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Abstract. In this note we show that the Novikov-Veselov equation (NV-equation) at positive energy (an analog of KdV in 2+1 dimensions) has no exponentially localized solitons in the two- dimensional sense.

1.Introduction and Theorem 1. We consider the following 2+1 - dimensional analog of the KdV equaion (Novikov-Veselov equation):

$$\begin{aligned} &\partial_t v = 4Re\,(4\partial_z^3 v + \partial_z(vw) - E\partial_z w),\\ &\partial_{\bar z} w = -3\partial_z v, \quad v = \bar v, \quad E \in \mathbb{R}, \\ &v = v(x,t), \quad w = w(x,t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \end{aligned} \tag{1}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big), \quad \partial_{\bar z} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big). \tag{2}$$

We assume that

$$\begin{aligned} &v \ \text{ is sufficiently regular and has sufficient decay as } \ |x| \to \infty, \\ &w \ \text{ is decaying as } \ |x| \to \infty. \end{aligned} \tag{3}$$

Equation (1) is contained implicitly in the paper of S.V.Manakov [M] as an equation possessing the following representation

$$\frac{\partial(L-E)}{\partial t} = [L-E, A] + B(L-E), \tag{4}$$

where $L = -\Delta + v(x,t)$, $\Delta = 4\partial_z\partial_{\bar z}$, A and B are suitable differential operators of the third and zero order respectively, $[\cdot,\cdot]$ denotes the commutator. Equation (1) was written in an explicit form by S.P.Novikov and A.P.Veselov in [NV1], [NV2], where higher analogs of (1) were also constructed.

For the case when

$$v(x_1,x_2,t), \ w(x_1,x_2,t) \ \text{ are independent of } \ x_2 \tag{5}$$

equation (1) is reduced to

$$\partial_t v = 2\partial_x^3 v - 12 v\partial_x v + 6E\partial_x v, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{6}$$

In terms of $u(x,t)$ such that

$$v(x,t) = u(-2t, x + 6Et), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{7}$$

equation (6) takes the standard form of the KdV equation (see [NMPZ]):

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{8}$$

It is well-known (see [NMPZ]) that (8) has the soliton solutions

$$u(x,t) = u_{\kappa,\varphi}(x - 4\kappa^2 t) = -\frac{2\kappa^2}{ch^2(\kappa(x - 4\kappa^2 t - \varphi))}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \ \kappa \in]0, +\infty[, \ \varphi \in \mathbb{R}. \tag{9}$$

In addition, one can see that

$$\begin{aligned} &u_{\kappa,\varphi} \in C^\infty(\mathbb{R}), \\ &\partial_x^j u_{\kappa,\varphi}(x) = O(e^{-2\kappa|x|}) \ \text{ as } \ x \to \infty, \ j = 0, 1, 2, 3, \ldots \end{aligned} \tag{10}$$

Properties (10) show, in particular, that the solitons of (9) are exponentially localized in x.

In the present note we obtain, in particular, the following result:

Theorem 1. *Let v, w satisfy (1) for $E = E_{fix} > 0$, where*

$$\begin{aligned} &v(x,t) = V(x - ct), \quad x \in \mathbb{R}^2, \quad c = (c_1, c_2) \in \mathbb{R}^2, \\ &V \in C^3(\mathbb{R}^2), \quad \partial_x^j V(x) = O(e^{-\alpha|x|}) \ \ for |x| \to \infty, \quad |j| \le 3 \ \ and\ some\ \alpha > 0, \end{aligned} \tag{11a}$$

(where $j = (j_1, j_2) \in (0 \cup \mathbb{N})^2$, $|j| = |j_1| + |j_2|$, $\partial_x^j = \partial^{j_1+j_2}/\partial x_1^{j_1}\partial x_2^{j_2}$),

$$w(\cdot, t) \in C(\mathbb{R}^2), \quad w(x,t) \to 0 \ \ as \ \ |x| \to \infty, \quad t \in \mathbb{R}. \tag{11b}$$

Then $V \equiv 0$, $v \equiv 0$, $w \equiv 0$.

Theorem 1 shows that equation (1) has no nonzero solitons (travel wave solutions) exponentially localized in x in the two-dimensional sense.

The proof of Theorem 1 is based on Proposition 1 and Proposition 2, see Section 4. In turn, Proposition 2 is based, in particular, on Lemma 1 and Lemma 2.

Lemma 1, Lemma 2 and Proposition 1 are recalled in Section 2. Proposition 2 is given in Section 3. It seems that the result of Proposition 2 (that sufficiently localized travel wave solutions for the NV-equation (1) for $E = E_{fix} > 0$ have zero scattering amplitude for the two-dimensional Schrödinger equation (12)) was not yet formulated in the literature.

