Credit Risk with asymmetric information on the default threshold

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International Conference on Stochastic Analysis and Applications,
Hammamet - October 12-17, 2009

Abstract

We study the impact of asymmetric information on the conditional default probabilities. We suppose that the default is triggered when a fundamental diffusion process of the firm passes below a random threshold. The managers of the firm have complete information on both the process and the default threshold, while other investors on the market only have partial observations. We specify the conditional survival probabilities under different information structures.

1 Introduction

In the credit risk analysis, the default or survival probability plays an important role in the pricing and the risk management of credit derivatives. To have a dynamic vision on these quantities, we need to specify the accessible information, as well as a suitable modelling framework.

How to model the occurrence of a default event is an important subject from both economic and financial point of view. There exist a large literature on this issue and mainly two modelling approaches: the structural one and the reduced-form one. In the structural approach, where the original idea goes back to the paper of Merton [19], the default is triggered when a fundamental process X of the firm passes below a threshold level L. The fundamental process may represent the asset value or the total cash flow of the firm where the debt value of the firm can also be taken into consideration. This provides a convincing economic interpretation for this approach. The default threshold L is in general supposed to be constant or deterministic. Its level is chosen by the managers of the firm according to some criterions — maximizing the equity value as in [18].

For an agent on the financial market, her vision on the default is quite different: on one hand, she possesses merely a limited information of the basic data (the process X for example) of the firm; on the other hand, to deal with financial products written on the firm, she needs to update the estimations of the default probability in a dynamic manner.

This leads to the reduced-form approach for default modelling where the default arrives in a more "surprising" way and the model parameters can be daily calibrated by using the market data such as the CDS spreads.

In this work, we aim to study the information concerning the default threshold L in the credit analysis, in addition to the partial observation of the process X. This is related to the insider's information problems. Indeed, when the managers make decisions on whether the firm will default or not, he has supplementary information on the default threshold L compared to an ordinary investor on the market. Financially speaking, this is also motivated by some recent "technical default events" during the crisis, where the bankruptcy occurs although the firm is still in a relatively healthy economic situation.

We present our model in the standard setting. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space where $\mathcal{A}$ is a σ-algebra of Ω representing the total information on the market. We consider a firm and model its default time as the first time that a continuous time process $(X_t)_{t\geq 0}$ (for example the assets value of the firm) reaches some default barrier L, i.e.,

$$\tau = \inf\{t : X_t \leq L\} \quad \text{where } X_0 > L \tag{1.1}$$

with the convention that $\inf \emptyset = +\infty$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by the process X, i.e., $\mathcal{F}_t = \sigma(X_s, s \leq t) \vee \mathcal{N}$ satisfying the usual conditions where $\mathcal{N}$ denotes the $\mathbb{P}$ null sets. Such construction of a default time adapts both to structural approach and to reduced form approach of the default modelling, according to the specification of the process X and the threshold L.

In the structural approach models, L is a constant or a deterministic function $L(t)$, then τ defined in (1.1) is an $\mathbb{F}$-stopping time as in the classical first passage models. In the reduced-form approach, the default barrier L is unknown and is described as a random variable in $\mathcal{A}$ (e.g. [17], [7]). We introduce the decreasing process X^* defined as $X_t^* = \inf\{X_s,\ s \leq t\}$. Then (1.1) can be rewritten as

$$\tau = \inf\{t : X_t^* = L\}. \tag{1.2}$$

This formulation gives a general reduced-form model of default. In particular, when the barrier L is supposed to be independent of $\mathcal{F}_\infty$, we may recover the Cox-process model. In addition, we have $\mathbb{P}(\tau > t|\mathcal{F}_\infty) = \mathbb{P}(X_t^* > L|\mathcal{F}_\infty)$. Note that the (H)-hypothesis is satisfied, that is, $\mathbb{P}(\tau > t|\mathcal{F}_\infty) = \mathbb{P}(\tau > t|\mathcal{F}_t)$.

The information asymmetry problems have been studied in the credit risk literature mostly concerning the process X (see [6, 4, 15, 11, 3, 2]). In a recent work [9], the authors have been interested in the information on the barrier L. In our paper, we shall consider several types of agents on the market who have different information on X and on L. Our approach is mainly based on the theory of enlargement of filtrations. We are interested in computing conditional default probabilities for these agents and we shall show that the information level is important for their estimations of default probabilities.

The rest of this paper is organized as follows. In Section 2, we introduce different information structures for various agents on the market. We shall distinguish the role of the managers who choose the default barrier L, the insiders who have a supplementary information on L and the investors who observe the occurrence of the default. We then make precise in Section 3 the mathematical hypothesis for these cases, using the language of enlargement of filtrations. Section 4 is devoted to the explicit calculations on the conditional default probabilities. We then give numerical examples in Section 5 for illustration.

