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Exponential instability in the Gel'fand inverse problem on the energy intervals

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Abstract

We consider the Gel'fand inverse problem and continue studies of [Mandache,2001]. We show that the Mandache-type instability remains valid even in the case of Dirichlet-to-Neumann map given on the energy intervals. These instability results show, in particular, that the logarithmic stability estimates of [Alessandrini,1988], [Novikov, Santacesaria,2010] and especially of [Novikov,2010] are optimal (up to the value of the exponent).

1 Introdution

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \tag{1.1}$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d,\ d \geq 2,\ \partial D \in C^2,\ v \in L^\infty(D). \tag{1.2}$$

Consider the map $\Phi(E)$ such that

$$\Phi(E)(\psi|_{\partial D}) = \frac{\partial \psi}{\partial \nu}|_{\partial D}. \tag{1.3}$$

for all sufficiently regular solutions ψ of (1.1) in $\bar{D} = D \cup \partial D$, where ν is the outward normal to ∂D. Here we assume also that

$$E \text{ is not a Dirichlet eigenvalue for operator } -\Delta + v \text{ in } D. \tag{1.4}$$

The map $\Phi(E)$ is called the Dirichlet-to-Neumann map and is considered as boundary measurements.

We consider the following inverse boundary value problem for equation (1.1).

Problem 1.1. Given $\Phi(E)$ on the union of the energy intervals $S = \bigcup\limits_{j=1}^{K} I_j$, find v.

Here we suppose that condition (1.4) is fulfilled for any $E \in S$.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation on the energy intervals (see [2], [6]).

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness for Problem 1.1 was obtained for the first time by Novikov (see Theorem 5.3 in [4]). Some global reconstruction method for Problem 1.1 was proposed for the first time in [4] also. Global uniqueness theorems and global reconstruction methods in the case of fixed energy were given for the first time in [6] in dimension $d \geq 3$ and in [9] in dimension $d = 2$.

Global stability estimates for Problem 1.1 were given for the first time in [1] in dimension $d \geq 3$ and in [8] in dimension $d = 2$. The Alessandrini result of [1] was recently improved by Novikov in [7]. In the case of fixed energy, Mandache showed in [3] that these logarithmic stability results are optimal (up to the value of the exponent). Mandache-type instability estimates for inverse inclusion and scattering problems are given in [12].

In the present work we extend studies of Mandache to the case of Dirichlet-to-Neumann map given on the energy intervals. The stability estimates and our instability results for Problem 1.1 are presented and discussed in Section 2. In Section 5 we prove the main results, using a ball packing and covering by ball arguments. In Section 3 we prove some basic properties of the Dirichlet-to-Neumann map, using some Lemmas about the Bessel functions wich we proved in Section 6.

2 Stability estimates and main results

As in [7] we assume for simplicity that

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^d,\ \partial D \in C^2, \\ &v \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d,\ \operatorname{supp} v \subset D,\ d \geq 2, \end{aligned} \tag{2.1}$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v :\ \partial^J v \in L^1(\mathbb{R}^d),\ |J| \leq m\},\ m \in \mathbb{N} \cup 0, \tag{2.2}$$

where

$$J \in (\mathbb{N} \cup 0)^d,\ |J| = \sum_{i=1}^{d} J_i,\ \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}. \tag{2.3}$$

Let

$$||v||_{m,1} = \max_{|J| \leq m} ||\partial^J v||_{L^1(\mathbb{R}^d)}. \tag{2.4}$$

We recall that if v_1, v_2 are potentials satisfying (1.4),(1.3), where E and D are fixed, then

$$\Phi_1 - \Phi_2 \text{ is a compact operator in } L^\infty(\partial D), \tag{2.5}$$

where Φ_1, Φ_2 are the DtN maps for v_1, v_2 respectively, see [6]. Note also that (2.1) $\Rightarrow$ (1.2).

Theorem 2.1 (variation of the result of [1], see [7]). *Let conditions (1.4), (2.1) hold for potentials v_1 and v_2, where E and D are fixed, $d \geq 3$. Let $||v_j||_{m,1} \leq N$, $j = 1, 2$, for some $N > 0$. Let Φ_1, Φ_2 denote DtN maps for v_1, v_2 respectively. Then*

$$||v_1 - v_2||_{L^\infty(D)} \leq c_1 (\ln(3 + ||\Phi_1 - \Phi_2||^{-1}))^{-\alpha_1}, \tag{2.6}$$

where $c_1 = c_1(N, D, m)$, $\alpha_1 = (m-d)/m$, $||\Phi_1-\Phi_2|| = ||\Phi_1-\Phi_2||_{L^\infty(\partial D)\to L^\infty(\partial D)}$.

An analog of stability estimate of [1] for $d = 2$ is given in [8].

A disadvantage of estimate (2.6) is that

$$\alpha_1 < 1 \text{ for any } m > d \text{ even if } m \text{ is very great.} \tag{2.7}$$

Theorem 2.2 (the result of [7])**.** *Let the assumptions of Theorem 2.1 hold. Then*

$$||v_1 - v_2||_{L^\infty(D)} \le c_2(\ln(3 + ||\Phi_1 - \Phi_2||^{-1}))^{-\alpha_2}, \tag{2.8}$$

where $c_2 = c_2(N, D, m)$, $\alpha_2 = m - d$, $||\Phi_1 - \Phi_2|| = ||\Phi_1 - \Phi_2||_{L^\infty(\partial D)\to L^\infty(\partial D)}$.

A principal advantage of estimate (2.8) in comparison with (2.6) is that

$$\alpha_2 \to +\infty \text{ as } m \to +\infty, \tag{2.9}$$

in contrast with (2.7). Note that strictly speaking Theorem 2.2 was proved in [7] for $E = 0$ with the condition that $\operatorname{supp} v \subset D$, so we cant make use of substitution $v_E = v - E$, since condition $\operatorname{supp} v_E \subset D$ does not hold.

We would like to mention that, under the assumptions of Theorems 2.1 and 2.2, according to the Mandache results of [3], estimate (2.8) can not hold with $\alpha_2 > m(2d - 1)/d$ for real-valued potentials and with $\alpha_2 > m$ for complex potentials.

