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**Global stability for the  
multi-channel Gel'fand-Calderón  
inverse problem in two  
dimensions**

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# GLOBAL STABILITY FOR THE MULTI-CHANNEL GEL'FAND-CALDERÓN INVERSE PROBLEM IN TWO DIMENSIONS

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ABSTRACT. We prove a global logarithmic stability estimate for the multi-channel Gel'fand-Calderón inverse problem on a two-dimensional bounded domain, i.e., the inverse boundary value problem for the equation  $-\Delta\psi + v\psi = 0$  on  $D$ , where  $v$  is a smooth matrix-valued potential defined on a bounded planar domain  $D$ .

## 1. INTRODUCTION

The Schrödinger equation at zero energy,

$$(1.1) \quad -\Delta\psi + v(x)\psi = 0 \text{ on } D \subset \mathbb{R}^2,$$

arises in quantum mechanics, acoustics and electrodynamics. The reconstruction of the complex-valued potential  $v$  in equation (1.1) through the Dirichlet-to-Neumann operator is one of the most studied inverse problems (see [11], [10], [4], [12], [13], [14] and references therein).

In this article we consider the multi-channel two-dimensional Schrödinger equation, i.e., equation (1.1) with matrix-valued potentials and solutions; this case was already studied in [15, 14]. One of the motivations for studying the multi-channel equation is that it comes up as a 2D-approximation for the 3D equation (see [14, Sec. 2]).

The main purpose of this paper is to give a global stability estimate for this inverse problem in the multi-channel case.

Let  $D$  be an open bounded domain in  $\mathbb{R}^2$  with  $C^2$  boundary and  $v \in C^1(\bar{D}, M_n(\mathbb{C}))$ , where  $M_n(\mathbb{C})$  is the set of the  $n \times n$  complex-valued matrices. The Dirichlet-to-Neumann map associated with  $v$  is the operator  $\Phi : C^1(\partial D, M_n(\mathbb{C})) \rightarrow L^p(\partial D, M_n(\mathbb{C}))$ ,  $p < \infty$ , defined by

$$(1.2) \quad \Phi(f) = \left. \frac{\partial\psi}{\partial\nu} \right|_{\partial D},$$

where  $f \in C^1(\partial D, M_n(\mathbb{C}))$ ,  $\nu$  is the outer normal of  $\partial D$  and  $\psi$  is the  $H^1(\bar{D}, M_n(\mathbb{C}))$ -solution of the Dirichlet problem

$$(1.3) \quad -\Delta\psi + v(x)\psi = 0 \text{ on } D, \quad \psi|_{\partial D} = f;$$

here we assume that

$$(1.4) \quad 0 \text{ is not a Dirichlet eigenvalue of the operator } -\Delta + v \text{ in } D.$$

This construction gives rise to the following inverse boundary value problem: given  $\Phi$ , find  $v$ .

This problem can be considered as the Gel'fand inverse boundary value problem for the multi-channel Schrödinger equation at zero energy (see [8], [11]) and can also be seen as a generalization of the Calderón problem for the electrical impedance tomography (see [5], [11]). Note also that we can think of this problem as a model for monochromatic ocean tomography (e.g., see [2] for similar problems arising in this type of tomography).

In the case of complex-valued potentials the global injectivity of the map  $v \rightarrow \Phi$  was first proved for  $D \subset \mathbb{R}^d$  with  $d \geq 3$  in [11] and for  $d = 2$  with  $v \in L^p$  in [4]: in particular, these results were obtained by the use of global reconstructions developed in the same papers. The first global uniqueness result (along with an exact reconstruction method) for matrix-valued potentials was given in [14], which deals with  $C^1$  matrix-valued potentials defined on a domain in  $\mathbb{R}^2$ . A global stability estimate for the Gel'fand-Calderón problem with  $d \geq 3$  was first found by Alessandrini in [1]; this result was recently improved in [12]. In the two-dimensional case the first global stability estimate was given in [13].

In this paper we extend the results of [13] to the matrix-valued case. We do not discuss global results for special real-valued potentials arising from conductivities: for this case the reader is referred to the references given in [1], [4], [10], [11], [12], [13].

