\mathcal{I})$ generated by the elements of $\mathcal{I}$. Recall that $\mathcal{L}(\mathcal{I})$ is the $\mathbb{R}$-vector space generated by the elements of $\mathcal{I}$ and their formal brackets, together with the relations of skew-symmetry and the Jacobi identity enforced (see [3] for more details). The *length* of an element I of a free Lie algebra $\mathcal{L}(\mathcal{I})$ is well defined via Eqs. (7) and (8), and is denoted by $|I|$. We use $\mathcal{L}^s(\mathcal{I})$ to denote the subspace generated by elements of $\mathcal{L}(\mathcal{I})$ of length not greater than s. Let $\widetilde{n}_s$ be the dimension of $\mathcal{L}^s(\mathcal{I})$.

A *P. Hall basis* of $\mathcal{L}(\mathcal{I})$ is a totally ordered set of elements $\mathcal{H} := \{I_j\}_{j\in\mathbb{N}}$ of $\mathcal{L}(\mathcal{I})$ defined as follows.

Definition 3.1 (*P. Hall basis*)**.** A subset $\mathcal{H} := \{I_j\}_{j\in\mathbb{N}}$ of $\mathcal{L}(\mathcal{I})$ is the *P. Hall basis* of $\mathcal{L}(\mathcal{I})$ if (H1), (H2), (H3), and (H4) are verified.

(H1) If $|I_i| < |I_j|$, then $I_i \prec I_j$;

(H2) $\{1, \ldots, m\} \subset \mathcal{H}$, and we impose that $1 \prec 2 \prec \cdots \prec m$;

(H3) every element of length 2 in $\mathcal{H}$ is in the form $[I_i, I_j]$ with $(I_i, I_j) \in \mathcal{I} \times \mathcal{I}$ and $I_i \prec I_j$;

(H4) an element $I_k \in \mathcal{L}(\mathcal{I})$ of length greater than 3 belongs to $\mathcal{H}$ if $I_k = [I_{k_1}, [I_{k_2}, I_{k_3}]]$ with $I_{k_1}, I_{k_2}, I_{k_3}$, and $[I_{k_2}, I_{k_3}]$ belonging to $\mathcal{H}$, $I_{k_2} \prec I_{k_3}$, $I_{k_2} \prec I_{k_1}$ or $I_{k_2} = I_{k_1}$, and $I_{k_1} \prec [I_{k_2}, I_{k_3}]$.

The elements of $\mathcal{H}$ form a basis of $\mathcal{L}(\mathcal{I})$, and " $\prec$ " defines a strict and total order over the set $\mathcal{H}$. In the sequel, we use I_k to denote the k^{th} element of $\mathcal{H}$ with respect to the order " $\prec$ ". Let $\mathcal{H}^s$ be the subset of $\mathcal{H}$ of all the elements of length not greater than s. The elements of $\mathcal{H}^s$ form a basis of $\mathcal{L}^s(\mathcal{I})$ and $\mathrm{Card}(\mathcal{H}^s) = \widetilde{n}_s$. We also consider the set $\mathcal{G}^s$ made of the elements in $\mathcal{H}$ of length equal to s. One has $\mathcal{G}^s = \mathcal{H}^s \setminus \mathcal{H}^{s-1}$. The cardinal of $\mathcal{G}^s$ will be denoted by $\widetilde{k}_s$.

By (H1)$-$(H4), every element $I_j \in \mathcal{H}$ can be expanded in a unique way as

$$I_j = [I_{k_1}, [I_{k_2}, \cdots, [I_{k_i}, I_k] \cdots]], \tag{16}$$

with $k_1 \geq \cdots \geq k_i$, $k_i < k$, and $k \in \{1, \ldots, \tilde{n}_1\}$. In that case, the element I_j is said to be a *direct descendent* of I_k, and we write $\phi(j) := k$. For $I_j \in \mathcal{H}^r$, the expansion (16) also associates with $I_j \in \mathcal{H}$ a sequence $\alpha_j = (\alpha_j^1, \ldots, \alpha_j^{\widetilde{n}_r})$ in $\mathbb{Z}^{\widetilde{n}_r}$ defined by

$$\alpha_j^\ell := \text{ Card } \{s \in \{1, \ldots, i\}, k_s = \ell\}.$$

By construction, one has $\alpha_j^\ell = 0$ for $\ell \geq j$, and $\alpha_j = (0, \ldots, 0)$ for $1 \leq j \leq \tilde{n}_1$.

The P. Hall basis $\mathcal{H}$ induces, via the *evaluation map*, a generating family of the control Lie algebra $L(X)$ associated with the vector fields $X_1, \ldots, X_m$ involved in System (4).

Definition 3.2 (*Evaluation map*)**.** The *evaluation map* E_X defined on $\mathcal{L}(\mathcal{I})$, with values in $L(X)$, assigns to every $I \in \mathcal{L}(\mathcal{I})$ the vector field $X_I := E_X(I)$ obtained by plugging in X_i, $i = 1, \ldots, m$, for the corresponding letter i.

Definition 3.3 (*P. Hall family*)**.** The *P. Hall family* H_X associated to the vector fields $X = \{X_1, \ldots, X_m\}$ is defined by

$$H_X := \{E_X(I), I \in \mathcal{H}\},$$

where E_X is the evaluation map and $\mathcal{H}$ is the P. Hall basis of the free Lie algebra $\mathcal{L}(\mathcal{I})$ constructed over $\mathcal{I} = \{1, \ldots, m\}$. Then, H_X also inherits the ordering and the numbering of the elements in $\mathcal{H}$ induced by (H1)–(H4).

Note that H_X is only a generating family of $L(X)$ and is not always a basis of $L(X)$.

3.2 Canonical form of a nilpotent free Lie algebra

We present in this subsection the construction of some canonical form of nilpotent free Lie algebra proposed by Grayson and Grossman in [9] and [10]. Similar results were also obtained by Sussmann in [21].

Definition 3.4 (*Free up to step s*)**.** Let s be a positive integer such that $1 \leq s \leq r$. A family of vector fields $\xi = \{\xi_1, \ldots, \xi_m\}$ defined on a subset $\widetilde{\Omega}$ of $\mathbb{R}^{\tilde{n}_r}$ is said to be *free up to step s* if, for every $\widetilde{x} \in \widetilde{\Omega}$, the growth vector $(n_1(\widetilde{x}), \ldots, n_s(\widetilde{x}))$ is equal to $(\tilde{n}_1, \ldots, \tilde{n}_s)$.

Remark 3.1. If ξ defined on $\widetilde{\Omega} \subset \mathbb{R}^{\tilde{n}_r}$ is free up to step r, every point of $\widetilde{\Omega}$ is *regular*.

Definition 3.5 (*Free weights*)**.** Let $\xi = \{\xi_1, \ldots, \xi_m\}$ be free up to step r on $\widetilde{\Omega} \subset \mathbb{R}^{\tilde{n}_r}$. The integers $\widetilde{w}_1, \ldots, \widetilde{w}_{\tilde{n}_r}$, where $\widetilde{w}_j = s$ if $n_{s-1}(\widetilde{x}) < j \leq n_s(\widetilde{x})$ for every $\widetilde{x} \in \widetilde{\Omega}$ are called the *free weights* at $\widetilde{x}$.

Let $v := \{v_1, \ldots, v_{\tilde{n}_r}\}$ be a system of coordinates in $\mathbb{R}^{\tilde{n}_r}$. For $j = 1, \ldots, \tilde{n}_r$, we assign to v_j the weight $\widetilde{w}_j$ at 0, and to ∂_{v_j} the weight $-w_j$ at 0. Then, the *weighted degree* of a monomial of the form $v_1^{\alpha_1} \ldots v_{\tilde{n}_r}^{\alpha_{\tilde{n}_r}}$ is equal to

$$\widetilde{w}(\alpha) := \widetilde{w}_1\alpha_1 + \cdots + \widetilde{w}_{\tilde{n}_r}\alpha_{\tilde{n}_r},$$

and the weighted degree of a monomial vector field $v_1^{\alpha_1} \ldots v_{\tilde{n}_r}^{\alpha_{\tilde{n}_r}} \partial_{v_j}$ is equal to $\widetilde{w}(\alpha) - \widetilde{w}_j$.

We now construct m vector fields $D := \{D_1, \ldots, D_m\}$ in coordinates v such that D is free up to step r.

For every $I_j \in \mathcal{H}^r$, let α_j be the sequence associated with I_j (see Subsection 3.1). Define the monomial $P_{k,j}$ associated with I_j by

$$P_{k,j}(v) := \frac{v^{\alpha_j}}{\alpha_j!}, \tag{17}$$

where $v^{\alpha_j} := \prod_{\ell} v_\ell^{\alpha_j^\ell}$, and $\alpha_j! := \prod_{\ell} \alpha_j^\ell!$. The monomial $P_{k,j}$ can also be defined inductively by the following formulas.

$$\begin{array}{lcll} P_{k,j} & := & 1 & \text{if } I_j \in \mathcal{H}^1 \text{ and } I_j = I_k; \\ P_{k,j} & := & \dfrac{v_{j_1}}{\alpha_{j_2}^{j_1} + 1} P_{k,j_2} & \text{if } I_j = [I_{j_1}, I_{j_2}] \text{ and } \phi(j) = k. \end{array} \tag{18}$$

We note that $P_{k,j} = 0$ in other cases.

Theorem 3.1 ([9, 10])**.** *With above notations, define m vector fields in coordinates v as follows*

$$\begin{aligned} D_1 &:= \partial_{v_1}, \\ D_2 &:= \partial_{v_2} + \sum_{\substack{2\leq |I_j|\leq r \\ \phi(j)=2}} P_{2,j}\partial_{v_j}, \\ &\vdots \\ D_m &:= \partial_{v_m} + \sum_{\substack{2\leq |I_j|\leq r \\ \phi(j)=m}} P_{m,j}\partial_{v_j}. \end{aligned}$$

Then, the Lie algebra generated by $D := \{D_1, \ldots, D_m\}$ is free to step r, and one has

$$D_{I_j}(0) = \partial_{v_j}, \quad \textit{for } I_j \in \mathcal{H}^r,$$

where $D_{I_j} := E_D(I_j)$, where E_D is used to denote the evaluation map with values in the Lie algebra generated by D.

The proof of Theorem 3.1 goes by induction on the length of elements in the Lie algebra generated by D. The reader is referred to [10] for a complete development.

Corollary 3.2. *For all $I_k \in \mathcal{H}^r$, D_{I_k} has the following form*

$$D_{I_k} = \partial_{v_k} + \sum_{I_j\in\mathcal{H}^r,\ |I_j|>|I_k|} P_j^k \partial_{v_j}, \tag{19}$$

where all non zero polynomials P_j^k are homogeneous of weighted degree equal to $|I_j| - |I_k|$.

Corollary 3.3. *For $i = 1, \ldots, m$, we define m derivations $\check{D}_i$ as follows*

$$\check{D}_i := \partial_{v_i} + \sum_{\substack{2\leq |I_k|\leq \mathcal{H}^{r-1} \\ \phi(k)=i}} P_{i,k}\partial_{v_k} + \sum_{\substack{I_j\in S \\ \phi(j)=i}} P_{i,k}\partial_{v_j},$$

where S is an arbitrary non-empty subset of $\mathcal{G}^r$. Then,

- *if $I_k \in \mathcal{H}^{r-1} \cup S$, we have*

$$\check{D}_{I_k} = \partial_{v_k} + \sum_{I_j\in\mathcal{H}^{r-1}\cup S,\ |I_j|>|I_k|} P_j^k \partial_{v_j};$$

- *if $I_k \in \mathcal{G}^r \setminus S$, we have $\check{D}_{I_k} = 0$.*

Definition 3.6 (*Canonical form*)**.** A family of vector fields $X = \{X_1, \ldots, X_m\}$ is said to be in *canonical form* in a system of coordinates v if for $i = 1, \ldots, m$, one has

$$v_* X_i = D_i,$$

where we use $v_* X_i$ to denote the push-forward of X_i by v.

Consider now the control system given by

$$\dot{v} = \sum_{i=1}^{m} u_i D_i(v), \quad \text{for } v \in \mathbb{R}^{\widetilde{n}_r}. \tag{20}$$

Writing (20) component by component, one has

$$\dot{v}_j = P_{k,j}(v_1, \dots, v_{j-1}) u_j \ \text{ if } \phi(j) = k, \text{ and } j = 1, \dots, \widetilde{n}_r, \tag{21}$$

or inductively,

$$\dot{v}_j = \frac{v_{j_1}}{\alpha_{j_2}^{j_1} + 1} \dot{v}_{j_2}, \quad \text{if } I_j = [I_{j_1}, I_{j_2}], \quad \text{and } j = 1, \dots, \widetilde{n}_r. \tag{22}$$

More explicitly, one has

$$\dot{v}_j = \frac{1}{k!} v_{j_1}^k \dot{v}_{j_2}, \quad \text{if } X_{I_j} = \mathrm{ad}^k_{X_{I_{j_1}}} X_{I_{j_2}}, \tag{23}$$

where $\mathrm{ad}^k_{X_{I_{j_1}}} X_{I_{j_2}} := \underbrace{[X_{I_{j_1}}, [X_{I_{j_1}}, \cdots, [X_{I_{j_1}}}_{k \text{ times}}, X_{I_{j_2}}]$, with $X_{I_{j_2}} = [X_{I_{j_3}}, X_{I_{j_4}}]$ and $I_{j_3} \prec I_{j_1}$. The inductive formula (23) will be used in Section 5.

Theorem 3.4 ([21])**.** *Assume that the family of vector fields $X = \{X_1, \dots, X_m\}$ generates a nilpotent free Lie algebra up to step r. Then, in the canonical coordinates of the second kind $(z_1, \dots, z_{\widetilde{n}_r})$ associated with the P. Hall basis H_X, the control system $\dot{x} = \sum_{i=1}^m u_i X_i(x)$ is in canonical form.*

Recall that the *canonical coordinates of the second kind* associated with H_X is the inverse of the local diffeomorphism

$$(z_1, \dots, z_{\widetilde{n}_r}) \longmapsto p\ e^{z_{\widetilde{n}_r} X_{I_{\widetilde{n}_r}}} \circ \cdots \circ e^{z_1 X_{I_1}}, \text{ with } p \in \mathbb{R}^{\widetilde{n}_r}, \tag{24}$$

where we use e^{zX_I} to denote the flow of X_I.

Remark 3.2. The canonical coordinates of the second kind require to determine the flow of the control vector fields, i.e. to integrate some differential equations. In general, there does not exist algebraic change of coordinates between an arbitrary system of coordinates and the canonical coordinates of the second kind.

3.3 Desingularization algorithm

Assume that we now work on a compact subset K of $\Omega \subset \mathbb{R}^n$. Let r be the maximum of the degree of nonholonomy of System (4) on K. Consider the P. Hall basis $\mathcal{H}^r$ of the free Lie algebra $\mathcal{L}^r(\mathcal{I})$ of step r with $\mathcal{I} = \{1, \dots, m\}$. Choose now a set $\mathcal{J} \subset \mathcal{H}^r$ of cardinal n as follows

$$\mathcal{J} := \{I_1, \dots, I_n |\ I_j \in \mathcal{H}^r \text{ for } j = 1, \dots, n, \text{ and } I_k < I_i \text{ for } 1 \le k < i \le n\}. \tag{25}$$

Define $\mathcal{J}^s := \{I_j \in \mathcal{J}, \text{ with } |I_j| = s\}$, and denote by k_s the cardinal of $\mathcal{J}^s$. We also define the domain $\mathcal{V}_{\mathcal{J}} \subset \Omega$ by

$$\mathcal{V}_{\mathcal{J}} := \{\ p \in \Omega \text{ such that } \det(X_{I_1}(p), \dots, X_{I_n}(p)) \neq 0\ \}, \tag{26}$$

where $X_{I_j} := E_X(I_j)$. This definition implies, in particular, that $\mathcal{V}_{\mathcal{J}}$ is open in Ω, possibly empty, and for every $p \in \mathcal{V}_{\mathcal{J}}$, the family of vectors $\{X_{I_1}(p), \ldots, X_{I_n}(p)\}$ forms a basis of the tangent space of $\mathcal{V}_{\mathcal{J}}$ at p.

Since K is compact, there exist $\mathcal{J}_1, \ldots, \mathcal{J}_M$ defined as in Eq. (25) such that

$$K = \bigcup_{i=1}^{M} \mathcal{V}_{\mathcal{J}_i}. \tag{27}$$

One deduces from (27) a compact covering of K in the form

$$K = \bigcup_{i=1}^{M} \mathcal{V}^{c}_{\mathcal{J}_i}, \tag{28}$$

where, for $i = 1, \ldots, M$, the set $\mathcal{V}^{c}_{\mathcal{J}_i} \subset \mathcal{V}_{\mathcal{J}_i}$ is compact.

Take one $\mathcal{J}$ among $\mathcal{J}_1, \ldots, \mathcal{J}_M$. Let a be a point of $\mathcal{V}_{\mathcal{J}}$. In the sequel, we construct, by induction on the length of elements in a free Lie algebra, m vector fields $\xi = \{\xi_1, \ldots, \xi_m\}$ defined on $\mathcal{V}_{\mathcal{J}} \times \mathbb{R}^{\tilde{n}_r - n}$ which is free up to step r. At the same time, we give in canonical form a nonholonomic first order approximation of ξ at $\tilde{a} := (a, 0) \in \mathcal{V}_{\mathcal{J}} \times \mathbb{R}^{\tilde{n}_r - n}$. For $s \geq 2$, we define $\mathcal{G}^s := \mathcal{H}^s \setminus \mathcal{H}^{s-1}$, and we will use $\tilde{k}_s$ to denote the cardinal of $\mathcal{G}^s$. Assume that the vector fields $\{X_1, \ldots, X_m\}$ are given in a system of coordinates x.

Desingularization Algorithm (DA)

- **Step 1:**

  (1-1) Define $\mathcal{V}^1 := \mathcal{V}_{\mathcal{J}} \times \mathbb{R}^{\tilde{k}_1 - k_1}$ and $\mathcal{K}^1 := \mathcal{H}^1 \cup (\mathcal{J} \setminus \mathcal{J}^1)$. Let v^1 be a system of coordinates in $\mathbb{R}^{\tilde{k}_1 - k_1}$; Let $a^1 := (a, 0) \in \mathcal{V}^1$;

  (1-2) define $\{\xi^1_1, \ldots, \xi^1_m\}$ on $\mathcal{V}^1$ in coordinates (x, v^1) as follows:

  $$\xi^1_i := X_i + \left\{ \begin{array}{ll} 0 & \text{for } i \in \mathcal{J}^1 \\ \partial_{v^1_i} & \text{for } i \in \mathcal{G}^1 \setminus \mathcal{J}^1 \end{array} \right. ;$$

  (1-3) make the linear change of coordinates y^1 on $\mathcal{V}^1$ (with values in $\mathbb{R}^n \times \mathbb{R}^{\tilde{k}_1 - k_1}$) defined by

  $$\partial_{y^1_j}|_{a^1} := \xi^1_{I_j}(a^1), \quad \text{for } I_j \in \mathcal{K}^1 ;$$

  (1-4) define the system of coordinates z^1 on $\mathcal{V}^1$ by

  $$\begin{aligned} z^1_j &:= y^1_j, \quad \text{for } j \in \mathcal{H}^1, \\ z^1_j &:= y^1_j - \sum_{k=1}^{\tilde{n}_1} (\xi^1_k \cdot y^1_k)(a^1) y^1_k, \quad \text{for } I_j \in \mathcal{K}^1 \setminus \mathcal{H}^1; \end{aligned}$$

  where I_j denotes the j^{th} element in $\mathcal{K}^1$.