As in [3] in what follows we fix $D = B(0, 1)$, where $B(x, r)$ is the open ball of radius r centred at x. We fix an orthonormal basis in $L^2(S^{d-1}) = L^2(\partial D)$

$$\begin{aligned} &\{f_{jp} : j \ge 0;\ 1 \le p \le p_j\},\\ &f_{jp} \text{ is a spherical harmonic of degree } j, \end{aligned} \tag{2.10}$$

where p_j is the dimension of the space of spherical harmonics of order j,

$$p_j = \binom{j + d - 1}{d - 1} - \binom{j + d - 3}{d - 1}, \tag{2.11}$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad \text{for } n \ge 0 \tag{2.12}$$

and

$$\binom{n}{k} = 0 \quad \text{for } n < 0. \tag{2.13}$$

The precise choice of f_{jp} is irrelevant for our purposes. Besides orthonormality, we only need f_{jp} to be the restriction of a homogeneous harmonic polynomial of degree j to the sphere and so $|x|^j f_{jp}(x/|x|)$ is harmonic. In the Sobolev spaces $H^s(S^{d-1})$ we will use the norm

$$||\sum_{j,p} c_{jp} f_{jp}||^2_{H^s} = \sum_{j,p}(1 + j)^{2s}|c_{jp}|^2. \tag{2.14}$$

The notation (a_{jpiq}) stands for a multiple sequence. We will drop the subscript

$$0 \le j,\ 1 \le p \le p_j,\ 0 \le i,\ 1 \le q \le p_i. \tag{2.15}$$

We use notations: $|A|$ is the cardinality of a set A, $[a]$ is the integer part of real number a and $(r, \omega) \in \mathbb{R}_+ \times S^{d-1}$ are polar coordinates for $r\omega = x \in \mathbb{R}^d$.

The interval $I = [a, b]$ will be referred as σ-regular interval if for any potential $v \in L^\infty(D)$ with $||v||_{L^\infty(D)} \le \sigma$ and any $E \in I$ condition (1.4) is fulfilled. Note that for any $E \in I$ and any Dirichlet eigenvalue λ for operator $-\Delta$ in D we have that

$$|E - \lambda| \ge \sigma. \tag{2.16}$$

It follows from the definition of σ-regular interval, taking $v \equiv E - \lambda$.

Theorem 2.3. *For $\sigma > 0$ and dimension $d \ge 2$ consider the union $S = \bigcup_{j=1}^{K} I_j$ of σ-regular intervals. Then for any $m > 0$ and any $s \ge 0$ there is a constant $\beta > 0$, such that for any $\epsilon \in (0, \sigma/3)$ and $v_0 \in C^m(D)$ with $||v_0||_{L^\infty(D)} \le \sigma/3$ and $\operatorname{supp} v_0 \subset B(0, 1/3)$ there are real-valued potentials $v_1, v_2 \in C^m(D)$, also supported in $B(0, 1/3)$, such that*

$$\begin{aligned}
&\sup_{E \in S} \Big(||\Phi_1(E) - \Phi_2(E)||_{H^{-s} \to H^s} \Big) \le \exp\Big(-\epsilon^{-\frac{1}{2m}} \Big),\\
&||v_1 - v_2||_{L^\infty(D)} \ge \epsilon,\\
&||v_i - v_0||_{C^m(D)} \le \beta, \quad i = 1, 2,\\
&||v_i - v_0||_{L^\infty(D)} \le \epsilon, \quad i = 1, 2,
\end{aligned} \tag{2.17}$$

where $\Phi_1(E)$, $\Phi_2(E)$ are the DtN maps for v_1 and v_2 respectively.

Remark 2.1. We can allow β to be arbitrarily small in Theorem 2.3, if we require $\epsilon \le \epsilon_0$ and replace the right-hand side in the instability estimate by $\exp(-c\epsilon^{-\frac{1}{2m}})$, with $\epsilon_0 > 0$ and $c > 0$, depending on β.

In addition to Theorem 2.3, we consider explicit instability example with a complex potential given by Mandache in [3]. We show that it gives exponential instability even in case of Dirichlet-to-Neumann map given on the energy intervals. Consider the cylindrical variables $(r_1, \theta, x') \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^{d-2}$, with $x' = (x_3, \ldots, x_d)$, $r_1 \cos\theta = x_1$ and $r_1 \sin\theta = x_2$. Take $\phi \in C^\infty(\mathbb{R}^2)$ with support in $B(0, 1/3) \cap \{x_1 > 1/4\}$ and with $||\phi||_{L^\infty} = 1$.

Theorem 2.4. *For $\sigma > 0$, $m > 0$, integer $n > 0$ and dimension $d \ge 2$ consider the union $S = \bigcup_{j=1}^{K} I_j$ of σ-regular intervals and define the complex potential*

$$v_{nm}(x) = \frac{\sigma}{3} n^{-m} e^{in\theta} \phi(r_1, |x'|). \tag{2.18}$$

Then $||v_{mn}||_{L^\infty(D)} = \frac{\sigma}{3} n^{-m}$ and for every $s \ge 0$ and $m > 0$ there are constants c, c' such that $||v_{mn}||_{C^m(D)} \le c$ and for every n

$$\sup_{E \in S} \Big(||\Phi_{mn}(E) - \Phi_0(E)||_{H^{-s} \to H^s} \Big) \le c' 2^{-n/4}, \tag{2.19}$$

where $\Phi_{mn}(E)$, $\Phi_0(E)$ are the DtN maps for v_{mn} and $v_0 \equiv 0$ respectively.

In some important sense, this is stronger than Theorem 2.3. Indeed, if we take $\epsilon = \frac{\sigma}{3} n^{-m}$ we obtain (2.17) with $\exp(-C\epsilon^{-1/m})$ in the right-hand side. An explicit real-valued counterexample should be difficult to find. This is due to nonlinearity of the map $v \to \Phi$.
Remark 2.2. Note that for sufficient large s one can see that

$$||\Phi_1 - \Phi_2||_{L^\infty(\partial D) \to L^\infty(\partial D)} \le C ||\Phi_1 - \Phi_2||_{H^{-s} \to H^s}. \tag{2.20}$$

So Theorem 2.3 and Theorem 2.4 imply, in particular, that the estimate

$$||v_1 - v_2||_{L^\infty(D)} \le c_3 \sup_{E \in S} \left(\ln(3 + ||\Phi_1(E) - \Phi_2(E)||^{-1}) \right)^{-\alpha_3}, \tag{2.21}$$

where $c_3 = c_3(N, D, m, S)$ and $||\Phi_1(E) - \Phi_2(E)|| = ||\Phi_1(E) - \Phi_2(E)||_{L^\infty(\partial D) \to L^\infty(\partial D)}$, can not hold with $\alpha_3 > 2m$ for real-valued potentials and with $\alpha_3 > m$ for complex potentials. Thus Theorem 2.3 and Theorem 2.4 show optimality of logarithmic stability results of Alessandrini and Novikov in considerably stronger sense that results of Mandache.