2 The informational structure

On the financial market, the available information for each agent is various. One market investor or practitioner may have different information compared to other investors. Furthermore, there is a strong information asymmetry between investors and the managers of the firm. One important point is that the managers may have information on whether the firm will default or not, or when the default may happen. We now describe the different information in these cases concerning the firm.

2.1 Manager and investor: knowledge on the default threshold

In this following, we suppose that the default threshold L in (1.2) is a random variable and recall that the process X^* is decreasing. We assume that L is chosen by the managers of the firm who hence have the total knowledge on L. The information of X^*_t is contained in the σ-algebra $\mathcal{F}_t$. However, the process X^* can not give us full information on $\mathcal{F}_t$.

We now distinguish two types of agents on the market. The first type is the manager who has complete information on X and on L. We call the full information on L the "initial (enlargement) information" on L and we shall precise some technical hypothesis in the next section. The filtration of the manager's information is then

$$\mathcal{G}^M_t := \mathcal{F}_t \vee \sigma(L).$$

The filtration $\mathbb{G}^M = (\mathcal{G}^M_t)_{t\geq 0}$ is in fact the initial enlargement of the filtration $\mathbb{F}$ with the random variable L and τ is a $\mathbb{G}^M$-stopping time.

In the credit analysis, the observable credit information on the market is often modelled by the progressive enlargement $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ of $\mathbb{F}$, that is, let $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ be the minimal filtration which makes τ a $\mathbb{D}$-stopping time, i.e. $\mathcal{D}_t = \mathcal{D}^0_{t+}$ with $\mathcal{D}^0_t = \sigma(\tau \wedge t)$, then $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$. In our model (1.2), this is interpreted as

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{L \leq X^*_t\})$$

and we call this information on L the "progressive information on L". If one observes $\mathcal{G}_t$ at time t, it means that he possesses information on $\mathcal{F}_t$, together with whether or not the default has occurred up to t. We see that the filtration $(\mathcal{G}^M_t)_{t\geq 0}$ represents indeed an insider's information of the manager, which is larger than $(\mathcal{G}_t)_{t\geq 0}$.

We now consider another type of agent: an investor whose observations on L is partial, described as the "noisy full information on L". That is, he observes $L_s = f(L, \epsilon_s)$ with ϵ being an independant noise perturbing the information L. The filtration of the investor's information is then

$$\mathcal{G}_t^I := \mathcal{F}_t \vee \sigma(L_s, s \leq t).$$

2.2 Partial observation on the underlying process

The process X driving the default risk is not totally observable for all agents. We suppose that at date 0, all investors are completely informed on the firm value. Later on, the agents are differently informed on the process X. Assume in the sequel that the process X is associated with a standard Brownian motion B (for example, X is a geometric Brownian motion or the solution of some SDE). Let $\mathcal{N}$ denotes the $\mathbb{P}$ null sets and we assume that $\mathcal{F}_t = \sigma(B_s, s \leq t) \vee \mathcal{N}$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ represents the information of an investor having complete information of the fundamental process X. Most investors on the market only have an incomplete observation described by an auxiliary filtration of $\mathbb{F}$. In the literature, there are several ways (noisy or delayed) to describe the incomplete information that we recall below.

Example 2.1 A structural type model with deterministic barrier is studied in [3]. The partial information is represented by an auxiliary process β depending on some noisy signal of the process X. The information of an investor observing the noisy signal of X is represented by the filtration $\mathcal{F}_t^\beta := \sigma(\beta_s, s \leq t) \vee \mathcal{N}$.

Example 2.2 The investors may have a delayed (continuous or discrete) observation of the fundamental process X, this type of models have been considered, among others, by [6, 4, 15, 11]. In this case, the observable information is characterized by a sub-filtration $(\mathcal{F}_t^D)_{t\geq 0}$ of $\mathbb{F}$, constructed by either a time change (continuously delayed filtration) or by a discretely delayed filtration.

In the following, we are particularly interested in the delayed information case. Let

$$\mathcal{F}_t^D = \begin{cases} \mathcal{F}_0 & \text{if } t \leq \delta(t), \\ \mathcal{F}_{t-\delta(t)} & \text{if } t > \delta(t), \end{cases}$$

where $\delta(t)$ is some function on t we shall precise. The above formulation covers the constant delay time model where $\delta(t) = \delta$ (see [4], [11]) and the discrete observation model where $\delta(t) = t - t_i^{(m)}$, $t_i^{(m)} \leq t < t_{i+1}^{(m)}$ where $0 = t_0^{(m)} < t_1^{(m)} < \cdots < t_m^{(m)} = T$ are the only discrete dates on which the $(\mathcal{F}_t)_{t\geq 0}$ information may be renewed (release dates of the accounting reports of the firm for example, see [6], [15]). In this case, the investor's information is represented by the filtration

$$\mathcal{G}_t^D := \mathcal{F}_t^D \vee \mathcal{D}_t.$$

3 Three types of side-information on L

In this section, we precise the hypothesis on the three types of side-information on L. Recall that the default barrier is fixed at date 0 by the manager as the realization of a random variable L.

