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The nonlinear Schrödinger equation with white noise dispersion

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THE NONLINEAR SCHRÖDINGER EQUATION WITH WHITE NOISE DISPERSION

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Abstract. Under certain scaling the nonlinear Schrödinger equation with random dispersion converges to the nonlinear Schrödinger equation with white noise dispersion. The aim of this work is to prove that this latter equation is globally well posed in L^2 or H^1. The main ingredient is the generalization of the classical Strichartz estimates. Additionally, we justify rigorously the formal limit described above.

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1. Introduction

The following nonlinear Schrödinger equation with random dispersion describes the propagation of a signal in an optical fibre with dispersion management (see [?],[?]):

$$\begin{cases} i\dfrac{dv}{dt} + \varepsilon m(t)\partial_{xx}v + \varepsilon^2 |v|^2 v = 0,\ x \in \mathbb{R},\ t > 0, \\ v(0,x) = v_0(x),\ x \in \mathbb{R}. \end{cases} \tag{1.1}$$

Recall that in the context of fibre optics, x corresponds to the retarded time while t corresponds to the distance along the fibre. The coefficient $\varepsilon m(t)$ accounts for the fact that ideally one would want a fibre with zero dispersion, in order to avoid chromatic dispersion of the signal. This is impossible to build in practise and engineers have proposed to build fibres with a small dispersion which varies along the fibre and has zero average. The case of a periodic deterministic dispersion has been studied in [?] where an averaged equation is derived. This averaged equation is then shown to possess ground states (see [?] for the case of positive residual dispersion, that is when $m(t)$ has positive average over a period, and [?] for the case of vanishing residual dispersion). Note that in this periodic setting, the nonlinear term is not of size ε^2 as such a nonlinear term would have no effect on the dynamics, the equation studied in [?] has in fact the coefficient ε in front on the nonlinearity.

In this article, we consider the case of a random dispersion, i.e. m is a centered stationary random process. As will be clear from our study, only a nonlinearity of order ε^2 is relevant in this context. In order to understand the limit as the small parameter ε goes to zero, it is natural to rescale the time variable by setting $u(t,x) = v(\frac{t}{\varepsilon}, x)$ and we obtain

$$\begin{cases} i\dfrac{du}{dt} + \dfrac{1}{\varepsilon} m\left(\dfrac{1}{\varepsilon^2}\right)\partial_{xx}u + |u|^2 u = 0,\ x \in \mathbb{R},\ t > 0, \\ u(0) = u_0,\ x \in \mathbb{R}. \end{cases} \tag{1.2}$$

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This model has been initially studied in [?] where a split step numerical scheme is proposed to simulate its solutions. Under classical ergodic assumptions on m, it is expected that the limiting model when ε goes to zero is the following stochastic nonlinear Schrödinger equation with white noise dispersion

$$\left\{ \begin{array}{l} idu + \sigma_0 \partial_{xx} u \circ d\beta + |u|^2 u = 0,\ x \in \mathbb{R},\ t > 0, \\ u(0) = u_0,\ x \in \mathbb{R}, \end{array} \right. \tag{1.3}$$

where β is a standard real valued Brownian motion, $\sigma_0^2 = 2\int_0^{+\infty} \mathbb{E}[m(0)m(t)]dt$, and $\circ$ is the Stratonovich product. In [?], the cubic nonlinearity is replaced by a nicer Lipschitz function so that the limiting equation can be easily studied using the fact that the evolution associated to the linear equation defines an isometry in all L^2 based Sobolev spaces. It is shown that the nonlinear Schrödinger equation with white noise dispersion is indeed the limit of the original problem and this result is used to prove that some numerical scheme produces good approximation result for a time step significantly higher than ε. Again, all this study is performed for an equation where a nice Lipschitz function replaces the power nonlinearity.

Our aim is to address the original equation with power nonlinearity. In fact, we study the more general equation for $\sigma > 0$:

$$\left\{ \begin{array}{l} idu + \sigma_0^2 \Delta u \circ d\beta + |u|^{2\sigma} u\, dt = 0,\ x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0,\ x \in \mathbb{R}^d. \end{array} \right.$$

Note that the sign in front of the nonlinear term $|u|^2 u$ is not important here, as it can be changed from $+1$ to -1 by changing β to $-\beta$ and u to its complex conjugate. Also, we will assume without loss of generality that $\sigma_0^2 = 1$.

We recall that the usual nonlinear Schrödinger equation

$$\left\{ \begin{array}{l} i\partial_t u + \Delta u + |u|^{2\sigma} u = 0,\ x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0,\ x \in \mathbb{R}^d. \end{array} \right. \tag{1.4}$$

preserves the Hamiltonian

$$H(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 dx - \frac{1}{2\sigma + 2} \int_{\mathbb{R}^d} |u|^{2\sigma+2} dx.$$

However, the varying dispersion destroys the Hamiltonian character of the equation. On the mathematical point of view, this implies the loss of the a priori estimate provided by the energy H and no a priori estimates in H^1 are available. On the contrary, the mass, equal to the square of the L^2 norm is still preserved. Thus, a L^2 theory is necessary to get global solutions. For equation (??), such a theory is possible thanks to Strichartz estimates which imply ultracontractivity of the linear group (see [?], [?], [?], [?]). We prove in Section 3 that Strichartz estimates can be generalized to the equation with white noise dispersion. This allows to construct local in time solutions for $\sigma < 2/d$, in L^2 or H^1, in section 4. Then in section 5, the conservation of the mass is used to prove global existence. Also, we prove that regularity is preserved so that if the initial state is H^1, then the L^2 global solution is a.s. continuous in time with values in H^1. Finally, in section 6, we show that Marty's method to prove convergence of solutions when ε goes to zero is easily generalized once the previous results are obtained.

In [?] some numerical simulations are given. It would be very interesting to do a more general and systematic numerical study on the equations considered here. For instance, the influence of the random dispersion on blow-up phenomena could be investigated (see [?], [?], [?] for such a study with different noises), eventhough this phenomenon is not present in fibre optics.

We finally note that all the results stated in Section 2 would still hold with a nonzero but small residual dispersion, i.e. if equation (**??**). is replaced by

$$\begin{cases} i\dfrac{dv}{dt} + \varepsilon m(t)\partial_{xx}v + \varepsilon^2\nu\partial_{xx}v + \varepsilon^2|v|^2v = 0,\ x\in\mathbb{R},\ t>0, \\ v(0,x) = v_0(x),\ x\in\mathbb{R}, \end{cases}$$

where $\nu \in \mathbb{R}$ is a constant. In this case, of course, the limit equation (**??**) should be replaced by

$$\begin{cases} idu + \sigma_0\partial_{xx}u\circ d\beta + \nu\partial_{xx}u + |u|^2u = 0,\ x\in\mathbb{R},\ t>0, \\ u(0) = u_0,\ x\in\mathbb{R}, \end{cases} \tag{1.5}$$

All the analysis made in the present paper applies to the above equation, the only difference being in the proof of Proposition **??** (see Remark **??**). However, the study of the complete model where residual, periodic and random dispersions are taken into account is more delicate, and will be the object of further studies. We refer to [?] for results on the complete model, using the physicists "collective coordinates" approach.

2. Preliminaries and main results

We consider the following stochastic nonlinear Schrödinger (NLS) equation

$$\begin{cases} idu + \Delta u\circ d\beta + |u|^{2\sigma}u\,dt = 0,\ x\in\mathbb{R}^d,\ t>0, \\ u(0) = u_0,\ x\in\mathbb{R}^d, \end{cases} \tag{2.1}$$

where the unknown u is a random process on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ depending on $t>0$ and $x\in\mathbb{R}^d$. The nonlinear term is a power law. The noise term involves a brownian motion β associated to a stochastic basis $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\geq 0})$. The product $\circ$ is a Stratonovich product. As usual, we do not consider this equation but its formally equivalent Itô form:

$$\begin{cases} idu + \dfrac{i}{2}\Delta^2u\,dt + \Delta u\,d\beta + |u|^{2\sigma}u\,dt = 0,\ x\in\mathbb{R}^d,\ t>0, \\ u(0) = u_0. \end{cases} \tag{2.2}$$

Note that, formally, the $L^2(\mathbb{R}^d)$ norm of a solution is a conserved quantity. However, the time dependent dispersion destroys the Hamiltonian character of the classical Nonlinear Schrödinger equation and there does not exist an energy here. We study this equation in the framework of the $L^2(\mathbb{R}^d)$ based Sobolev spaces. We also use the spaces $L^p(\mathbb{R}^d)$ to treat the nonlinear term thanks the Strichartz estimates. In order to lighten the presentation, we use the following notations

$$H^s_x = H^s(\mathbb{R}^d),\ L^p_x = L^p(\mathbb{R}^d),\ p\geq 1,$$

and, when the time interval I does not need to be specified or is obvious from the context:

$$L^r_tL^p_x = L^r(I;L^p(\mathbb{R}^d)),\ r,p\geq 1.$$

Note that, in all the article, these are spaces of complex valued functions. The norm of a Banach space K is simply denoted by $|\cdot|_K$. When we consider moments with respect to the random parameter $\omega\in\Omega$, we sometimes write

$$L^p_\omega(K) = L^p(\Omega;K),\ p\geq 1.$$

For spaces of predictible time dependent processes, we use the subscript $\mathcal{P}$. For instance $L^r_{\mathcal{P}}(\Omega; L^p(0,T;K))$ is the subspace of $L^r(\Omega; L^p(0,T;K))$ consisting of predictible processes. We will denote associate conjugate exponents using "prime" upperscripts, that is if $p \geq 1$, then p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Our first main result is the following.