3 Some basic properties of Dirichlet-to-Neumann map

We continue to consider $D = B(0,1)$ and also to use polar coordinates $(r, \omega) \in \mathbb{R}_+ \times S^{d-1}$, with $x = r\omega$. Solutions of equation $-\Delta\psi = E\psi$ in D can be expressed by the Bessel functions J_α and Y_α with integer or half-integer order α, see definitions of Section 6. Here we state some Lemmas about these functions (Lemma 3.1, Lemma 3.2 and Lemma 3.3).

Lemma 3.1. *Suppose $k \neq 0$ and k^2 is not a Dirichlet eigenvalue for operator $-\Delta$ in D. Then*

$$\psi_0(r, \omega) = r^{-\frac{d-2}{2}} \frac{J_{j+\frac{d-2}{2}}(kr)}{J_{j+\frac{d-2}{2}}(k)} f_{jp}(\omega) \tag{3.1}$$

is the solution of equation (1.1) with $v \equiv 0$, $E = k^2$ and boundary condition $\psi|_{\partial D} = f_{jp}$.

Remark 3.1. Note that the assumptions of Lemma 3.1 imply $J_{j+\frac{d-2}{2}}(k) \neq 0$.

Lemma 3.2. *Let the assumptions of Lemma 3.1 hold. Then system of functions*

$$\{\psi_{jp}(r, \omega) = R_j(k, r) f_{jp}(\omega) : j \ge 0; 1 \le p \le p_j\}, \tag{3.2}$$

where

$$R_j(k, r) = r^{-\frac{d-2}{2}} \Big(Y_{j+\frac{d-2}{2}}(kr) J_{j+\frac{d-2}{2}}(k) - J_{j+\frac{d-2}{2}}(kr) Y_{j+\frac{d-2}{2}}(k) \Big), \tag{3.3}$$

is complete orthogonal system (in the sense of L_2) in the space of solutions of equation (1.1) in $D' = B(0,1) \setminus B(0,1/3)$ with $v \equiv 0$, $E = k^2$ and boundary condition $\psi|_{r=1} = 0$.

Lemma 3.3. *For any $C > 0$ and integer $d \geq 2$ there is a constant $N > 3$ depending on C such that for any integer $n \geq N$ and any $|z| \leq C$*

$$\frac{1}{2}\frac{(|z|/2)^{\alpha}}{\Gamma(\alpha+1)} \leq |J_{\alpha}(z)| \leq \frac{3}{2}\frac{(|z|/2)^{\alpha}}{\Gamma(\alpha+1)}, \tag{3.4}$$

$$|J'_{\alpha}(z)| \leq 3\frac{(|z|/2)^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.5}$$

$$\frac{1}{2\pi}(|z|/2)^{-\alpha}\Gamma(\alpha) \leq |Y_{\alpha}(z)| \leq \frac{3}{2\pi}(|z|/2)^{-\alpha}\Gamma(\alpha) \tag{3.6}$$

$$|Y'_{\alpha}(z)| \leq \frac{3}{\pi}(|z|/2)^{-\alpha-1}\Gamma(\alpha+1) \tag{3.7}$$

where $'$ denotes derivation with respect to z, $\alpha = n + \frac{d-2}{2}$ and $\Gamma(x)$ is the Gamma function.

Proofs of Lemma 3.1, Lemma 3.2 and Lemma 3.3 are given in Section 6.

Lemma 3.4. *Consider a compact $W \subset \mathbb{C}$. Suppose, that v is bounded, $\operatorname{supp} v \subset B(0, 1/3)$ and condition (1.4) is fulfilled for any $E \in W$ and potentials v and v_0, where $v_0 \equiv 0$. Denote $\Lambda_{v,E} = \Phi(E) - \Phi_0(E)$. Then there is a constant $\rho = \rho(W, d)$, such that for any $0 \leq j, 1 \leq p \leq p_j$, $0 \leq i, 1 \leq q \leq p_i$, we have*

$$|\langle \Lambda_{v,E} f_{jp}, f_{iq}\rangle| \leq \rho\, 2^{-\max(j,i)} ||v||_{L^{\infty}(D)} ||(-\Delta + v - E)^{-1}||_{L^2(D)}, \tag{3.8}$$

where $\Phi(E)$, $\Phi_0(E)$ are the DtN maps for v and v_0 respectively and $(-\Delta + v - E)^{-1}$ is considered with the Dirichlet boundary condition.

Proof of Lemma 3.4. For simplicity we give first a proof under the additional assumtions that $0 \notin W$ and there is a holomorphic germ $\sqrt{E}$ for $E \in W$. Since W is compact there is $C > 0$ such that for any $z \in W$ we have $|z| \leq C$. We take N from Lemma 3.3 for this C. We fix indeces j, p. Consider solutions $\psi(E)$, $\psi_0(E)$ of equation (1.1) with $E \in W$, boundary condition $\psi|_{\partial D} = f_{jp}$ and potentials v and v_0 respectively. Then $\psi(E) - \psi_0(E)$ has zero boundary values, so it is domain of $-\Delta + v - E$, and since

$$(-\Delta + v - E)\,(\psi(E) - \psi_0(E)) = -v\psi_0(E) \ \text{ in } D, \tag{3.9}$$

we obtain that

$$\psi(E) - \psi_0(E) = -(-\Delta + v - E)^{-1} v\psi_0(E). \tag{3.10}$$

If $j \geq N$ from Lemma 3.1 and Lemma 3.3 we have that

$$\begin{aligned} ||\psi_0(E)||^2_{L^2(B(0,1/3))} &= ||f_{jp}||^2_{L^2(S^{d-1})} \int_0^{1/3} \left| r^{-\frac{d-2}{2}} \frac{J_{j+\frac{d-2}{2}}(\sqrt{E}\,r)}{J_{j+\frac{d-2}{2}}(\sqrt{E})} \right|^2 r^{d-1} dr \leq \\ &\leq \int_0^{1/3} \left(\frac{3}{2} \frac{(|E|^{1/2} r/2)^{j+\frac{d-2}{2}}}{\Gamma(j+\frac{d-2}{2}+1)} \right)^2 \Big/ \left(\frac{1}{2} \frac{(|E|^{1/2}/2)^{j+\frac{d-2}{2}}}{\Gamma(j+\frac{d-2}{2}+1)} \right)^2 r\, dr = \\ &= 3\int_0^{1/3} r^{2j+d-1} dr = \frac{3}{2j+d}\left(\frac{1}{3}\right)^{2j+d} < 2^{-2j}. \end{aligned} \tag{3.11}$$