3.1 Full information

We suppose in this subsection that the manager knows the barrier L ω-wise from the beginning. Thus his information is given as

$$\mathbb{G}^M = (\mathcal{G}_t^M)_{t\geq 0} \ \text{ with } \ \mathcal{G}_t^M := \mathcal{F}_t \vee \sigma(L).$$

As we have mentioned previously, this corresponds to the initial enlargement of the filtration $\mathbb{F}$ with respect to the random variable L. Let us make more precise the nature of this initial information.

Assumption 3.1 *We assume that L is an $\mathcal{A}$-measurable random variable with values in a Polish space $(E, \mathcal{E})$ which satisfies the assumption :*
$\mathbb{P}(L \in \cdot|\mathcal{F}_t)(\omega) \sim \mathbb{P}(L \in \cdot)$ *for all t for $\mathbb{P}$ almost all $\omega \in \Omega$.*

Remark: Assumption 3.1 is satisfied if L is independent of $\mathcal{F}_\infty$.

Assumption 3.1 is the standard assumption by Jacod [13, 14]. We denote by $P_t^L(\omega, dx)$ a regular version of the conditional law of L given $\mathcal{F}_t$ and by P^L the law of L. According to Jacod, there exists a measurable version of the conditional density $p_t(x)(\omega) = \frac{dP_t^L}{dP^L}(\omega, x)$ which is a $(\mathbb{F}, \mathbb{P})$-martingale and for all t, $p_t(L) > 0$ $\mathbb{P}$ almost surely.
Grorud et al. [10] proved that Assumption 3.1 is equivalent to the existence of a probability measure equivalent to $\mathbb{P}$ and under which for any $t \geq 0$, $\mathcal{F}_t$ and $\sigma(L)$ are independent. We consider the only one, denoted $\mathbb{Q}^L$, that is identical to $\mathbb{P}$ on $\mathcal{F}_\infty$. $\mathbb{Q}^L$ is characterized by the density process

$$E_{\mathbb{Q}^L}[\frac{d\mathbb{P}}{d\mathbb{Q}^L}|\mathcal{G}_t^M] = p_t(L). \tag{3.1}$$

3.2 Noisy full information

In this subsection, the investors information on the barrier L changes through time. His knowledge is perturbed by an independent noise, and is getting to him clearer as time evolves.

Assumption 3.2
$\forall t, \mathcal{G}_t^I = \cap_{u>t}(\mathcal{F}_u \vee \sigma(L_s, s \leq u))$ *where* $L_s = f(L, \epsilon_s)$ *with*

- *$f:\mathbb{R}^2\to\mathbb{R}$ is a given measurable function.*
- *$\epsilon=\{\epsilon_t, t\geq 0\}$ is independent of $\mathcal{F}_\infty\vee\sigma(L)$.*
- *$\mathbb{P}(L\in\cdot|\mathcal{F}_t)(\omega)\sim\mathbb{P}(L\in\cdot)$ for all t for $\mathbb{P}$ almost all $\omega\in\Omega$.*

If we work on a finite horizon T, the last two assumptions are
$\epsilon=\{\epsilon_t, t\leq T\}$ is independent of $\mathcal{F}_T\vee\sigma(L)$ and
$\mathbb{P}(L\in\cdot|\mathcal{F}_t)(\omega)\sim\mathbb{P}(L\in\cdot)$ for all $t\in[0,T[$ for $\mathbb{P}$ almost all $\omega\in\Omega$.

ϵ represents an additional noise that perturbs the knowledge of the barrier L. Therefore one expects in general that the variance of the noise decreases to zero as time t goes to infinity.

3.3 Progressive and delayed information

In this subsection, we concentrate on investors who know at each time t whether or not default has occured. Thus their information is given as the progressive enlargement of filtration of L with respect to $\mathbb{F}$:

$$\mathbb{G}=(\mathcal{G}_t)_{t\geq 0}\ \text{ with }\ \mathcal{G}_t=\mathcal{F}_t\vee\sigma(\{\tau\leq s\}, s\leq t).$$

Remark: If L is independent of $\mathcal{F}_\infty$, the standard (H)-hypothesis is satisfied:
every $(\mathbb{F},\mathbb{P})$ local martingale is also a $(\mathbb{G},\mathbb{P})$ local martingale.

The delayed information is in fact the progressive enlargement of filtration of L with respect to $\mathbb{F}^D$:

$$\mathbb{G}^D=(\mathcal{G}_t^D)_{t\geq 0}\ \text{ with }\ \mathcal{G}_t^D=\mathcal{F}_t^D\vee\sigma(\{\tau\leq s\}, s\leq t).$$

4 Conditional law of default

Our aim is to compute the conditional probabilities of default with respect to different filtrations. More precisely, we compute $\mathbb{P}(\tau>\theta|\mathcal{H}_t)$ for all t, for all θ, where the filtration $(\mathcal{H}_t)_{t\geq 0}$ describes the accessible information for the investors.