Theorem 2.1. *Assume $\sigma < \frac{2}{d}$; let $u_0 \in L^2_x$ a.s.be $\mathcal{F}_0$-measurable, then there exists a unique solution u to (**??**) with paths a.s. in $L^r_{loc}(0,\infty; L^p(\mathbb{R}^d))$, with $p = 2\sigma+2 \leq r < \frac{4(\sigma+1)}{d\sigma}$; moreover, u has paths in $C(\mathbb{R}^+; L^2_x)$, a.s. and*

$$|u(t)|_{L^2_x} = |u_0|_{L^2_x}, \quad \text{a.s.}$$

u also has the additional integrability properties :

- *$u \in L^\rho_{loc}(0,+\infty; L^\infty(\mathbb{R}))$ a.s. for any $\rho < 4$ if $d = 1$.*
- *$u \in L^\rho_{loc}(0,+\infty; L^q(\mathbb{R}^d))$ a.s. for any (ρ, q) with $2 \leq q < \frac{2d}{d-2}$, and $2 \leq \rho < \frac{4q}{d(q-2)}$ if $d \geq 2$.*

If in addition $u_0 \in H^1_x$, then u has paths a.s. in $C(\mathbb{R}^+; H^1_x)$.

Remark 2.2. *In the case $\frac{2}{d} \leq \sigma \leq \frac{2}{d-2}$ (or $\frac{2}{d} \leq \sigma < +\infty$ if $d = 1$ or 2), it is possible to prove a local existence result of solutions with paths a.s. in $C([0,\tau]; H^1_x)$ provided $u_0 \in H^1_x$, using similar argument as those used in the present paper, but with a cut-off at fixed time in $L^{2\sigma+2}_x$ norm (see Section 4 for the necessity of the use of a cut-off). However, because no energy conservation is available for equation (**??**), only in the case $\sigma < 2/d$ global existence may be obtained, thanks to the conservation of L^2 norm and Strichartz estimates.*

The result of Theorem **??** is used to justify rigorously the convergence of the solution of the random equation (**??**) to the solution of (**??**) with $\sigma = 1$, $d = 1$. In order to state the result precisely, we assume the following.

Assumption 1. *The real valued centered stationary random process $m(t)$ is continous and such that for any $T > 0$, the process $t \mapsto \varepsilon \int_0^{t/\varepsilon^2} m(s)ds$ converges in distribution to a standard real valued Brownian motion in $C([0,T])$.*

Let us recall classical conditions on m ensuring that the above Assumption 1 is satisfied. This holds e.g. if m is a Markov process with a unique and ergodic invariant measure and its generator satisfies the Fredholm alternative; for instance, m can satisfy Doeblin's condition. Assumption 1 also holds under some mixing conditions on m. We refer to [**?**], [**?**], [**?**], [**?**] and [**?**] for more general and precise conditions.

To our knowledge, Strichartz estimates are not available for equation (**??**). Hence we cannot get solutions in $L^2(\mathbb{R})$. Since the equation is set in space dimension 1, a local existence existence result can be easily proved in $H^1(\mathbb{R})$ but since no energy is available we do not know if the solutions are global in time. In the following result, we prove that the lifetime of the solutions converges to infinity when ε goes to zero, and that solutions of (**??**) converge in distribution to the solutions of the white noise driven equation (**??**).

Theorem 2.3. *Suppose that m satisfies the above assumption. Then, for any $\varepsilon > 0$ and $u_0 \in H^1(\mathbb{R})$, there exists a unique solution u_ε of equation (??) with continuous paths in $H^1(\mathbb{R})$ which is defined on a random interval $[0, \tau_\varepsilon(u_0))$. Moreover, for any $T > 0$*

$$\lim_{\varepsilon \to 0} \mathbb{P}(\tau_\varepsilon(u_0) \le T) = 0,$$

and the process $u_\varepsilon \mathbb{1}_{[\tau_\varepsilon > T]}$ converges in distribution to the solution u of (??) in $C([0,T]; H^s(\mathbb{R}))$ for any $s < 1$.

3. The linear equation and Strichartz type estimates

It is important to understand the properties of the linear part of equation (??). Indeed, in the case of the deterministic NLS equation, the linear part possesses ultracontractivity properties which are extremely helpful to solve the nonlinear equation (see for instance [?]). We use below this equation starting from an initial data at various initial times. We therefore consider in this section the following stochastic linear Schrödinger equation:

$$\begin{cases} idu + \Delta u \circ d\beta = 0, \ t \ge s, \\ u(s) = u_s. \end{cases} \tag{3.1}$$

We interpret this equation in the Itô sense and consider the following equation which is formally equivalent to (??):

$$\begin{cases} idu + \dfrac{i}{2}\Delta^2 u \, dt + \Delta u \, d\beta = 0, \ t \ge s, \\ u(s) = u_s. \end{cases} \tag{3.2}$$

As was noticed in [?], we have an explicit formula for the solutions of (??).

Proposition 3.1. *For any $s \le T$ and $u_s \in \mathcal{S}'(\mathbb{R}^n)$, there exists a unique solution of (??) almost surely in $C([s,T]; \mathcal{S}'(\mathbb{R}^n))$ and adapted. Its Fourier transform in space is given by*

$$\hat{u}(t,\xi) = e^{-i|\xi|^2(\beta(t)-\beta(s))}\hat{u}_s(\xi), \ t \ge s, \ \xi \in \mathbb{R}^d.$$

Moreover, if $u_s \in H^\sigma_x$ for some $\sigma \in \mathbb{R}$, then $u(\cdot) \in C([0,T]; H^\sigma_x)$ a.s. and $|u(t)|_{H^\sigma} = |u_s|_{H^\sigma}$, a.s. for $t \ge s$.

If $u_s \in L^1_x$, the solution u of (??) has the expression

$$u(t) = S(t,s)u_s := \frac{1}{(4i\pi\,(\beta(t)-\beta(s)))^{d/2}} \int_{\mathbb{R}^d} \exp\left(i\frac{|x-y|^2}{4(\beta(t)-\beta(s))}\right) u_s(y)dy, \ t \in [s,T]. \tag{3.3}$$

Proof. The proof is the same as in the deterministic case (see for instance [?]). It suffices to take the Fourier transform in space of equation (??). □

Proposition 3.2 leads to the following spatial estimates for the solution $S(t,s)u_s$.

Lemma 3.2. *For any $p \ge 2$ and $s \le t$, $S(t,s)$ maps $L^{p'}_x$ into L^p_x and there exists a constant C_p depending only on p such that*

$$|S(t,s)u_s|_{L^p_x} \le \frac{C_p}{|\beta(t)-\beta(s)|^{d(\frac{1}{2}-\frac{1}{p})}} |u_s|_{L^{p'}}, \text{ for any } u_s \in L^{p'}.$$

Proof. It is easily seen from (**??**) and a density argument that $S(t,s)$ is an isometry on L^2_x. Thus, the result is true for $p=2$ with $C_2=1$. Also, for $p=\infty$, we obtain the result from (**??**) with $C_\infty = \frac{1}{(4\pi)^{d/2}}$. The general result follows from the Riesz-Thorin interpolation theorem. □

Lemma **??** is the preliminary step to get Strichartz type estimates. Contrary to the classical deterministic case, we cannot immediately deduce from Lemma 3.2 space-time estimates on the mapping $f \mapsto \int_0^{\cdot} S(\cdot,s)f(s)ds$. This is due to the fact that formula (**??**) defining $S(t,s)u_s$ is not in terms of $t-s$ and the Hausdorff-Young inequality for convolution cannot be used here in order to get estimates in time. We need the following result.