- **Step s, $2 \leq s \leq r$:**

  (s-1) Define $\mathcal{V}^s := \mathcal{V}^{s-1} \times \mathbb{R}^{\tilde{k}_s - k_s}$ and $\mathcal{K}^s := \mathcal{K}^{s-1} \cup (\mathcal{G}^s \setminus \mathcal{J}^s)$. Let v^s be a system of coordinates in $\mathbb{R}^{\tilde{k}_s - k_s}$; Let $a^s := (a, 0) \in \mathcal{V}^s$;

(s-2) define $\{\xi_1^s, \dots, \xi_m^s\}$ on $\mathcal{V}^s$ in coordinates (z^{s-1}, v^s) as follows:

$$\xi_i^s := \xi_i^{s-1} + \sum_{I_k \in \mathcal{G}^s \setminus \mathcal{J}^s} P_{i,k}(z^{s-1}) \partial_{v_k^s} \ ;$$

(s-3) make the linear change of coordinates y^s on $\mathcal{V}^s$ defined by

$$\partial_{y^s_{\phi(I)}}|_{a^s} = \xi_I^s(a^s), \quad \text{for } I \in \mathcal{K}^s \ ;$$

(s-4) define the system of coordinates $\tilde{z}^s$ on $\mathcal{V}^s$ by the following recursive formulas:

s-4-(a) for $I_j \in \mathcal{H}^s$,

$$\tilde{z}_j^s := y_j^s + \sum_{k=2}^{|I_j|-1} r_k(y_1^s, \dots, y_{j-1}^s), \tag{29}$$

where, for $k = 2, \dots, |I_j| - 1$,

$$\begin{aligned} & r_k(y_1^s, \dots, y_{j-1}^s) \\ = & - \sum_{\substack{|\beta|=k \\ \omega(\beta)<|I_j|}} [(\xi_{I_1}^s)^{\beta_1} \cdots (\xi_{I_{j-1}}^s)^{\beta_{j-1}} (y_j^s + \sum_{q=2}^{k-1} r_q)(a^s)] \frac{(y_1^s)^{\beta_1}}{\beta_1!} \cdots \frac{(y_{j-1}^s)^{\beta_{j-1}}}{\beta_{j-1}!} \ ; \end{aligned}$$

s-4-(b) for $I_j \in \mathcal{K}^s \setminus \mathcal{H}^s$,

$$\tilde{z}_j^s := y_j^s + \sum_{k=2}^{s} r_k(y_1^s, \dots, y_{\widetilde{n}_s}^s), \tag{30}$$

where, for $k = 2, \dots, s$,

$$r_k(y_1^s, \dots, y_{\widetilde{n}_s}^s) = - \sum_{\substack{|\beta|=k \\ \omega(\beta)\leq s}} [(\xi_{I_1}^s)^{\beta_1} \cdots (\xi_{I_{\widetilde{n}_s}}^s)^{\beta_{\widetilde{n}_s}} (y_j^s + \sum_{q=2}^{s} r_q)(a^s)] \frac{(y_1^s)^{\beta_1}}{\beta_1!} \cdots \frac{(y_{j-1}^s)^{\beta_{\widetilde{n}_s}}}{\beta_{\widetilde{n}_s}!};$$

(s-5) construct the system of coordinates z^s as follows:

s-5-(a) for $j > \widetilde{n}_s$, set $z_j^s := \tilde{z}_j^s$;

s-5-(b) for $j = 1, \dots, \widetilde{n}_s$, set $z_j^s := \Phi_j^s(\tilde{z}_1^s, \dots, \tilde{z}_{j-1}^s)$, where Φ_j^s is a homogeneous polynomial of weighted degree equal to w_j, and in the coordinates z^s, the $\widetilde{n}_s$ first components of ξ_i^s are in the form

$$\xi_{i,j}^s = P_{i,j}(z_1^s, \dots, z_{j-1}^s) + R_{i,j}(z^s), \tag{31}$$

where $\xi_{i,j}^s$ denotes the j^{th} component of ξ_i^s in coordinates z^s, and $\mathrm{ord}_{a^s}^s(P_{i,j}) = w_j - 1$, and $\mathrm{ord}_{a^s}^s(R_{i,j}) \geq w_j$, with $\mathrm{ord}_{a^s}^s(\cdot)$ denoting the nonholonomic order defined by $\{\xi_1^s, \dots, \xi_m^s\}$ at a^s.

Theorem 3.5. *Let $\xi_i := \xi_i^r$, for $i = 1, \dots, m$, and $z_j := z_j^r$, for $j = 1, \dots, \widetilde{n}_r$, where ξ_i^r and z_j^r are given by the desingularization algorithm. Then,*

- *the family of vector fields $\xi := \{\xi_1, \dots, \xi_m\}$ defined on $\Omega \times \mathbb{R}^{\widetilde{n}_r - n}$ is free up to step r;*

- *the system of coordinates* $z := (z_1, \dots, z_{\widetilde{n}_r})$ *is a system of privileged coordinates at* $\tilde{a}$*;*
- *the nonholonomic first order approximation* $\widehat{\xi} := \{\widehat{\xi}_1, \dots, \widehat{\xi}_m\}$ *of* ξ *at* $\tilde{a}$ *in the coordinates* z *is in canonical form:*

$$\widehat{\xi}_i = \partial_{z_i} + \sum_{\substack{2 \le |I_j| \le \widetilde{n}_r \\ i \prec j}} P_{i,j}(z_1, \dots, z_{j-1}) \partial_{z_j}, \quad \text{for } i = 1, \dots, m. \tag{32}$$

Remark 3.3. As the lifted system $\{\xi_1, \dots, \xi_m\}$ is regular on $\widetilde{\Omega}$, the motion planning algorithm presented Section 4 is globally convergent for the extended control system (15). Due to the particular form of ξ, the projection of the trajectories of (15) on Ω gives rise to trajectories of the original control system (4). Therefore, as mentioned at the beginning of Section 3, in order to steer the system (4) from p to q with $(p, q) \in \Omega \times \Omega$, it suffices to determine an input u steering the extended system (15) from $(p, 0)$ to $(q, 0)$, and the same input will steer System (4) from p to q.

3.4 Proof of Theorem 3.5

The proof of Theorem 3.5 is based on the following proposition.

Proposition 3.6. *The desingularization algorithm is feasible from* $s = 1$ *to* $s = r$*. At each stage* s *of the construction* $(s = 1, \dots, r)$*, the following properties hold true:*

(A1) *The family of vectors* $\{\ \xi_I^s(a^s)\ \}_{I \in \mathcal{K}^s}$ *is linearly independent;*

(A2) *if* $|I_j| \le s$*, then* $\mathrm{ord}_{a^s}^s(\tilde{z}_j^s) = |I_j|$*, and* $\mathrm{ord}_{a^s}^s(z_j^s) = |I_j|$*;*

(A3) *if* $|I_j| > s$*, then* $\mathrm{ord}_{a^s}^s(z_j^s) > s$*;*

(A4) *the change of coordinates* $(\Phi_j^s)_{j=1,\dots,\widetilde{n}_s}$ *exists;*

(A5) *in coordinates* z^s*, for* $I_k \in \mathcal{K}^s$*, the vector fields* $\xi_{I_k}^s$ *has the following form*

$$\xi_{I_k}^s = \sum_{I_j \in \mathcal{H}^s} (P_j^k + R_j^k) \partial_{z_j^s} + \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q_\ell^k \partial_{z_\ell^s} \tag{33}$$

with $\mathrm{ord}_{a^s}^s(R_j^k) > |I_j| - |I_k|$*, and* $\mathrm{ord}_{a^s}^s(Q_\ell^k) > s - |I_k|$*, and* P_j^k *given by Eq. (19).*

More precisely, if one defines $\check{\xi}_i^s$ *by*

$$\check{\xi}_i^s := \sum_{\substack{I_j \in \mathcal{H}^s \\ \phi(j) = i}} P_{i,j} \partial_{z_j^s},$$

then, one has

$$\check{\xi}_{I_k}^s = \sum_{I_j \in \mathcal{H}^s} P_j^k \partial_{z_j^s},$$

where the polynomials P_j^k *verify the following properties*

- *if* $I_k \in \mathcal{H}^s$*, then*

  - *for* $|I_j| < |I_k|$, $P_j^k = 0$;
  - *for* $|I_j| = |I_k|$, $P_j^j = 1$, *and* $P_j^k = 0$ *if* $k \neq j$;
  - *for* $|I_j| > |I_k|$, $\mathrm{ord}^s_{a^s}(P_j^k) = |I_j| - |I_k|$;
- *if* $I_k \in \mathcal{K}^s \setminus \mathcal{H}^s$, $P_j^k = 0$ *for all* $j = 1, \dots, \widetilde{n}_s$.

Remark 3.4. Property (A1) implies that Step (s-3) is feasible, which, in turn, guarantees that Steps s-4-(a) and s-4-(b) are well defined, and $\tilde{z}^s$ is a system of coordinates because the differential of the application $y^s \mapsto \tilde{z}^s$ at 0 is equal to the identity map. Property (A4) guarantees that Step s-5-(b) is feasible. Property (A2) ensures that, at the end of the algorithm, the system of coordinates z^r is a system of privileged coordinates. Property (A5) finally ensures that for $s = r$, the approximation $\widehat{\xi}$ of ξ is in canonical form.

By Remark 3.4, Theorem 3.5 is a consequence of Proposition 3.6. It remains to prove Proposition 3.6. The proof goes by induction on s.

Proof of Proposition 3.6. We begin by showing that Properties (A1)-(A5) hold true for $s = 1$.

Claim 1. *The family of vectors* $\{\xi_I^1(a^1)\}_{I \in \mathcal{K}^1}$ *is linearly independent, i.e., Property* (A1) *holds true for* $s = 1$.

Proof of Claim 1. By construction, for every $I \in \mathcal{J}$, one has $\xi_I^1(a^1) = X_I(a)$ which belongs to $\mathbb{R}^n \times \{0\}$. For $i \in \mathcal{G}^1 \setminus \mathcal{J}^1$, the vector $\xi_i^1(a^1)$ belongs to $\mathbb{R}^n \times \mathbb{R}^{\widetilde{k}_1 - k_1}$, and the family of vectors $\{\xi_i^1(a^1)\}_{i \in \mathcal{G}^1 \setminus \mathcal{J}^1}$ is linearly independent. Therefore, the family of vectors $\{\xi_I^1(a^1)\}_{I \in \mathcal{K}^1}$ is linearly independent and Claim 1 holds true.

□

Claim 2. *For* $j = 1, \dots, \widetilde{n}_1$, *one has* $\mathrm{ord}^1_{a^1}(z_j^1) = 1$, *i.e., Property* (A2) *holds true for* $s = 1$.

Proof of Claim 2. For $j = 1, \dots, \widetilde{n}_1$, one has by construction $\xi_j^1 \cdot z_j^1(a^1) = 1$. Thus, $\mathrm{ord}^1_{a^1}(z_j^1) \leq 1$. Since z^1 is a system of coordinates centered at a^1, one has $z_j^1(a^1) = 0$, then $\mathrm{ord}^1_{a^1}(z_j^1) > 0$. Therefore, one has $\mathrm{ord}^1_{a^1}(z_j^1) = 1$ and Claim 2 holds true.

□

Claim 3. *For* $I_j \in \mathcal{K}^1$ *with* $|I_j| > 1$, *one has* $\mathrm{ord}^1_{a^1}(z_j^1) > 1$, *i.e., Property* (A3) *holds true for* $s = 1$.

Proof of Claim 3. For $|I_j| \geq 2$, i.e. $I_j \in \mathcal{K}^1 \setminus \mathcal{J}^1$, one computes $\xi_k^1 \cdot z_j^1$ at a^1 for every $k \in \{1, \dots, \widetilde{n}_1\}$.

$$\begin{aligned} \xi_k^1 \cdot z_j^1(a^1) &= \xi_k^1 \cdot y_j^1(a^1) - \sum_{i=1}^{\widetilde{n}_1} (\xi_i^1 \cdot y_j^1)(a^1)(\xi_k^1 \cdot y_i^1)(a^1) \\ &= \xi_k^1 \cdot y_j^1(a^1) - \xi_k^1 \cdot y_j^1(a^1) = 0. \end{aligned}$$

Then, by definition, one has $\mathrm{ord}^1_{a^1}(z_j^1) > 1$ for $|I_j| > 1$ and Claim 3 holds true.

□

Claim 4. *For* $i = 1, \dots, m$, *and* $j = 1, \dots, \widetilde{n}_1$, *the* j^{th} *component of* ξ_i^1 *in coordinates* z^1 *is equal to* 1 *if* $i = j$, *and equal to* 0 *otherwise. In other words, for* $i = 1, \dots, m$, *the* $\widetilde{n}_1$ *first components of* ξ_i^1 *verify Eq. (31). Properties* (A4) *and* (A5) *hold true for* $s = 1$.

Proof of Claim 4. By Claim 1, $\{\xi_1^1(a^1), \ldots, \xi_{\widetilde{n}_1}^1(a^1)\}$ is a basis of $\mathbb{R}^{\widetilde{n}_1}$, then the linear change of coordinates y^1 exists. As $\partial_{y_j^1}(0) = \xi_j^1(a^1)$, and $z_j^1 = y_j^1$ for $j = 1, \ldots, \widetilde{n}_1$. Claim 4 holds true. $\square$

Therefore, Properties (A1)-(A5) hold true for $s = 1$. Assume now that Properties (A1)-(A5) hold true for $1 \le s < r$, show that they still hold true for $s+1$.

Claim 5. *The vector fields $\{\xi_i^{s+1}\}_{i=1,\ldots,m}$ are well defined. Moreover, one has $\mathrm{ord}^s_{a^{s+1}}(P_{i,k}) = s$.*

Proof of Claim 5. Consider $I_k \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}$, then one has $I_k = [I_{k_1}, I_{k_2}]$. Assume that $\phi(I_k) = i$. By Eq. (18), one has

$$P_{i,k}(z^s) := \frac{z^s_{k_1}}{\alpha^{k_1}_{k_2} + 1} P_{i,k_2}(z^s).$$

Since $|I_{k_1}| \le s$ and $|I_{k_2}| \le s$, we have $k_1 \le \widetilde{n}_s$ and $k_2 \le \widetilde{n}_s$, thus the right-hand side of the above equation is well defined in coordinates $z^s = (z_1^s, \ldots, z^s_{\widetilde{n}_s})$. Therefore, the new vector fields $\{\xi_i^{s+1}\}_{i=1,\ldots,m}$ are well defined.

Since $\mathrm{ord}^s_{a^{s+1}}(z^s_{k_1} P_{i,k_2}) = \mathrm{ord}^s_{a^{s+1}}(z^s_{k_1}) + \mathrm{ord}^s_{a^{s+1}}(P_{i,k_2})$, and by inductive hypothesis (namely (A2) holds true at step s), one has $\mathrm{ord}^s_{a^{s+1}}(z^s_{k_1}) = |I_{k_1}|$, and $\mathrm{ord}^s_{a^{s+1}}(P_{i,k_2}) = |I_{k_2}| - 1$, therefore $\mathrm{ord}^s_{a^{s+1}}(P_{i,k}) = |I_{k_1}| + |I_{k_2}| - 1 = s$. $\square$

Claim 6. *For $I_k \in \mathcal{K}^{s+1}$ with $|I_k| \le s+1$, one has*

$$\xi^{s+1}_{I_k} = \xi^s_{I_k} + \sum_{I_j \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}} \tilde{P}^k_j(z^s) \partial_{v_j}, \tag{34}$$

where

$$\tilde{P}^k_j(z^s) = P^k_j(z^s_1, \ldots, z^s_{\widetilde{n}_s}) + \tilde{R}^k_j(z^s),$$

with $\mathrm{ord}^s_{a^{s+1}}(P^k_j) = |I_j| - |I_k|$ *and* $\mathrm{ord}^s_{a^{s+1}}(\tilde{R}^k_j) > |I_j| - |I_k|$.

Proof of Claim 6. The proof goes by induction on the length $|I_k|$. For $|I_k| = 1$, one has (by construction)

$$\xi^{s+1}_k := \xi^s_k + \sum_{\substack{I_j \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1} \\ \phi(j) = k}} P_{k,j} \partial_{v_j}.$$

By Claim 5, $\mathrm{ord}^s_{a^{s+1}}(P_{k,j}) = s = |I_j| - |I_k|$. Claim 6 holds true for $|I_k| = 1$.

Assume that Claim 6 holds true for every $I \in \mathcal{K}^{s+1}$ of length less than s_1. Consider $I_k \in \mathcal{K}^{s+1}$ with $|I_k| = s_1 + 1$. One has

$$\begin{aligned} \xi^{s+1}_{I_k} &= [\xi^{s+1}_{I_{k_1}}, \xi^{s+1}_{I_{k_2}}] = [\xi^s_{I_{k_1}} + \sum_{I_i \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}} (P^{k_1}_i + \tilde{R}^{k_1}_i) \partial_{v_i^{s+1}}, \xi^s_{I_{k_2}} + \sum_{I_i \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}} (P^{k_2}_i + \tilde{R}^{k_2}_i) \partial_{v_i^{s+1}}] \\ &= [\xi^s_{I_{k_1}}, \xi^s_{I_{k_2}}] + \sum_{I_i \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}} \{\xi^s_{I_{k_2}} \cdot (P^{k_1}_i + \tilde{R}^{k_1}_i) - \xi^s_{I_{k_1}} \cdot (P^{k_2}_i + \tilde{R}^{k_2}_i)\} \partial_{v_i^{s+1}}. \end{aligned}$$

Since (A5) holds true up to step s, one has

$$
\begin{aligned}
&\xi^s_{I_{k_1}} \cdot (P_i^{k_2} + \tilde{R}_i^{k_2}) = \left[\sum_{I_j \in \mathcal{H}^s} (P_j^{k_1} + R_j^{k_1}) \partial_{z_j^s} + \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q_\ell^{k_1} \partial_{z_\ell^s}\right] \cdot (P_i^{k_2} + \tilde{R}_i^{k_2}) \\
= \;& \sum_{I_j \in \mathcal{H}^s} (P_j^{k_1} + R_j^{k_1}) \partial_{z_j^s} P_i^{k_2} + \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q_\ell^{k_1} \partial_{z_\ell^s} P_i^{k_2} \\
& + \sum_{I_j \in \mathcal{H}^s} (P_j^{k_1} + R_j^{k_1}) \partial_{z_j^s} \tilde{R}_i^{k_2} + \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q_\ell^{k_1} \partial_{z_\ell^s} \tilde{R}_i^{k_2} \\
= \;& \sum_{I_j \in \mathcal{H}^s} P_j^{k_1} \partial_{z_j^s} P_i^{k_2} + \left[\sum_{I_j \in \mathcal{H}^s} R_j^{k_1} \partial_{z_j^s} P_i^{k_2} + \sum_{I_j \in \mathcal{H}^s} (P_j^{k_1} + R_j^{k_1}) \partial_{z_j^s} \tilde{R}_i^{k_2} + \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q_\ell^{k_1} \partial_{z_\ell^s} \tilde{R}_i^{k_2}\right] \\
:= \;& \sum_{I_j \in \mathcal{H}^s} P_j^{k_1} \partial_{z_j^s} P_i^{k_2} + \mathcal{R}_{i,1}.
\end{aligned}
$$

We first show that every term in $\mathcal{R}_{i,1}$ has, at a^{s+1}, an order strictly greater than $s+1-|I_k|$. Indeed, for $I_j \in \mathcal{H}^s$, since $\mathrm{ord}^s_{a^{s+1}}(z_j) = |I_j|$, $\mathrm{ord}^s_{a^{s+1}}(P_i^{k_2}) = |I_i| - |I_{k_2}|$, and $\mathrm{ord}^s_{a^{s+1}}(R_j^{k_1}) > |I_j| - |I_{k_1}|$, then one has

$$
\mathrm{ord}^s_{a^{s+1}}(R_j^{k_1} \partial_{z_j^s} P_i^{k_2}) > |I_j| - |I_{k_1}| + (|I_i| - |I_{k_2}|) - |I_j| = |I_i| - |I_k|, \text{ with } |I_i| = s+1.
$$

Note that $\mathrm{ord}^s_{a^{s+1}}((P_j^{k_1} + R_j^{k_1}) \partial_{z_j^s} \tilde{R}_i^{k_2}) = \mathrm{ord}^s_{a^{s+1}}(P_j^{k_1} \partial_{z_j^s} \tilde{R}_i^{k_2})$. Since $\mathrm{ord}^s_{a^{s+1}}(P_j^{k_1}) = |I_j| - |I_{k_1}|$, and $\mathrm{ord}^s_{a^{s+1}}(\tilde{R}_i^{k_2}) > |I_i| - |I_{k_1}|$, one has

$$
\mathrm{ord}^s_{a^{s+1}}(P_j^{k_1} \partial_{z_j^s} \tilde{R}_i^{k_2}) > |I_j| - |I_{k_2}| + |I_i| - |I_{k_1}| - |I_j| = |I_i| - |I_k|.
$$

Recall that, by definition, all the functions have positive order. Therefore, one gets

$$
\mathrm{ord}^s_{a^{s+1}}(Q_\ell^{k_1} \partial_{z_\ell^s} \tilde{R}_i^{k_2}) \geq \mathrm{ord}^s_{a^{s+1}}(Q_\ell^{k_1}) > s - |I_{k_1}| = s - (|I_k| - |I_{k_2}|) \geq s + 1 - |I_k|.
$$

In conclusion, $\mathrm{ord}^s_{a^{s+1}}(\mathcal{R}_{i,1}) > s + 1 - |I_k|$.