Our main result is the following:

**Theorem 1.1.** *Let  $D \subset \mathbb{R}^2$  be an open bounded domain with a  $C^2$  boundary,  $v_1, v_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$  two matrix-valued potentials which satisfy (1.4), with  $\|v_j\|_{C^2(\bar{D})} \leq N$  for  $j = 1, 2$ , and  $\Phi_1, \Phi_2$  the corresponding Dirichlet-to-Neumann operators. For simplicity we also assume that  $v_1|_{\partial D} = v_2|_{\partial D}$  and  $\frac{\partial}{\partial \nu} v_1|_{\partial D} = \frac{\partial}{\partial \nu} v_2|_{\partial D}$ . Then there exists a constant  $C = C(D, N, n)$  such that*

$$(1.5) \quad \|v_2 - v_1\|_{L^\infty(D)} \leq C (\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{-\frac{3}{4}} (\log(3 \log(3 + \|\Phi_2 - \Phi_1\|^{-1})))^2,$$

where  $\|\cdot\|$  is the induced operator norm on  $L^\infty(\partial D, M_n(\mathbb{C}))$  and  $\|v\|_{L^\infty(D)} = \max_{1 \leq i, j \leq n} \|v_{i,j}\|_{L^\infty(D)}$  (likewise for  $\|v\|_{C^2(\bar{D})}$ ) for a matrix-valued potential  $v$ .

This is the first global stability result for the multi-channel ( $n \geq 2$ ) Gel'fand-Calderón inverse problem in two dimensions. In addition, Theorem 1.1 is new also for the scalar case, as the estimate obtained in [13] is weaker. We remark, in particular, that this result is true in the special case when  $v_1 \equiv v_2 \equiv \Lambda \in M_n(\mathbb{C})$  in a neighborhood of  $\partial D$  (situation which appears in the approximation of the 3D equation, see [14, Remark 3 and Section 2]).

Instability estimates complementing the stability estimates of [1], [12], [13] and of the present work are given in [10], [9].

The proof of Theorem 1.1 is based on results obtained in [13], [14], which take inspiration mostly from [4] and [1]. In particular, for  $z_0 \in D$  we use the existence and uniqueness of a family of solutions  $\psi_{z_0}(z, \lambda)$  of equation (1.1) where in particular  $\psi_{z_0} \rightarrow e^{\lambda(z-z_0)^2} I$ , for  $\lambda \rightarrow \infty$  (where  $I$  is the identity matrix). Then, using an appropriate matrix-valued version of Alessandrini's identity along with stationary phase techniques, we obtain the result. Note that this matrix-valued identity is one of the new results of this paper.

A generalizations of Theorem 1.1 in the case where we do not assume that  $v_1|_{\partial D} = v_2|_{\partial D}$  and  $\frac{\partial}{\partial \nu} v_1|_{\partial D} = \frac{\partial}{\partial \nu} v_2|_{\partial D}$ , is given in section 5.

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## 2. PRELIMINARIES

In this section we introduce and give details on the above-mentioned family of solutions of equation (1.1), which will be used throughout the paper.

We identify  $\mathbb{R}^2$  with  $\mathbb{C}$  and use the coordinates  $z = x_1 + ix_2$ ,  $\bar{z} = x_1 - ix_2$  where  $(x_1, x_2) \in \mathbb{R}^2$ . Let us define the function spaces  $C_{\bar{z}}^1(\bar{D}) = \{u : u, \frac{\partial u}{\partial \bar{z}} \in C(\bar{D}, M_n(\mathbb{C}))\}$  with the norm  $\|u\|_{C_{\bar{z}}^1(\bar{D})} = \max(\|u\|_{C(\bar{D})}, \|\frac{\partial u}{\partial \bar{z}}\|_{C(\bar{D})})$ , where  $\|u\|_{C(\bar{D})} = \sup_{z \in \bar{D}} |u|$  and  $|u| = \max_{1 \leq i, j \leq n} |u_{i,j}|$ ; we also define  $C_z^1(\bar{D}) = \{u : u, \frac{\partial u}{\partial z} \in C(\bar{D}, M_n(\mathbb{C}))\}$  with an analogous norm. Following [13], [14], we consider the functions:

$$(2.1) \quad G_{z_0}(z, \zeta, \lambda) = e^{\lambda(z-z_0)^2} g_{z_0}(z, \zeta, \lambda) e^{-\lambda(\zeta-z_0)^2},$$

$$(2.2) \quad g_{z_0}(z, \zeta, \lambda) = \frac{e^{\lambda(\zeta-z_0)^2 - \bar{\lambda}(\bar{\zeta} - \bar{z}_0)^2}}{4\pi^2} \int_D \frac{e^{-\lambda(\eta-z_0)^2 + \bar{\lambda}(\bar{\eta} - \bar{z}_0)^2}}{(z-\eta)(\bar{\eta} - \bar{\zeta})} d\text{Re}\eta d\text{Im}\eta,$$

$$(2.3) \quad \psi_{z_0}(z, \lambda) = e^{\lambda(z-z_0)^2} \mu_{z_0}(z, \lambda),$$

$$(2.4) \quad \mu_{z_0}(z, \lambda) = I + \int_D g_{z_0}(z, \zeta, \lambda) v(\zeta) \mu_{z_0}(\zeta, \lambda) d\operatorname{Re}\zeta d\operatorname{Im}\zeta,$$

$$(2.5) \quad h_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} v(z) \mu_{z_0}(z, \lambda) d\operatorname{Re}z d\operatorname{Im}z,$$

where  $z, z_0, \zeta \in D$ ,  $\lambda \in \mathbb{C}$  and  $I$  is the identity matrix. In addition, equation (2.4) at fixed  $z_0$  and  $\lambda$ , is considered as a linear integral equation for  $\mu_{z_0}(\cdot, \lambda) \in C_{\bar{z}}^1(\bar{D})$ . The functions  $G_{z_0}(z, \zeta, \lambda)$ ,  $g_{z_0}(z, \zeta, \lambda)$ ,  $\psi_{z_0}(z, \lambda)$ ,  $\mu_{z_0}(z, \lambda)$  defined above, satisfy the following equations (see [13], [14]):

$$(2.6) \quad 4 \frac{\partial^2}{\partial z \partial \bar{z}} G_{z_0}(z, \zeta, \lambda) = \delta(z - \zeta),$$

$$(2.7) \quad 4 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} G_{z_0}(z, \zeta, \lambda) = \delta(\zeta - z),$$

$$(2.8) \quad 4 \left( \frac{\partial}{\partial z} + 2\lambda(z - z_0) \right) \frac{\partial}{\partial \bar{z}} g_{z_0}(z, \zeta, \lambda) = \delta(z - \zeta),$$

$$(2.9) \quad 4 \frac{\partial}{\partial \bar{\zeta}} \left( \frac{\partial}{\partial \zeta} - 2\lambda(\zeta - z_0) \right) g_{z_0}(z, \zeta, \lambda) = \delta(\zeta - z),$$

$$(2.10) \quad -4 \frac{\partial^2}{\partial z \partial \bar{z}} \psi_{z_0}(z, \lambda) + v(z) \psi_{z_0}(z, \lambda) = 0,$$

$$(2.11) \quad -4 \left( \frac{\partial}{\partial z} + 2\lambda(z - z_0) \right) \frac{\partial}{\partial \bar{z}} \mu_{z_0}(z, \lambda) + v(z) \mu_{z_0}(z, \lambda) = 0,$$

where  $z, z_0, \zeta \in D$ ,  $\lambda \in \mathbb{C}$ ,  $\delta$  is the Dirac delta. (In addition, we assume that (2.4) is uniquely solvable for  $\mu_{z_0}(\cdot, \lambda) \in C_{\bar{z}}^1(\bar{D})$  at fixed  $z_0$  and  $\lambda$ .)

We say that the functions  $G_{z_0}$ ,  $g_{z_0}$ ,  $\psi_{z_0}$ ,  $\mu_{z_0}$ ,  $h_{z_0}$  are the Bukhgeim-type analogues of the Faddeev functions (see [14]). We recall that the history of these functions goes back to [7] and [3].