Proposition 3.3. *Let $\alpha\in[0,1)$, there exists a constant c_α depending only on α such that for any $T\ge 0$ and $f\in L^2_{\mathcal{P}}(\Omega;L^2(0,T))$*

$$\mathbb{E}\left(\int_0^T\left(\int_0^t \frac{1}{|\beta(t)-\beta(s)|^\alpha}|f(s)|ds\right)^2 dt\right) \le c_\alpha T^{2-\alpha}\mathbb{E}\left(\int_0^T |f(s)|^2 ds\right)$$

Proof. The result is clear for $\alpha=0$ so that by an interpolation argument, it suffices to consider the case $\alpha\in(1/2,1)$. Let us write

$$\begin{aligned}\left(\int_0^t \frac{1}{|\beta(t)-\beta(s)|^\alpha}|f(s)|ds\right)^2 &= \int_0^t\int_0^t \frac{|f(s_1)|\,|f(s_2)|}{|\beta(t)-\beta(s_1)|^\alpha|\beta(t)-\beta(s_2)|^\alpha}ds_1ds_2\\ &= 2\int_0^t\int_0^{s_1} \frac{|f(s_1)|\,|f(s_2)|}{|\beta(t)-\beta(s_1)|^\alpha|\beta(t)-\beta(s_2)|^\alpha}ds_2ds_1.\end{aligned}$$

Since f is adapted, and $|\beta(t)-\beta(s_1)|$ is independent of $\mathcal{F}_{s_1}$, we may write

$$\begin{aligned}I &= \mathbb{E}\int_0^T\left(\int_0^t \frac{1}{|\beta(t)-\beta(s)|^\alpha}|f(s)|ds\right)^2 dt\\ &= 2\,\mathbb{E}\int_0^T\int_0^t\int_0^{s_1} \frac{|f(s_1)|\,|f(s_2)|}{|\beta(t)-\beta(s_1)|^\alpha|(\beta(t)-\beta(s_1))+(\beta(s_1)-\beta(s_2))|^\alpha}ds_2ds_1dt\\ &= 2\int_0^T\int_0^t\int_0^{s_1}\mathbb{E}\left(\left(\int_{\mathbb{R}}\frac{1}{|x|^\alpha|x+(\beta(s_1)-\beta(s_2))|^\alpha}\mu(dx)\right)|f(s_1)|\,|f(s_2)|\right)ds_2ds_1dt\end{aligned}$$

where $\mu=\mathcal{N}(0,t-s_1)$ is the law of $\beta(t)-\beta(s_1)$. We have

$$\begin{aligned}&\int_{\mathbb{R}}\frac{1}{|x|^\alpha|x+(\beta(s_1)-\beta(s_2))|^\alpha}\mu(dx)\\ &= \frac{1}{(2\pi(t-s_1))^{1/2}}\int_{\mathbb{R}}\frac{1}{|x|^\alpha|x+(\beta(s_1)-\beta(s_2))|^\alpha}e^{-\frac{|x|^2}{2(t-s_1)}}dx\\ &= \frac{1}{(2\pi)^{1/2}}(t-s_1)^{-\alpha}\int_{\mathbb{R}}\frac{1}{|x|^\alpha\left|x+\dfrac{\beta(s_1)-\beta(s_2)}{(t-s_1)^{1/2}}\right|^\alpha}e^{-\frac{|x|^2}{2}}dx.\end{aligned}$$

We need the following Lemma.

Lemma 3.4. *Let $\alpha \in (1/2,1)$, there exists a constant c_α depending on α such that for any $\gamma \in \mathbb{R}$, $\gamma \neq 0$,*

$$\int_{\mathbb{R}} \frac{e^{-\frac{|x|^2}{2}}}{|x|^\alpha |x-\gamma|^\alpha} dx \leq \begin{cases} c_\alpha |\gamma|^{1-2\alpha}, \ |\gamma| \in (0,1), \\ \\ c_\alpha, \ |\gamma| \geq 1. \end{cases}$$

Proof. By symmetry, we may assume $\gamma > 0$.

For $\gamma \in (0,1)$, we split the integral on the disjoint intervals $(-\infty,-1)$, $[-1,\gamma+1]$ and $(\gamma+1,+\infty)$ and majorize the integrand to obtain

$$\begin{aligned} \int_{\mathbb{R}} \frac{e^{-\frac{|x|^2}{2}}}{|x|^\alpha |x-\gamma|^\alpha} dx &\leq \int_{-\infty}^{-1} e^{-\frac{|x|^2}{2}} dx + \int_{-1}^{\gamma+1} \frac{1}{|x|^\alpha |x-\gamma|^\alpha} dx + \int_{\gamma+1}^{\infty} e^{-\frac{|x|^2}{2}} dx \\ &\leq \gamma^{1-2\alpha} \int_{-\frac{1}{2}-\gamma^{-1}}^{\frac{1}{2}+\gamma^{-1}} \frac{1}{|y-\frac{1}{2}|^\alpha |y+\frac{1}{2}|^\alpha} dy + (2\pi)^{1/2} \\ &\leq 2\gamma^{1-2\alpha} \max\left\{ \int_{\mathbb{R}} \frac{1}{|y-\frac{1}{2}|^\alpha |y+\frac{1}{2}|^\alpha} dy ; (2\pi)^{1/2} \right\}. \end{aligned}$$

For $\gamma \geq 1$, we have

$$\begin{aligned} &\int_{\mathbb{R}} \frac{e^{-\frac{|x|^2}{2}}}{|x|^\alpha |x-\gamma|^\alpha} dx \\ \leq \ & \frac{1}{2^{2\alpha}} \int_{(-\infty,-\frac{1}{2})\cup(\frac{1}{2},\gamma-\frac{1}{2})\cup(\gamma+\frac{1}{2},\infty)} e^{-\frac{|x|^2}{2}} dx + \frac{1}{2^\alpha} \int_{-1/2}^{1/2} \frac{1}{|x|^\alpha} dx + \frac{1}{2^\alpha} \int_{\gamma-\frac{1}{2}}^{\gamma+\frac{1}{2}} \frac{1}{|x-\gamma|^\alpha} dx \\ \leq \ & \frac{(2\pi)^{1/2}}{2^{2\alpha}} dx + \frac{2}{1-\alpha}. \end{aligned}$$

□

We now proceed with the estimate of I. For $|\beta(s_1)-\beta(s_2)| \leq |t-s_1|^{1/2}$, we deduce from Lemma **??**:

$$\begin{aligned} (t-s_1)^{-\alpha} \int_{\mathbb{R}} \frac{e^{-\frac{|x|^2}{2}}}{|x|^\alpha \left| x + \dfrac{\beta(s_1)-\beta(s_2)}{(t-s_1)^{1/2}} \right|^\alpha} dx &\leq c_\alpha |t-s_1|^{-1/2} |\beta(s_1)-\beta(s_2)|^{1-2\alpha} \\ &\leq c_\alpha |t-s_1|^{-\alpha/2} |\beta(s_1)-\beta(s_2)|^{-\alpha}. \end{aligned}$$

On the other hand, if $|\beta(s_1)-\beta(s_2)| > |t-s_1|^{1/2}$,

$$(t-s_1)^{-\alpha} \int_{\mathbb{R}} \frac{e^{-\frac{|x|^2}{2}}}{|x|^\alpha \left| x + \dfrac{\beta(s_1)-\beta(s_2)}{(t-s_1)^{1/2}} \right|^\alpha} dx \leq c_\alpha (t-s_1)^{-\alpha}.$$

It follows

$$\begin{aligned}
I &\leq 2c_\alpha \int_0^T \int_0^t \int_0^{s_1} \mathbb{E}\left[|t-s_1|^{-\alpha/2}|\beta(s_1)-\beta(s_2)|^{-\alpha} + (t-s_1)^{-\alpha}\right] |f(s_1)|\,|f(s_2)| ds_2 ds_1 dt \\
&\leq \frac{2c_\alpha}{1-\alpha/2} T^{1-\alpha/2} \mathbb{E} \int_0^T |f(s_1)| \int_0^{s_1} |\beta(s_1)-\beta(s_2)|^{-\alpha} |f(s_2)| ds_2 ds_1 \\
&\quad + \frac{2c_\alpha}{1-\alpha} T^{1-\alpha} \mathbb{E} \int_0^T |f(s_1)| \int_0^{s_1} |f(s_2)| ds_2 ds_1 \\
&\leq \frac{2c_\alpha}{1-\alpha/2} T^{1-\alpha/2} \left(\mathbb{E} \int_0^T |f(s_1)|^2 ds_1 \right)^{1/2} I^{1/2} + \frac{2c_\alpha}{1-\alpha} T^{2-\alpha} \mathbb{E} \int_0^T |f(s_1)|^2 ds_1 \\
&\leq c'_\alpha T^{2-\alpha} \mathbb{E} \int_0^T |f(s_1)|^2 ds_1 + \frac{1}{2} I,
\end{aligned}$$

from which we deduce the result. □

Remark 3.5. *The reader may easily convince himself that the estimate of Proposition* **??** *is still true with the same bound on the right hand side if* $|\beta(t)-\beta(s)|^\alpha$ *on the left hand side is replaced by* $|\beta(t)-\beta(s)+\nu(t-s)|^\alpha$*. This is the only change to be made to apply all the analysis of the paper to equation* (**??**).