For $j < N$ we use fact that $||\psi_0(E)||_{L^2(B(0,1))}$ is continuous function on compact W and, since N depends only on W, we get that there is a constant $\rho_1 = \rho_1(W,d)$ such that

$$||\psi_0(E)||_{L^2(B(0,1/3))} \le \rho_1 2^{-j}. \tag{3.12}$$

Since v has support in $B(0,1/3)$ from (3.10) we get that

$$||\psi(E) - \psi_0(E)||_{L^2(B(0,1))} \le \rho_1 2^{-j}||v||_{L^\infty(D)}||(-\Delta + v - E)^{-1}||_{L^2(D)}. \tag{3.13}$$

Note that $\psi(E) - \psi_0(E)$ is the solution of equation (1.1) in $D' = B(0,1) \setminus B(0,1/3)$ with potential $v_0 \equiv 0$ and boundary condition $\psi|_{r=1} = 0$. From Lemma 3.2 we have that

$$\psi(E) - \psi_0(E) = \sum_{0\le i, 1\le q\le p_i} c_{iq}(E)\psi_{iq}(E) \ \text{ in } D' \tag{3.14}$$

for some c_{iq}, where

$$\psi_{iq}(E)(r,\omega) = R_i(\sqrt{E}, r) f_{iq}(\omega). \tag{3.15}$$

Since $R_i(\sqrt{E},1) = 0$

$$\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1} = \left.\frac{\partial\left(r^{\frac{d-2}{2}}R_i(\sqrt{E},r)\right)}{\partial r}\right|_{r=1}. \tag{3.16}$$

For $i \ge N$ from Lemma 3.3 we have that

$$\begin{aligned}\left|\frac{\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}}{Y_\alpha(\sqrt{E})J_\alpha(\sqrt{E})}\right| &= |E|^{1/2}\left|\frac{Y'_\alpha(\sqrt{E})}{Y_\alpha(\sqrt{E})} - \frac{J'_\alpha(\sqrt{E})}{J_\alpha(\sqrt{E})}\right| \le \\ \le 6|E|^{1/2}&\left(\frac{(|E|^{1/2}/2)^{-\alpha-1}\Gamma(\alpha+1)}{(|E|^{1/2}/2)^{-\alpha}\Gamma(\alpha)} + \frac{(|E|^{1/2}/2)^{\alpha-1}\Gamma(\alpha+1)}{(|E|^{1/2}/2)^{\alpha}\Gamma(\alpha)}\right) = 6\alpha,\end{aligned} \tag{3.17}$$

$$\begin{aligned}\left(\frac{||r^{-\frac{d-2}{2}}Y_\alpha(\sqrt{E}r)||_{L^2(\{1/3<|x|<2/5\})}}{|Y_\alpha(\sqrt{E})|}\right)^2 &\ge \int_{1/3}^{2/5}\left(\frac{1}{3}\frac{(|E|^{1/2}r/2)^{-\alpha}\Gamma(\alpha)}{(|E|^{1/2}/2)^{-\alpha}\Gamma(\alpha)}\right)^2 r\,dr \\ &\ge \left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\left(\frac{1}{3}(5/2)^\alpha\right)^2,\end{aligned} \tag{3.18}$$

$$\begin{aligned}\left(\frac{||r^{-\frac{d-2}{2}}J_\alpha(\sqrt{E}r)||_{L^2(\{1/3<|x|<2/5\})}}{|J_\alpha(\sqrt{E})|}\right)^2 &\le \int_{1/3}^{2/5}\left(3\frac{(|E|^{1/2}r/2)^{\alpha}\Gamma(\alpha)}{(|E|^{1/2}/2)^{\alpha}\Gamma(\alpha)}\right)^2 r\,dr \\ &\le \left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\left(3(2/5)^\alpha\right)^2,\end{aligned} \tag{3.19}$$

where $\alpha = i + \frac{d-2}{2}$. Since $N > 3$ we have that $\alpha > 3$. Using (3.18) and (3.19) we get that

$$\frac{||\psi_{iq}(E)||_{L^2(\{1/3<|x|<2/5\})}}{\left|Y_\alpha(\sqrt{E})J_\alpha(\sqrt{E})\right|} \geq \left(\left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\right)^{1/2}\left(\frac{1}{3}(5/2)^\alpha - 3(2/5)^\alpha\right) \geq \frac{1}{1000}(5/2)^\alpha. \tag{3.20}$$

For $i \geq N$ we get that

$$\left|\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}\right| \leq 1000\alpha(5/2)^{-\alpha}||\psi_{iq}(E)||_{L^2(\{1/3<|x|<1\})}. \tag{3.21}$$

For $i < N$ we use the fact that $\left|\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}\right| / ||\psi_{iq}(E)||_{L^2(\{1/3<|x|<1\})}$ is continuous function on compact W and get that for any $i \geq 0$ there is a constant $\rho_2 = \rho_2(W,d)$ such that

$$\left|\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}\right| \leq \rho_2\, 2^{-i}||\psi_{iq}(E)||_{L^2(\{1/3<|x|<1\})}. \tag{3.22}$$

Proceeding from (3.14) and using the Cauchy-Schwarz inequality we get that

$$|c_{iq}(E)| = \left|\frac{\left\langle \psi(E)-\psi_0(E), \psi_{iq}(E)\right\rangle_{L^2(\{1/3<|x|<1\})}}{||\psi_{iq}(E)||^2_{L^2(\{1/3<|x|<1\})}}\right| \leq \frac{||\psi(E)-\psi_0(E)||_{L^2(B(0,1))}}{||\psi_{iq}(E)||_{L^2(\{1/3<|x|<1\})}}. \tag{3.23}$$

Taking into account

$$\langle \Lambda_{v,E} f_{jp}, f_{iq}\rangle = \left\langle \left.\frac{\partial(\psi(E)-\psi_0(E))}{\partial\nu}\right|_{\partial D}, f_{iq}\right\rangle = c_{iq}(E)\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1} \tag{3.24}$$

and combining (3.22) and (3.23) we obtain that

$$|\langle \Lambda_{v,E} f_{jp}, f_{iq}\rangle| \leq \rho_2 2^{-i}||\psi(E)-\psi_0(E)||_{L^2(B(0,1))}. \tag{3.25}$$

From (3.13) and (3.25) we get (3.8).

For the general case we consider two compacts

$$W_\pm = W \cap \{z \mid \pm \mathrm{Im} z \geq 0\}. \tag{3.26}$$

Note that $\frac{J_{j+\frac{d-2}{2}}(\sqrt{E}r)}{J_{j+\frac{d-2}{2}}(\sqrt{E})}$ and $\frac{Y_{j+\frac{d-2}{2}}(\sqrt{E}r)}{Y_{j+\frac{d-2}{2}}(\sqrt{E})}$ have removable singularity in $E=0$ or, more precisely,

$$\begin{aligned} \frac{J_{j+\frac{d-2}{2}}(\sqrt{E}r)}{J_{j+\frac{d-2}{2}}(\sqrt{E})} &\longrightarrow r^{j+\frac{d-2}{2}}, \\ \frac{Y_{j+\frac{d-2}{2}}(\sqrt{E}r)}{Y_{j+\frac{d-2}{2}}(\sqrt{E})} &\longrightarrow r^{-j-\frac{d-2}{2}} \end{aligned} \tag{3.27}$$

as $E \longrightarrow 0$.