A similar computation shows that

$$
\begin{aligned}
&\xi^s_{I_{k_2}} \cdot (P_i^{k_1} + \tilde{R}_i^{k_1}) \\
= \;& \sum_{I_j \in \mathcal{H}^s} P_j^{k_2} \partial_{z_j^s} P_i^{k_1} + \left[\sum_{I_j \in \mathcal{H}^s} R_j^{k_2} \partial_{z_j^s} P_i^{k_1} + \sum_{I_j \in \mathcal{H}^s} (P_j^{k_2} + R_j^{k_2}) \partial_{z_j^s} \tilde{R}_i^{k_1} + \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q_\ell^{k_2} \partial_{z_\ell^s} \tilde{R}_i^{k_1}\right] \\
:= \;& \sum_{I_j \in \mathcal{H}^s} P_j^{k_2} \partial_{z_j^s} P_i^{k_1} + \mathcal{R}_{i,2}, \text{ with } \mathrm{ord}^s_{a^{s+1}}(\mathcal{R}_{i,2}) > s + 1 - |I_k|.
\end{aligned}
$$

Therefore, one gets

$$
\begin{aligned}
\xi^{s+1}_{I_k} \;=\; & \xi^s_{I_k} + \sum_{I_i \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}} \{\sum_{I_j \in \mathcal{H}^s} (P_j^{k_1} \partial_{z_j^s} P_i^{k_2} - P_j^{k_2} \partial_{z_j^s} P_i^{k_1})\} \partial_{v_i} \\
& + \sum_{I_i \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}} (\mathcal{R}_{i,1} + \mathcal{R}_{i,2}) \partial_{v_i}
\end{aligned}
$$

with $\mathrm{ord}^s_{a^{s+1}}(\mathcal{R}_{i,1}+\mathcal{R}_{i,2}) \geq \min(\mathrm{ord}^s_{a^{s+1}}(\mathcal{R}_{i,1}),\mathrm{ord}^s_{a^{s+1}}(\mathcal{R}_{i,2})) > s+1-|I_k|$.
Since

$$\sum_{I_j \in \mathcal{H}^s} (P_j^{k_1} \partial_{z_j^s} P_i^{k_2} - P_j^{k_2} \partial_{z_j^s} P_i^{k_1}) = P_i^k,$$

and $\mathrm{ord}^s_{a^{s+1}}(P_i^k) = |I_i| - |I_k|$ by Corollary 3.3, one gets

$$\xi_{I_k}^{s+1} = \xi_{I_k}^s + \sum_{I_i \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}} (P_i^k + \tilde{R}_i^k) \partial_{v_i},$$

with $\mathrm{ord}^s_{a^{s+1}}(P_i^k) = s+1-|I_k|$, and $\mathrm{ord}^s_{a^{s+1}}(\tilde{R}_i^k) > s+1-|I_k|$.

Therefore, Claim 6 still holds true for $I_k \in \mathcal{K}^{s+1}$ with $|I_k| = s_1 + 1$. This terminates the induction, and Claim 6 is now proved.

$\square$

Claim 7. *The family of vectors $\{\xi_{I_k}^{s+1}(a^{s+1})\}_{I_k \in \mathcal{K}^{s+1}}$ is linearly independent, i.e.,* (A1) *holds true at step $s+1$.*

Proof of Claim 7. Claim 6 implies that for all $I_k \in \mathcal{K}^s$, one has $\xi_{I_k}^{s+1}(a^{s+1}) = \xi_{I_k}^s(a^s) \in \mathbb{R}^{\widetilde{n}_s} \times \{0\}$. Corollary 3.3 implies that for all $I_k \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}$, one has $\xi_{I_k}^{s+1}(a^{s+1}) = \xi_{I_k}^s(a^s) + \partial_{v_k} \in \mathbb{R}^{\widetilde{n}_s} \times \mathbb{R}^{\widetilde{k}_{s+1} - k_{s+1}}$. Therefore, by (A1) at step s, the vectors $\{\xi_{I_k}^{s+1}(a^{s+1})\}_{I_k \in \mathcal{K}^{s+1}}$ are linearly independent.

$\square$

Claim 8. *After performing* (s+1)-4-(a) *and* (s+1)-4-(b) *in the desingularization algorithm, one has that for every $I_j \in \mathcal{H}^{s+1}$, $\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{z}_j^{s+1}) = |I_j|$, and for every $I_j \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}$, $\mathrm{ord}^{s+1}_{a^{s+1}}(z_j^{s+1}) > s+1$.*

The proof of Claim 8 is based on the following result due to Bellaïche [1, Lemma 4.12].

Lemma 3.7. *Let $\{X_1, \ldots, X_m\}$ be a family vector fields defined on Ω. Consider $\{W_1, \ldots, W_n\}$ a basis adapted to the flag $L^1(x_a) \subset \cdots \subset L^r(x_a) = T_{x_a}\Omega$ at $x_a \in \Omega$. A function f is of order $> s$ at x_a is and only if*

$$(W_1^{\alpha_1} \ldots W_n^{\alpha_n} f)(x_a) = 0,$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $w(\alpha) \leq s$.

Proof of Claim 8. Claim 7 guarantees that $\{\xi_{I_k}^{s+1}\}_{I_k \in \mathcal{H}^{s+1}}$ is a basis adapted to the flag

$$L^1(a^{s+1}) \subset \cdots \subset L^{s+1}(a^{s+1}).$$

Complete $\{\xi_{I_k}^{s+1}\}_{I_k \in \mathcal{H}^{s+1}}$ with other brackets to get a basis adapted to the flag

$$L^1(a^{s+1}) \subset \cdots L^{s+1}(a^{s+1}) \subset \cdots \subset L^r(a^{s+1}).$$

For $I_j \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}$, (s+1)-4-(b) ensures that

$$((\xi_{I_1}^{s+1})^{\beta_1} \cdots (\xi_{I_{\widetilde{n}_{s+1}}}^{s+1})^{\beta_{\widetilde{n}_{s+1}}} \cdot \tilde{z}_j^{s+1})(a^{s+1}) = 0,$$

for all $\beta = (\beta_1, \ldots, \beta_{\widetilde{n}_{s+1}})$ such that $w(\beta) \leq s+1$. By Lemma 3.7, one has

$$\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{z}_j^{s+1}) > s+1, \quad \text{for } I_j \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}.$$

For $I_j \in \mathcal{H}^{s+1}$, since (s+1)-4-(a) implies that

$$((\xi_{I_1}^{s+1})^{\beta_1} \cdots (\xi_{I_{j-1}}^{s+1})^{\beta_{j-1}} \cdot \tilde{z}_j^{s+1})(a^{s+1}) = 0,$$

for all $\beta = (\beta_1, \ldots, \beta_{j-1})$ such that $w(\beta) \leq |I_j| - 1$ and by Lemma 3.7, one has

$$\mathrm{ord}_{a^{s+1}}^{s+1}(\tilde{z}_j^{s+1}) > |I_j| - 1, \quad \text{for } I_j \in \mathcal{H}^{s+1}.$$

By construction, one already has that $\mathrm{ord}_{a^{s+1}}^{s+1}(\tilde{z}_j^{s+1}) \leq w_j = |I_j|$. Therefore, one gets

$$\mathrm{ord}_{a^{s+1}}^{s+1}(\tilde{z}_j^{s+1}) = |I_j|, \quad \text{for } I_j \in \mathcal{H}^{s+1}.$$

Claim 8 is now proved. Note that this proof only involves the family $\{\xi_I^{s+1}\}_{I \in \mathcal{H}^{s+1}}$.

□

Claim 9. *The change of coordinates* $(\Phi_j^{s+1})_{j=1,\ldots,\widetilde{n}_{s+1}}$ *exists, i.e., Property* (A4) *holds true.*

Proof of Claim 9. Note first that after performing Steps (s+1)-4-(a) and (s+1)-4-(b), one obtains a new system of coordinates $\tilde{z}^{s+1}$. Then, one can write ξ_i^{s+1} in coordinates $\tilde{z}^{s+1}$ as follows

$$\xi_i^{s+1}(\tilde{z}^{s+1}) = \partial_{\tilde{z}_i^{s+1}} + \sum_{\substack{I_j \in \mathcal{H}^{s+1} \\ |I_j| \geq 2, \phi(j) = i}} (\tilde{P}_{i,j} + \tilde{R}_{i,j}) \partial_{\tilde{z}_j^{s+1}} + \sum_{I_\ell \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}} \tilde{Q}_{i,\ell} \partial_{\tilde{z}_\ell^{s+1}},$$

where $\mathrm{ord}_{a^{s+1}}^{s+1}(\tilde{P}_{i,j}) = w_j - 1$, $\mathrm{ord}_{a^{s+1}}^{s+1}(\tilde{R}_{i,j}) \geq w_j$, and $\mathrm{ord}_{a^{s+1}}^{s+1}(\tilde{Q}_{i,\ell}) > s$. Since $\mathrm{ord}_{a^{s+1}}^{s+1}(\tilde{z}_j^{s+1}) = w_j$ for $I_j \in \mathcal{H}^{s+1}$, and $\mathrm{ord}_{a^{s+1}}^{s+1}(\tilde{z}_j^{s+1}) > s+1$ for $I_j \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}$, the polynomials $\tilde{P}_{i,j}$ are homogeneous of weighted degree equal to $w_j - 1$, thus contain only variables of weight not greater than $w_j - 1$.

Let us now show that there exists a change of coordinates Φ^{s+1} which transforms coordinates $\tilde{z}^{s+1}$ into new coordinates z^{s+1} such that $\mathrm{ord}_{a^{s+1}}^{s+1}(z_j^{s+1}) = w_j$ for $I_j \in \mathcal{H}^{s+1}$, $\mathrm{ord}_{a^{s+1}}^{s+1}(z_j^{s+1}) > s+1$ for $I_j \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}$, and in the new coordinates, the $\widetilde{n}_{s+1}$ first components of ξ_i^{s+1} are in the form

$$\xi_{i,j}^{s+1}(z^{s+1}) = P_{i,j}(z_1^{s+1}, \ldots, z_{j-1}^{s+1}) + R_{i,j}(z^{s+1}),$$

where $\xi_{i,j}^{s+1}$ denotes the j^{th} component of ξ_i^{s+1} in coordinates z^{s+1}, one has $\mathrm{ord}_{a^{s+1}}^{s+1}(P_{i,j}) = w_j - 1$, and $\mathrm{ord}_{a^{s+1}}^{s+1}(R_{i,j}) \geq w_j$. Note that, once one has $\mathrm{ord}_{a^{s+1}}^{s+1}(z_j^{s+1}) = w_j$ for $I_j \in \mathcal{H}^{s+1}$ and $\mathrm{ord}_{a^{s+1}}^{s+1}(z_j^{s+1}) > s+1$ for $I_j \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}$, the order of $P_{i,j}$ will be equal to its weighted degree, and thus automatically equal to $w_j - 1$ by construction of these polynomials.

Consider now $\check{\xi}_i^{s+1}$ defined in coordinates $\tilde{z}^{s+1}$ by

$$\check{\xi}_i^{s+1}(\tilde{z}^{s+1}) = \partial_{\tilde{z}_i^{s+1}} + \sum_{\substack{I_j \in \mathcal{H}^{s+1} \\ \phi(j) = i}} \tilde{P}_{i,j} \partial_{\tilde{z}_j^{s+1}}.$$

Recall that, by construction, the vector fields $\{\check{\xi}_i\}_{i=1,\ldots,m}$ generate a free nilpotent Lie algebra of step $s+1$. Moreover, in the canonical coordinates of the second kind $(z_1^{s+1}, \ldots, z_{\widetilde{n}_{s+1}}^{s+1})$ associated with $\{\check{\xi}_{I_k}^{s+1}\}_{I_k \in \mathcal{H}^{s+1}}$, the vector fields $\check{\xi}_i^{s+1}$ are in the canonical form, i.e.

$$\check{\xi}_i^{s+1}(z^{s+1}) = \partial_{\tilde{z}_i^{s+1}} + \sum_{\substack{I_j \in \mathcal{H}^{s+1} \\ \phi(j) = i}} P_{i,j} \partial_{z_j^{s+1}}.$$

By definition of a system of coordinates, there exist $\widetilde{n}_{s+1}$ smooth functions $(\Phi_1^{s+1}, \ldots, \Phi_{\widetilde{n}_{s+1}}^{s+1})$ such that for $j = 1, \ldots, \widetilde{n}_{s+1}$, one has

$$z_j^{s+1} = \Phi_j^{s+1}(\tilde{z}_1^{s+1}, \ldots, \tilde{z}_{\widetilde{n}_{s+1}}^{s+1}).$$

Claim 9 is now proved.

□

Remark 3.5. The change of coordinates $(\Phi_j^{s+1})_{j=1,\ldots,\widetilde{n}_{s+1}}$ is computed by identification. Indeed, since $\mathrm{ord}_{a^{s+1}}^{s+1}(z_j^{s+1}) = w_j$, and the nonholonomic order does not depend on any system of coordinates, then Φ_j^{s+1} is a function of order w_j at a^{s+1}, i.e., the Taylor expansion of Φ_j^{s+1} at a^{s+1} contains only monomials of weighted degree equal to w_j, which are finite. Therefore, the function Φ_j^{s+1} is necessarily in the following form

$$\Phi_j^{s+1}(\tilde{z}^{s+1}) = \sum_{w(\alpha)=w_j} \varphi_j^\alpha (\tilde{z}_1^{s+1})^{\alpha_1} \ldots (\tilde{z}_{\widetilde{n}_{s+1}}^{s+1})^{\alpha_{\widetilde{n}_{s+1}}}, \quad \text{where } \varphi_j^\alpha \text{ are real numbers.} \tag{35}$$

Eq. (35) is a finite sum and therefore the scalar coefficients (φ_j^α) can be obtained by identification. Claim 9 guarantees that such a set of real numbers (φ_j^α) exists. Note also that, due to the constraint on the weight, Eq. (35) only involves variables $\tilde{z}_k^{s+1}$ of weight less than w_j, implying that the change of coordinates $(\Phi_j^{s+1})_{j=1,\ldots,\widetilde{n}_{s+1}}$ is naturally triangular.

Remark 3.6. Let us now illustrate Remark 3.5 with a simple example. Consider here a nilpotent system of step 2 generated by two vector fields (ξ_1, ξ_2). We have $\xi_{I_1} = \xi_1$, $\xi_{I_2} = \xi_2$ and $\xi_{I_3} = [\xi_1, \xi_2]$. In coordinates $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$, ξ_1 and ξ_2 are necessarily in the form $\xi_1 = (1, 0, \alpha_1\tilde{z}_1 + \alpha_2\tilde{z}_2)$, and $\xi_2 = (0, 1, \beta_1\tilde{z}_1 + \beta_2\tilde{z}_2)$, where α_1, α_2, β_1 and β_2 are real numbers verifying $\beta_1 - \alpha_2 = 1$. As mentioned in Remark 3.5, in the change of coordinates (Φ_1, Φ_2, Φ_3), every Φ_j is a homogeneous polynomial of weighted degree equal to w_j. Set

$$z = (\Phi_1(\tilde{z}), \Phi_2(\tilde{z}), \Phi_3(\tilde{z})) = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 + a\tilde{z}_1\tilde{z}_2 + b\tilde{z}_1^2 + c\tilde{z}_2^2),$$

with a, b, and c to be determined. One imposes th at $\xi_2(z) = (0, 1, z_1)$. By computation, one gets

$$\begin{aligned}(\alpha_1 + 2b)\tilde{z}_1 + (\alpha_2 + a)\tilde{z}_2 &= 0,\\ (\beta_1 + a)\tilde{z}_1 + (\beta_2 + 2c)\tilde{z}_2 &= z_1 = \tilde{z}_1.\end{aligned}$$

By identification, one gets $a = -\alpha_2$, $b = -\frac{\alpha_1}{2}$, $c = -\frac{\beta_2}{2}$, and in that case, $\beta_1 + a = \beta_1 - \alpha_2 = 1$ is automatically verified. Then, the triangular change of coordinates

$$(z_1, z_2, z_3) = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 - \alpha_2\tilde{z}_1\tilde{z}_2 - \frac{\alpha_1}{2}\tilde{z}_1^2 - \frac{\beta_2}{2}\tilde{z}^2)$$

puts ξ_1 and ξ_2 into the canonical form.

Claim 10. *Property* (A5) *holds true at step* $s + 1$.

Proof of Claim 10. The proof goes by induction on the length of $I_k \in \mathcal{K}^{s+1}$. It is similar to the one of Claim 6.