Corollary 3.6. *Let* $\alpha = 0$*,* $r \leq \infty$ *or* $\alpha \in (0,1)$*,* $2 \leq r < \frac{2}{\alpha}$ *and* ρ *be such that* $r' \leq \rho \leq r$*; then there exists* $C_{\alpha,\rho,r}$ *such that, for any* $T \geq 0$ *and* $f \in L^\rho_{\mathcal{P}}(\Omega; L^{r'}(0,T))$*,*

$$\left| \int_0^t |\beta(t)-\beta(s)|^{-\alpha} |f(s)| ds \right|_{L^\rho_\omega(L^r(0,T))} \leq C_{\alpha,\rho,r} T^{\frac{2}{r}-\frac{\alpha}{2}} |f|_{L^\rho(\Omega;L^{r'}(0,T))}.$$

Proof. The result is clear for $\alpha = 0$ and $\rho \leq r = \infty$. For $\alpha < 1$ and $\rho = r = 2$, it is the statement of Proposition **??**. We obtain the general result by an interpolation argument. □

Corollary **??** is exactly what we need to replace Hausdorff-Young inequality in order to get Strichartz type estimates. Note that in the deterministic case, *i.e.* if $\beta(t)$ is replaced by t, the limiting case $r = \frac{2}{\alpha}$ is allowed. We state an immediate consequence of Lemma **??** and Corollary **??**.

Proposition 3.7. *Let* $2 \leq r < \infty$ *and* $2 \leq p \leq \infty$ *be such that* $\frac{2}{r} > d\left(\frac{1}{2}-\frac{1}{p}\right)$ *or* $r = \infty$ *and* $p = 2$*. Let* ρ *be such that* $r' \leq \rho \leq r$*; there exists a constant* $c_{\rho,r,p} > 0$ *such that for any* $s \in \mathbb{R}$*,* $T \geq 0$ *and* $f \in L^\rho_{\mathcal{P}}(\Omega; L^{r'}(s,s+T;L^{p'}_x))$

$$\left| \int_s^\cdot S(\cdot,\sigma) f(\sigma) d\sigma \right|_{L^\rho(\Omega;L^r(s,s+T;L^p_x))} \leq c_{\rho,r,p} T^\beta \, |f|_{L^\rho(\Omega;L^{r'}(s,s+T;L^{p'}_x))}$$

with $\beta = \frac{2}{r} - \frac{d}{2}\left(\frac{1}{2}-\frac{1}{p}\right)$.

Remark 3.8. *This result is very similar to the classical Strichartz estimates. However, we need* $\frac{2}{r} > d\left(\frac{1}{2} - \frac{1}{p}\right)$ *whereas in the classical case, one can choose* $\frac{2}{r} = d\left(\frac{1}{2} - \frac{1}{p}\right)$. *A pair of numbers* (r, p) *satisfying this latter condition is often called an admissible pair. We believe that in the stochastic case considered here the result is still true for* $\frac{2}{r} = d\left(\frac{1}{2} - \frac{1}{p}\right)$ *but our proof does not cover this case. Note also that the exponent* β *is much bigger than in the classical case where one would have* $\beta = \frac{1}{r} - \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$.

By analogy with the deterministic theory we define admissible pairs.

Definition 3.9. *A pair of real numbers is called an admissible pair if* $r = \infty$ *and* $p = 2$ *or if the following conditions are satisfied:*

$$2 \le r < \infty, \quad 2 \le p \le \infty \textit{ and } \quad \frac{2}{r} > d\left(\frac{1}{2} - \frac{1}{p}\right).$$

Proof of Proposition ??. let (r, p) be an admissible pair, let ρ be such that $r' \le \rho \le r$ and let $f \in L^\rho_{\mathcal{P}}(\Omega; L^{r'}(s, s+T; L^{p'}_x))$. By Lemma **??**

$$\begin{aligned}\left|\int_s^t S(t,\sigma) f(\sigma) d\sigma\right|_{L^p_x} &\le \int_s^t |S(t,\sigma) f(\sigma)|_{L^p_x} d\sigma \\ &\le c\int_s^t \frac{1}{|\beta(t) - \beta(\sigma)|^{d(\frac{1}{2} - \frac{1}{p})}} |f(\sigma)|_{L^{p'}_x} d\sigma.\end{aligned}$$

By Corollary **??** with $\alpha = d(\frac{1}{2} - \frac{1}{p}) \in [0, 1)$, we deduce

$$\mathbb{E}\left(\left|\int_s^{\cdot} S(\cdot,\sigma) f(\sigma) d\sigma\right|^\rho_{L^r(s,s+T;L^p_x)}\right) \le c T^{\rho(\frac{2}{r} - \frac{d}{2}(\frac{1}{2} - \frac{1}{p}))} |f|^\rho_{L^\rho(\Omega; L^{r'}(s,s+T;L^{p'}_x))},$$

which is the result. $\square$

Using a duality argument, we then have :

Proposition 3.10. *Let* $2 \le r \le \infty$ *and* $2 \le p \le \infty$ *be such that* $\frac{2}{r} > d\left(\frac{1}{2} - \frac{1}{p}\right)$ *or* $r = \infty$ *and* $p = 2$*; there exists a constant* $c_{r,p} > 0$ *such that for any* $s \in \mathbb{R}$, $T \ge 0$ *and* $u_s \in L^r(\Omega; L^2_x)$, $\mathcal{F}_s$*-measurable,* $S(\cdot, s) u_s \in L^r_{\mathcal{P}}(\Omega; L^r(s, s+T; L^p_x))$ *and*

$$|S(\cdot, s) u_s|_{L^r\left(\Omega; L^r(s,s+T;L^p_x)\right)} \le c_{r,p} T^{\beta/2} |u_s|_{L^r_\omega(L^2_x)}$$

with $\beta = \frac{2}{r} - \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$.

Proof. Note that $S(t, s)^* = S(s, t)$, where the adjoint is taken with respect to the L^2_x inner product. Thus for $u_s \in L^r(\Omega; L^2_x)$, $\mathcal{F}_s$-measurable, and $f \in L^2_{\mathcal{P}}(\Omega \times [s, s+T] \times \mathbb{R}^d)$ we have

$$\begin{aligned}\int_s^{s+T} (S(t,s) u_s, f(t)) dt &= \int_s^{s+T} (u_s, S(s,t) f(t)) dt \\ &\le |u_s|_{L^2_x} \left|\int_s^{s+T} S(s,t) f(t) dt\right|_{L^2_x}.\end{aligned}$$

Moreover

$$\begin{aligned}
\left|\int_s^{s+T} S(s,t)f(t)dt\right|^2_{L^2_x} &= \int_s^{s+T}\int_s^{s+T}(f(t),S(t,\sigma)f(\sigma))dtd\sigma \\
&= \int\int_{s\le\sigma\le t\le s+T}(f(t),S(t,\sigma)f(\sigma))dtd\sigma \\
&\quad+\int\int_{s\le t\le\sigma\le s+T}(S(\sigma,t)f(t),f(\sigma))dtd\sigma \\
&= 2\int\int_{s\le\sigma\le t\le s+T}(f(t),S(t,\sigma)f(\sigma))dtd\sigma \\
&= 2\int_s^{s+T}(f(t),\int_s^t S(t,\sigma)f(\sigma)d\sigma)dt \\
&\le 2\,|f|_{L^{r'}(s,s+T;L^{p'}_x)}\left|\int_s^{\cdot} S(\cdot,\sigma)f(\sigma)d\sigma\right|_{L^r(s,s+T;L^p_x)}.
\end{aligned}$$

It follows from Proposition **??**,

$$\begin{aligned}
\mathbb{E}\int_s^{s+T}(S(t,s)u_s,f(t))dt &\le 2^{1/2}\mathbb{E}\left(|u_s|_{L^2_x}\,|f|^{1/2}_{L^{r'}(s,s+T;L^{p'}_x)}\left|\int_s^{\cdot} S(\cdot,\sigma)f(\sigma)d\sigma\right|^{1/2}_{L^r(s,s+T;L^p_x)}\right) \\
&\le 2^{1/2}|u_s|_{L^r_\omega(L^2_x)}\,|f|^{1/2}_{L^{r'}_\omega(L^{r'}(s,s+T;L^{p'}_x))}\left|\int_s^{\cdot} S(\cdot,\sigma)f(\sigma)d\sigma\right|^{1/2}_{L^{r'}_\omega(L^r(s,s+T;L^p_x))} \\
&\le c\,T^{\beta/2}|u_s|_{L^r_\omega(L^2_x)}\,|f|_{L^{r'}_\omega(L^{r'}(s,s+T;L^{p'}_x))}
\end{aligned}$$

with $\beta=\frac{2}{r}-\frac{d}{2}\left(\frac12-\frac1p\right)$. This implies the result. □

In the deterministic case, it is well known that Strichartz estimates still hold with different admissible pairs in the left and right hand sides. We also have such results here. These will be useful later to prove regularity properties of solutions of the nonlinear equation and to prove rigorously that these are indeed limits of solutions of equation (**??**) when ε goes to 0.