4.1 Full information

Proposition 4.1 *If $\mathcal{H}_t=\mathcal{G}_t^M$ is the full information, then under Assumption 3.1 we have*

$$\begin{aligned}\mathbb{P}(\tau>\theta|\mathcal{G}_t^M) &= \tfrac{1}{p_t(L)}[E_{\mathbb{P}}(p_\theta(x)1_{X_\theta^*>x}|\mathcal{F}_t)]_{x=L} && \textit{if } \theta>t\\ &= 1_{\tau>\theta} && \textit{otherwise}\end{aligned}$$

where $p_t(x)(\omega)=\frac{dP_t^L}{dP^L}(\omega,x)$, $P_t^L(\omega,dx)$ being a regular version of the conditional law of L given $\mathcal{F}_t$ and P^L being the law of L.

Proof: The result is trivial for $\theta \leq t$. Otherwise, using the facts that $\mathcal{F}_\theta$ and $\sigma(L)$ are independent under $\mathbb{Q}^L$, that $E_{\mathbb{Q}^L}[\frac{d\mathbb{P}}{d\mathbb{Q}^L}|\mathcal{G}_t^M] = p_t(L)$, and that $\mathbb{Q}^L$ is identical to $\mathbb{P}$ on $\mathcal{F}_\infty$, we have

$$\begin{aligned}
\mathbb{P}(\tau > \theta|\mathcal{G}_t^M) &= E_{\mathbb{P}}(1_{X_\theta^* > L}|\mathcal{F}_t \vee \sigma(L)) \\
&= \frac{1}{p_t(L)} E_{\mathbb{Q}^L}(p_\theta(L) 1_{X_\theta^* > L}|\mathcal{F}_t \vee \sigma(L)) \\
&= \frac{1}{p_t(L)} [E_{\mathbb{Q}^L}(p_\theta(x) 1_{X_\theta^* > x}|\mathcal{F}_t)]_{x=L} \\
&= \frac{1}{p_t(L)} [E_{\mathbb{P}}(p_\theta(x) 1_{X_\theta^* > x}|\mathcal{F}_t)]_{x=L}.
\end{aligned}$$

Remark: If $\mathcal{F}_\theta$ and $\sigma(L)$ are independent under $\mathbb{P}$ we obtain the simple formula

$$\mathbb{P}(\tau > \theta|\mathcal{G}_t^M) = P_t^{X_\theta^*}(]L, +\infty[),$$

$P_t^{X_\theta^*}(dy)$ being the regular conditional probability of X_θ^* given $\mathcal{F}_t$.

4.2 Noisy full information

In the sequel, $\mathcal{H}_t = \mathcal{G}_t^I$ and we consider the particular but useful case in finite horizon time T where $L_t = L + \epsilon_t$, $\epsilon_t = Z_{T-t}$, Z being a continuous process with independent increments whose marginal has density q_t (example introduced in Corcuera et al. [5]). For example, $\epsilon_t = W_{g(T-t)}$ with W an independent Brownian motion, and $g : [0, T] \to [0, +\infty)$ a strictly increasing bounded function with $g(0) = 0$.

Proposition 4.2 *We assume that $\mathcal{H}_t = \mathcal{G}_t^I$ is the progressive strong information with $L_t = L + \epsilon_t$, $\epsilon_t = Z_{T-t}$, Z being a continuous process with independent increments whose marginal has density q_t. Then we have*

$$\begin{aligned}
\mathbb{P}(\tau > \theta|\mathcal{G}_t^I) &= \frac{\int_{\mathbb{R}} \frac{1}{p_t(l)} E_{\mathbb{P}}(p_\theta(l) 1_{X_\theta^* > l}|\mathcal{F}_t) q_{T-t}(L_t - l) P_t^L(dl)}{\int_{\mathbb{R}} q_{T-t}(L_t - l) P_t^L(dl)} \qquad \textit{if } \theta > t \\
&= \frac{\int_{\mathbb{R}} 1_{X_\theta^* > l} q_{T-t}(L_t - l) P_t^L(dl)}{\int_{\mathbb{R}} q_{T-t}(L_t - l) P_t^L(dl)} \qquad \textit{otherwise}
\end{aligned}$$

where P_t^L is a regular version of the conditional law of L given $\mathcal{G}_t$.