Considering the limit as $E \to 0$ we get that (3.13), (3.25) and consequently (3.8) are valid for $W_{\pm}$. To complete proof we can take $\rho = \max\{\rho_+, \rho_-\}$. ∎

Remark 3.2. From (3.1) and (3.10) we get that

$$\langle \Lambda_{v,E} f_{jp}, f_{iq} \rangle \quad \text{is holomorphic function in } W. \tag{3.28}$$

4 A fat metric space and a thin metric space

Definition 4.1. Let $(X, dist)$ be a metric space and $\epsilon > 0$. We say that a set $Y \subset X$ is an ϵ-net for $X_1 \subset X$ if for any $x \in X_1$ there is $y \in Y$ such that $dist(x, y) \leq \epsilon$. We call ϵ-entropy of the set X_1 the number $\mathcal{H}_\epsilon(X_1) := \log_2 \min\{|Y| : Y \text{ is an } \epsilon\text{-net fot } X_1\}$.

A set $Z \subset X$ is called ϵ-discrete if for any distinct $z_1, z_2 \in Z$, we have $dist(z_1, z_2) \geq \epsilon$. We call ϵ-capacity of the set X_1 the number $\mathcal{C}_\epsilon := \log_2 \max\{|Z| : Z \subset X_1 \text{ and } Z \text{ is } \epsilon\text{-discrete}\}$.

The use of ϵ-entropy and ϵ-capacity to derive properties of mappings between metric spaces goes back to Vitushkin and Kolmogorov (see [10] and references therein). One notable application was HilbertŠs 13th problem (about representing a function of several variables as a composition of functions of a smaller number of variables). In essence, Lemma 4.1 and Lemma 4.2 are parts of the Theorem XIV and the Theorem XVII in [10].

Lemma 4.1. *Let $d \geq 2$ è $m > 0$. For $\epsilon, \beta > 0$, consider the real metric space*

$$X_{m\epsilon\beta} = \{f \in C^m(D) \mid supp\, f \subset B(0, 1/3),\ ||f||_{L^\infty(D)} \leq \epsilon,\ ||f||_{C^m(D)} \leq \beta\},$$

with the metric induced by L^∞. Then there is a $\mu > 0$ such that for any $\beta > 0$ and $\epsilon \in (0, \mu\beta)$, there is an ϵ-discrete set $Z \subset X_{m\epsilon\beta}$ with at least $\exp\left(2^{-d-1}(\mu\beta/\epsilon)^{d/m}\right)$ elements.

Lemma 4.1 was also formulated and proved in [3].

Lemma 4.2. *For the interval $I = [a, b]$ with $a < b$ and $\gamma > 0$ consider ellipse $W_{I,\gamma} \in \mathbb{C}$*

$$W_{I,\gamma} = \{\frac{a+b}{2} + \frac{a-b}{2} \cos z \mid |Im\, z| \leq \gamma\}. \tag{4.1}$$

Then there is a constant $\nu = \nu(C, \gamma) > 0$, such that for every $\delta \in (0, e^{-1})$, there is a δ-net for the space functions on I with L^∞-norm, having holomorphic continuation to $W_{I,\gamma}$ with module bounded above on $W_{I,\gamma}$ by the constant C, with at most $\exp(\nu(\ln \delta^{-1})^2)$ elements.

Proof of Lemma 4.2. Theorem XVII in [10] provides asymptotic behaviour of the entropy of this space with respect to $\delta \to 0$. Here we get upper estimate of it. Suppose $g(z)$ is holomorphic function in $W_{I,\gamma}$ with module bounded above by the constant C. Consider the function $f(z) = g(\frac{a+b}{2} + \frac{a-b}{2} \cos z)$. By the choise of $W_{I,\gamma}$ we get that $f(z)$ is 2π-periodic holomorphic function in the stripe $|\mathrm{Im}\, z| \le \gamma$. Then for any integer n

$$|c_n| = \left| \int_0^{2\pi} e^{inx} f(x) dx \right| \le \int_0^{2\pi} e^{-|n|\gamma} C dx \le 2\pi C e^{-|n|\gamma}. \tag{4.2}$$

Let n_δ be the smallest natural number such that $2\pi C e^{-n\gamma} \le 6\pi^{-2}(n+1)^{-2}\delta$ for any $n \ge n_\delta$. Taking natural logarithm and using $\ln \delta^{-1} \ge 1$, we get that

$$n_\delta \le C' \ln \delta^{-1}, \tag{4.3}$$

where C' depends only on C and γ. We denote $\delta' = 3\pi^{-2}(n_\delta+1)^{-2}\delta$. Consider the set

$$Y_\delta = \delta'\mathbb{Z} \bigcap [-2\pi C, 2\pi C] + i \cdot \delta'\mathbb{Z} \bigcap [-2\pi C, 2\pi C]. \tag{4.4}$$

Using (4.3), we have that

$$|Y_\delta| = (1 + 2[2\pi C/\delta'])^2 \le C''\delta^{-2} \ln^4 \delta^{-1}, \tag{4.5}$$

with C'' depending only on C and γ. We set

$$Y = \left\{ \sum_{n=0}^{\infty} d_n \cos\left(n \arccos \frac{x - \frac{a+b}{2}}{\frac{a-b}{2}} \right) \mid d_n \in Y_\delta \text{ for } n \le n_\delta,\ d_n = 0 \text{ otherwise} \right\}. \tag{4.6}$$

For given $f(z)$ in case of $n \le n_\delta$ we take d_n to be one of the closest elements of Y_δ to c_n. Since $|c_n| \le 2\pi C$, this ensures $|c_n - d_n| \le 2\delta'$. For $n > n_\delta$ we take $d_n = 0$. We have then

$$|c_n - d_n| \le 6\pi^{-2}(n+1)^{-2}\delta. \tag{4.7}$$

For $n > n_\delta$ this is true by the construction of n_δ, otherwise by the choise of δ'. Since $f(x)$ is 2π-periodic even function, we get $g_Y(x) \in Y$ such that

$$||g(x) - g_Y(x)||_{L^\infty(a,b)} \le \sum_{n=0}^{\infty} |c_n - d_n| \le 6\pi^{-2}\delta \sum_{n=1}^{\infty} \frac{1}{n^2} = \delta. \tag{4.8}$$