Proposition 3.11. *Let (r,p) and (γ,δ) be two admissible pairs such that*

$$\frac{1}{\gamma}=\frac{1-\lambda}{r},\ \frac{1}{\delta}=\frac{\lambda}{2}+\frac{1-\lambda}{p}, \tag{3.4}$$

with $\lambda\in[0,1]$, and ρ be such that $\max\{\rho,\rho'\}\le r$; then there exists a constant $c(r,p,\gamma,\delta,\rho)$ such that for any $s\in\mathbb{R},\ T\ge0$,

$$\left|\int_s^{\cdot} S(\cdot,\sigma)f(\sigma)d\sigma\right|_{L^\rho(\Omega;L^r(s,s+T;L^p_x))}\le c(r,p,\gamma,\delta,\rho)T^{\tilde\beta}\,|f|_{L^\rho(\Omega;L^{\gamma'}(s,s+T;L^{\delta'}_x))} \tag{3.5}$$

if $f \in L^\rho_{\mathcal{P}}(\Omega; L^{\gamma'}(s, s+T; L^{\delta'}_x))$ *and*

$$\left| \int_s^\cdot S(\cdot,\sigma) f(\sigma) d\sigma \right|_{L^\rho(\Omega; L^\gamma(s,s+T; L^\delta_x))} \leq c(r,p,\gamma,\delta,\rho) T^{\tilde{\beta}} \, |f|_{L^\rho(\Omega; L^{r'}(s,s+T; L^{p'}_x))} \tag{3.6}$$

if $f \in L^\rho_{\mathcal{P}}(\Omega; L^{r'}(s, s+T; L^{p'}_x))$. *In this latter case, we also have*

$$\int_s^\cdot S(\cdot,\sigma) f(\sigma) d\sigma \in L^\rho(\Omega; C([s, s+T]; L^2_x)). \tag{3.7}$$

Here, $\tilde{\beta} = \left(\frac{2}{r} - \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right)\right)\left(1 - \frac{\lambda}{2}\right)$.

Proof. We first consider the case $\lambda = 1$ in (**??**) and prove that given (r,p) an admissible pair, $r' \leq \rho' \leq r$ and $f \in L^\rho_{\mathcal{P}}(\Omega; L^1(s, s+T; L^2_x))$, we have

$$\left| \int_s^\cdot S(\cdot,\sigma) f(\sigma) d\sigma \right|_{L^\rho(\Omega; L^r(s,s+T; L^p_x))} \leq c T^{\beta/2} \, |f|_{L^\rho(\Omega; L^1(s,s+T; L^2_x))} \tag{3.8}$$

with $\beta = \frac{2}{r} - \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$.

In order to prove this, we consider $\varphi \in L^{\rho'}_{\mathcal{P}}(\Omega; L^{r'}(s, s+T; L^{p'}_x))$ and write

$$\begin{aligned}
& \mathbb{E}\left(\int_s^{s+T}\left(\int_s^t S(t,\sigma) f(\sigma) d\sigma, \varphi(t)\right) dt\right) \\
= \;& \mathbb{E}\left(\int_s^{s+T}\int_s^t \left(f(\sigma), S(\sigma,t)\varphi(t)\right) d\sigma dt\right) \\
= \;& \mathbb{E}\left(\int_s^{s+T}\left(f(\sigma), \int_\sigma^{s+T} S(\sigma,t)\varphi(t) dt\right) d\sigma\right) \\
\leq \;& \mathbb{E}\left(|f|_{L^1_t L^2_x} \sup_{\sigma\in[s,s+T]}\left|\int_\sigma^{s+T} S(\sigma,t)\varphi(t) dt\right|_{L^2_x}\right).
\end{aligned}$$

We need to bound the second factor. For any $\sigma \in [s, s+T]$, we have

$$\begin{aligned}
&\left| \int_\sigma^{s+T} S(\sigma,t)\varphi(t)dt \right|_{L_x^2}^2 \\
= & \int_\sigma^{s+T} \int_\sigma^{s+T} \left(S(\sigma,t)\varphi(t), S(\sigma,\theta)\varphi(\theta) \right) dt\, d\theta \\
= & \; 2 \int \int_{\sigma \le t \le \theta \le s+T} \left(S(\theta,t)\varphi(t), \varphi(\theta) \right) dt\, d\theta \\
= & \; 2 \int_\sigma^{s+T} \left(\int_\sigma^\theta S(\theta,t)\varphi(t)dt, \varphi(\theta) \right) d\theta \\
\le & \; 2 \left| \varphi \right|_{L^{r'}(\sigma,s+T;L_x^{p'})} \left| \int_\sigma^{\cdot} S(\cdot,t)\varphi(t)dt \right|_{L^r(\sigma,s+T;L_x^p)} \\
\le & \; 2 \left| \varphi \right|_{L^{r'}(s,s+T;L_x^{p'})} \left| \int_\sigma^{\cdot} \left| S(\cdot,t)\varphi(t) \right|_{L_x^p} dt \right|_{L^r(\sigma,s+T)} \\
\le & \; 2 \left| \varphi \right|_{L^{r'}(s,s+T;L_x^{p'})} \left| \int_s^{\cdot} \left| S(\cdot,t)\varphi(t) \right|_{L_x^p} dt \right|_{L^r(s,s+T)} .
\end{aligned}$$

Therefore

$$\sup_{\sigma \in [s,s+T]} \left| \int_\sigma^{s+T} S(\sigma,t)\varphi(t)dt \right|_{L_x^2}^2 \le 2 \left| \varphi \right|_{L_t^{r'} L_x^{p'}} \left| \int_s^{\cdot} \left| S(\cdot,t)\varphi(t) \right|_{L_x^p} dt \right|_{L_t^r}$$

and

$$\begin{aligned}
&\mathbb{E} \left(\int_s^{s+T} \left(\int_s^t S(t,\sigma) f(\sigma) d\sigma, \varphi(t) \right) dt \right) \\
\le & \; \sqrt{2} \mathbb{E} \left(\left| f \right|_{L_t^1 L_x^2} \left| \varphi \right|_{L_t^{r'} L_x^{p'}}^{1/2} \left| \int_s^{\cdot} \left| S(\cdot,t)\varphi(t) \right|_{L_x^p} dt \right|_{L_t^r}^{1/2} \right) \\
\le & \; \sqrt{2} \left| f \right|_{L_\omega^\rho L_t^1 L_x^2} \left| \varphi \right|_{L_\omega^{\rho'} L_t^{r'} L_x^{p'}}^{1/2} \left| \int_s^{\cdot} \left| S(\cdot,t)\varphi(t) \right|_{L_x^p} dt \right|_{L_\omega^{\rho'} L_t^r}^{1/2} \\
\le & \; c\, T^{\beta/2} \left| f \right|_{L_\omega^\rho L_t^1 L_x^2} \left| \varphi \right|_{L_\omega^{\rho'} L_t^{r'} L_x^{p'}}
\end{aligned}$$

if $r' \le \rho' \le r$, or equivalently if $r' \le \rho \le r$, and with $\beta = \frac{2}{r} - \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$ by the same argument as for Proposition **??**. Claim (**??**) follows.

By Proposition **??**, we have

$$\left|\int_s^{\cdot} S(\cdot,\sigma)f(\sigma)d\sigma\right|_{L^\rho(\Omega;L^r(s,s+T;L^p_x))} \leq cT^\beta\,|f|_{L^\rho(\Omega;L^{r'}(s,s+T;L^{p'}_x))} \tag{3.9}$$

if $r' \leq \rho \leq r$. Interpolation between (**??**) and (**??**) leads to (**??**).