Proof: Let $A_\theta \in \mathcal{F}_\theta$ and h be a bounded measurable function. Using the independence of $\mathcal{F}_{\theta \vee t} \vee \sigma(L)$ and Z, we have

$$\begin{aligned}
\mathbb{E}\left(h(L) 1_{A_\theta}|\mathcal{G}_t^I\right) &= \mathbb{E}\left(h(L) 1_{A_\theta}|\mathcal{F}_t \vee \sigma(L_t) \vee \sigma((\epsilon_t - \epsilon_s), s \leq t))\right) \\
&= \mathbb{E}\left(h(L) 1_{A_\theta}|\mathcal{F}_t \vee \sigma(L + \epsilon_t)\right)
\end{aligned}$$

Let $P_t^L(dl)$ be the regular conditional probability of L given $\mathcal{F}_t$. Then for $C \in \mathcal{B}(\mathbb{R}^2)$,

$$\mathbb{P}\left((L, L+\epsilon_t) \in C|\mathcal{F}_t\right) = \int_{\mathbb{R}^2} 1_C(l,x) q_{T-t}(x-l) P_t^L(dl)dx.$$

Therefore

$$\mathbb{E}\left(h(L)|\mathcal{G}_t^I\right) = \frac{\int_{\mathbb{R}} h(l) q_{T-t}(L_t - l) P_t^L(dl)}{\int_{\mathbb{R}} q_{T-t}(L_t - l) P_t^L(dl)}. \tag{4.1}$$

Hence, if $\theta \leq t$ we have

$$\mathbb{P}\left(\tau > \theta|\mathcal{G}_t^I\right) = \frac{\int_{\mathbb{R}} 1_{X_\theta^* > l} q_{T-t}(L_t - l) P_t^L(dl)}{\int_{\mathbb{R}} q_{T-t}(L_t - l) P_t^L(dl)}.$$

If $\theta > t$, we use the following successive conditional expectations

$$\mathbb{P}\left(\tau > \theta|\mathcal{F}_t \vee \sigma(L+\epsilon_t)\right) = \mathbb{P}\left(\mathbb{P}\left(\tau > \theta|\mathcal{F}_t \vee \sigma(L+\epsilon_t) \vee \sigma(L)\right)|\mathcal{F}_t \vee \sigma(L+\epsilon_t)\right).$$

Using the fact that ϵ is independent to $\mathcal{F}_T \vee \sigma(L)$, we have

$$\mathbb{P}\left(\tau > \theta|\mathcal{F}_t \vee \sigma(L+\epsilon_t) \vee \sigma(L)\right) = \mathbb{P}\left(X_\theta^* > L|\mathcal{F}_t \vee \sigma(\epsilon_t) \vee \sigma(L)\right) = \mathbb{P}\left(X_\theta^* > L|\mathcal{F}_t \vee \sigma(L)\right) =: h_t(L)$$

where $h_t(L) = \frac{1}{p_t(L)}[E_{\mathbb{P}}(p_\theta(x) 1_{X_\theta^* > x}|\mathcal{F}_t)]_{x=L}$ corresponds to the conditional default probability for the full information. Therefore

$$\mathbb{P}(\tau > \theta|\mathcal{G}_t^I) = \frac{\int_{\mathbb{R}} \frac{1}{p_t(l)} E_{\mathbb{P}}(p_\theta(l) 1_{X_\theta^* > l}|\mathcal{F}_t) q_{T-t}(L_t - l) P_t^L(dl)}{\int_{\mathbb{R}} q_{T-t}(L_t - l) P_t^L(dl)}.$$

Remark: This demonstration can be extended at others examples in infinite horizon. For example, let $\epsilon_t = W_{g(\frac{1}{t+1})}$ with W an independent Brownian motion, and $g : [0,1] \to [0,+\infty)$ a strictly increasing bounded function with $g(0) = 0$. ϵ_t is a centered gausian process with independent increments. Let q_t be the density of ϵ_t. We have

$$\mathbb{P}\left(\tau > \theta|\mathcal{G}_t^I\right) = \frac{\int_{\mathbb{R}} \frac{1}{p_t(l)} E_{\mathbb{P}}(p_\theta(l) 1_{X_\theta^* > l}|\mathcal{F}_t) q_t(L_t - l) P_t^L(dl)}{\int_{\mathbb{R}} q_t(L_t - l) P_t^L(dl)}.$$

4.3 Progressive information and the delayed case

The computation in the progressive information case is classical in the literature (e.g.[16, 8, 1]), which we recall below.

Proposition 4.3 *If $\mathcal{H}_t = \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ is the progressive information, we have*

$$\begin{aligned} \mathbb{P}(\tau > \theta|\mathcal{G}_t) &= 1_{\tau > t} \frac{\mathbb{E}(P_\theta^L(X_\theta^*)|\mathcal{F}_t)}{P_t^L(X_t^*)} && \text{if } \theta > t \\ &= 1_{\tau > \theta} && \text{otherwise.} \end{aligned}$$

Proof: It is trivial if $\theta \le t$. Otherwise, one has by classical computation in the progressive enlargement that

$$\mathbb{P}(\tau > \theta|\mathcal{G}_t) = 1_{\tau>t}\frac{\mathbb{P}(\tau > \theta|\mathcal{F}_t)}{\mathbb{P}(\tau > t|\mathcal{F}_t)} = 1_{\tau>t}\frac{\mathbb{E}(S_\theta|\mathcal{F}_t)}{S_t}, \qquad \theta > t$$

where $S_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$. In our model, S is given explicitly by $S_t = \mathbb{P}(X_t^* > L|\mathcal{F}_t) = P_t^L(X_t^*)$ with P_t^L being the conditional law of L given $\mathcal{F}_t$. This leads to the result.