Now we state some fundamental lemmata. Let

$$(2.12) \quad g_{z_0, \lambda} u(z) = \int_D g_{z_0}(z, \zeta, \lambda) u(\zeta) d\operatorname{Re}\zeta d\operatorname{Im}\zeta, \quad z \in \bar{D}, \quad z_0, \lambda \in \mathbb{C},$$

where  $g_{z_0}(z, \zeta, \lambda)$  is defined by (2.2) and  $u$  is a test function.

**Lemma 2.1** ([13]). *Let  $g_{z_0, \lambda} u$  be defined by (2.12). Then, for  $z_0, \lambda \in \mathbb{C}$ , the following estimates hold:*

$$(2.13) \quad g_{z_0, \lambda} u \in C_{\bar{z}}^1(\bar{D}), \quad \text{for } u \in C(\bar{D}),$$

$$(2.14) \quad \|g_{z_0, \lambda} u\|_{C^1(\bar{D})} \leq c_1(D, \lambda) \|u\|_{C(\bar{D})}, \quad \text{for } u \in C(\bar{D}),$$

$$(2.15) \quad \|g_{z_0, \lambda} u\|_{C_{\bar{z}}^1(\bar{D})} \leq \frac{c_2(D)}{|\lambda|^{\frac{1}{2}}} \|u\|_{C_{\bar{z}}^1(\bar{D})}, \quad \text{for } u \in C_{\bar{z}}^1(\bar{D}), \quad |\lambda| \geq 1.$$

Given a potential  $v \in C_{\bar{z}}^1(\bar{D})$  we define the operator  $g_{z_0, \lambda} v$  simply as  $(g_{z_0, \lambda} v)u(z) = g_{z_0, \lambda} w(z)$ ,  $w = vu$ , for a test function  $u$ . If  $u \in C_{\bar{z}}^1(\bar{D})$ , by

Lemma 2.1 we have that  $g_{z_0, \lambda} v : C_{\bar{z}}^1(\bar{D}) \rightarrow C_{\bar{z}}^1(\bar{D})$ ,

$$(2.16) \quad \|g_{z_0, \lambda} v\|_{C_{\bar{z}}^1(\bar{D})}^{op} \leq 2n \|g_{z_0, \lambda}\|_{C_{\bar{z}}^1(\bar{D})}^{op} \|v\|_{C_{\bar{z}}^1(\bar{D})},$$

where  $\|\cdot\|_{C_{\bar{z}}^1(\bar{D})}^{op}$  denotes the operator norm in  $C_{\bar{z}}^1(\bar{D})$ ,  $z_0, \lambda \in \mathbb{C}$ . In addition,  $\|g_{z_0, \lambda}\|_{C_{\bar{z}}^1(\bar{D})}^{op}$  is estimated in Lemma 2.1. Inequality (2.16) and Lemma 2.1 imply the existence and uniqueness of  $\mu_{z_0}(z, \lambda)$  (and thus also of  $\psi_{z_0}(z, \lambda)$ ) for  $|\lambda| > \rho(D, K, n)$ , where  $\|v\|_{C_{\bar{z}}^1(\bar{D})} < K$ .

Let

$$\begin{aligned} \mu_{z_0}^{(k)}(z, \lambda) &= \sum_{j=0}^k (g_{z_0, \lambda} v)^j I, \\ h_{z_0}^{(k)}(\lambda) &= \int_D e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} v(z) \mu_{z_0}^{(k)}(z, \lambda) d\operatorname{Re}z d\operatorname{Im}z, \end{aligned}$$

where  $z, z_0 \in D$ ,  $\lambda \in \mathbb{C}$ ,  $k \in \mathbb{N} \cup \{0\}$ .