For $|I_k| = 1$, one has

$$\xi_i^{s+1} = \sum_{\substack{I_j \in \mathcal{H}^{s+1} \\ \phi(j)=i}} (P_{i,j} + R_{i,j}) \partial_{z_j^{s+1}} + \sum_{I_\ell \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}} Q_{i,\ell} \partial_{z_\ell^{s+1}},$$

with $\mathrm{ord}_{a^{s+1}}^{s+1}(P_{i,j}) = |I_j| - 1$, $\mathrm{ord}_{a^{s+1}}^{s+1}(R_{i,j}) > |I_j| - 1$, and $\mathrm{ord}_{a^{s+1}}^{s+1}(Q_{i,\ell}) > s$. Claim 10 holds true for $|I_k| = 1$.
Assume that Claim 10 holds for brackets of length less than s_1. We show that it still holds true for brackets of length $s_1 + 1$. Consider $I_k \in \mathcal{K}^{s+1}$ with $|I_k| = s_1 + 1$.

$$\begin{aligned}
& \xi_{I_k}^{s+1} = [\xi_{I_{k_1}}^{s+1}, \xi_{I_{k_2}}^{s+1}] \\
= & [\sum_{I_j \in \mathcal{H}^{s+1}} (P_j^{k_1} + R_j^{k_1}) \partial_{z_j^{s+1}} + \sum_{I_\ell \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}} Q_\ell^{k_1} \partial_{z_\ell^{s+1}}, \\
& \sum_{I_j \in \mathcal{H}^{s+1}} (P_j^{k_2} + R_j^{k_2}) \partial_{z_j^{s+1}} + \sum_{I_\ell \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}} Q_\ell^{k_2} \partial_{z_\ell^{s+1}}] \\
= & \sum_{I_j \in \mathcal{H}^{s+1}} [\sum_{I_i \in \mathcal{H}^{s+1}} P_i^{k_1} \partial_{z_i^{s+1}} P_j^{k_2} - P_i^{k_2} \partial_{z_i^{s+1}} P_j^{k_1}] \partial_{z_j^{s+1}} \\
+ & \sum_{I_j \in \mathcal{H}^{s+1}} [\sum_{I_i \in \mathcal{H}^{s+1}} \{R_i^{k_1} \partial_{z_i^{s+1}} (P_j^{k_2} + R_j^{k_2}) - R_i^{k_2} \partial_{z_i^{s+1}} (P_j^{k_1} + R_j^{k_1})\} + \{P_i^{k_1} \partial_{z_i^{s+1}} R_j^{k_2} - P_i^{k_2} \partial_{z_i^{s+1}} R_j^{k_1}\} \\
& + \sum_{I_\ell \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}} Q_\ell^{k_1} \partial_{z_\ell^{s+1}} (P_j^{k_2} + R_j^{k_2}) - Q_\ell^{k_2} \partial_{z_\ell^{s+1}} (P_j^{k_1} + R_j^{k_1})] \partial_{z_j^{s+1}} \\
+ & \sum_{I_j \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}} [\sum_{I_i \in \mathcal{H}^{s+1}} (P_i^{k_1} + R_i^{k_1}) \partial_{z_i^{s+1}} Q_j^{k_2} - (P_i^{k_2} + R_i^{k_2}) \partial_{z_i^{s+1}} Q_j^{k_1} \\
& + \sum_{I_\ell \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}} Q_\ell^{k_1} \partial_{z_\ell^{s+1}} Q_j^{k_2} - Q_\ell^{k_2} \partial_{z_\ell^{s+1}} Q_j^{k_1}] \partial_{z_j^{s+1}}.
\end{aligned}$$

By the inductive hypothesis, one gets that

- since $\mathrm{ord}_{a^{s+1}}^{s+1}(R_i^{k_1}) > |I_i| - |I_{k_1}|$, and $\mathrm{ord}_{a^{s+1}}^{s+1}(\partial_{z_i^{s+1}}(P_j^{k_2} + R_j^{k_2})) \geq |I_j| - |I_{k_2}| - |I_i|$, then $\mathrm{ord}_{a^{s+1}}^{s+1} R_i^{k_1} \partial_{z_i^{s+1}}(P_j^{k_2} + R_j^{k_2}) > |I_j| - |I_k|$. By a similar argument, $\mathrm{ord}_{a^{s+1}}^{s+1} R_i^{k_2} \partial_{z_i^{s+1}}(P_j^{k_1} + R_j^{k_1}) > |I_j| - |I_k|$.
  Therefore, one gets
  $$\mathrm{ord}_{a^{s+1}}^{s+1}(R_i^{k_1} \partial_{z_i^{s+1}}(P_j^{k_2} + R_j^{k_2}) - R_i^{k_2} \partial_{z_i^{s+1}}(P_j^{k_1} + R_j^{k_1})) > |I_j| - |I_k|.$$

- Since $\mathrm{ord}_{a^{s+1}}^{s+1}(P_i^{k_1}) = |I_i| - |I_{k_1}|$, and $\mathrm{ord}_{a^{s+1}}^{s+1}(\partial_{z_i^{s+1}} R_j^{k_2}) > |I_j| - |I_{k_2}| - |I_i|$, then $\mathrm{ord}_{a^{s+1}}^{s+1}(P_i^{k_1} \partial_{z_i^{s+1}} R_j^{k_2}) > |I_j| - |I_k|$. By a similar argument, $\mathrm{ord}_{a^{s+1}}^{s+1}(P_i^{k_2} \partial_{z_i^{s+1}} R_j^{k_1}) > |I_j| - |I_k|$.
  Therefore, one gets
  $$\mathrm{ord}_{a^{s+1}}^{s+1}(P_i^{k_1} \partial_{z_i^{s+1}} R_j^{k_2} - P_i^{k_2} \partial_{z_i^{s+1}} R_j^{k_1}) > |I_j| - |I_k|.$$

- Since $\mathrm{ord}^{s+1}_{a^{s+1}}(\partial_{z^{s+1}_\ell}(P^{k_2}_j+R^{k_2}_j)) > |I_j|-|I_{k_2}|-(s+1)$, and $\mathrm{ord}^{s+1}_{a^{s+1}}(Q^{k_1}_\ell) > s+1-|I_{k_1}|$, then $\mathrm{ord}^{s+1}_{a^{s+1}}(Q^{k_1}_\ell\partial_{z^{s+1}_\ell}(P^{k_2}_j+R^{k_2}_j)) > |I_j|-|I_k|$. By a similar argument, $\mathrm{ord}^{s+1}_{a^{s+1}}Q^{k_2}_\ell\partial_{z^{s+1}_\ell}(P^{k_1}_j+R^{k_1}_j > |I_j|-|I_k|$.

  Therefore, one gets

$$\mathrm{ord}^{s+1}_{a^{s+1}}(Q^{k_1}_\ell\partial_{z^{s+1}_\ell}(P^{k_2}_j+R^{k_2}_j)-Q^{k_2}_\ell\partial_{z^{s+1}_\ell}(P^{k_1}_j+R^{k_1}_j)) > |I_j|-|I_k|.$$

- Since $\mathrm{ord}^{s+1}_{a^{s+1}}(P^{k_1}_i+R^{k_1}_i) = |I_i|-|I_{k_1}|$, and $\mathrm{ord}^{s+1}_{a^{s+1}}(\partial_{z^{s+1}_i}Q^{k_2}_j) > s+1-|I_{k_2}|-|I_i|$, then $\mathrm{ord}^{s+1}_{a^{s+1}}((P^{k_1}_i+R^{k_1}_i)\partial_{z^{s+1}_i}Q^{k_2}_j) > s+1-|I_k|$. By a similar argument, $\mathrm{ord}^{s+1}_{a^{s+1}}((P^{k_2}_i+R^{k_2}_i)\partial_{z^{s+1}_i}Q^{k_1}_j) > s+1-|I_k|$.

  Therefore, one gets

$$\mathrm{ord}^{s+1}_{a^{s+1}}((P^{k_1}_i+R^{k_1}_i)\partial_{z^{s+1}_i}Q^{k_2}_j-(P^{k_2}_i+R^{k_2}_i)\partial_{z^{s+1}_i}Q^{k_1}_j) > s+1-|I_k|.$$

- Since $\partial_{z^{s+1}_\ell}Q^{k_2}_j$ is a function, one knows by definition that $\mathrm{ord}^{s+1}_{a^{s+1}}(\partial_{z^{s+1}_\ell}Q^{k_2}_j) \geq 0$. As $\mathrm{ord}^{s+1}_{a^{s+1}}(Q^{k_1}_\ell) > s+1-|I_{k_1}|$, one has

$$\mathrm{ord}^{s+1}_{a^{s+1}}(Q^{k_1}_\ell\partial_{z^{s+1}_\ell}Q^{k_2}_j) > s+1-|I_{k_1}| = s+1-(|I_k|-|I_{k_2}|) > s+1-|I_k|.$$

  By a similar argument, $\mathrm{ord}^{s+1}_{a^{s+1}}(Q^{k_2}_\ell\partial_{z^{s+1}_\ell}Q^{k_1}_j) > s+1-|I_k|$.

  Therefore, one gets

$$\mathrm{ord}^{s+1}_{a^{s+1}}(Q^{k_1}_\ell\partial_{z^{s+1}_\ell}Q^{k_2}_j-Q^{k_2}_\ell\partial_{z^{s+1}_\ell}Q^{k_1}_j) > s+1-|I_k|.$$

Summing up the above terms, one gets, for $I_k \in \mathcal{K}^{s+1}$ of length s_1+1, that the bracket $\xi^{s+1}_{I_k}$ can be written in the form

$$\xi^{s+1}_{I_k} = \sum_{I_j\in\mathcal{H}^{s+1}}(P^k_j+R^k_j)\partial_{z^{s+1}_j} + \sum_{I_j\in\mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1}}Q^k_j\partial_{z^{s+1}_j},$$

with $\mathrm{ord}^{s+1}_{a^{s+1}}(P^k_j) = |I_j|-|I_k|$, $\mathrm{ord}^{s+1}_{a^{s+1}}(R^k_j) > |I_j|-|I_k|$, and $\mathrm{ord}^{s+1}_{a^{s+1}}(Q^k_\ell) > s+1-|I_k|$. Claim 10 is now proved.

□

Therefore, Properties (A1)-(A5) still hold true at step $s+1$ in the desingularization algorithm. The induction step is established, which concludes the proof of Proposition 3.6.

□

4 Global Steering Method for Regular Systems

Assume, in this section, that the family of vectors fields $X = \{X_1,\ldots,X_m\}$ is free to step r (cf Definition 3.4). Recall that, in that case, every point $x \in \Omega$ is regular and the growth vector is constant on Ω. We present in Subsection 4.1 an algebraic construction of privileged coordinates converting the nonholonomic first order approximation of X into the canonical form. For regular systems, this construction also provides a continuously varying system of privileged coordinates. We then propose in Subsection 4.2 a global motion planning algorithm for regular systems.

4.1 Construction of the approximate system

Let x_a be a point in Ω.

Construction of a Nonholonomic First Order Approximation at x_a

Step (1) Choose an adapted frame $W_1, \dots, W_n$ at x_a;

Step (2) choose a system of coordinates $y = (y_1, \dots, y_n)$ centered at x_a such that $\partial_{y_i}|_{x_a} = W_i(x_a)$;

Step (3) build the system of privileged coordinates $\widetilde{z} = (\widetilde{z}_1, \dots, \widetilde{z}_n)$ by the following iterative formula: for $j = 1, \dots, n$,

$$\widetilde{z}_j := y_j + \sum_{k=2}^{w_j - 1} h_k(y_1, \dots, y_{j-1}), \tag{36}$$

where, for $k = 2, \dots, w_j - 1$,

$$h_k(y_1, \dots, y_{j-1}) = - \sum_{\substack{|\alpha|=k \\ w(\alpha)<w_j}} W_1^{\alpha_1} \dots W_{j-1}^{\alpha_{j-1}} \cdot (y_j + \sum_{q=2}^{k-1} h_q(y))(x_a) \frac{y_1^{\alpha_1}}{\alpha_1!} \cdots \frac{y_{j-1}^{\alpha_{j-1}}}{\alpha_{j-1}!},$$

where $|\alpha| := \alpha_1 + \cdots \alpha_n$;

Step (4) express the dynamics of the original system in the privileged coordinates $\widetilde{z}$:

$$\dot{\widetilde{z}} = \sum_{i=1}^{m} X_i(\widetilde{z}) u_i,$$

where by abuse of notation, we use $X_i(\widetilde{z})$ to denote $\widetilde{z}_* X_i(x)$;

Step (5) for $j = 1, \dots, m$, compute the Taylor expansion of the vector fields $X_i(\widetilde{z})$ at 0, and express every vector field as a sum of homogeneous vector fields with respect to the weighted degree:

$$X_i(\widetilde{z}) = X_i^{(-1)}(\widetilde{z}) + X_i^{(0)}(\widetilde{z}) + \cdots,$$

where we use $X_i^{(k)}(\widetilde{z})$ to denote the sum of all the terms of weighted degree equal to k; let $\widehat{X}_i(\widetilde{z}) := X_i^{(-1)}(\widetilde{z})$;

Step (7) construct a new system of privileged coordinates $z := (z_1, \dots, z_n)$ by setting, for $j = 1, \dots, n$, $z_j := \Phi_j(\widetilde{z}_1, \dots, \widetilde{z}_{j-1})$, where Φ_j is a homogeneous polynomial of weighted degree equal to w_j such that $\widehat{X}(z) = \{z_* \widehat{X}_1(\widetilde{z}), \dots, z_* \widehat{X}_m(\widetilde{z})\}$ (the approximate system in the coordinates z) is the canonical form.

Remark 4.1. For Step (1), on can use for example elements in the *P. Hall family* generated by $\{X_1, \dots, X_m\}$. The reader is referred to Subsection 3.1 for more details about the *P. Hall family*. A system of coordinates y considered in Step 2 is called *linearly adapted coordinates*. It can be obtained by an affine change of coordinates from the original system of coordinates x.

Remark 4.2. Steps (1)-(3) construct a system of privileged coordinates $\widetilde{z}$. The proof that z is a system of privileged coordinates is essentially based on Lemma 3.7. Roughly speaking, the idea to obtain z_j from y_j goes as follows: for every $\alpha = (\alpha_1, \dots, \alpha_n)$ with $w(\alpha) < w_j$ (so $\alpha_j = \cdots = \alpha_n = 0$), compute $W_1^{\alpha_1} \cdots W_{j-1}^{\alpha_{j-1}} \cdot y_j(x_a)$. If it is not equal to zero, then replace y_j by

$$y_j - (W_1^{\alpha_1} \cdots W_{j-1}^{\alpha_{j-1}} \cdot y_j(x_a)) \frac{y_1^{\alpha_1}}{\alpha_1!} \cdots \frac{y_{j-1}^{\alpha_{j-1}}}{\alpha_{j-1}!}.$$

With that new value of y_j, one gets $W_1^{\alpha_1} \cdots W_{j-1}^{\alpha_{j-1}} \cdot y_j(x_a) = 0$. Therefore, by Lemma 3.7, one has $\operatorname{ord}_{x_a}(z_j) \geq w_j$ for $j = 1, \dots, n$. On the other hand, since Step (3) of the construction does not modify the linear part, the system of coordinates $\widetilde{z}$ remains adapted. By Remark 2.6, one also has $\operatorname{ord}_{x_a}(\widetilde{z}_j) \leq w_j$. Therefore, $\operatorname{ord}_{x_a}(\widetilde{z}_j) = w_j$.

Remark 4.3. The existence of Φ_j involving in Step (4) is guaranteed by an adaptation of Claim 9. See also Remarks 3.5 and 3.6.

Remark 4.4. We will propose in Section 5 an effective and exact method for steering general nilpotent systems given in the canonical form.

It results from [1] that, for *regular systems*, the mapping $\Phi : (x_a, x) \to z$ is a continuously varying system of privileged coordinates on Ω. Note also that the coordinates z are obtained from y by expressions of the form

$$\begin{aligned} z_1 &= y_1 \\ z_2 &= y_2 + \operatorname{pol}_2(y_1) \\ &\vdots \\ z_n &= y_n + \operatorname{pol}_n(y_1, \dots, y_{n-1}), \end{aligned}$$

where, for $j = 1, \dots, n$, the function $\operatorname{pol}_j(\cdot)$ is a polynomial which does not contain constant nor linear terms. Due to the triangular form of this change of coordinates, the inverse change of coordinates from z to y bears exactly the same form. Therefore, the mapping $z = \Phi(x_a, \cdot)$ is defined on the whole $\mathbb{R}^n$, i.e., Φ has an infinite injectivity radius. We also note that the continuity of the mapping $\mathcal{A} : (x_a, x) \to \{\widehat{X}_1^{x_a}(z), \dots, \widehat{X}_m^{x_a}(z)\}$ results from the one of the mapping $\Phi : (x_a, x) \to z$. Therefore, for *regular* systems, the construction provides a continuous approximation of X on Ω.

4.2 Global approximate steering algorithm for regular systems

In this subsection, we devise an algorithm to steer System (4) from any $x_0 \in \Omega$ to the origin denoted by 0. That algorithm, as described in Fig. 1. does not require any a priori knowledge on the critical distance ε_K. Note that this algorithm bears similarities with *trust-region* methods (see [2] for more details).

The parameterized path $t \mapsto \delta_{0,t}(x)$ is defined by

$$\delta_{0,t}(x) := (t^{w_1} z_1(x), \dots, t^{w_n} z_n(x)), \quad \text{for } x \in \Omega,$$

where $z := \Phi(0, \cdot)$, and $(w_1, \dots, w_n)$ are the weights at 0. Note that $\delta_{0,t}$ is the (weighted) *dilatation* in privileged coordinates at the origin with parameter t.

The function Subgoal is the following.

Subgoal$(\overline{x}, \eta_i, j)$

1. $t_j := \max(0, 1 - \frac{j\eta_i}{\|z(\overline{x})\|_0})$;
2. $x^d := \delta_{0,t_j}(\overline{x})$

The formula for generating t_j guarantees that $\|z(x^d) - z(x^d_{j-1})\|_0 \leq \eta_i$ and that $x^d = 0$ for j large enough.

Global$(x_0, 0)$

Step 1.	$i := 0$; $j := 1$;	
Step 2.	$x_i := x_0$; $\overline{x} := x_0$;	
Step 3.	$\eta_i := \|z(x_0)\|_0$;	*initial choice of the maximum step size;*
Step 4.	while $\|z(x_i)\|_0 > e$	*while the pseudonorm at 0 of the state is above a given tolerance e...;*
Step 5.	$x^d :=$ Subgoal$(\overline{x}, \eta_i, j)$;	*choose the subgoal x^d at a distance η_i from x^d_{j-1};*
Step 6.	$x :=$ AppSteer(x_{i-1}, x^d);	*steer the system from x_{i-1} using an approximate steering control with target x^d;*
Step 7.	if $\|z(x)\|_{x^d} > \frac{1}{2}\|z(x_{i-1})\|_{x^d}$	*if the system is not approaching the subgoal,*
Step 8.	then $\eta_i := \frac{\eta_i}{2}$;	*reduce the maximum step size;*
	$\overline{x} := x_{i-1}$; $j := 1$;	*and change the path $\delta_{0,t}(\overline{x})$;*
Step 9.	else $x_i := x$; $x^d_i := x^d$;	
	$i := i + 1$; $j := j + 1$;	

Figure 1: The approximate steering algorithm.

The global convergence of the approximate steering algorithm (Fig.1) is established in the following result. For sake of simplicity, we assume to work on a compact set $K \subset \Omega$. Alternatively, this condition can be guaranteed by adding a step in the algorithm as indicated at the end of Section 4.2.

Theorem 4.1. *Assume that the sequences $(x_i)_{i\geq 0}$ and $(x^d_i)_{i\geq 0}$ by the algorithm* Global$(x_0, 0)$ *both belong to a compact set $K \subset \Omega$. Then the algorithm terminates in a finite number of steps for any choice of the tolerance $e > 0$.*

Proof of Theorem 4.1. Note first that, if the conditional statement of Step 7 is not true for every i greater than some i_0, then $x^d_i = 0$ after a finite number of iterations. In this case, the error $\|z(x_i)\|_0$ is reduced at each iteration and the algorithm stops when it becomes smaller than the given tolerance e. This happens in particular if $d(x_{i-1}, x^d) < \varepsilon_K$ for all i greater than i_0 because condition (14) is verified. Another preliminary remark is that, due to the continuity of the control distance and of the function $\|\cdot\|_0$, there exists $\overline{\eta} > 0$ such that, for every pair $(x, y) \in K \times K$, one has

$$\|z(x) - z(y)\|_0 < \overline{\eta} \implies d(x, y) < \frac{\varepsilon_K}{2}. \tag{37}$$

In the following, we will prove by induction that if, for some i_0, one has $\eta_{i_0} < \overline{\eta}$, then, for all $i > i_0$, one has

$$d(x_{i-1}, x_i^d) < (1/2 + \cdots + (1/2)^{i-i_0})\varepsilon_K < \varepsilon_K.$$

We assume without loss of generality that $i_0 = 0$ and $\overline{x} = x_0$. For $i = 1$, by construction, $x^d = \text{Subgoal}(x_0, \eta_0, 1)$ and

$$\|z(x_0) - z(x^d)\|_0 \leq \eta_0 < \overline{\eta}.$$

In view of (37), one then has $d(x_0, x^d) < \varepsilon_K/2$, and so $x_1^d = x^d$ by (14). Therefore $d(x_0, x_1^d) < \varepsilon_K/2$.