The delayed case is computed similarly.

Corollary 4.4 *If $\mathcal{H}_t = \mathcal{G}_t^D = \mathcal{F}_t^D \vee \mathcal{D}_t$, we have*

$$\begin{aligned}\mathbb{P}(\tau > \theta|\mathcal{G}_t^D) &= 1_{\tau>t}\frac{\mathbb{E}(P_\theta^L(X_\theta^*)|\mathcal{F}_t^D)}{\mathbb{E}(P_t^L(X_t^*)|\mathcal{F}_t^D)} && \text{if } \theta > t\\ &= 1_{\tau>\theta} && \text{otherwise.}\end{aligned}$$

Proof: It's trivial for $\theta \le t$. If $\theta > t$, we have similarly as in the previous case

$$\mathbb{P}(\tau > \theta|\mathcal{G}_t^D) = 1_{\tau>t}\frac{\mathbb{P}(\tau > \theta|\mathcal{F}_t^D)}{\mathbb{P}(\tau > t|\mathcal{F}_t^D)} = 1_{\tau>t}\frac{\mathbb{E}(S_\theta|\mathcal{F}_t^D)}{\mathbb{E}(S_t|\mathcal{F}_t^D)} = 1_{\tau>t}\frac{\mathbb{E}(P_\theta^L(X_\theta^*)|\mathcal{F}_t^D)}{\mathbb{E}(P_t^L(X_t^*)|\mathcal{F}_t^D)}.$$

5 Geometrical brownian example and numerical illustrations

We consider in this section explicit examples of independent and dependent default threshold L, where the asset values process X satisfies the Black Scholes model :

$$\frac{dX_t}{X_t} = \mu dt + \sigma dB_t, \quad t \ge 0$$

where μ and σ are real constants and B is an $\mathbb{F}$-Brownian motion. For $t \ge 0$ and $h, l > 0$, one has ([1, p.69])

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(1_{X_t^*>l} - 1_{X_{t+h}^*>l}|\mathcal{F}_t) &= 1_{X_t^*>l}\left(\Phi\Big(\frac{-Y_t^l - \nu h}{\sigma\sqrt{h}}\Big) + e^{2\nu\sigma^{-2}Y_t}\Phi\Big(\frac{-Y_t^l + \nu h}{\sigma\sqrt{h}}\Big)\right)\\ &=: 1_{X_t^*>l}\Phi_{t,h}(l)\end{aligned}$$

where Φ is the standard Gaussian cumulative distribution function and

$$Y_t^l = \nu t + \sigma B_t + \ln\frac{X_0}{l}, \quad \text{with} \quad \nu = \mu - \frac{1}{2}\sigma^2.$$

5.1 Example of an independent default threshold

Corollary 5.1 *We assume that the default threshold L independent of $\mathcal{F}_T$. Then if the asset process X satisfies the Black Scholes model, we have*

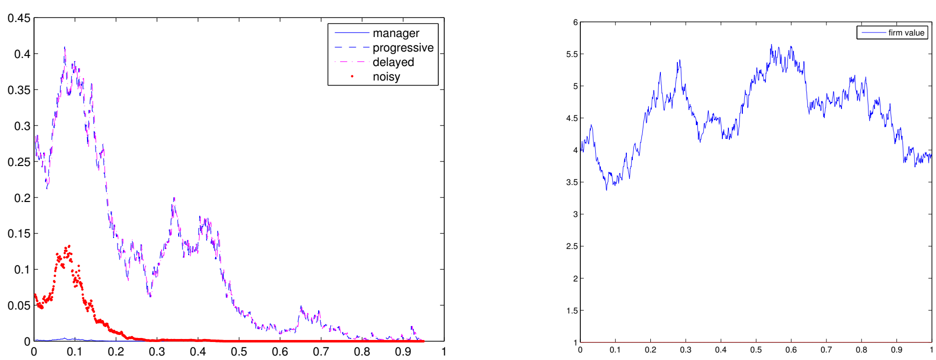
- $\mathbb{P}(t+h \geq \tau > t | \mathcal{G}_t^M) = 1_{\tau > t} \Phi_{t,h}(L)$

- $\mathbb{P}(t+h \geq \tau > t | \mathcal{G}_t^I) = \frac{\int 1_{X_t^* > l} \Phi_{t,h}(l) q_{T-t}(L_t - l) P_t^L(dl)}{\int q_{T-t}(L_t - l) P_t^L(dl)}$

- $\mathbb{P}(t+h \geq \tau > t | \mathcal{G}_t) = 1_{\tau > t} \frac{\int \Phi_{t,h}(l) P^L(dl)}{\int 1_{X_t^* > l} P^L(dl)}$