We have that $|Y| = |Y_\delta|^{n_\delta}$. Taking into account (4.3),(4.5) and $\ln \delta^{-1} \ge 1$, we get

$$|Y| \le (C''\delta^{-2} \ln^4 \delta^{-1})^{C' \ln \delta^{-1}} \le \exp\left(C''' \ln \delta^{-1} C' \ln \delta^{-1}\right) \le \exp(\nu (\ln \delta^{-1})^2). \tag{4.9}$$

■

Remark 4.1. The assertion is valid even in the case of $a = b$. As δ-net we can take

$$Y = \frac{\delta}{2}\mathbb{Z} \bigcap [-C, C] + i \cdot \frac{\delta}{2}\mathbb{Z} \bigcap [-C, C]. \tag{4.10}$$

Consider an operator $A : H^{-s}(S^{d-1}) \to H^s(S^{d-1})$. We denote its matrix elements in the basis $\{f_{jp}\}$ by $a_{jpiq} = \langle Af_{jp}, f_{iq}\rangle$. From [3] we have that

$$||A||_{H^{-s}\to H^s} \le 4 \sup_{j,p,i,q} (1 + \max(j,i))^{2s+d} |a_{jpiq}|. \tag{4.11}$$

Consider system $S = \bigcup_{j=1}^{K} I_j$ of σ-regular intervals. We introduce the Banach space

$$X_{S,s} = \left\{ \Big(a_{jpiq}(E)\Big) \mid \Big\| \Big(a_{jpiq}(E)\Big) \Big\|_{X_{S,s}} < \infty \right\}, \tag{4.12}$$

where

$$\Big\| \Big(a_{jpiq}(E)\Big) \Big\|_{X_{S,s}} = \sup_{j,p,i,q} \left((1 + \max(j,i))^{2s+d} \sup_{E\in S} |a_{jpiq}(E)| \right). \tag{4.13}$$

Denote by B^∞ the ball of centre 0 and radius $2\sigma/3$ in $L^\infty(B(0,1/3))$. We identify in the sequel an operator $A(E) : H^{-s}(S^{d-1}) \to H^s(S^{d-1})$ with its matrix $\Big(a_{jpiq}(E)\Big)$. Note that the estimate (4.11) implies that

$$\sup_{E\in S} \|A(E)\|_{H^{-s}\to H^s} \le 4 \Big\| \Big(a_{jpiq}(E)\Big) \Big\|_{X_{S,s}}. \tag{4.14}$$

We consider operator $\Lambda_{v,E}$ from Lemma 3.4 as

$$\Lambda : B^\infty \to \left\{ \Big(a_{jpiq}(E)\Big) \right\}, \tag{4.15}$$

where $a_{jpiq}(E)$ are matrix elements in the basis $\{f_{jp}\}$ of operator $\Lambda_{v,E}$.

Lemma 4.3. *Λ maps B^∞ into $X_{S,s}$ for any s. There is a constant $\eta = \eta(S,s,d) > 0$ such that for every $\delta \in (0, e^{-1})$ there is a δ-net Y for $\Lambda(B^\infty)$ in $X_{S,s}$ with at most $\exp(\eta(\ln\delta^{-1})^{2d})$ elements.*

Proof of Lemma 4.3. For simplicity we give first a proof in case of S consists of only one σ-regular interval I. From (4.1) we take $W_I = W_{I,\gamma}$, where constant $\gamma > 0$ is such as for any $E \in W_I$ there is E_I in I such as $|E - E_I| < \sigma/6$. From (2.16) we get that

$$|E - \lambda| \ge |E_I - \lambda| - |E - E_I| \ge 5\sigma/6, \tag{4.16}$$

with λ being Dirichlet eigenvalue for operator $-\Delta$ in D which is closest to E. Then for potential $v \in B^\infty$ and $E \in W_I$ we have that

$$||(-\Delta + v - E)^{-1}||_{L^2(D)} \le (|\lambda - E| - 2\sigma/3)^{-1} \le (5\sigma/6 - 2\sigma/3)^{-1} = 6/\sigma \tag{4.17}$$

and

$$||v||_{L^\infty(D)} ||(-\Delta + v - E)^{-1}||_{L^2(D)} \le (2\sigma/3)(6/\sigma) = 4, \tag{4.18}$$

where $(-\Delta+v-E)^{-1}$ is considered with the Dirichlet boundary condition. We obtain from Lemma 3.4 that

$$|a_{jpiq}(E)| \le 4\rho\, 2^{-\max(j,i)}, \tag{4.19}$$

where $\rho=\rho(W_I,d)$. Hence $||(a_{jpiq}(E))||_{X_{S,s}} \le \sup_l (1+l)^{2s+d} 4\rho\, 2^{-l} < \infty$ for any s and d and so the first assertion of the Lemma 4.3 is proved.

Let $l_{\delta s}$ be the smallest natural number such that $(1+l)^{2s+d}4\rho\, 2^{-l} \le \delta$ for any $l \ge l_{\delta s}$. Taking natural logarithm and using $\ln\delta^{-1} \ge 1$, we get that

$$l_{\delta s} \le C' \ln\delta^{-1}, \tag{4.20}$$

where the constant C' depends only on s, d and I. Denote Y_{jpiq} is δ_{jpiq}-net from Lemma 4.2 with constant $C=\sup_l(1+l)^{2s+d}4\rho\, 2^{-l}$, where $\delta_{jpiq}=(1+\max(j,i))^{-2s-d}\delta$. We set

$$Y=\left\{(a_{jpiq}(E)) \mid a_{jpiq}(E)\in Y_{jpiq} \text{ for } \max(j,i)\le l_{\delta s},\ a_{jpiq}(E)=0 \text{ otherwise}\right\}. \tag{4.21}$$

For any $(a_{jpiq}(E))\in\Lambda(B^\infty)$ there is an element $(b_{jpiq}(E))\in Y$ such that

$$(1+\max(j,i))^{2s+d}|a_{jpiq}(E)-b_{jpiq}(E)| \le (1+\max(j,i))^{2s+d}\delta_{jpiq}=\delta, \tag{4.22}$$

in case of $\max(j,i)\le l_{\delta s}$ and

$$(1+\max(j,i))^{2s+d}|a_{jpiq}(E)-b_{jpiq}(E)| \le (1+\max(j,i))^{2s+d}2\rho\, 2^{-\max(j,i)} \le \delta, \tag{4.23}$$

otherwise.