Assume now that for $i > 1$ one has:

$$d(x_{i-2}, x_{i-1}^d) < (1/2 + \cdots + (1/2)^{i-1})\varepsilon_K. \tag{38}$$

Let $x^d = \text{Subgoal}(\overline{x}, \eta_i, j)$. One can write:

$$d(x_{i-1}, x^d) \leq d(x_{i-1}, x_{i-1}^d) + d(x_{i-1}^d, x^d).$$

By construction, it is

$$\|z(x_{i-1}^d) - z(x^d)\|_0 \leq \eta_i < \overline{\eta},$$

which implies $d(x_{i-1}^d, x^d) < \varepsilon_K/2$. The induction hypothesis (38) implies that

$$d(x_{i-1}, x_{i-1}^d) \leq \frac{1}{2} d(x_{i-2}, x_{i-1}^d).$$

Finally, one gets

$$\begin{aligned} d(x_{i-1}, x^d) &\leq \frac{1}{2} d(x_{i-2}, x_{i-1}^d) + d(x_{i-1}^d, x^d) \\ &\leq (1/2 + \cdots + (1/2)^i)\varepsilon_K. \end{aligned}$$

In view of (14), the conditional statement of Step 7 is not true, and so $x_i^d = x^d$.

Notice that, for some i, $\eta_i \geq \overline{\eta}$, the conditional statement of Step 7 could be false. In this case, η_i is decreased as in Step 8. The updating law of η_i guarantees that after a finite number of iterations of Step 8, there holds $\eta_i < \overline{\eta}$. This ends the proof. □

When the working space Ω is equal to the whole $\mathbb{R}^n$, the assumption that the algorithm stays in a compact set can be removed. This requires a simple modification of Step 9 of the algorithm.

We choose a real number R close to one, precisely $(\frac{1}{2})^{1/(r+1)^2} < R < 1$, where r is the maximum value of the degree of nonholonomy of System (4). For every non-negative integer k, we set $R_k = 1 + R + \cdots + R^k$. The algorithm is modified as follows. Introduce first a new

Step 9'. else

9'.1. if $\|z(x)\|_0 \geq R_{k+1}\|z(x_0)\|_0 \quad \eta_i := \frac{\eta_i}{2}$;

9'.2. if $R_k\|z(x_0)\|_0 \leq \|z(x)\|_0 < R_{k+1}\|z(x_0)\|_0$

$x_i := x$; $x_i^d := x^d$; $i := i+1$; $j := j+1$;

$\eta_i := \frac{\eta_{i-1}}{2}$; $k := k+1$;

9'.3. if $\|z(x)\|_0 \leq R_k\|z(x_0)\|_0$

$x_i := x$; $x_i^d := x^d$; $i := i+1$; $j := j+1$;

Figure 2: Step 9'

variable k, and add the initialization $k := 0$ to Step 1. Replace then Step 9 by Step 9' below (Figure 2).

Step 9' guarantees that the sequences $(x_i)_{i\geq 0}$ and $(x_i^d)_{i\geq 0}$ of the algorithm both belong to the compact set

$$K = \{x \in \mathbb{R}^n \ : \ \|z(x)\|_0 \leq \frac{1}{1-R}\|z(x_0)\|_0\}.$$

Moreover, at each iteration of the algorithm, the new variable k is such that

$$\|z(x_i)\|_0 \geq R_k\|z(x_0)\|_0 \quad \Rightarrow \quad \eta_i \leq \frac{\|z(x_0)\|_0}{2^k}.$$

Proposition 4.2. *The modified algorithm* Global *(with Step 9' instead of Step 9) terminates in a finite number of iterations for any choice of x_0 and of the tolerance e.*

Proof of Proposition 4.2. Notice that Step 9'.3 is identical to Step 9. It is therefore enough to show that, after a finite number of iterations, only Step 9'.3 occurs in Step 9'. Another preliminary remark is that the distance $\|\cdot\|_0$ give a rough estimate of the sub-Riemannian distance. Indeed it follows from Theorem 2.3 that, for every pair of close enough points $(x,y) \in K \times K$, one has

$$\frac{1}{C_0}\|z(x)-z(y)\|_0^{r+1} \leq d(x,y) \leq C_0\|z(x)-z(y)\|_0^{1/(r+1)}, \tag{39}$$

where C_0 is a positive constant. As a consequence, Eq. (37) holds true if $\overline{\eta} \leq (\varepsilon_K/(2C_0))^{r+1}$.

Let us choose a positive $\overline{\eta}$ smaller than $(\varepsilon_K/(2C_0))^{r+1}$. We next show that if, for some i_0, $\eta_{i_0} < \overline{\eta}$, then Steps 9'.1 and 9'.2 occur only in a finite number of iterations. Recall first that, from the proof of Theorem 4.1, one gets, for every $i > i_0$,

$$\|z(x_i^d)\|_0 \leq \|z(x_{i_0})\|_0 \quad \text{and} \quad d(x_{i-1}, x_i^d) \leq \varepsilon_K.$$

In view of (39), an obvious adaptation of the latter proof yields, for every $i > i_0$, $d(x_{i-1}, x_i^d) \leq 2C_0\eta_{i_0}^{1/(r+1)}$, and so $\|z(x_{i-1}) - z(x_i^d)\|_0 \leq (2C_0^2)^{1/(r+1)}\eta_{i_0}^{1/(r+1)^2}$. Finally one gets

$$\|z(x_i)\|_0 \leq \|z(x_{i+1}^d)\|_0 + \|z(x_i) - z(x_{i+1}^d)\|_0 \leq \|z(x_{i_0})\|_0 + (2C_0^2)^{1/(r+1)}\eta_{i_0}^{1/(r+1)^2}. \tag{40}$$

On the other hand, there exists an integer k_0 such that $\eta_{i_0} \geq \frac{\|z(x_0)\|_0}{2^{k_0}}$. This implies that $\|z(x_{i_0})\|_0 \leq R_{k_0}\|z(x_0)\|_0$. Up to reducing $\overline{\eta}$, and so increasing k_0, assume

$$(2C_0^2)^{1/(r+1)}(\frac{\|z(x_0)\|_0}{2^{k_0}})^{1/(r+1)^2} \leq R^{k_0+1}\|z(x_0)\|_0,$$

since one has chosen $R > (\frac{1}{2})^{1/(r+1)^2}$. Using (40), it holds, for every $i \geq i_0$,

$$\|z(x_i)\|_0 \leq R_{k_0}\|z(x_0)\|_0 + R^{k_0+1}\|z(x_0)\|_0 = R_{k_0+1}\|z(x_0)\|_0.$$

Therefore, Steps 9'.1 and 9'.2 can occur in at most $k_0 + 1$ iterations.

Applying again the arguments of the proof of Theorem 4.1, the conclusion follows.

□

5 Exact Steering Method for Nilpotent Systems

In this section, we give an exact steering method for nilpotent systems. In particular, this method can be applied for controlling the approximate system (32). For practical uses, we require that the control laws give rise to smooth trajectories which are not too “complex" in the sense that, during the control process, we do not want the system to stop too many times or to make a large number of maneuvers.

Several algorithms were proposed for controlling nilpotent systems. In [15], the authors make use of piecewise constant controls and obtain smooth controls by imposing some special parameterization (namely by requiring the control system to stop during the control process). In that case, the smoothness of the inputs is recovered by using a reparameterization of the time, which cannot prevent in general the occurrence of cusps or corners for the corresponding trajectories. However, smoothness of the *trajectories* is generally mandatory for robotic applications. Therefore, the method proposed in [15] is not adapted to such applications. In [16], the proposed controls are polynomial (in time), but an algebraic system must be inverted in order to access to these inputs. Moreover, the size and the degree of this algebraic system increase exponentially with respect to the dimension of state space, and there does not exist a general efficient exact method to solve it. Even the existence of solutions is a non trivial issue. Furthermore, the methods [15] and [16] both make use of exponential coordinates which are not explicit, and thus require in general numerical integrations of nonlinear differential equations. That prevents the use of these methods in an iterative scheme such as Algorithm 1. Let us also mention the path approximation method by Liu and Sussmann [18], which uses unbounded sequences of sinusoids. Even though this method bears similar theoretical aspects with our method, it is not adapted from a numerical point of view to the motion planning issue since it relies on a limit process of highly oscillating inputs.

In this section, we assume that $\{X_1, \ldots, X_m\}$ generate a free Lie algebra up to step r and they are in the canonical form in coordinates x. The components of x will be numbered by the elements of $\mathcal{H}^r$, i.e., for $I \in \mathcal{H}^r$, the component x_I corresponds to the element X_I. Recall that

$$\begin{array}{rcll} \dot{x}_i & = & u_i, & \text{if } i = 1, \ldots, m; \\ \dot{x}_I & = & \frac{1}{k!}x_{I_L}\dot{x}_{I_R}, & \text{if } X_I = \mathrm{ad}^k_{X_{I_L}} X_{I_R}, \quad \text{with } I_L, I_R \in \mathcal{H}^r. \end{array} \tag{41}$$

5.1 Steering by sinusoids

We consider input functions in the form of linear combinations of sinusoids with integer frequencies. In [19], authors used this family of inputs to control the chained-form systems.

We first note that if every component of the input $u = (u_1, \ldots, u_m)$ in Eq. (41) is a linear combination of sinusoids with integer frequencies, then the dynamics of every component in Eq. (41) is also a linear combination of sinusoids with integer frequencies which are linear combinations of frequencies involved in the input u. One may therefore expect to move some components during one $2\pi-$period without modifying others if the frequencies in u are properly chosen. Due to the triangular form of Eq. (41), it is reasonable to expect to move the components of x one after another according to the order $'' \prec ''$ induced by P. Hall basis. In that case, one must ensure that all the components already moved to their preassigned values return to the same values after each $2\pi-$period of control process while the component under consideration arrives to its preassigned position. However, all the components cannot be moved independently by using sinusoids. We introduce the following notion of *equivalence*.

Definition 5.1 (*Equivalence*)**.** Two elements X_I and X_J in a P. Hall family are said to be *equivalent* if $\Delta_i(X_I) = \Delta_i(X_J)$ for $i = 1, \ldots, m$, where we use $\Delta_i(X_I)$ to denote the number of times X_i occurs in X_I. We write $X_I \sim X_J$ if X_I and X_J are equivalent and *equivalence classes* will be denoted by

$$\mathcal{E}_X(\ell_1, \ldots, \ell_m) := \{X_I \mid \Delta_i(X_I) = l_i, \text{ for } i = 1, \ldots, m\}.$$

We say that the components x_I and x_J are *equivalent* if the corresponding brackets X_I and X_J are equivalent and *equivalent classes for components* are defined as follows,

$$\mathcal{E}_x(\ell_1, \ldots, \ell_m) := \{x_I \mid X_I \in \mathcal{E}_X(\ell_1, \ldots, \ell_m)\}.$$

Remark 5.1. We will see in the following subsections that the frequencies occurring in the dynamics of x_I only depend on the equivalence class of x_I, and not on the structure of the bracket X_I. Therefore, the equivalent components (in the sense of Definition 5.1) cannot be moved separately by using sinusoids.

Definition 5.2 (*Ordering of equivalence classes*)**.** Let $\mathcal{E}_x(\ell_1, \ldots, \ell_m)$ and $\mathcal{E}_x(\tilde{\ell}_1, \ldots, \tilde{\ell}_m)$ be two equivalence classes. $\mathcal{E}_x(\ell_1, \ldots, \ell_m)$ is said to be smaller than $\mathcal{E}_x(\tilde{\ell}_1, \ldots, \tilde{\ell}_m)$ if the smallest element (in the sense of "$\prec$") in $\mathcal{E}_x(\ell_1, \ldots, \ell_m)$ is smaller than the one in $\mathcal{E}_x(\tilde{\ell}_1, \ldots, \tilde{\ell}_m)$, and we write (by abuse of notation) $\mathcal{E}_x(\ell_1, \ldots, \ell_m) \prec \mathcal{E}_x(\tilde{\ell}_1, \ldots, \tilde{\ell}_m)$.

Let $\{\mathcal{E}_x^1, \mathcal{E}_x^2, \ldots, \mathcal{E}_x^{\widetilde{N}}\}$ be the partition of the set of the components of x induced by Definition 5.1. Assume that, for every pair $(i, j) \in \{1, \ldots, \widetilde{N}\}^2$ with $i < j$, one has $\mathcal{E}_x^i \prec \mathcal{E}_x^j$. Our control strategy consists in displacing these equivalence classes one after another according to the ordering "$\prec$" by using sinusoidal inputs. The key point is, for every $j = 1, \ldots, \widetilde{N}$, to determine how to construct an input u^j defined on $[0, 2\pi]$ such that the two following conditions are verified:

(C1) under the action of u^j, every element of $\mathcal{E}_x^j$ reaches its preassigned value at $t = 2\pi$;

(C2) under the action of u^j, for every $i < j$, every element of $\mathcal{E}_x^i$ returns at $t = 2\pi$ to its value taken at $t = 0$.

Remark 5.2. Once one knows how to construct an input u^j verifying (C1) and (C2) for every $j = 1, \dots, \widetilde{N}$, it suffices to *concatenate* them to control the complete system. Moreover, we will see that it is possible make smooth concatenations such that the inputs, as well as the corresponding trajectories, are not only piecewise smooth, but globally smooth.

5.2 Choice of frequencies

In this subsection, we fix an equivalence class $\mathcal{E}_x^j$. We choose frequencies in u^j such that Conditions (C1) and (C2) are verified. For sake of clarity, we first treat the case $m = 2$ in Paragraphs 5.2.1 and 5.2.2., and we show, in Paragraph 5.2.3, how to adapt the method to greater values of m.

5.2.1 A simple case: $m = 2$ and Card $(\mathcal{E}_x^j) = 1$

Let x_I be the only element of $\mathcal{E}_x^j$, and X_I the corresponding bracket. Let $m_1 := \Delta_1(X_I)$, and $m_2 := \Delta_2(X_I)$.

Proposition 5.1. *Consider three positive integers ω_1, ω_2, ω_3, and $\varepsilon \in \{0, 1\}$ such that*

$$\begin{cases} \omega_3 = m_1\omega_1 + (m_2 - 1)\omega_2, \\ \varepsilon = m_1 + m_2 - 1 \pmod 2, \end{cases} \tag{42}$$

and

$$\omega_2 > (m_1 + m_2)m_1. \tag{43}$$

By choosing properly ζ, the control

$$u_1 = \cos\omega_1 t,\ \ u_2 = \cos\omega_2 t + \zeta\cos(\omega_3 t - \varepsilon\frac{\pi}{2}), \tag{44}$$

steers, during $[0, 2\pi]$, the component x_I from any initial value to any preassigned final value without modifying any component x_J, with $J \prec I$. Moreover, $x_I(2\pi) - x_I(0)$ gives rise to a non zero linear function of ζ, where ζ is the coefficient in front of $\cos(\omega_3 t - \varepsilon\frac{\pi}{2})$ in Eq. (44).

The key point is to understand the frequencies occurring in the dynamics $\dot{x}_I$.

Lemma 5.2. *For $J \leq I$, the dynamics $\dot{x}_J$ is a linear combination of cosine functions of the form*

$$\cos\{(\ell_1\omega_1 + \ell_2\omega_2 + \ell_3\omega_3)t - (\ell_3\varepsilon + \ell_1 + \ell_2 + \ell_3 - 1)\frac{\pi}{2}\}, \tag{45}$$

where $\ell_1, \ell_2, \ell_3 \in \mathbb{Z}$ satisfy $|\ell_1| \leq m_1$, $|\ell_2| + |\ell_3| \leq m_2$.

In particular, the term

$$\cos[(m_1\omega_1 + (m_2 - 1)\omega_2 - \omega_3)t - (-\varepsilon + m_1 + m_2 - 1)\frac{\pi}{2}]$$

occurs in $\dot{x}_I$ with a zero coefficient depending linearly on ζ.

Proof of Lemma 5.2. The proof goes by induction on $|J|$.

- $|J| = 1$, the result is true since $\dot{x}_{I_1} = u_1$ and $\dot{x}_{I_2} = u_2$.

- *Inductive step:*
  Assume that the result holds true for all $\tilde{J}$ such that $|\tilde{J}| < s$. We show that it remains true for J such that $|J| = s$.

  By construction, we have $X_J = \mathrm{ad}^k_{X_{J_1}} X_{J_2}$ with $|J_1| < s$ and $|J_2| < s$. Then,

$$\dot{x}_J = \frac{1}{k!} x^k_{J_1} \dot{x}_{J_2}, \tag{46}$$

  $\dot{x}_{J_2}$ is given by the inductive hypothesis and x_{I_1} is obtained by integration of Eq. (45). By using product formulas for cosine function, the result still holds true for J of length s. This ends the proof of Lemma 5.2.

□

Proof of Proposition 5.1. First note that integrating between 0 and 2π a function of the form $\cos(\gamma t + \bar{\gamma}\frac{\pi}{2})$ with $(\gamma, \bar{\gamma}) \in \mathbb{N}^2$ almost always gives 0 except for $(\gamma, \bar{\gamma}) = (0, 0)$. Therefore, in order to obtain a non trivial contribution for x_I, $\dot{x}_I$ must contains some cosine functions verifying the following condition

$$\begin{cases} \ell_1\omega_1 + \ell_2\omega_2 + \ell_3\omega_3 = 0, \\ \ell_3\varepsilon + \ell_1 + m_2 + \ell_3 - 1 \equiv 0 \pmod 2, \end{cases} \tag{47}$$

and this condition must not be verified by $J \prec I$ in order to avoid a change in the component x_J.
Under conditions (42) and (43), we claim that

(1) $(m_1, m_2 - 1, -1, \varepsilon)$ is the only 4-tuple verifying (47) for x_I, and $x_I(2\pi) - x_I(0)$ is a non zero *linear* function of ζ;

(2) Eq. (47) is never satisfied for x_J with $J < I$.

Indeed, consider $(\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3$ verifying $|\ell_1| \leq m_1$, $|\ell_2| + |\ell_3| \leq m_2$. One has

$$\begin{aligned} & \ell_1\omega_1 + \ell_2\omega_2 + \ell_3\omega_3 \\ = \;& \ell_1\omega_1 + \ell_2\omega_2 + \ell_3((m_2 - 1)\omega_2 + m_1\omega_1) \\ = \;& (\ell_3(m_2 - 1) + \ell_2)\omega_2 + (\ell_1 + \ell_3 m_1)\omega_1. \end{aligned} \tag{48}$$

Assume that $\omega_2 > (m_1 + m_2)m_1\omega_1$. Then, except for the 4−tuple $(m_1, m_2, m_3, \varepsilon)$ verifying Eq. (42), the only possibility to have Eq. (48) equal to 0 is $\ell_1 = \ell_2 = \ell_3 = 0$. In that case,

$$\ell_1 + \ell_2 + \ell_3 \neq 1 \pmod 2.$$

Then, Eq. (47) is not satisfied, and **(2)** is proved.