- $\mathbb{P}(t+h \geq \tau > t | \mathcal{G}_t^D) = 1_{\tau > t} \frac{\int (\Phi_{t-\delta(t), h+\delta(t)}(l) - \Phi_{t-\delta(t), \delta(t)}(l)) P^L(dl)}{\int (1 - \Phi_{t-\delta(t), \delta(t)}(l)) P^L(dl)}$

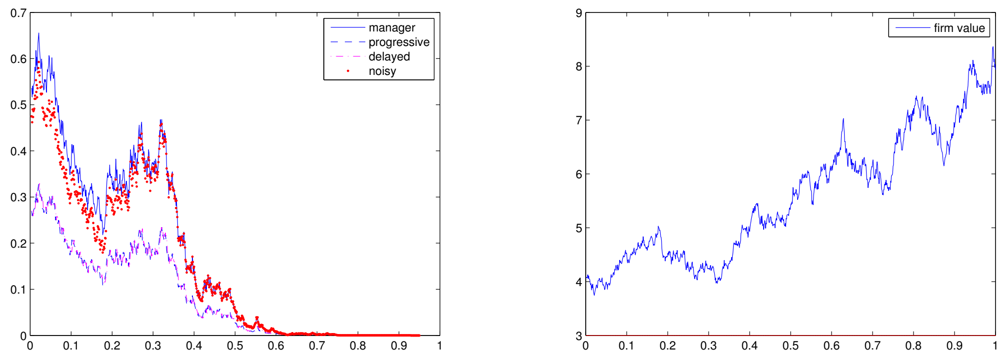
We give numerical comparisons of the conditional probability of default for different information, in the following binomial example: L independent, $L = l_i$ with probability α, $L = l_s$ with probability $(1-\alpha)$, $0 < \alpha < 1$, $l_i \leq l_s$. In the simulation, we take the numerical values: $l_i = 1, l_s = 3, \alpha = \frac{1}{2}$.



$t \to \mathbb{P}(T \geq \tau > t | \mathcal{H}_t)$ firm value

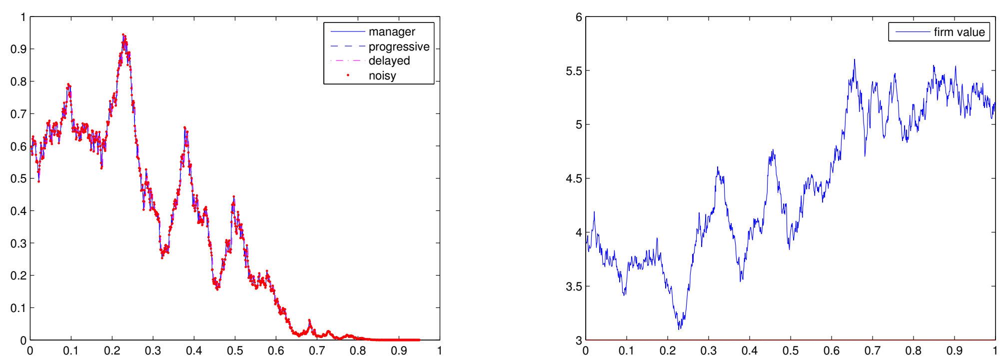
Figure 1: $L = l_i$

Comments : The probabilities of default for a full or noisy full information are quite different from the ones with respect to the progressive or the delayed information. More precisely, if $L = l_i$, the manager has fixed the lower value for the default threshold and thus the probability of default will be lower for the full information than for the progressive information (see Figure 1). Conversely if $L = l_s$ (see Figure 2). In both cases, the estimation of the default probability for the noisy full information is between the estimations for the full and the progressive information. If L is constant ($l_i = l_s$), the probabilities of default are the same, whatever the information we consider (see Figure 3). We observe that the variation of the default probabilities is closely related to the variation of the firm value.



$t \to \mathbb{P}(T \geq \tau > t|\mathcal{H}_t)$ firm value

Figure 2: $L = l_s$



$t \to \mathbb{P}(T \geq \tau > t|\mathcal{H}_t)$ firm value

Figure 3: $l_i = l_s$: L constant

5.2 Example of a dependent default threshold

Let

$$L = l_i 1_{[a,+\infty[}(X_A) + l_s 1_{[0,a[}(X_A), \quad A > T, \quad l_i \leq l_s. \tag{5.1}$$

The manager chooses the level of L according to a constant threshold a and to the value of the asset process X on some given date A ($A > T$ where T is a fixed horizon time, for example the maturity of the credit derivatives we consider). If $X_A \geq a$, the manager judge the firm on healthy situation and choose the lower barrier l_i, otherwise, he will choose the higher barrier to accelerate the default.