The second inequality is proved similarly : we have by similar arguments as above

$$\left|\int_s^t S(t,\sigma)f(\sigma)d\sigma\right|^2_{L^2_x} \leq 2\,|f|_{L^{r'}(s,s+T;L^{p'}_x)}\left|\int_s^{\cdot} |S(\cdot,\sigma)f(\sigma)|_{L^p_x}\,d\sigma\right|_{L^r(s,s+T)},$$

for any $t \in [s;s+T]$. Therefore, by Cauchy-Schwarz inequality,

$$\begin{aligned}\left|\int_s^{\cdot} S(\cdot,\sigma)f(\sigma)d\sigma\right|^2_{L^\rho_\omega L^\infty_t L^2_x} &\leq c\,|f|^{\frac12}_{L^\rho_\omega L^{r'} L^{p'}_x}\left|\int_s^{\cdot} |S(\cdot,\sigma)f(\sigma)|_{L^p_x}\,d\sigma\right|^{\frac12}_{L^\rho_\omega L^r_t}\\ &\leq cT^{\beta/2}\,|f|_{L^\rho_\omega L^{r'} L^{p'}_x}.\end{aligned}$$

The fact that $\int_s^{\cdot} S(\cdot,\sigma)f(\sigma)d\sigma$ has a.s. continuous paths with values in L^2_x follows from a density argument and the preceding estimate. Again, (**??**) follows by interpolation between the above inequality and (**??**). □

4. A truncated equation

We now construct a local solution of equation (**??**). We use a similar cut-off of the nonlinearity as in [**?**] and [**?**]. Let $\theta \in C_0^\infty(\mathbb{R})$ be such that $\theta = 1$ on $[0,1]$, $\theta = 0$ on $[2,\infty)$. For $s \in \mathbb{R}$, $u \in L^r_{loc}(s,\infty;L^p_x)$, $R \geq 1$ and $t \geq 0$, we set

$$\theta^s_R(u)(t) = \theta\left(\frac{|u|_{L^r(s,s+t;L^p_x)}}{R}\right).$$

For $s=0$, we set $\theta^0_R = \theta_R$. We take in this section $p = 2\sigma+2$ and r such that $2\sigma+2 \leq r < \frac{4(\sigma+1)}{d\sigma}$. Note that such a r exists, since we have assumed $\sigma < \frac{2}{d}$.

We consider the following truncated form of equation (**??**)

$$\left\{\begin{array}{l} idu^R + \Delta u^R \circ d\beta + \theta_R(u^R)|u^R|^{2\sigma}u^R\,dt = 0,\\ u^R(0) = u_0.\end{array}\right. \tag{4.1}$$

More precisely, we consider the truncation of its Itô form

$$\left\{\begin{array}{l} idu^R + \frac{i}{2}\Delta^2 u^R\,dt + \Delta u^R d\beta + \theta_R(u^R)|u^R|^{2\sigma}u^R\,dt = 0,\\ u^R(0) = u_0.\end{array}\right. \tag{4.2}$$

We interpret it in the mild sense

$$u^R(t) = S(t,0)u_0 + i\int_0^t S(t,s)\theta_R(u^R)(s)|u^R(s)|^{2\sigma}u^R(s)ds. \tag{4.3}$$

Theorem 4.1. *Let $\sigma < \frac{2}{d}$, $p = 2\sigma + 2$ and r be such that $2\sigma + 2 \leq r < \frac{4(\sigma+1)}{d\sigma}$. For any $\mathcal{F}_0$-measurable $u_0 \in L^r_\omega(L^2_x)$, there exists a unique u^R in $L^r_{\mathcal{P}}(\Omega \times [0,T]; L^p_x)$ for any $T > 0$, solution of (**??**). Moreover u^R is a weak solution of (**??**) in the sense that for any $\varphi \in C_0^\infty(\mathbb{R}^d)$ and any $t \geq 0$,*

$$\begin{aligned} & i(u^R(t) - u_0, \varphi)_{L^2_x} \\ = \quad & -\frac{i}{2}\int_0^t (u^R, \Delta^2\varphi)_{L^2_x} ds - \int_0^t \theta_R(u^R)(|u^R|^{2\sigma}u^R, \varphi)_{L^2_x} ds - \int_0^t (u^R, \Delta\varphi)_{L^2_x} d\beta(s), \ a.s. \end{aligned}$$

Finally, the L^2_x norm is conserved:

$$|u^R(t)|_{L^2_x} = |u_0|_{L^2_x}, \ t \geq 0, \ a.s.$$

and $u \in C([0,T]; L^2_x)$ a.s.

Proof. In order to lighten the notations we omit the R dependence in this proof. By Proposition **??**, we know that $S(\cdot, 0)u_0 \in L^r_{\mathcal{P}}(\Omega \times [0,T]; L^p_x)$. Then, by Proposition **??**, for $u, v \in L^r_{\mathcal{P}}(\Omega \times [0,T]; L^p_x)$,

$$\begin{aligned} & \left| \int_0^t S(t,s) \left(\theta(u)(s)|u(s)|^{2\sigma}u(s) - \theta(v)(s)|v(s)|^{2\sigma}v(s) \right) ds \right|_{L^r(\Omega\times[0,T];L^p_x)} \\ \leq \quad & cT^\beta \left| \theta(u)|u|^{2\sigma}u - \theta(v)|v|^{2\sigma}v \right|_{L^r_\omega(L^{r'}([0,T];L^{p'}_x))} \end{aligned}$$

with $\beta = \frac{2}{r} - \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$. Moreover, by standard arguments (see [**?**]),

$$\left| \theta(u)|u|^{2\sigma}u - \theta(v)|v|^{2\sigma}v \right|_{L^r_\omega(L^{r'}([0,T];L^{p'}_x))} \leq c\, T^\gamma R^{2\sigma} |u - v|_{L^r(\Omega\times[0,T];L^p_x)}$$

with $\gamma = 1 - \frac{2\sigma+2}{r}$. It follows that

$$\mathcal{T}^R : u \mapsto S(t,0)u_0 + i\int_0^t S(t,s)\theta(u(s))|u(s)|^{2\sigma}u(s)ds \tag{4.4}$$

defines a strict contraction on $L^r_{\mathcal{P}}(\Omega \times [0,T]; L^p_x)$ provided $T \leq T_0$ where T_0 depends only on R. Iterating this construction, one easily ends the proof of the first statement. The proof that u is in fact a weak solution is classical.

Let $M \geq 0$ and $u_M = P_M u$ be a regularization of the solution u defined by a truncation in Fourier space: $\hat{u}_M(t,\xi) = \theta\left(\frac{|\xi|}{M}\right)\hat{u}(t,\xi)$. We deduce from the weak form of the equation that

$$idu_M + \frac{i}{2}\Delta^2 u_M\, dt + \Delta u_M d\beta + P_M\left(\theta(u)|u|^{2\sigma}u\right)\, dt = 0.$$

We apply Itô formula to $|u_M|^2_{L^2_x}$ and obtain

$$|u_M(t)|^2_{L^2_x} = |u_0|^2_{L^2_x} + Re\left(i\int_0^t \left(\theta(u)|u|^{2\sigma}u, P_M u_M\right) ds \right), \ t \in [0,T].$$

We know that $u \in L^{2\sigma+2}([0,T] \times \mathbb{R}^d)$ a.s. Since

$$\lim_{M\to\infty} P_M u_M = u \text{ in } L^{2\sigma+2}([0,T] \times \mathbb{R}^d),$$

we may let M go to infinity in the above equality and obtain

$$\lim_{M\to\infty} |u_M(t)|_{L^2_x} = |u_0|_{L^2_x},\ t \in [0,T],\ a.s.$$

This implies $u(t) \in L^2_x$ for any $t \in [0,T]$ and $|u(t)|_{L^2_x} = |u_0|_{L^2_x}$. In particular $u \in L^\infty(0,T;L^2_x)$. As easily seen from the weak form of the equation, u is almost surely continuous with values in H^{-4}_x. It follows that u is weakly continuous with values in L^2_x. Finally the continuity of $t \mapsto |u(t)|_{L^2_x}$ implies $u \in C([0,T];L^2_x)$ and $|u(t)|_{L^2_x} = |u_0|_{L^2_x}$ a.s. □

5. Proof of Theorem ??

We use the solution of the truncated problem obtained in Section 4 to construct a solution to the original equation (**??**). There is no loss of generality in assuming that $u_0 \in L^2_x$ is deterministic. Uniqueness is clear since two solutions are solutions of the truncated equation on a random interval.

Let us define

$$\tau_R = \inf\{t \in [0,T],\ |u^R|_{L^r(0,t;L^p_x)} \geq R\}.$$

Clearly u^R is a solution of (**??**) on $[0,\tau_R]$. In order to see that τ_R cannot be too small, we need to prove that the $L^r_t L^p_x$ norm of u^R can be controlled. Recall that $p = 2\sigma + 2$ and $2\sigma + 2 \leq r \leq \frac{4(\sigma+1)}{d\sigma}$.