Due to Eq. (44), the power of ζ is equal to the number of times ω_3 occurs in the resonance condition (42). The latter is clearly equal to 1. Thus, $x_I(2\pi) - x_I(0)$ gives rise to a linear function of ζ. It remains to show that the coefficient in front of ζ is not equal to zero. By Lemma 5.2, one knows that

$$\begin{aligned} \dot{x}_I \;=\;& g_I \cos\{(m_1\omega_1 + m_2\omega_2)t - (m_1 + m_2 - 1)\frac{\pi}{2}\} \\ & + f_I a \cos\{(m_1\omega_1 + (m_2 - 1)\omega_2 - \omega_3)t - (m_1 + m_2 - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}, \end{aligned} \tag{49}$$

where we gathered all other terms into $\mathcal{R}$. Note that the numerical coefficients f_I and g_I depend on the frequencies ω_1, ω_2, and ω_3. The goal is to show that f_I is not equal to zero if we want to move the component x_I, i.e. when $\omega_3 = (m_2-1)\omega_2 + m_1\omega_1$. If we consider f_I as a function of ω_1, ω_2, and ω_3, it suffices to show that this function is not identically equal to zero over the hyperplane of $\mathbb{R}^3$ defined by the resonance condition $\omega_3 = (m_2-1)\omega_2 + m_1\omega_1$. We assume that the next lemma holds true, and we will provide an argument immediately after finishing the proof of Proposition 5.1.

Lemma 5.3. *For all $J \leq I$, let $m_1^J := \Delta_1(X_J)$ and $m_2^J := \Delta_2(X_J)$. If f_J is the coefficient in front of the term $\cos\{(m_1^J\omega_1 + (m_2^J-1)\omega_2 - \omega_3)t - (m_1^J + m_2^J - 1 - \varepsilon)\frac{\pi}{2}\}$, and g_J the one in front of the term $\cos\{(m_1^J\omega_1 + m_2^J\omega_2)t - (m_1^J + m_2^J - 1)\frac{\pi}{2}\}$. Then, the quotient $\alpha_J := f_J/g_J$ verifies the following inductive formula.*

- *If $X_J = X_1$, $\alpha_J = 0$; If $X_J = X_2$, $\alpha_J = 1$;*
- *If $X_J = [X_{J_1}, X_{J_2}]$, α_J is defined by*

$$\alpha_J = \frac{m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2}{m_1^{J_1}\omega_1 + (m_2^{J_1}-1)\omega_2 - \omega_3}\alpha_{J_1} + \alpha_{J_2}.$$

  where $m_i^{J_1} = \Delta_i(X_{J_1})$ for $i = 1,\ 2$.

Let us take $\omega_3 = -\omega_2$. It results from Lemma 5.3 that, for every $J \leq I$, one has

$$\alpha_J \ = \ \alpha_{J_1} + \alpha_{J_2}, \ \text{ if } X_J = [X_{J_1}, X_{J_2}].$$

Since $\alpha_1 = 0$ and $\alpha_2 = 1$, then, over the hyperplane of $\mathbb{R}^3$ defined by $\omega_3 = -\omega_2$, the function α_J is a strictly positive number independent of ω_1 and ω_2.

Let us show now that $\alpha_J(\omega_1,\omega_2,\omega_3)$ is not identically equal to zero over the hyperplane of $\mathbb{R}^3$ defined by $\omega_3 = m_1\omega_1 + (m_2-1)\omega_2$. Let $\hat\omega_2 := -m_1\omega_1/m_2$. One has

$$m_1\omega_1 + (m_2-1)\hat\omega_2 = -\hat\omega_2.$$

It implies that

$$\alpha_I(\omega_1, \hat\omega_2, m_1\omega_1 + (m_2-1)\hat\omega_2) = \alpha_I(\omega_1, \hat\omega_2, -\hat\omega_2).$$

Since the function $\alpha_I(\omega_1,\omega_2,-\omega_2)$ is never equal to zero, and it coincides with the function $\alpha_I(\omega_1,\omega_2, m_1\omega_1 + (m_2-1)\omega_2)$ at the point $(\omega_1,\hat\omega_2)$, the latter is not identically equal to zero.

Therefore, $f_I(\omega_1,\omega_2,\omega_3)$ is not identically equal to zero over the hyperplane $\omega_3 = (m_2 - 1)\omega_2 + m_1\omega_1$. Moreover, as it is a non trivial rational function, it eventually vanishes at a finite number of integer points. Then, we obtain a non zero linear function of ζ, and **(1)** is now proved. Proposition 5.1 results from **(1)** and **(2)**.

□

Proof of Lemma 5.3. The proof goes by induction on $|I|$. Since $\dot x_1 = u_1$, and $\dot x_2 = u_2$, by Eq. (44), one has $\alpha_1 = 0$ and $\alpha_2 = 1$.

Assume that $|J| \geq 2$. By construction, one has $X_J = [X_{J_1}, X_{J_2}]$ with $|J_1| \leq |J_2| < |J|$. According to the inductive hypothesis, one has

$$\begin{aligned}
\dot{x}_{J_1} &= g_{J_1} \cos\{(m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2)t - (m_1^{J_1} + m_2^{J_1} - 1)\frac{\pi}{2}\} \\
&\quad + f_{J_1} \cos\{(m_1^{J_1}\omega_1 + (m_2^{J_1} - 1)\omega_2 - \omega_3)t - (m_1^{J_1} + m_2^{J_1} - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_{J_1}, \\
\dot{x}_{J_2} &= g_{J_2} \cos\{(m_1^{J_2}\omega_1 + m_2^{J_2}\omega_2)t - (m_1^{J_2} + m_2^{J_2} - 1)\frac{\pi}{2}\} \\
&\quad + f_{J_2} \cos\{(m_1^{J_2}\omega_1 + (m_2^{J_2} - 1)\omega_2 - \omega_3)t - (m_1^{J_2} + m_2^{J_2} - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_{J_2}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\dot{x}_J &= \Big(\frac{1}{m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2} g_{J_1} \cos\{(m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2)t - (m_1^{J_1} + m_2^{J_1})\frac{\pi}{2}\} \\
&\quad + \frac{1}{m_1^{J_1} + (m_2^{J_1} - 1)\omega_2 - \omega_3} f_{J_1} a \cos\{(m_1^{J_1}\omega_1 + (m_2^{J_1} - 1)\omega_2 - \omega_3)t \\
&\quad - (m_1^{J_1} + m_2^{J_2} - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_{J_1}\Big) \Big(g_{J_2} \cos\{(m_1^{J_2}\omega_1 + m_2^{J_2}\omega_2)t - (m_1^{J_2} + m_2^{J_2} - 1)\frac{\pi}{2}\} \\
&\quad + f_{J_2} a \cos\{(m_1^{J_2}\omega_1 + (m_2^{J_2} - 1)\omega_2 - \omega_3)t - (m_1^{J_2} + m_2^{J_2} - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_{J_2}\Big) \\
&= \frac{1}{2} \frac{g_{J_1} g_{J_2}}{m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2} \cos\{(m_1^J\omega_1 + m_2^J\omega_2)t - (m_1^J + m_2^J - 1)\frac{\pi}{2}\} \\
&\quad + \frac{1}{2}\left(\frac{g_{J_1} f_{J_2}}{m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2} + \frac{g_{J_2} f_{J_1}}{m_1^{J_1}\omega_1 + (m_2^{J_1} - 1)\omega_2 - \omega_3}\right) \\
&\quad \cos\{(m_1^J\omega_1 + m_2^J\omega_2 - \omega_3)t - (m_1^J + m_2^J - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_J \\
&= g_J \cos\{(m_1^J\omega_1 + m_2^J\omega_2)t - (m_1^J + m_2^J - 1)\frac{\pi}{2}\} \\
&\quad + f_J \cos\{(m_1^J\omega_1 + m_2^J\omega_2 - \omega_3)t - (m_1^J + m_2^J - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_J.
\end{aligned}$$

Therefore, one obtains

$$\alpha_J = \frac{m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2}{m_1^{J_1}\omega_1 + (m_2^{J_1} - 1)\omega_2 - \omega_3} \alpha_{J_1} + \alpha_{J_2}.$$

$\square$

5.2.2 A more general case: $m = 2$ and Card $(\mathcal{E}_x^j) > 1$

In general, given a pair (m_1, m_2), the equivalence class $\mathcal{E}_x(m_1, m_2)$ has more than one element. This situation first occurs for brackets of length 5. For instance, given the pair $(3, 2)$, one has both $X_I = [X_2, [X_1, [X_1, [X_1, X_2]]]]$ and $X_J = [[X_1, X_2], [X_1, [X_1, X_2]]]$. By Lemma 5.2, if one chooses a 4-tuple verifying the resonance condition (42) for $\dot{x}_I$, the same resonance occurs in $\dot{x}_J$. Such two components cannot be independently steered by using resonance. The idea is to move simultaneously these components. For instance, one can choose (u_1, u_2) as follows:

$$\begin{aligned}
u_1(t) &= \cos \omega_1 t \\
u_2(t) &= \cos \omega_2 t + a_I \cos \omega_3 t + \cos \omega_4 t + a_J \cos \omega_5 t,
\end{aligned}$$

where $\omega_1 = 1$, ω_2 is chosen according to Eq. (43), $\omega_3 = (m_2 - 1)\omega_2 + m_1\omega_1$ and $\omega_5 = (m_2 - 1)\omega_4 + m_1\omega_1$, with ω_4 large enough to guarantee a non-resonance condition. After explicit integration of Eq. (41), one obtains

$$\begin{pmatrix} f_I(\omega_1, \omega_2) & f_I(\omega_1, \omega_4) \\ f_J(\omega_1, \omega_2) & f_J(\omega_1, \omega_4) \end{pmatrix} \begin{pmatrix} a_I \\ a_J \end{pmatrix} = A \begin{pmatrix} a_I \\ a_J \end{pmatrix} = \begin{pmatrix} x_I(2\pi) - x_I(0) \\ x_J(2\pi) - x_J(0) \end{pmatrix},$$

where f_I and f_J are two rational functions of frequencies. Thus, the pair (u_1, u_2) controls exactly and simultaneously x_I and x_J, provided that the matrix A is invertible. We generalize this strategy in the following paragraphs. Assume that $\mathcal{E}_x^j(m_1, m_2) = \{x_{I_1}, \ldots, x_{I_N}\}$. The main result is given next.

Proposition 5.4. *Consider*

$$\{\omega_{11}^1, \ldots, \omega_{11}^{m_1}\}, \ldots, \{\omega_{1N}^1, \ldots, \omega_{1N}^{m_1}\}, \{\omega_{21}^1, \ldots, \omega_{21}^{m_2-1}, \omega_{21}^*\}, \ldots, \{\omega_{2N}^1, \ldots, \omega_{2N}^{m_2-1}, \omega_{2N}^*\}$$

belonging to $\mathbb{N}^{m_1 N} \times \mathbb{N}^{m_2 N}$ *such that*

$$\begin{cases} \forall\ j = 1 \ldots N, \quad \omega_{2j}^* = \sum_{i=1}^{m_1} \omega_{1j}^i + \sum_{i=1}^{m_2-1} \omega_{2j}^i, \\ \varepsilon = m_1 + m_2 - 1 \pmod 2, \end{cases} \tag{50}$$

and

$$\forall j = 1 \ldots N-1, \quad \begin{cases} \omega_{11}^1 & \in & \mathbb{N}\ ; & \\ \omega_{1j}^{i+1} & > & m_1\omega_{1j}^i\ ; & i = 1 \ldots m_1, \\ \omega_{2j}^1 & > & m_1\omega_{1j}^{m_1}\ ; & \\ \omega_{2j}^i & > & m_2\omega_{2j}^{i-1} + m_1\omega_{1j}^{m_1}\ ; & i = 2 \ldots m_2 - 1, \\ \omega_{1j+1}^1 & > & m_2\omega_{2j}^{m_2-1} + m_1\omega_{1j}^{m_1}\ ; & \end{cases} \tag{51}$$

then, the control

$$\begin{cases} u_1 & := & \sum_{j=1}^{N}\sum_{i=1}^{m_1} \cos\omega_{1j}^i t, \\ u_2 & := & \sum_{j=1}^{N}\sum_{i=1}^{m_2-1} \cos\omega_{2j}^i t + a_j \cos(\omega_{2j}^* t - \varepsilon\frac{\pi}{2}), \end{cases} \tag{52}$$

steers the components $(x_{I_1}, \ldots, x_{I_N})$ *from an arbitrary initial condition* $(x_{I_1}(0), \ldots, x_{I_N}(0))$ *to an arbitrary final one* $(x_{I_1}(2\pi), \ldots, x_{I_N}(2\pi))$, *without modifying any other component having been previously moved to its final value.*

This result generalizes Proposition 5.1. The proof is decomposed in two parts. In the first one, we show that, if (51) holds and the control functions are of the form (52), then (50) is the *only* resonance occurring in $(\dot{x}_{I_1}, \ldots, \dot{x}_{I_N})$. The resonance gives rise to a system of linear equations on $(a_1, \ldots, a_N)$. In a second part, we recover the invertibility of this system by choosing suitable frequencies in the control function (52).

Part I Frequencies and Resonance

Consider inputs of the form (52). Generalizing Lemma 5.2, we give a general form of frequencies involved in $\dot{x}_J$.

Lemma 5.5. *The dynamics $\dot{x}_J$ is a linear combination of cosine functions of the form*

$$(\ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell_2^* \cdot \omega_2^*)t - (\ell_1 + \ell_2 + m_2^* - 1 + \ell_2^* \varepsilon)\frac{\pi}{2}, \tag{53}$$

where

$$\ell_1 \cdot \omega_1 = \sum_{j=1}^{N} \sum_{i=1}^{m_1} \ell_{1j}^i \omega_{1j}^i,\ \ \ell_2 \cdot \omega_2 = \sum_{j=1}^{N} \sum_{i=1}^{m_2-1} \ell_{2j}^i \omega_{2j}^i,\ \ \ell_2^* \cdot \omega_2^* = \sum_{j=1}^{N} \ell_{2j}^* \omega_{2j}^*, \tag{54}$$

$$\ell_1 = \sum_{j=1}^{N} \sum_{i=1}^{m_1} \ell_{1j}^i,\ \ \ell_2 = \sum_{j=1}^{N} \sum_{i=1}^{m_2-1} \ell_{2j}^i,\ \ \ell_2^* = \sum_{j=1}^{N} m_{2j}^*, \tag{55}$$

with $(\ell_{1j}^i, \ell_{2j}^i, \ell_{2j}^*) \in \mathbb{Z}^3$.

Let

$$|\ell_1| = \sum_{j=1}^{N} \sum_{i=1}^{m_1} |\ell_{1j}^i|,\ \ |\ell_2| = \sum_{j=1}^{N} \sum_{i=1}^{m_2-1} |\ell_{2j}^i|,\ \ \textit{and}\ \ |\ell_2^*| = \sum_{j=1}^{N} |\ell_{2j}^*|,$$

then, one has $|\ell_1| \leq \Delta_1(X_J)$, $|\ell_2| + |\ell_2^*| \leq \Delta_2(X_J)$.

Proof of Lemma 5.5. The proof goes by induction on $|J|$.

- $|J| = 1$: the result is true since $\dot{x}_1 = u_1$ and $\dot{x}_2 = u_2$.
- *Inductive step:*
  Assume that the result holds true for all $x_{\tilde{J}}$ such that $1 \leq |\tilde{J}| < s$. We show that it remains true for x_J with $|J| = s$. By construction, we have $X_J = \mathrm{ad}_{X_{J_1}}^k X_{J_2}$, and

  $$\dot{x}_J = \frac{1}{k!} x_{J_1}^k \dot{x}_{J_2}, \tag{56}$$

  with $|J_1| < |J|$, $|J_2| < |J|$, and $k|J_1| + |J_2| = |J|$.

  Then, by the inductive hypothesis, we have

  $$\begin{aligned} \dot{x}_{J_1} &= \mathrm{LinCom}\left\{\cos\{(\ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell_2^* \cdot \omega_2^*)t - (\ell_1 + \ell_2 + \ell_2^* - 1 + \ell_2^* \varepsilon)\frac{\pi}{2}\}\right\}, \quad (57) \\ \dot{x}_{J_2} &= \mathrm{LinCom}\left\{\cos\{(\tilde{\ell}_1 \cdot \omega_1 + \tilde{\ell}_2 \cdot \omega_2 + \tilde{\ell}_2^* \cdot \omega_2^*)t - (\tilde{\ell}_1 + \tilde{\ell}_2 + \tilde{\ell}_2^* - 1 + \tilde{\ell}_2^* \varepsilon)\frac{\pi}{2}\}\right\}, \quad (58) \end{aligned}$$

  where LinCom$\{\cdot\}$ stands "linear combination".

  Eq. (57) implies that

  $$\begin{aligned} x_{J_1} &= \mathrm{LinCom}\left\{\cos\{(\ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell_2^* \cdot \omega_2^*)t - (\ell_1 + \ell_2 + \ell_2^* - 1 + \ell_2^* \varepsilon)\frac{\pi}{2} - \frac{\pi}{2}\}\right\} \\ &= \mathrm{LinCom}\left\{\cos\{(\ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell_2^* \cdot \omega_2^*)t - (\ell_1 + \ell_2 + \ell_2^* + \ell_2^* \varepsilon)\frac{\pi}{2}\}\right\}. \quad (59) \end{aligned}$$

  For notational ease, we will only write down the case $\dot{x}_J = x_{J_1} \dot{x}_{J_2}$.

  Using product formulas for cosine function, we have

  $$\begin{aligned} \dot{x}_J = \mathrm{LinCom}\Big\{&\cos\{[(\ell_1 \pm \tilde{\ell}_1) \cdot \omega_1 + (\ell_2 \pm \tilde{\ell}_2) \cdot \omega_2 + (\ell_2^* \pm \tilde{\ell}_2^*) \cdot \omega_2^*]t \\ &-[(\ell_1 \pm \tilde{\ell}_1) + (\ell_2 \pm \tilde{\ell}_2) + (\ell_2^* \pm \tilde{\ell}_2^*) - 1 + (\ell_2^* \pm \tilde{\ell}_2^*)\varepsilon]\frac{\pi}{2}\}\Big\}. \quad (60) \end{aligned}$$

Moreover, according to the inductive hypothesis, we have $|\ell_1| \le \Delta_1(X_{J_1})$, $|\ell_2| + |\ell_2^*| \le \Delta_2(X_{J_1})$, and $|\tilde{\ell}_1| \le \Delta_1(X_{J_2})$, $|\tilde{\ell}_2| + |\tilde{\ell}_2^*| \le \Delta_2(X_{J_2})$. Then, we get

$$|\tilde{\ell}_1 \pm \tilde{\ell}_1| \le \Delta_1(X_J), \text{ and} |\tilde{\ell}_2 \pm \tilde{\ell}_2| + |\ell_2^* \pm \tilde{\ell}_2^*| \le \Delta_2(X_J).$$

This concludes the proof of Lemma 5.5.

□

By Lemma 5.5, one gets a non trivial contribution for x_J if the resonance condition

$$\begin{cases} \ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell_2^* \cdot \omega_2^* = 0, \\ \ell_2^* \varepsilon + \ell_1 + \ell_2 + \ell_2^* - 1 \equiv 0 \pmod 2, \end{cases} \tag{61}$$

is verified by the frequencies of some cosine functions involved in $\dot{x}_J$.