2. Lemma 1, Lemma 2 and Proposition 1. Consider the equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^2, \quad E = E_{fix} > 0, \tag{12}$$

where

$$\begin{aligned} & v(x) = \overline{v(x)}, \quad x \in \mathbb{R}^2, \\ & (1+|x|)^{2+\varepsilon} v(x) \in L^\infty(\mathbb{R}^2) \ \ \text{(as a function of } \ x \in \mathbb{R}^2) \ \ \text{for some} \ \ \varepsilon > 0. \end{aligned} \tag{13}$$

It is known that for any $k \in \mathbb{R}^2$, such that $k^2 = E$, there exists an unique bounded solution $\psi^+(x,k)$ of equation (12) with the following asymptotics:

$$\psi^+(x,k) = e^{ikx} - i\pi\sqrt{2\pi}e^{-i\pi/4} f(k, |k|\frac{x}{|x|})\frac{e^{i|k||x|}}{\sqrt{|k||x|}} + o(\frac{1}{\sqrt{|x|}}) \ \ \text{as} \ \ |x| \to \infty. \tag{14}$$

This solution describes scattering of incident plane wave e^{ikx} on the potential v. The function f on

$$\mathcal{M}_E = \{k \in \mathbb{R}^2, \ l \in \mathbb{R}^2 : \ k^2 = l^2 = E\} \tag{15}$$

arising in (14) is the scattering amplitude for v in the framework of equation (12). Under assumptions (13), it is known, in particular, that

$$f \in C(\mathcal{M}_E). \tag{16}$$

Lemma 1. *Let v satisfy (13) and v_y, $y \in \mathbb{R}^2$, be defined by*

$$v_y(x) = v(x-y), \quad x \in \mathbb{R}^2. \tag{17}$$

Then the scattering amplitude f for v and the scattering amplitude f_y for v_y are related by the formula

$$f_y(k,l) = f(k,l)e^{iy(k-l)}, \quad (k,l) \in \mathcal{M}_E, \quad y = (y_1, y_2) \in \mathbb{R}^2. \tag{18}$$

Lemma 1 follows, for example, from the definition of the scattering amplitude by means of (14) and the fact that $\psi^+(x-y,k)$ solves (12) for v replaced by v_y, where $k^2 = E$.

Lemma 1 was given, for example, in [N3].

Lemma 2. *Let v, w satisfy (1), (3), where $E = E_{fix} > 0$. Then the scattering amplitude $f(\cdot,\cdot,t)$ for $v(\cdot,t)$ and the scattering amplitude $f(\cdot,\cdot,0)$ for $v(\cdot,0)$ are related by*

$$f(k,l,t) = f(k,l,0)\exp[2it(k_1^3 - 3k_1k_2^2 - l_1^3 + 3l_1l_2^2)], \quad (k,l) \in \mathcal{M}_E, \quad t \in \mathbb{R}. \tag{19}$$

Lemma 2 was given for the first time in [N1].

Note that in the framework of Lemma 2 properties (3) can be specified as follows:

$$\begin{aligned} & v, w \in C(\mathbb{R}^2 \times \mathbb{R}) \ \ \text{and for each} \ \ t \in \mathbb{R} \ \ \text{the following properties are fulfiled :} \\ & v(\cdot,t) \in C^3(\mathbb{R}^2), \ \ \partial_x^j v(x,t) = O(|x|^{-2-\varepsilon}) \ \ \text{for} \ \ |x| \to \infty, \ \ |j| \le 3 \ \ \text{and some} \ \ \varepsilon > 0, \\ & w(x,t) \to 0 \ \ \text{for} \ \ |x| \to \infty. \end{aligned} \tag{20}$$

Proosition 1. *Let*

$$v(x)=\overline{v(x)},\quad e^{\alpha|x|}v(x)\in L^{\infty}(\mathbb{R}^2)\ \ \text{(as a function of } x\text{) for some } \alpha>0 \tag{21}$$

and the scattering amplitude $f\equiv 0$ on $\mathcal{M}_E$ for this potential for some $E=E_{fix}>0$. Then $v\equiv 0$ in $L^{\infty}(\mathbb{R}^2)$.

In the general case the result of Proposition 1 was given for the first time in [GN]. Under the additional assumption that v is sufficiently small (in comparison with E) the result of Proposition 1 was given for the first time in [N2]-[N4].

3. Transparency of solitons. In this section we show that sufficiently localized solitons (travel wave solutions) for the NV-equation (1) for $E=E_{fix}>0$ have zero scattering amplitude for the two-dimensional Schrödinger equation (12).