It remains to count the elements of Y. Using again the fact that $\ln\delta^{-1}\ge 1$ and (4.20) we get for $\max(j,i)\le l_{\delta s}$

$$|Y_{jpiq}| \le \exp(\nu(\ln\delta_{jpiq}^{-1})^2) \le \exp(\nu'(\ln\delta^{-1})^2). \tag{4.24}$$

From [3] we have that $n_{\delta s}\le 8(1+l_{\delta s})^{2d-2}$, where $n_{\delta s}$ is the number of fourtuples (j,p,i,q) with $\max(j,i)\le l_{\delta s}$. Taking η to be big enough we get that

$$\begin{aligned}|Y| &\le \left(\exp(\nu'(\ln\delta^{-1})^2)\right)^{n_{\delta s}}\\ &\le \exp\left(\nu'(\ln\delta^{-1})^2 8(1+C'\ln\delta^{-1})^{2d-2}\right)\\ &\le \exp\left(\eta(\ln\delta^{-1})^{2d}\right).\end{aligned} \tag{4.25}$$

For $S=\bigcup_{j=1}^{K} I_j$ assertion follows immediately, taking η to be in K times more and Y as composition $(Y_1,\dots,Y_K)$ of δ-nets for each interval. ∎

5 Proofs of the main results

In this section we give proofs of Theorem 2.3 and Theorem 2.4.

Proof of Theorem 2.3. Take $v_0 \in L^\infty(B(0,1/3))$, $||v_0||_{L^\infty(D)} \le \sigma/3$ and $\epsilon \in (0,\sigma/3)$. By Lemma 4.1, the set $v_0 + X_{m\epsilon\beta}$ has an ϵ-discrete subset $v_0 + Z$. Since for $\epsilon \in (0,\sigma/3)$ we have $v_0 + X_{m\epsilon\beta} \subset B^\infty$, where B^∞ is the ball of centre 0 and radius $2\sigma/3$ in $L^\infty(B(0,1/3))$. The set Y constructed in Lemma 4.3 is also δ-net for $\Lambda(v_0 + X_{m\epsilon\beta})$. We take δ such that $8\delta = \exp\left(-\epsilon^{-\frac{1}{2m}}\right)$. Note that inequalities of (2.17) follow from

$$|v_0 + Z| > |Y|. \tag{5.1}$$

In fact, if $|v_0 + Z| > |Y|$, then there are two potentials $v_1, v_2 \in v_0 + Z$ with images under Λ in the same $X_{S,s}$-ball radius δ centered at a point of Y, so we get from (4.14)

$$\sup_{E\in S} ||\Phi_1(E) - \Phi_2(E)||_{H^{-s}\to H^s} \le 4||\Lambda_{v_1,E} - \Lambda_{v_2,E}||_{X_{S,s}} \le 8\delta = \exp\left(-\epsilon^{-\frac{1}{2m}}\right). \tag{5.2}$$

It remains to find β such as (5.1) is fullfiled. By Lemma 4.3

$$|Y| \le \exp\left(\eta\left(\ln 8 + \epsilon^{-\frac{1}{2m}}\right)^{2d}\right) \le \max\Big(\exp\left((2\ln 8)^{2d}\eta\right), \exp\left(2^{2d}\eta\epsilon^{-d/m}\right)\Big). \tag{5.3}$$

Now we take

$$\beta > \mu^{-1}\max\left(\sigma/3, \eta^{m/d}2^{3m}, \frac{\sigma}{3}\eta^{m/d}2^m(2\ln 8)^{2m}\right) \tag{5.4}$$

This fulfils requirement $\epsilon < \mu\beta$ in Lemma 4.1, which gives

$$\begin{aligned}|v_0 + Z| = |Z| \ge \exp\left(2^{-d-1}(\mu\beta/\epsilon)^{d/m}\right) \overset{(5.4)}{>} \\ > \max\Big(\exp\left(2^{-d-1}(\eta^{m/d}2^{3m}/\epsilon)^{d/m}\right), \exp\left(2^{-d-1}(\eta^{m/d}2^m(2\ln 8)^{2m})^{d/m}\right)\Big) \overset{(5.3)}{\ge} |Y|.\end{aligned} \tag{5.5}$$

■

Proof of Theorem 2.4. In a similar way with the proof of Theorem 2 of [3] we obtain that

$$\langle(\Phi_{mn}(E) - \Phi_0(E))\, f_{jp}, f_{iq}\rangle = 0 \tag{5.6}$$

for $j, i \le \left[\frac{n-1}{2}\right]$. The only difference is that instead of the operator $-\Delta$ we consider the operator $-\Delta - E$. From (4.11), (4.19) and (5.6) we get

$$||\Phi_{mn}(E) - \Phi_0(E)||_{H^{-s}\to H^s} \le 16\rho \sup_{l\ge n/2}(1+l)^{2s+d}2^{-l} \le c' 2^{-n/4}. \tag{5.7}$$

The fact that $||v_{mn}||_{C^m(D)}$ is bounded as $n \to \infty$ is also a part of Theorem 2 of [3]. ■

6 Bessel functions

In this section we prove Lemma 3.1, Lemma 3.2 and Lemma 3.3 about the Bessel functions. Consider the problem of finding solutions of the form $\psi(r,\omega) = R(r)f_{jp}(\omega)$ of equation (1.1) with $v \equiv 0$. We have that

$$\Delta = \frac{\partial^2}{(\partial r)^2} + (d-1)r^{-1}\frac{\partial}{\partial r} + r^{-2}\Delta_{S^{d-1}}, \tag{6.1}$$

where $\Delta_{S^{d-1}}$ is Laplace-Beltrami operator on S^{d-1}. We have that

$$\Delta_{S^{d-1}} f_{jp} = -j(j+d-2)f_{jp}. \tag{6.2}$$

Then we have the following equation for $R(r)$:

$$-R'' - \frac{d-1}{r}R' + \frac{j(j+d-2)}{r^2}R = ER. \tag{6.3}$$

Taking $R(r) = r^{-\frac{d-2}{2}}\tilde{R}(r)$, we get

$$r^2\tilde{R}'' + r\tilde{R}' + \left(Er^2 - \left(j + \frac{d-2}{2}\right)^2\right)\tilde{R} = 0. \tag{6.4}$$

This equation is known as Bessel's equation. For $E = k^2 \neq 0$ it has two linearly independent solutions $J_{j+\frac{d-2}{2}}(kr)$ and $Y_{j+\frac{d-2}{2}}(kr)$, where

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\alpha}}{\Gamma(m+1)\Gamma(m+\alpha+1)}, \tag{6.5}$$

$$Y_\alpha(z) = \frac{J_\alpha(z)\cos\pi\alpha - J_{-\alpha}(z)}{\sin\pi\alpha} \text{ for } \alpha \notin \mathbb{Z}, \tag{6.6}$$

and

$$Y_\alpha(z) = \lim_{\alpha'\to\alpha} Y_{\alpha'}(z) \text{ for } \alpha \in \mathbb{Z}. \tag{6.7}$$

The following Lemma is called the Nielsen inequality. A proof can be found in [5]

Lemma 6.1.

$$\begin{aligned} J_\alpha(z) &= \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}(1+\theta), \\ |\theta| &< \exp\left(\frac{|z|^2/4}{|\alpha_0+1|}\right) - 1, \end{aligned} \tag{6.8}$$

where $|\alpha_0+1|$ *is the least of numbers* $|\alpha+1|, |\alpha+2|, |\alpha+3|, \ldots$.