Lemma 5.6. *Under conditions* (50) *and* (51) *in Proposition 5.4, one gets a non trivial contribution on x_{I_j} depending linearly on a_j for all $j = 1\ldots, N$.*

Proof of Lemma 5.6. It is clear that the resonance condition (61) holds for

$$\{\omega_{11}^1, \ldots, \omega_{11}^{m_1}\}, \ldots, \{\omega_{1N}^1, \ldots, \omega_{1N}^{m_1}\}, \{\omega_{21}^1, \ldots, \omega_{21}^{m_2-1}, \omega_{21}^*\}, \ldots, \{\omega_{2N}^1, \ldots, \omega_{2N}^{m_2-1}, \omega_{2N}^*\},$$

and $\varepsilon \in \{0, 1\}$ verifying (50). We show that it is the only resonance occurring in $\dot{x}_{I_j}$. Indeed, by Lemma 5.5, the integer part of frequencies in $\dot{x}_{I_j}$ is in the following form

$$\begin{aligned} & \ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell_2^* \cdot \omega_2^* \\ = & \sum_{j=1}^{N} \sum_{i=1}^{m_1} \ell_{1j}^i \omega_{1j}^i + \sum_{j=1}^{N} \sum_{i=1}^{m_2-1} \ell_{2j}^i \omega_{2j}^i + \sum_{j=1}^{N} \ell_{2j}^* \omega_{2j}^* \\ = & \sum_{j=1}^{N} \sum_{i=1}^{m_1} \ell_{1j}^i \omega_{1j}^i + \sum_{j=1}^{N} \sum_{i=1}^{m_2-1} \ell_{2j}^i \omega_{2j}^i + \sum_{j=1}^{N} \ell_{2j}^* \left(\sum_{i=1}^{m_1} \omega_{1j}^i + \sum_{i=1}^{m_2-1} \omega_{2j}^i \right) \\ = & \sum_{j=1}^{N} \sum_{i=1}^{m_1} (\ell_{1j}^i + \ell_{2j}^*) \omega_{1j}^i + \sum_{j=1}^{N} \sum_{i=1}^{m_2-1} (\ell_{2j}^i + \ell_{2j}^*) \omega_{2j}^i. \end{aligned} \tag{62}$$

By Condition (51), Eq. (62) is equal to zero if and only if

$$\begin{aligned} \ell_{1j}^i + \ell_{2j}^* &= 0, \quad \text{for } i = 1, \ldots, m_1, \\ \ell_{2j}^i + \ell_{2j}^* &= 0, \quad \text{for } i = 1, \ldots, m_2 - 1. \end{aligned}$$

Then, one has

$$\begin{aligned} |\ell_1| &= \sum_{j=1}^{N} \sum_{i=1}^{m_1} |\ell_{1j}^i| = \sum_{j=1}^{N} \sum_{i=1}^{m_1} |\ell_{2j}^*| = m_1 \sum_{j=1}^{N} |\ell_{2j}^*|, \\ |\ell_2| &= \sum_{j=1}^{N} \sum_{i=1}^{m_2-1} |\ell_{2j}^i| = \sum_{j=1}^{N} \sum_{i=1}^{m_2-1} |\ell_{2j}^*| = (m_2 - 1) \sum_{j=1}^{N} |\ell_{2j}^*|. \end{aligned}$$

However, by Lemma 5.5, one knows that $|\ell_1| \le m_1$ and $|\ell_2| + |\ell_2^*| \le m_2$. Then, one necessarily has $m_{2j}^* = 0$ for all $j = 1, \ldots, N$. In that case, one obtains $\ell_2^* \varepsilon + \ell_1 + \ell_2 + \ell_2^* - 1 = -1 \neq 0$

(mod 2). In conclusion, the resonance condition (61) does not hold for any 4−tuple $(\ell_1, \ell_2, \ell_2^*, \varepsilon)$ different from $(m_1, m_2 - 1, -1, m_1 + m_2 - 1 \pmod 2)$.

By Eq. (52), the power of a_j is equal to the number of times ω_{2j}^* occurs in the resonance condition (42). Since the latter is equal to 1, we obtain a linear function of a_j. This ends the proof of Lemma 5.6.

□

Lemma 5.7. *If $x_J \in \mathcal{E}_x^i$ and $i < j$, then $x_J(2\pi) - x_J(2\pi) = 0$.*

Proof of Lemma 5.7. We first note that Eq. (62) still holds true. Recall its expression here.

$$\begin{aligned} & \ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell_2^* \cdot \omega_2^* \\ = \ & \sum_{j=1}^{N}\sum_{i=1}^{m_1}(\ell_{1j}^i + \ell_{2j}^*) \cdot \omega_{1j}^i + \sum_{j=1}^{N}\sum_{i=1}^{m_2-1}(\ell_{2j}^i + \ell_{2j}^*) \cdot \omega_{2j}^i \end{aligned} \tag{63}$$

By condition (51) in Proposition 5.4, Eq. (63) is equal to zero if and only if $\ell_{1j}^i + \ell_{2j}^* = 0$ for $i = 1, \ldots, m_1$, $j = 1, \ldots, N$ and $\ell_{2j}^i + \ell_{2j}^* = 0$ for $i = 1, \ldots, m_2 - 1$, $j = 1, \ldots, N$. In that case, one has

$$\begin{aligned} |\ell_1| &= m_1 \sum_{j=1}^{N} |\ell_{2j}^*|, \\ |\ell_2| + |\ell_2^*| &= m_2 \sum_{j=1}^{N} |\ell_{2j}^*|. \end{aligned}$$

One also knows that $|\ell_1| \leq \Delta_1(X_J)$, $|\ell_2| + |\ell_2^*| \leq \Delta_2(X_J)$ with $\Delta_1(X_J) < m_1$ or $\Delta_2(X_J) < m_2$. Therefore, one has $\ell_{2j}^* = 0$ for all $j = 1, \ldots, N$. This implies that

$$\ell_2^*\varepsilon + \ell_1 + \ell_2 + \ell_2^* - 1 = -1 \neq 0 \pmod 2.$$

In conclusion, the resonance condition (61) does not hold true. This ends the proof of Lemma 5.7.

□

Part II Invertibility

As a consequence of Lemma 5.6, one has

$$\begin{aligned} \begin{pmatrix} x_{I_1}(2\pi) - x_{I_1}(0) \\ \vdots \\ x_{I_N}(2\pi) - x_{I_N}(0) \end{pmatrix} &= A(\omega_{11}^1, \ldots, \omega_{2N}^{m_2-1}, \omega_{2N}^*) \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \\ = \begin{pmatrix} f_{I_1}^X(\omega_{11}^1, \ldots, \omega_{21}^1, \omega_{21}^*), & \cdots & , f_{I_1}^X(\omega_{1N}^1, \ldots, \omega_{2N}^1, \ldots, \omega_{2N}^*) \\ \vdots & \ddots & \vdots \\ f_{I_N}^X(\omega_{11}^1, \ldots, \omega_{21}^1, \ldots, \omega_{21}^*), & \cdots & , f_{I_N}^X(\omega_{1N}^1, \ldots, \omega_{2N}^1, \ldots, \omega_{2N}^*) \end{pmatrix} & \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}, \end{aligned} \tag{64}$$

where $f_{I_j}^X : \mathbb{R}^m \to \mathbb{R}$ are rational functions of frequencies, and every ω_{2j}^* verifies Eq. (50) for $j = 1, \ldots, N$. We show in the sequel that it is possible to choose integer frequencies

$$\{\omega_{11}^1, \ldots, \omega_{11}^{m_1}\}, \ldots, \{\omega_{1N}^1, \ldots, \omega_{1N}^{m_1}\}, \{\omega_{21}^1, \ldots, \omega_{21}^{m_2-1}, \omega_{21}^*\}, \ldots, \{\omega_{2N}^1, \ldots, \omega_{2N}^{m_2-1}, \omega_{2N}^*\},$$

so that the invertibility of the matrix A involved in Eq. (64) is guaranteed, as well as the non-resonance of every component x_J belonging to a class smaller than $\mathcal{E}_x^j$.

For $j = 1, \dots, N$, we use P_j to denote the hyperplane in $\mathbb{R}^M$ with $M := m_1 + m_2$ defined by Eq. (50), which we recall the expression next,

$$\omega_{2j}^* = \sum_{i=1}^{m_1} \omega_{1j}^i + \sum_{i=1}^{m_2-1} \omega_{2j}^i.$$

We begin by showing that the function $\det A(\omega_{11}^1, \dots, \omega_{2N}^*)$ is not identically equal to zero on $\cap_{j=1}^N P_j$. This is a consequence of the following lemma.

Lemma 5.8. *The family of functions*

$$\{f_{I_1}^X(\omega_1^1, \dots, \omega_1^{m_1}, \omega_2^1, \dots, \omega_2^{m_2-1}, \omega_2^*), \dots, f_{I_N}^X(\omega_1^1, \dots, \omega_1^{m_1}, \omega_2^1, \dots, \omega_2^{m_2-1}, \omega_2^*)\}$$

is linearly independent on the hyperplane P in $\mathbb{R}^M$ defined by the equation $\omega_2^* = \sum_{i=1}^{m_1} \omega_1^i + \sum_{i=1}^{m_2-1} \omega_2^i$.

Proof of Lemma 5.8. The first part of the argument consists in considering a family of M indeterminates $Y = \{Y_1, \dots, Y_m\}$ and the associated control system

$$\dot{y} = \sum_{i=1}^{M} v_i Y_i. \tag{65}$$

Let H_Y be a *P. Hall family* over Y. Consider all the elements $\{Y_{J_1}, \dots, Y_{J_{\tilde{N}}}\}$ in H_Y of length M such that $\Delta_i(Y_{J_j}) = 1$ for all $i = 1 \dots M$, $j = 1, \dots, \tilde{N}$, and the corresponding components $\{y_{J_1}, \dots, y_{J_{\tilde{N}}}\}$ in exponential coordinates.

If we apply one control of the form $\{v_i = \cos \nu_i t\}_{i=1 \dots M}$, with $\nu_m = \sum_{i=1}^{m-1} \nu_i$, to System (65), then, by explicit integration, there exists, for each component y_{J_j}, a fractional function $f_{J_j}^Y : \mathbb{R}^m \to \mathbb{R}$ such that

$$y_{J_j}(2\pi) - y_{J_j}(0) = f_{J_j}^Y(\nu_1, \dots, \nu_M), \quad \text{for } \nu_M = \sum_{i=1}^{M-1} \nu_i. \tag{66}$$

Claim 11. *The family of functions $\{f_{J_1}^Y, \dots, f_{I_{\tilde{N}}}^Y\}$ is linearly independent on the hyperplane in $\mathbb{R}^M$ defined by $\nu_M = \sum_{i=1}^{M-1} \nu_i$.*

Proof of Claim 11. We first define $\tilde{f}_{J_j}^Y$ by

$$\tilde{f}_{J_j}^Y(\nu_1, \dots, \nu_M) = f_{J_j}^Y(\nu_1, \dots, -\nu_M). \tag{67}$$

Then, it is easy to see that $\tilde{f}_{J_j}^Y$ verifies the following inductive formula.

1. For $J = i = 1 \dots M$, $\tilde{f}_J^Y(\nu_i) = \frac{1}{\nu_i}$;

2. for $|J| > 1$, $Y_J = [Y_{J_1}, Y_{J_2}]$, there exists an injective function $\sigma_J : \{1, \dots, m^J\} \to \{1, \dots, M\}$ such that

$$\tilde{f}_J^Y(\nu_{\sigma_J(1)}, \dots, \nu_{\sigma_J(m^J)}) = \frac{\tilde{f}_{J_1}^Y(\nu_{\sigma_J(1)}, \dots, \nu_{\sigma_J(m^{J_1})})}{\sum_{i=1}^{m^{J_1}} \nu_{\sigma_J(i)}} \tilde{f}_{J_2}^Y(\nu_{\sigma_J(m^{J_1}+1)}, \dots, \nu_{\sigma_J(m^J)}), \tag{68}$$

   where $m^J := \Delta(Y_J)$, $m^{J_1} := \Delta(Y_{J_1})$, and $m^{J_2} := \Delta(Y_{J_2})$.

We note that the family of rational functions $\tilde{f}_J^Y$ is well defined for all brackets Y_J such that $\Delta_i(Y_J) \le 1$, $i = 1, \dots, M$. The algebraic construction could be extended to all the Lie brackets, but it is not necessary for our purpose. We also note that Lemma 11 is equivalent to the fact that the family of fractional functions

$$\{\tilde{f}_{J_j}^Y(\nu_1, \dots, \nu_M)\}_{j=1,\dots,\tilde{N}}$$

is linearly independent over the hyperplane $\sum_{i=1}^M \nu_i = 0$.

Recall that every element Y_{J_j} in the family $\{Y_{J_1}, \dots, Y_{J_{\tilde{N}}}\}$ writes uniquely as

$$Y_{J_j} = [Y_{J_{j_1}}, Y_{J_{j_2}}]. \tag{69}$$

Definition 5.3 (*Left and right factors*)**.** For $J \in \{J_1, \dots, J_{\tilde{N}}\}$, the *left factor* $L(J)$ and the *right factor* $R(J)$ of J are defined in such a way that $Y_J = [Y_{L(J)}, Y_{R(J)}]$.

Let L^* be defined by

$$L^* := \max_{j=1,\dots,\tilde{N}} \{L(J_j)\}. \tag{70}$$

The integer L^* is well defined since a P. Hall family is totally ordered. Thus, there exists $J^* \in \{J_1, \dots, J_{\tilde{N}}\}$ such that $L^* = L(J^*)$. Then, define $R^* := R(J^*)$ and set $m^* = |L^*|$. Let $\Lambda = \Lambda_L \cup \Lambda_R$ and $\bar{\Lambda} = \{1, \dots, \tilde{N}\} \setminus \Lambda$ with Λ_L and Λ_R defined by

$$\begin{aligned} \Lambda_L &:= \{j \in \{1, \dots, \tilde{N}\}, \text{ such that } Y_{L(J_j)} \sim Y_L\}, && (71) \\ \Lambda_R &:= \{j \in \{1, \dots, \tilde{N}\}, \text{ such that } Y_{L(J_j)} \sim Y_R\}. && (72) \end{aligned}$$

Then, for all $j \in \Lambda$, there exists an injection function $\sigma_j : \{1, \dots, M\} \to \{1, \dots, M\}$ such that one has

$$\begin{aligned} \tilde{f}_{J_j}^Y(\nu_1, \dots, \nu_M) &= \frac{\tilde{f}_{L(J_j)}^Y(\nu_{\sigma_j(1)}, \dots, \nu_{\sigma_j(m^*)})}{\sum_{i=1}^{m^*} \nu_{\sigma_j(i)}} \tilde{f}_{R(J_j)}^Y(\nu_{\sigma_j(m^*+1)}, \dots, \nu_{\sigma_j(M)}), \quad \text{if } j \in \Lambda_L, && (73) \\ \tilde{f}_{J_j}^Y(\nu_1, \dots, \nu_M) &= \frac{\tilde{f}_{L(J_j)}^Y(\nu_{\sigma_j(m^*+1)}, \dots, \nu_{\sigma_j(M)})}{\sum_{i=m^*+1}^{M} \nu_{\sigma_j(i)}} \tilde{f}_{R(J_j)}^Y(\nu_{\sigma_j(1)}, \dots, \nu_{\sigma(m^*)}), \quad \text{if } j \in \Lambda_R. && (74) \end{aligned}$$

Note that for all j_1 and j_2 in Λ_L, one has $\{\nu_{\sigma_{j_1}(1)}, \dots, \nu_{\sigma_{j_1}(m^*)}\} = \{\nu_{\sigma_{j_2}(1)}, \dots, \nu_{\sigma_{j_2}(m^*)}\}$. Denote by Ξ_L the set of variables involved in $\tilde{f}_{L(J_j)}^Y$ with $j \in \Lambda_L$. A similar property holds for Λ_R. For all j_1 and j_2 in Λ_R, we have $\{\nu_{\sigma_{j_1}(m^*+1)}, \dots, \nu_{\sigma_{j_1}(M)}\} = \{\nu_{\sigma_{j_2}(m^*+1)}, \dots, \nu_{\sigma_{j_2}(M)}\}$. Denote by Ξ_R the set of all variables occurring in $\tilde{f}_{L(J_j)}^Y$ with $j \in \Lambda_R$. We have $\Xi_L \cup \Xi_R = \{\nu_1, \dots, \nu_M\}$. By abuse of notation, we re-write Eqs. (73) and (74) in the following form:

$$\begin{aligned} \tilde{f}_{J_j}^Y(\nu_1, \dots, \nu_M) &= \frac{\tilde{f}_{L(J_j)}^Y(\Xi_L)}{\sum_{\tilde{\nu}_k \in \Xi_L} \tilde{\nu}_k} \tilde{f}_{R(J_j)}^Y(\Xi_R), \quad \text{if } j \in \Lambda_L; && (75) \\ \tilde{f}_{J_j}^Y(\nu_1, \dots, \nu_M) &= \frac{\tilde{f}_{L(J_j)}^Y(\Xi_R)}{\sum_{\tilde{\nu}_k \in \Xi_R} \tilde{\nu}_k} \tilde{f}_{R(J_j)}^Y(\Xi_L), \quad \text{if } j \in \Lambda_R. && (76) \end{aligned}$$

Moreover, by the resonance condition $\sum_{i=1}^M \nu_i = 0$, Eq. (76) becomes

$$\tilde{f}_{J_j}^Y(\nu_1, \dots, \nu_M) = \frac{\tilde{f}_{L(J_j)}^Y(\Xi_R)}{-\sum_{\tilde{\nu}_k \in \Xi_L} \tilde{\nu}_k} \tilde{f}_{R(J_j)}^Y(\Xi_L), \quad \text{if } j \in \Lambda_R. \tag{77}$$

We now prove that the family of fractional functions $\{\tilde{f}^Y_{J_j}(\nu_1,\ldots,\nu_M)\}_{j=1,\ldots,\tilde{N}}$ is linearly independent over the hyperplane $\sum_{i=1}^M \nu_i = 0$. The proof goes by induction over the length of the Lie brackets under consideration. For the brackets of length 1, the result is obviously true. Assume that the result holds for all brackets of length $\leq M-1$.