We fix a T_0 and explain how to construct a solution of (**??**) on $[0,T_0]$.

Lemma 5.1. *There exist constants c_1, c_2 such that if*

$$T^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma} \leq c_1\, R^{-2r\sigma}$$

then

$$\mathbb{P}(\tau_R \leq T) \leq \frac{c_2 |u_0|^r_{L^2_x}}{R^r}$$

Proof. Let us write

$$u^R(t)\mathbb{1}_{[0,\tau_R]}(t) = S(t,0)u_0\mathbb{1}_{[0,\tau_R]}(t) + i\int_0^t S(t,s)|u^R|^{2\sigma}u^R\mathbb{1}_{[0,\tau_R]}(s)ds\mathbb{1}_{[0,\tau_R]}(t). \tag{5.1}$$

Thus for $T \leq T_0$

$$|u^R\mathbb{1}_{[0,\tau_R]}|_{L^r(0,T;L^p_x)} \leq |S(\cdot,0)u_0\mathbb{1}_{[0,\tau_R]}|_{L^r(0,T;L^p_x)} + |\int_0^t S(t,s)|u^R|^{2\sigma}u^R\mathbb{1}_{[0,\tau_R]}(s)ds|_{L^r(0,T;L^p_x)}.$$

Proposition **??** and Proposition **??** yield

$$\mathbb{E}\left(|u^R\mathbb{1}_{[0,\tau_R]}|^r_{L^r(0,T;L^p_x)}\right) \leq c(r,T_0)|u_0|^r_{L^2_x} + c\,T^{2-\frac{dr\sigma}{4(\sigma+1)}}\mathbb{E}(||u^R|^{2\sigma+1}\mathbb{1}_{[0,\tau_R)}|^r_{L^{r'}(L^{p'})}).$$

Then, by Hölder inequality,

$$\begin{aligned}\mathbb{E}\left(|u^R\mathbb{1}_{[0,\tau_R]}|^r_{L^r(0,T;L^p_x)}\right) &\leq c(r,T_0)|u_0|^r_{L^2_x} + c\,T^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma}\mathbb{E}(|u^R\mathbb{1}_{[0,\tau_R)}|^{r(2\sigma+1)}_{L^r(L^p)}) \\ &\leq c(r,T_0)|u_0|^r_{L^2_x} + c\,T^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma}R^{2r\sigma}\mathbb{E}(|u^R\mathbb{1}_{[0,\tau_R)}|^r_{L^r(L^p)})\end{aligned}$$

Hence, if $c\,T^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma}R^{2r\sigma} \leq \dfrac{1}{2}$,

$$\mathbb{E}\left(|u^R\mathbb{1}_{[0,\tau_R]}|^r_{L^r(0,T;L^p_x)}\right) \leq 2c(r,T_0)|u_0|^r_{L^2_x}$$

and by Markov inequality

$$\mathbb{P}(\tau_R \le T) \le \frac{2c(r,T_0)|u_0|^r_{L^2_x}}{R^r}.$$

□

In order to construct a solution to (**??**) on $[0,T_0]$, we iterate the local construction. We fix $R>0$ and have a local solution on $[0,\tau_R]$. We then consider the equation for u:

$$u(t+\tau_R) = S(t+\tau_R,\tau_R)u(\tau^R) + \int_0^t S(t+\tau_R, s+\tau_R)\theta_R^{\tau_R}(u)(s)|u(s+\tau_R)|^{2\sigma}u(s+\tau_R)ds$$

All the arguments of Section 4 can be reproduced. We obtain a unique global solution of this equation, that we denote by u_R^2. Moreover setting

$$\tau_R^2 = \inf\{t\in[0,T],\ |u_R^2|_{L^r(\tau_R,t+\tau_R;L^p_x)} \ge R\}$$

we obtain a solution of the non truncated equation on $[\tau_R,\tau_R+\tau_R^2]$ and thus on $[0,\tau_R+\tau_R^2]$. We also have by Lemma **??** and the conservation of the L^2_x norm

$$\mathbb{P}(\tau_R^2 \le T|\mathcal{F}_{\tau_R}) \le \frac{c_2|u(\tau_R)|^r_{L^2_x}}{R^r} = \frac{c_2|u_0|^r_{L^2_x}}{R^r},$$

provided that $T^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma} \le c_1\ R^{-2r\sigma}$. We continue this construction recursively and obtain a solution on $[0,T_R^n]$, where $T_R^n = \tau_R + \cdots + \tau_R^n$, with

$$\mathbb{P}(\tau_R^n \le T|\mathcal{F}_{T_R^{n-1}}) \le \frac{c_2|u_0|^r_{L^2_x}}{R^r},$$

provided $T^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma} \le c_1\ R^{-2r\sigma}$. Note that

$$\mathbb{P}\left(\lim_{n\to+\infty}\tau_R^n = 0\right) = \lim_{\varepsilon\to 0}\lim_{N\to+\infty}\mathbb{P}(\tau_R^n\le\varepsilon,\ \forall n\ge N).$$

For R large enough and $\varepsilon^{2-\frac{dr}{2}(\frac{1}{2}-\frac{1}{p})+r-2\sigma-2} \le c_1\ R^{-2r\sigma}$,

$$\mathbb{P}(\tau_R^n\le\varepsilon|\mathcal{F}_{T_R^{n-1}}) \le \frac{1}{2},$$

and we deduce that

$$\mathbb{P}(\tau_R^n\le\varepsilon,\ \forall n\ge N) \le \lim_{M\to\infty}\mathbb{E}\Big(\prod_{N\le n\le M-1}\mathbb{1}_{\{\tau_R^n\le\varepsilon\}}\mathbb{P}(\mathbb{1}_{\{\tau_R^M\le\varepsilon\}}|\mathcal{F}_{T_R^{M-1}})\Big) \le \lim_{M\to\infty}\frac{1}{2^{M-N}} = 0.$$

Hence, $\mathbb{P}(\lim_{n\to+\infty}\tau_R^n = 0) = 0$ so that T_R^n goes to infinity, a.s. and we have constructed a global solution.

The conservation of the L^2-norm and the fact that $u\in C(\mathbb{R}^+;L^2_x)$ a.s. was proved in Theorem **??**. In order to obtain the extra-integrability properties given in the statement of Theorem **??**, we apply Proposition **??** and (**??**) of Proposition **??** with (ρ,q) on the left hand side ($q=+\infty$ if $d=1$) and with $\gamma=r$, $\delta=2\sigma+2$ to equation (**??**). Note that (ρ,q) is an admissible pair thanks to the conditions $\rho \le 4$ if $d=1$ and $2 \le q < \frac{2d}{d-2}$, $2 \le \rho < \frac{4d}{d(q-2)}$ if $d \ge 2$. This gives, setting $q=+\infty$ if $d=1$:

$$\begin{aligned} & |u^R \mathbb{1}_{[0,\tau_R]}|_{L^\rho_\omega(L^\rho(0,T;L^q_x))} \\ \le\ & c(\rho,q,T_0)|u_0|_{L^2_x} + c'(\rho,q,T_0)|(u^R)^{2\sigma}u^R\mathbb{1}_{[0,\tau_R]}|_{L^\rho_\omega(L^{\gamma'}(0,T;L^{\delta'}_x))} \\ \le\ & c(\rho,q,T_0)|u_0|_{L^2_x} + c'(\rho,q,T_0)|u^R\mathbb{1}_{[0,\tau_R]}|^{2\sigma+1}_{L^{(2\sigma+1)\rho}_\omega(L^r(0,T;L^{2\sigma+2}_x))} \\ \le\ & c(\rho,q,T_0)|u_0|_{L^2_x} + c'(\rho,q,T_0,R) \end{aligned}$$

where R is chosen as above. Estimates on other intervals of the form $[T^n_R, T^{n+1}_R]$ are obtained similarly.