Lemma 6.1 implies that $r^{-\frac{d-2}{2}}J_{j+\frac{d-2}{2}}(kr)$ has removable singularity at $r = 0$. Then, using the boundary conditions $R(1) = 1$ and $R(1) = 0$, one can obtain assertions of Lemma 3.1 and Lemma 3.2, respectively.

Proof of Lemma 3.3 Formula (3.4) follows immediately from Lemma 6.1. We have from [5] that

$$J'_\alpha(z) = J_{\alpha-1}(z) - \frac{\alpha}{z} J_\alpha(z). \tag{6.9}$$

Further, taking α big enough we get

$$|J'_\alpha(z)| \le |J_{\alpha-1}(z)| + |\frac{\alpha}{z} J_\alpha(z)| \le \frac{3}{2}\frac{(|z|/2)^{\alpha-1}}{\Gamma(\alpha)} + \frac{3\alpha}{2|z|}\frac{(|z|/2)^{\alpha}}{\Gamma(\alpha+1)} \le 3\frac{(|z|/2)^{\alpha-1}}{\Gamma(\alpha)}. \tag{6.10}$$

For $\alpha = n+1/2$ we have $Y_\alpha = (-1)^{n+1}J_{-\alpha}$. Consider its series expansion, see (6.5).

$$J_{-\alpha}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m-\alpha}}{m!\,\Gamma(m-\alpha+1)} = \sum_{m=0}^{\infty} c_m (z/2)^{2m-\alpha}. \tag{6.11}$$

Note that $|c_m/c_{m+1}| = (m+1)|m-\alpha+1| \ge n/2$. As corollary we obtain that

$$\begin{aligned} |Y_\alpha(z)| &= \frac{(|z|/2)^{-\alpha}}{|\Gamma(-\alpha+1)|}(1+\theta) = \frac{1}{\pi}(|z|/2)^{-\alpha}\Gamma(\alpha)(1+\theta), \\ |\theta| &\le \sum_{m=1}^{\infty}\left(\frac{|z|^2}{2n}\right)^{2m} \le \frac{|z|^2/2n}{1-|z|^2/2n}. \end{aligned} \tag{6.12}$$

For $\alpha = n$ we have from [5] that

$$\begin{aligned} Y_n(z) &= \frac{2}{\pi}J_n(z)\ln\left(\frac{z}{2}\right) - \frac{1}{\pi}\sum_{m=0}^{n-1}\left(\frac{z}{2}\right)^{2m-n}\frac{(n-m-1)!}{m!} - \\ &-\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m (z/2)^{2m+n}}{m!(m+n)!}\left(\frac{\Gamma'(m+1)}{\Gamma(m+1)} + \frac{\Gamma'(m+n+1)}{\Gamma(m+n+1)}\right) = \\ &= \frac{2}{\pi}J_n(z)\ln\left(\frac{z}{2}\right) - \frac{1}{\pi}\sum_{m=0}^{n-1}\tilde{c}_m (z/2)^{2m-n} - \frac{1}{\pi}\sum_{m=0}^{\infty} b_m (z/2)^{2m+n}. \end{aligned} \tag{6.13}$$

Using well-known equality $\Gamma'(x)/\Gamma(x) < \ln x$, $x > 1$, see [11], we get following estimation for the coefficients b_m are defined in (6.13).

$$|b_m| < \frac{\ln(m+1) + \ln(n+m+1)}{m!(n+m)!} < \frac{2(n+m)}{m!(n+m)!} < \frac{1}{m!}. \tag{6.14}$$

Note also that $|\tilde{c}_m/\tilde{c}_{m+1}| = (m+1)(n-m-1) \ge n/2$. Combining it with (6.13) and (6.14), we obtain that

$$\begin{aligned} |Y_n(z)| &= \frac{1}{\pi}(|z|/2)^{-n}\Gamma(n)(1+\theta), \\ |\theta| &\le 3\frac{(|z|/2)^{2n}|\ln(z/2)|}{\Gamma(n)} + \sum_{m=1}^{n-1}\left(\frac{|z|^2}{2n}\right)^{2m} + \frac{(|z|/2)^{2n}}{\Gamma(n)}\sum_{m=0}^{\infty}\frac{(|z|/2)^{2m}}{m!} \le \\ &\le 3\pi\frac{\max\left(1, (|z|/2)^{2n+1}\right)}{\Gamma(n)} + \frac{|z|^2/2n}{1-|z|^2/2n} + \frac{(|z|/2)^{2n}e^{|z|^2/4}}{\Gamma(n)}. \end{aligned} \tag{6.15}$$

Formula (3.6) follows from (6.12) and (6.15). We have from [5] that

$$Y'_\alpha(z) = Y_{\alpha-1}(z) - \frac{\alpha}{z} Y_\alpha(z). \tag{6.16}$$

Taking n big enough, we get that

$$\begin{gathered} |Y'_\alpha(z)| \le |Y_{\alpha-1}(z)| + |\frac{\alpha}{z} Y_\alpha(z)| \le \\ \le \frac{3}{2\pi}\left((|z|/2)^{-\alpha+1}\,\Gamma(\alpha-1) + \frac{\alpha}{|z|}(|z|/2)^{\alpha}\Gamma(\alpha)\right) \le \frac{3}{\pi}(|z|/2)^{-\alpha-1}\Gamma(\alpha+1). \end{gathered} \tag{6.17}$$

Combining reqirements for n, stated above, we get that for any $n \ge N+1$ all inequalities of Lemma 3.3 are fullfiled, where N such that

$$\begin{cases} N > 3, \\ \exp\left(\dfrac{C^2/4}{N+1}\right) - 1 \le 1/2, \\ 3\pi \dfrac{\max\left(1, (C/2)^{2N+1}\right)}{\Gamma(N)} + \dfrac{C^2}{2N - C^2} + \dfrac{(C/2)^{2N} e^{C^2/4}}{\Gamma(N)} \le 1/2. \end{cases} \tag{6.18}$$

■

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