Assume that there exist $\ell_j \in \mathbb{R}^{\tilde{N}}$ such that

$$\sum_{j=1}^{\tilde{N}} \ell_j \tilde{f}^Y_{J_j}(\nu_1,\ldots,\nu_M) = 0, \quad \text{with } \sum_{i=1}^{M} \nu_i = 0. \tag{78}$$

One has

$$\begin{aligned}
&\sum_{j=1}^{\tilde{N}} \ell_j \tilde{f}^Y_{J_j}(\nu_1,\ldots,\nu_M) = \sum_{j\in\Lambda} \ell_j \tilde{f}^Y_{J_j}(\nu_1,\ldots,\nu_M) + \sum_{j\in\bar{\Lambda}} \ell_j \tilde{f}^Y_{J_j}(\nu_1,\ldots,\nu_M)\\
= \;& \sum_{j\in\Lambda_L} \ell_j \frac{\tilde{f}^Y_{L(J_j)}(\Xi_L)}{\sum_{\tilde{\nu}_k\in\Xi_L}\tilde{\nu}_k} \tilde{f}^Y_{R(J_j)}(\Xi_R) - \sum_{j\in\Lambda_R} \ell_j \frac{\tilde{f}^Y_{L(J_j)}(\Xi_R)}{\sum_{\tilde{\nu}_k\in\Xi_L}\tilde{\nu}_k} \tilde{f}^Y_{R(J_j)}(\Xi_L) + \sum_{j\in\bar{\Lambda}} \ell_j \tilde{f}^Y_{J_j}(\nu_1,\ldots,\nu_M)\\
= \;& 0.
\end{aligned} \tag{79}$$

Multiplying Eq. (79) by the factor $\sum\limits_{\tilde{\nu}_k\in\Xi_L}\tilde{\nu}_k$, we get

$$\begin{aligned}
&\sum_{j\in\Lambda_L} \ell_j \tilde{f}^Y_{L(J_j)}(\Xi_L)\tilde{f}^Y_{R(J_j)}(\Xi_R) - \sum_{j\in\Lambda_R} \ell_j \tilde{f}^Y_{L(J_j)}(\Xi_R)\tilde{f}^Y_{R(J_j)}(\Xi_L) + (\sum_{\tilde{\nu}_k\in\Xi_L}\tilde{\nu}_k)\sum_{j\in\bar{\Lambda}} \ell_j \tilde{f}^Y_{J_j}(\nu_1,\ldots,\nu_M)\\
= \;& 0.
\end{aligned} \tag{80}$$

Since L^* is the maximal element among the left factors of Lie brackets of length M, the fraction $\tilde{f}^Y_{J_j}$ does not contain the factor $\sum\limits_{\tilde{\nu}_k\in\Xi_L}\tilde{\nu}_k$ for all $j\in\bar{\Lambda}$. Therefore, on the hyperplane of $\mathbb{R}^{m^*}$ defined by $\sum\limits_{\tilde{\nu}_k\in\Xi_L}\tilde{\nu}_k = 0$, we have

$$\sum_{j\in\Lambda_L} \ell_j \tilde{f}^Y_{L(J_j)}(\Xi_L)\tilde{f}^Y_{R(J_j)}(\Xi_R) - \sum_{j\in\Lambda_R} \ell_j \tilde{f}^Y_{L(J_j)}(\Xi_R)\tilde{f}^Y_{R(J_j)}(\Xi_L) = 0. \tag{81}$$

Fixing variables belonging to Ξ_R, Eq. (81) is a linear combination of elements of the family $\{\tilde{f}^Y_{L(J_j)}(\Xi_L)\}_{j\in\Lambda_L} \cup \{\tilde{f}^Y_{R(J_j)}(\Xi_L)\}_{j\in\Lambda_R}$ associated to elements of length m^* in P. Hall family. By the inductive hypothesis, this family is linearly independent over the hyperplane of $\mathbb{R}^{m^*}$ defined by $\sum\limits_{\tilde{\nu}_k\in\Xi_L}\tilde{\nu}_k = 0$. We therefore obtain that

$$\ell_j \tilde{f}^Y_{R(J_j)}(\Xi_R) = 0, \qquad \text{for all } j\in\Lambda_L, \tag{82}$$

$$\ell_j \tilde{f}^Y_{L(J_j)}(\Xi_R) = 0, \qquad \text{for all } j\in\Lambda_R. \tag{83}$$

Since Eqs. (82) and (83) hold true over the whole hyperplane of $\mathbb{R}^{M-m^*}$ defined by $\sum\limits_{\tilde{\nu}_k\in\Xi_R}\tilde{\nu}_k = 0$, this implies that $\ell_j = 0$ for every $j\in\Lambda$.

Therefore, Eq. (79) becomes

$$\sum_{j\in\bar{\Lambda}} \ell_j \tilde{f}^Y_{J_j}(\nu_1,\ldots,\nu_M) = 0. \tag{84}$$

Consider now the maximum left factor for $j \in \bar{\Lambda}$ and iterate the same reasoning used for Eq. (78). We deduce that $\ell_j = 0$ for every $j \in \bar{\Lambda}$. Therefore, the family $\{\tilde{f}^Y_{J_j}(\nu_1,\ldots,\nu_M)\}_{j=1,\ldots,\tilde{N}}$ is linearly independent over the hyperplane $\sum_{i=1}^M \nu_i = 0$ and this concludes the proof of Claim 11.

□

We are now in a position to proceed with the argument of Lemma 5.8. Let X_I be an element of $\mathcal{E}_X(m_1,m_2)$, $M := m_1+m_2$ and $N :=$Card $\mathcal{E}_X(m_1,m_2)$. Consider also another family of M indeterminates $Y = \{Y_1,\ldots,Y_M\}$ and let H_Y be the *P. Hall family* over Y. Finally, consider all the elements of the class $\mathcal{E}_Y(1,\ldots,1) = \{Y_{J_1},\ldots,Y_{J_{\tilde{N}}}\}$ in H_Y.

Let Π be the algebra homomorphism from $L(Y)$ to $L(X)$ defined by

$$\Pi(Y_i) = X_1, \quad \text{for} \quad i = 1,\ldots,m_1, \tag{85}$$
$$\Pi(Y_i) = X_2, \quad \text{for} \quad i = m_1+1,\ldots,M. \tag{86}$$

Π is surjective from $\mathcal{E}_Y$ onto $\mathcal{E}_X$. Consider the following vector fields

$$V_Y = \{v_1Y_1 + \cdots + v_MY_M\},$$

where

$$v_i = \cos\omega_i t,\ \text{for}\ i = 1\ldots M-1,\ \text{and}\ v_M = \cos(\omega_M t + \varepsilon\frac{\pi}{2}), \tag{87}$$

with $\omega_M = \sum_{i=1}^{M-1}\omega_i$, and ω_i verifying the non-resonance conditions.

Then, the non autonomous flow of V_Y between 0 and 2π is given by

$$\overrightarrow{\exp}(V_Y)(0,2\pi) \;=\; e^{f^Y_{J_1}Y_{J_1}} \circ \cdots \circ e^{f^Y_{J_{\tilde{N}}}Y_{J_{\tilde{N}}}} \circ \prod_{J>J_1} e^{f^Y_J Y_J}. \tag{88}$$

Let us now apply Π to V_Y, we get

$$\Pi(V_Y) := V^X = \{v_1\Pi(Y_1) + \cdots + v_m\Pi(Y_m)\} = \{u_1X_1 + u_2X_2\}, \tag{89}$$

where

$$u_1 \;=\; \sum_{i=1}^{m_1} v_i = \sum_{i=1}^{m_1}\cos\omega_i t, \tag{90}$$
$$u_2 \;=\; \sum_{i=m_1+1}^{m} v_i = \sum_{i=m_1+1}^{m-1}\cos\omega_i t + \cos(\omega_m t + \varepsilon\frac{\pi}{2}). \tag{91}$$

Then, the non autonomous flow of V_X between 0 and 2π is given by

$$\begin{aligned}\overrightarrow{\exp}(V_X)(0,2\pi) &= e^{f^Y_{J_1}\Pi(Y_{J_1})} \circ \cdots \circ e^{f^Y_{J_{\tilde{N}}}\Pi(Y_{J_{\tilde{N}}})} \circ \prod_{J>J_1} e^{f^Y_J\Pi(Y_J)} \\ &= e^{\sum_{j=1}^{\tilde{N}} f^Y_{J_j}\Pi(Y_{J_j})} \circ \prod_{J>J_1} e^{\bar{f}^Y_J\Pi(Y_J)}\end{aligned} \tag{92}$$

We also know that

$$\begin{aligned}\overrightarrow{\exp}(V_X)(0,2\pi) &= e^{f^X_{I_1}X_{I_1}}\circ\cdots\circ e^{f^X_{I_N}X_{I_N}}\circ\prod_{I>I_1}e^{f^X_I X_I}\\ &= e^{\sum_{j=1}^N f^X_{I_j}X_{I_j}}\circ\prod_{I>I_1}e^{\bar{f}^X_I X_I}. \qquad (93)\end{aligned}$$

Recall that Π is surjective from $\mathcal{E}_Y(1,\dots,1)$ onto $\mathcal{E}_X(m_1,m_2)$. Therefore, by identifying Eqs. (92) and (93), we obtain that for all $j=1,\dots,N$, $f^X_{I_j}$ is a linear combination of $f^Y_{J_i}$ with $i=1,\dots,\tilde{N}$, i.e.

$$f^X_{I_j}=\sum_{i=1}^{\tilde{N}}\alpha_i^j f^Y_{J_i}. \qquad (94)$$

Since the family $(f^Y_{J_i})_{i=1,\dots,\tilde{N}}$ is linearly independent, and the matrix $\mathcal{A}:=(\alpha_i^j)_{i=1,\dots,N;j=1,\dots,\tilde{N}}$ is surjective, we conclude that the family $(f^X_{I_j})_{j=1,\dots,N}$ is also linearly independent. This ends the proof of Lemma 5.8.

□

A consequence of Lemma 5.8 is the following.

Corollary 5.9. *With the above notations, the function* $\det A$ *is identically equal to zero on* $\cap_{j=1}^N P_j$.

Proof of Corollary 5.9. For $j=1,\dots,N$, we define the vector L_j by

$$L_j=\Big(f^X_{I_j}(\omega^1_{11},\dots,\omega^{m_1}_{11},\omega^1_{21},\dots,\omega^{m_2-1}_{21},\omega^*_{21}),\cdots,f^X_{I_j}(\omega^1_{1N},\dots,\omega^{m_1}_{1N},\omega^1_{2N},\dots,\omega^{m_2-1}_{2N},\omega^*_{2N})\Big)^T.$$

Assume that $\sum_{j=1}^N \ell_j L_j=0$ with $l_j\in\mathbb{R}$. Then, for all $i=1,\dots,N$, we have

$$\sum_{j=1}^N \ell_j f^X_{I_j}(\omega^1_{1i},\dots,\omega^{m_1}_{1i},\omega^1_{2i},\dots,\omega^{m_2-1}_{2i},\omega^*_{2i})=0. \qquad (95)$$

By Lemma 5.8, we have $l_j=0$ for $j=1,\dots,N$. Then, the family $(L_j)_{j=1,\dots,N}$ is linearly independent. We conclude that $\det A$ is not equal to zero. This ends the proof of Corollary 5.9.

□

We still need another technical lemma which guarantees that there exists integer frequencies such that Eq. (51) is satisfied and the matrix A in Eq. (64) is invertible.

Lemma 5.10. *There exists integer frequencies such that* (51) *is satisfied and* $\det A$ *is not equal to zero.*

Proof of Lemma 5.10. For $j=1,\dots,N$, we set

$$f_j(\omega_1,\dots,\omega_{m-1})=f^X_{I_j}(\omega_1,\dots,\omega_{m-1},\sum_{i=1}^{m-1}\omega_i), \qquad (96)$$

then, we have

$$
\begin{aligned}
\det A &= \begin{vmatrix} f_1(\omega_{11}^1, \ldots, \omega_{11}^{m_1}, \omega_{21}^1, \ldots, \omega_{21}^{m_2-1}), & \cdots & , f_1(\omega_{1N}^1, \ldots, \omega_{1N}^{m_1}, \omega_{2N}^1, \ldots, \omega_{2N}^{m_2-1}) \\ \vdots & \ddots & \vdots \\ f_N(\omega_{11}^1, \ldots, \omega_{11}^{m_1}, \omega_{21}^1, \ldots, \omega_{21}^{m_2-1}), & \cdots & , f_N(\omega_{1N}^1, \ldots, \omega_{1N}^{m_1}, \omega_{2N}^1, \ldots, \omega_{2N}^{m_2-1}) \end{vmatrix} \\
&= \frac{P(\omega_{11}^1, \ldots, \omega_{11}^{m_1}, \omega_{21}^1, \ldots, \omega_{21}^{m_2-1}, \ldots, \omega_{1N}^1, \ldots, \omega_{1N}^{m_1}, \omega_{2N}^1, \ldots, \omega_{2N}^{m_2-1})}{Q(\omega_{11}^1, \ldots, \omega_{11}^{m_1}, \omega_{21}^1, \ldots, \omega_{21}^{m_2-1}, \ldots, \omega_{1N}^1, \ldots, \omega_{1N}^{m_1}, \omega_{2N}^1, \ldots, \omega_{2N}^{m_2-1})}, \qquad (97)
\end{aligned}
$$

where P and Q are two polynomials of $(m-1)N$ variables.

We first note that Q never vanishes over integer frequencies. Assume, by contradiction, that P is always equal to zero for integer frequencies verifying Eq. (51). Consider P as a polynomial in one variable $\omega_{2N}^{m_2-1}$, i.e.,

$$
\begin{aligned}
& P(\omega_{11}^1, \ldots, \omega_{11}^{m_1}, \omega_{21}^1, \ldots, \omega_{21}^{m_2-1}, \ldots, \omega_{1N}^1, \ldots, \omega_{1N}^{m_1}, \omega_{2N}^1, \ldots, \omega_{2N}^{m_2-1}) \\
= & \sum_{j=0}^{M} P_j(\omega_{11}^1, \ldots, \omega_{11}^{m_1}, \omega_{21}^1, \ldots, \omega_{21}^{m_2-1}, \ldots, \omega_{1N}^1, \ldots, \omega_{1N}^{m_1}, \omega_{2N}^1, \ldots, \omega_{2N}^{m_2-2})(\omega_{2N}^{m_2-1})^j. \qquad (98)
\end{aligned}
$$

Given integer frequencies $(\omega_{11}^1, \ldots, \omega_{11}^{m_1}, \omega_{21}^1, \ldots, \omega_{21}^{m_2-1}, \ldots, \omega_{1N}^1, \ldots, \omega_{1N}^{m_1}, \omega_{2N}^1, \ldots, \omega_{2N}^{m_2-2})$, if Eq. (98) is not identically equal to zero, then this polynomial in the variable $\omega_{2N}^{m_2-1}$ most has a finite number of roots. However, for a given choice of $(m-1)N-1$ first frequencies, there exist an infinite number of $\omega_{2N}^{m_2-1}$ verifying (51). Then, $P_j = 0$ over all integer frequencies, and P_M is not identically equal to zero. We note that all P_j are polynomials of $(m-1)N-1$ variables. Proceeding by induction on the number of variables, it is easy to see that, at the end, we obtain a polynomial in the variable ω_{11}^1 which is equal to zero over all integer ω_{11}^1, and which is not identically equal to zero according to Corollary 5.9. That contradiction ends the proof of Lemma 5.10.

□

5.2.3 General case: $m > 2$

Notice that the proof of Theorem 5.4 does not really depend on the number of vector fields involved in the control system (4). Indeed, for $m > 2$, if the control functions are linear combination of sinusoids with integer frequencies, then the state variables in the canonical form are also linear combinations of sinusoids so that the frequencies are $\mathbb{Z}-$linear combinations of the frequencies occurring in the control functions. The proof is the same as that of Lemma 5.5, up to extra notation. Since Lemma 11 depends only on the length of the Lie brackets, but not on the number of vector fields, the proof of Lemma 5.8 does not depend on m, either. In order to prove a similar result for $m > 2$, we just need to re-project Eqs. (85) and (86) to m vector fields instead of 2.

5.3 Numerical implementation issues

In this paragraph, we explain how Proposition 5.4 can be used in practice for controlling nilpotent systems in the canonical form. The numerical implementation of this strategy is divided into off-line and on-line computations. The off-line ones consist in choosing frequencies for each equivalence class and computing the corresponding matrix involved in Eq. (64). We note that Proposition 5.4 only gives sufficient conditions to prevent resonance (by choosing widely spaced frequencies, cf. Eq. (51)) and guarantee the invertibility of the corresponding matrix (by using a sufficiently large number of independent frequencies). These conditions tend to produce high frequencies while it is desirable to find smaller ones for practical use. We can prove that two independent frequencies suffice to steer one component (cf. Section 5.2.1), and we conjecture that $2N$ independent frequencies suffice to control one equ ivalence class of cardinal N by producing an invertible matrix. One can implement a searching algorithm for finding the optimal frequencies for each equivalence class such that they prevent all resonances in smaller classes and produce an invertible matrix for the class under consideration. Proposition 5.4 guarantees the finiteness of such an algorithm. Moreover, one can construct once for all a table containing the choice of frequencies and the corresponding matrices for each equivalence class in the free canonical system. Once the frequencies and matrices are obtained, the on-line computations needed to determine the scalar coefficients in front of the corresponding sinusoids is only one matrix multiplication for each equivalence class.

Remark 5.3. Recall that the key point in our control strategy consists in choosing suitable frequencies such that, during each $2\pi-$period, the corresponding input function displaces components of one equivalence class to their preassigned positions while all the components of smaller classes (according to the ordering in Definition 5.2) return at the end of this control period to the values taken at the beginning of the period. In order to achieve the previous task, special resonance conditions must be verified by the appropriate components, and these conditions must not hold for all the other smaller components (according to the ordering in Definition 5.2). Note that two categories of frequencies have been picked in Proposition 5.4: the basic frequencies $\{\omega_{ij}^k\}$, and the resonance frequencies $\{\omega_{ij}^*\}$. Since frequencies occurring in the dynamics of the state variables are just $\mathbb{Z}-$linear combinations of $\{\omega_{ij}^k\} \cup \{\omega_{ij}^*\}$, and the resonance frequencies $\{\omega_{ij}^*\}$ are chosen to be special $\mathbb{Z}-$linear combinations of basic frequencies (resonance condition), then the frequencies in the dynamics of the state variables are special $\mathbb{Z}-$linear combinations of $\{\omega_{ij}^k\}$.

6 User's Guide

For the reader's convenience, the global motion planning strategy developed in this paper is summarized in this section.

$\boxed{\textbf{User's Guide}}$

Let (Σ) be a driftless control-affine system defined on a compact domain K, x_0 be the starting point, and x_1 be the goal.

1. Decompose K into compacts sets $\mathcal{V}^c_{\mathcal{J}_i}$, with $i = 1, \dots, M$ (Subsection 3.3). Without loss of generality, one can assume that $x_0 \in \mathcal{V}^c_{\mathcal{J}_1}$ and $x_1 \in \mathcal{V}^c_{\mathcal{J}_M}$.
2. Choose a sequence $(x^i)_{i=1,\dots,M-1}$ such that $x^i \in \mathcal{V}^c_{\mathcal{J}_i} \cap \mathcal{V}^c_{\mathcal{J}_{i+1}}$. Without loss of generality, one can assume that $x_0 := x^i$ and $x_1 := x^{i+1}$ for some $i \in \{1, \dots, M-1\}$.
3. Apply Desingularization Algorithm (Subsection 3.3) at $x_a := x_0$ with $\mathcal{J} := \mathcal{J}_{i+1}$. One obtains a regular control system ξ, a system of privileged coordinates z at x_a, and an approximation $\widehat{\xi}$ in canonical form.
4. Apply Global $(\widetilde{x}_0, \widetilde{x}_1)$ (Subsection 4.2) with $\widetilde{x}_0 := (x_0, 0)$ and $\widetilde{x}_1 := (x_1, 0)$.

   At each iteration,

   (a) construct an approximation of ξ at the current point (Subsection 4.1);

   (b) compute $\hat{u}$ (Section 5);

   (c) construct AppSteer (Definition 2.10).

7 Conclusion

In this paper, an effective framework has been proposed for solving the motion planning problem for driftless control-affine systems. First, an iterative steering algorithm based on the nonholonomic approximation techniques has been devised. This algorithm is globally convergent for regular systems, and it does not require a priori knowledge on any critical distance. Second, for general systems which contain singular points, an explicit desingularization procedure involving only explicit polynomial transformations has been proposed. This construction gives rise to a "lifted" system which is regular, and it has been shown that, in order to steer the original system from one state to another, it suffices to solve a motion planning problem for the lifted system. Finally, an exact method using sinusoidal controls for steering general nilpotent systems has been proposed. In particular, it can be used to control exactly approximate systems involved in our general planning algorithm.This method gives rise to smooth trajectories, leading to possible dynamical extensions.

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