Finally, assume that $u_0 \in H^1_x$. Then going back to $\mathcal{T}^R$ defined in (**??**), and applying the same estimates as in the proof of Lemma **??**, after having taken first order space derivatives, lead to

$$\begin{aligned} & |\mathcal{T}^R u|_{L^r(\Omega\times[0,T];W^{1,p})} \\ & \le C T_0^{\beta/2}|u_0|_{H^1} + C' T^{\tilde\beta} R^{2r\sigma} |u|_{L^r(\Omega\times[0,T];W^{1,p})} \end{aligned}$$

with $\tilde\beta = r - 2\sigma - \frac{dr\sigma}{4(\sigma+1)}$. This proves that if $B = B(0,R_0)$ is the (closed) ball of radius R_0 in $L^r(\Omega\times[0,T];W^{1,p})$, then $\mathcal{T}^R B \subset B$ provided $T \le \tilde T_0$, where $\tilde T_0$ depends only on R and not on R_0. Since closed balls of $L^r(\Omega\times[0,T];W^{1,p})$ are closed in $L^r(\Omega\times[0,T];L^p_x)$, this implies that the fixed point of $\mathcal{T}^R$, which is the solution u^R of (**??**), is in $L^r(\Omega\times[0,T];W^{1,p})$. Applying then Proposition **??**, and (**??**) in Proposition **??** to equation (**??**) (or (**??**)), again after having taken first order space derivatives, gives the result. □

6. Equation (??) as limit of NLS equation with random dispersion

To prove Theorem **??**, we use the same argument as in [**?**]. Let us recall its main lines. Note that we introduce a slight modification since we work with $H^1(\mathbb{R})$ functions instead of $H^2(\mathbb{R})$ as in [**?**]. Consider the following nonlinear Schrödinger equation:

$$\begin{cases} i\dfrac{du}{dt} + \dot n(t)\partial_{xx}u + F(|u|^2)u = 0,\ x\in\mathbb{R},\ t>0, \\ u(0) = u_0, x\in\mathbb{R}, \end{cases} \tag{6.1}$$

where F is a smooth function with compact support and n is a real valued function. Note that, using the mild form

$$u_n(t) = S_n(t)u_0 + i\int_0^t S_n(t,s)F(|u(s)|^2)u(s)ds,$$

where we have denoted by $S_n(t,s)$ the evolution operator associated to the linear equation

$$i\frac{dv}{dt} + \dot n(t)\partial_{xx}v = 0,\ x\in\mathbb{R},\ t>0,$$

whose solution can be written down explicitly thanks to spatial Fourier transform, one can give a meaning to the solution u of equation (**??**) as soon as n is a continuous function of t. Indeed, for each $t,s\in\mathbb{R}$, $S_n(t,s)$ is an isometry on any Sobolev space $H^s(\mathbb{R})$. Since the nonlinear term has bounded derivatives, a fixed point argument can be used in $C([0,T];L^2(\mathbb{R}))$ and a global solution u_n is obtained in this space if $u_0 \in L^2(\mathbb{R})$. Moreover, the solutions belongs to $C([0,T];H^1(\mathbb{R}))$ if $u_0 \in H^1(\mathbb{R})$.

Using Fourier transform, we see that, for $n_1,\ n_2 \in C([s,s+T])$, we have, for $s\in[0,1]$,

$$|(S_{n_1}(\cdot,s)-S_{n_2}(\cdot,s))u_s|_{L^\infty(s,s+T;H^s_x)} \le 2|n_1-n_2|^{(1-s)/2}_{C([s,s+T])}|u_s|_{H^1_x}.$$

Proceeding as in the proof of Theorem 3.7 in [?], we deduce, for $s\in(\frac12,1)$,

$$|u_{n_1}-u_{n_2}|_{C([0,T];H^s_x)} \le c|n_1-n_2|^{(1-s)/2}_{C([0,T])}|u_0|_{H^1_x}$$

where the constant c depends on T and F. It follows that for $u_0\in H^1(\mathbb{R})$ the mapping

$$\begin{array}{ccc} n & \longmapsto & u_n \\ C([0,T]) & \longrightarrow & C([0,T];H^s(\mathbb{R})) \end{array}$$

is continuous for $s\in(\frac12,1)$. Since our assumption on the process m says that the process $t\mapsto \int_0^t \frac{1}{\varepsilon}m(\frac{s}{\varepsilon^2})ds$ converges in distribution in $C([0,T])$ to a brownian motion, we deduce that the solution of

$$\left\{\begin{array}{l} i\dfrac{du}{dt}+\dfrac{1}{\varepsilon}m(\dfrac{t}{\varepsilon^2})\partial_{xx}u+F(|u|^2)u=0,\ x\in\mathbb{R},\ t>0, \\ \\ u(0)=u_0,\ x\in\mathbb{R}, \end{array}\right. \tag{6.2}$$

converges in distribution in $C([0,T];H^s(\mathbb{R}))$ to the solution of

$$\left\{\begin{array}{l} idu+\Delta u\circ d\beta+F(|u|^2)u\,dt=0,\ x\in\mathbb{R},\ t>0, \\ u(0)=u_0,\ x\in\mathbb{R}, \end{array}\right.$$

for $s\in(\frac12,1)$. We now want to extend this result to the original power nonlinear term. Let us introduce the truncated equations, where θ is as in section 4,

$$\left\{\begin{array}{l} i\dfrac{du}{dt}+\dfrac{1}{\varepsilon}m(\dfrac{t}{\varepsilon^2})\partial_{xx}u+\theta\left(\dfrac{|u|^2}{M}\right)|u|^2u=0,\ x\in\mathbb{R},\ t>0, \\ u(0)=u_0,\ x\in\mathbb{R}, \end{array}\right. \tag{6.3}$$

and

$$\left\{\begin{array}{l} idu+\Delta u\circ d\beta+\theta\left(\dfrac{|u|^2}{M}\right)|u|^2u\,dt=0,\ x\in\mathbb{R},\ t>0, \\ u(0)=u_0,\ x\in\mathbb{R}. \end{array}\right. \tag{6.4}$$

We denote by u^M_ε and u^M their respective solutions. By the previous arguments, these solutions exist and are unique in $C([0,T];H^1(\mathbb{R}))$. Note that setting

$$\widetilde{\tau}^M_\varepsilon=\inf\{t\ge0:\ |u^M_\varepsilon(t)|_{L^\infty_x}\ge M\}$$

and $u_\varepsilon=u^M_\varepsilon$ on $[0,\widetilde{\tau}^M_\varepsilon]$, defines a unique local solution u_ε of equation (**??**) on $[0,\tau_\varepsilon)$ with $\tau_\varepsilon=\lim_{M\to\infty}\tau^M_\varepsilon$.

We also set

$$\widetilde{\tau}^M=\inf\{t\ge0:\ |u^M(t)|_{L^\infty_x}\ge M\}.$$

By the above result, for each M, u^M_ε converges to u^M in distribution in $C([0,T];H^s(\mathbb{R}))$ for $s\in(\frac12,1)$. By Skohorod Theorem, after a change of probability space, we can assume that for each M the convergence of u_ε^M to u^M holds almost surely in $([0,T];H^s(\mathbb{R}))$. To conclude, let us notice that for $0<\delta\le 1$, if

$$\widetilde{\tau}_{M-1}\ge T \text{ and } |u_\varepsilon^M-u^M|_{C([0,T];H^s(\mathbb{R}))}\le\delta$$

then $u^M=u$, the solution of (**??**), on $[0,T]$. Moreover, by the Sobolev embedding $H^s(\mathbb{R})\subset L^\infty(\mathbb{R})$, we have

$$|u_\varepsilon^M-u^M|_{C([0,T];L^\infty(\mathbb{R}))}\le c\delta$$

for some $c>0$. We deduce $|u_\varepsilon^M|_{C([0,T];L^\infty(\mathbb{R}))}\le M$ provided δ is small enough. Therefore

$$\tau_\varepsilon>\widetilde{\tau}_\varepsilon^M\ge T \text{ and } u_\varepsilon^M=u_\varepsilon \text{ on } [0,T].$$

It follows that for $\delta>0$ small enough,

$$\begin{aligned}&\mathbb{P}(\tau_\varepsilon(u_0)\le T)+\mathbb{P}(\tau_\varepsilon(u_0)>T \text{ and } |u_\varepsilon-u|_{C([0,T];H^s(\mathbb{R}))}>\delta)\\ \le\;&\mathbb{P}(|u_\varepsilon^M-u^M|_{C([0,T];H^s(\mathbb{R}))}>\delta)+\mathbb{P}(\tilde{\tau}_{M-1}<T).\end{aligned}$$

Since $u_0\in H^1(\mathbb{R})$, we know that u is almost surely in $C(\mathbb{R}^+;H^1(\mathbb{R}))$; we deduce

$$\lim_{M\to\infty}\mathbb{P}(\tilde{\tau}_{M-1}<T)=0.$$

Choosing first M large and then ε small we obtain

$$\lim_{\varepsilon\to 0}\mathbb{P}(\tau_\varepsilon(u_0)\le T)=0$$

and

$$\lim_{\varepsilon\to 0}\mathbb{P}(\tau_\varepsilon(u_0)>T \text{ and } |u_\varepsilon-u|_{C([0,T];H^s(\mathbb{R}))}>\delta)=0$$

The result follows. □

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