We have explicitely the conditional law of L given $\mathcal{F}_t$ for $t < A$,

$$\mathbb{P}(L = l_s|\mathcal{F}_t) = \mathbb{P}(X_A < a|\mathcal{F}_t) = \Phi(k_t), \quad \mathbb{P}(L = l_i|\mathcal{F}_t) = 1 - \Phi(k_t)$$

where Φ is the cumulative distribution function of the normal distribution $N(0, 1)$ and

$$k_t = \frac{\ln a - \ln X_0 - \nu A - \sigma B_t}{\sigma\sqrt{A - t}}, \ \nu = \mu - \frac{1}{2}\sigma^2.$$

Hence

$$p_t(l_s) = \frac{\Phi(k_t)}{\Phi(k_0)}, \quad p_t(l_i) = \frac{1 - \Phi(k_t)}{1 - \Phi(k_0)}.$$

To compute $\mathbb{P}(\tau > t+h|\mathcal{G}_t^M) = \frac{1}{p_t(L)}[E_{\mathbb{P}}(p_{t+h}(l)1_{X^*_{t+h}>l}|\mathcal{F}_t)]_{l=L}$, we use the following lemma [1]

Lemma 5.2 *For $y \geq 0$, on the set $\{\tau > t\}$*

$$\mathbb{P}\left(Y^l_{t+h} \geq y, 1_{X^*_{t+h}>l}|\mathcal{F}_t\right) = \Phi\big(\frac{-y + Y^l_t + \nu h}{\sigma\sqrt{h}}\big) - e^{2\nu\sigma^{-2}Y_t}\Phi\big(\frac{-y - Y^l_t + \nu h}{\sigma\sqrt{h}}\big)$$

where Φ is the standard Gaussian cumulative distribution function, $Y^l_t = \nu t + \sigma B_t + \ln\frac{X_0}{l}$ and $\nu = \mu - \frac{1}{2}\sigma^2$.

We then deduce

- the conditional joint law of $(Y^l_{t+h}, 1_{X^*_{t+h}>l}|\mathcal{F}_t)$
- the conditional joint law $(p_{t+h}(l), 1_{X^*_{t+h}>l}|\mathcal{F}_t)$

More precisely, for $\theta > t$,

$$\mathbb{P}(\tau > \theta|\mathcal{G}_t^M) = 1_{\{L=l_s\}}\frac{1}{\Phi(k_t)}\mathbb{E}(\Phi(k_\theta)1_{\{X^*_\theta>l_s\}}|\mathcal{F}_t)+1_{\{L=l_i\}}\frac{1}{1 - \Phi(k_t)}\mathbb{E}((1-\Phi(k_\theta))1_{\{X^*_\theta>l_i\}}|\mathcal{F}_t).$$

$\Phi(k_t) = g_{l_s}(Y^{l_s}_t)$ where $g_{l_s}(x) = \Phi\left(\frac{\ln\frac{a}{l_s}-x-\nu(A-t)}{\sigma\sqrt{A-t}}\right)$. So

$$\mathbb{E}(\Phi(k_\theta)1_{X^*_\theta>l_s}|\mathcal{F}_t) = \mathbb{E}(g_{l_s}(Y^{l_s}_\theta)1_{(Y^{l_s})^*_\theta>0}|\mathcal{F}_t).$$

Denote by $F_{t,\theta,l_s}(y) := \mathbb{P}(Y_\theta^{l_s} \geq y, 1_{X_\theta^* > l_s} | \mathcal{F}_t)$ and $f_{t,\theta,l_s}(y) = \frac{\partial}{\partial y} F_{t,\theta,l_s}(y)$, then

$$\mathbb{E}(\Phi(k_\theta) 1_{X_\theta^* > l_s} | \mathcal{F}_t) = \mathbb{E}(g_{l_s}(Y_\theta^{l_s}) 1_{(Y^{l_s})_\theta^* > 0} | \mathcal{F}_t) = 1_{X_t^* > l_s} \int_0^\infty g_{l_s}(y) f_{t,\theta,l_s}(y) dy. \tag{5.2}$$

This gives the conditional default probability for the full information. The result for the noisy information is then straightforward, using Proposition 4.2. The progressive and delayed case is a classical computation. We have similar observations to those of the previous section with the numerical results.

6 Conclusions

We have investigated the impact of different information levels on the conditional default probabilities. The conditional survival probability plays an important role in the pricing of credit derivatives (we refer the interested reader to a forthcoming work of Hillairet and Jiao [12]). For example let us consider a defaultable bond with zero recovery, that is, the buyer of the bond receives 1 euro if there is no default and zero otherwise. Then the price of such a product is given exactly as the conditional survival probability with respect to the accessible information. Furthermore, in the credit risk analysis, one often calls the default intensity (or the default spread) as the instantaneous default probability

$$\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}(t < \tau \leq t + \Delta t | \mathcal{H}_t), \quad a.s.$$

We observe immediately that the conditional default probability is the key term to compute this quantity.

In the literature, the information on the value process of the firm has been thoroughly studied. However, only few works concentrate on the default threshold. Our results show that the information on the default threshold also have a significant impact in the credit risk analysis and deserve to be studied in more details.

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