Proposition 2. *Let v,w satisfy (1) for $E=E_{fix}>0$, where*

$$\begin{aligned}&v(x,t)=V(x-ct),\quad x\in\mathbb{R}^2,\quad c=(c_1,c_2)\in\mathbb{R}^2,\\&V\in C^3(\mathbb{R}^2),\quad \partial_x^j V(x)=O(|x|^{-2-\varepsilon})\ \ for|x|\to\infty,\quad |j|\le 3\ \ and\ some\ \varepsilon>0,\end{aligned} \tag{22a}$$

$$w(\cdot,t)\in C(\mathbb{R}^2),\quad w(x,t)\to 0\ \ as\ \ |x|\to\infty,\quad t\in\mathbb{R}. \tag{22b}$$

Then

$$f\equiv 0\ \ \text{on}\ \ \mathcal{M}_E, \tag{23}$$

where f is the scattering amplitude for $v(x)=V(x)$ in the framework of the Schrödinger equation (12).

The proof of Proposition 2 consists in the following.

We consider

$$T=\{\lambda\in\mathbb{C}:\ \ |\lambda|=1\}. \tag{24}$$

We use that

$$\mathcal{M}_E\approx T\times T,\quad E=E_{fix}>0, \tag{25}$$

where diffeomorphism (25) is given by the formulas:

$$\lambda=\frac{k_1+ik_2}{\sqrt{E}},\quad \lambda'=\frac{l_1+il_2}{\sqrt{E}},\quad (k,l)\in\mathcal{M}_E, \tag{26}$$

$$\begin{aligned}&k_1=\frac{\sqrt{E}}{2}\big(\lambda+\frac{1}{\lambda}\big),\quad k_2=\frac{i\sqrt{E}}{2}\big(\frac{1}{\lambda}-\lambda\big),\\&l_1=\frac{\sqrt{E}}{2}\big(\lambda'+\frac{1}{\lambda'}\big),\quad l_2=\frac{i\sqrt{E}}{2}\big(\frac{1}{\lambda'}-\lambda'\big),\quad (\lambda,\lambda')\in T\times T.\end{aligned} \tag{27}$$

We use that in the variables λ, λ' formulas (18), (19) take the form

$$f_y(\lambda,\lambda',E)=f(\lambda,\lambda',E)\exp\big[\frac{i}{2}\sqrt{E}\big(\lambda\bar{y}+\frac{y}{\lambda}-\lambda'\bar{y}-\frac{y}{\lambda'}\big)\big], \tag{28}$$

where $(\lambda,\lambda')\in T\times T$, y is considered as $y=y_1+iy_2$,

$$f(\lambda,\lambda',E,t)=f(\lambda,\lambda',E,0)\exp\bigl[iE^{3/2}t\bigl(\lambda^3+\frac{1}{\lambda^3}-(\lambda')^3-\bigl(\frac{1}{\lambda'}\bigr)^3\bigr)\bigr], \tag{29}$$

where $(\lambda,\lambda')\in T\times T$, $t\in\mathbb{R}$.

The assumptions of Proposition 2 and Lemmas 1 and 2 (with (18), (19) written as (28), (29)) imply that

$$\begin{aligned} &f(\lambda,\lambda',E)\exp\bigl[\frac{i}{2}\sqrt{E}t\bigl(\lambda\bar c+\frac{c}{\lambda}-\lambda'\bar c-\frac{c}{\lambda'}\bigr)\bigr]=\\ &f(\lambda,\lambda',E)\exp\bigl[iE^{3/2}t\bigl(\lambda^3+\frac{1}{\lambda^3}-(\lambda')^3-\bigl(\frac{1}{\lambda'}\bigr)^3\bigr)\bigr]\end{aligned} \tag{30}$$

for $(\lambda,\lambda')\in T\times T$, $t\in\mathbb{R}$, where f is the scattering amplitude for $v(x,0)=V(x)$, c is considered as $c=c_1+ic_2$.

Property (16), identity (30) and the fact that λ^3, λ^{-3}, λ, λ^{-1}, 1 are linear independent on each nonempty open subset of T imply (23).

4. Proof of Theorem 1 and final remark. Theorem 1 follows from Proposition 1 and Proposition 2.

Finally, note that the result of Theorem 1 does not hold, in general, without the assumption that $V(x)=O(e^{-\alpha|x|})$ as $|x|\to\infty$ for some $\alpha>0$: "counter examples" to Theorem 1 with rational bounded V decaying at infinity as $O(|x|^{-2})$ are implicitly contained in [G].

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