**Lemma 2.2** ([13]). *For  $v \in C_{\bar{z}}^1(\bar{D})$  such that  $v|_{\partial D} = 0$  the following formula holds:*

$$(2.17) \quad v(z_0) = \frac{2}{\pi} \lim_{\lambda \rightarrow \infty} |\lambda| h_{z_0}^{(0)}(\lambda), \quad z_0 \in D.$$

*In addition, if  $v \in C^2(\bar{D})$ ,  $v|_{\partial D} = 0$  and  $\frac{\partial v}{\partial \bar{v}}|_{\partial D} = 0$  then*

$$(2.18) \quad \left| v(z_0) - \frac{2}{\pi} |\lambda| h_{z_0}^{(0)}(\lambda) \right| \leq c_3(D, n) \frac{\log(3|\lambda|)}{|\lambda|} \|v\|_{C^2(\bar{D})},$$

*for  $z_0 \in D$ ,  $\lambda \in \mathbb{C}$ ,  $|\lambda| \geq 1$ .*

Let

$$W_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} w(z) d\operatorname{Re}z d\operatorname{Im}z,$$

where  $z_0 \in \bar{D}$ ,  $\lambda \in \mathbb{C}$  and  $w$  is some  $M_n(\mathbb{C})$ -valued function on  $\bar{D}$ . (One can see that  $W_{z_0} = h_{z_0}^{(0)}$  for  $w = v$ .)

**Lemma 2.3** ([13]). *For  $w \in C_{\bar{z}}^1(\bar{D})$  the following estimate holds:*

$$(2.19) \quad |W_{z_0}(\lambda)| \leq c_4(D) \frac{\log(3|\lambda|)}{|\lambda|} \|w\|_{C_{\bar{z}}^1(\bar{D})}, \quad z_0 \in \bar{D}, \quad |\lambda| \geq 1.$$

**Lemma 2.4** ([14]). *For  $v \in C_{\bar{z}}^1(\bar{D})$  and for  $\|g_{z_0, \lambda} v\|_{C_{\bar{z}}^1(\bar{D})}^{op} \leq \delta < 1$  we have that*

$$(2.20) \quad \|\mu_{z_0}(\cdot, \lambda) - \mu_{z_0}^{(k)}(\cdot, \lambda)\|_{C_{\bar{z}}^1(\bar{D})} \leq \frac{\delta^{k+1}}{1-\delta},$$

$$(2.21) \quad |h_{z_0}(\lambda) - h_{z_0}^{(k)}(\lambda)| \leq c_5(D, n) \frac{\log(3|\lambda|)}{|\lambda|} \frac{\delta^{k+1}}{1-\delta} \|v\|_{C_{\bar{z}}^1(\bar{D})},$$

*where  $z_0 \in D$ ,  $\lambda \in \mathbb{C}$ ,  $|\lambda| \geq 1$ ,  $k \in \mathbb{N} \cup \{0\}$ .*

The proofs of Lemmata 2.1-2.4 can be found in the references given.

We will also need the following two new lemmata.

**Lemma 2.5.** *Let  $g_{z_0, \lambda} u$  be defined by (2.12), where  $u \in C_{\bar{z}}^1(\bar{D})$ ,  $z_0, \lambda \in \mathbb{C}$ . Then the following estimate holds:*

$$(2.22) \quad \|g_{z_0, \lambda} u\|_{C(\bar{D})} \leq c_6(D) \frac{\log(3|\lambda|)}{|\lambda|} \|u\|_{C_{\bar{z}}^1(\bar{D})}, \quad |\lambda| \geq 1.$$

**Lemma 2.6.** *The expression*

$$(2.23) \quad W(u, v)(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} u(z) (g_{z_0, \lambda} v)(z) d\operatorname{Re}z d\operatorname{Im}z,$$

*defined for  $u, v \in C_{\bar{z}}^1(\bar{D})$  with  $\|u\|_{C_{\bar{z}}^1(\bar{D})}, \|v\|_{C_{\bar{z}}^1(\bar{D})} \leq N_1$ ,  $\lambda \in \mathbb{C}$ ,  $z_0 \in D$ , satisfies the estimate*

$$(2.24) \quad |W(u, v)(\lambda)| \leq c_7(D, N_1, n) \frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}}, \quad |\lambda| \geq 1.$$

The proofs of Lemmata 2.5, 2.6 are given in section 4.

### 3. PROOF OF THEOREM 1.1

We begin with a technical lemma, which will prove useful when generalising Alessandrini's identity.

**Lemma 3.1.** *Let  $v \in C^1(\bar{D}, M_n(\mathbb{C}))$  be a matrix-valued potential which satisfies condition (1.4) (i.e., 0 is not a Dirichlet eigenvalue for the operator  $-\Delta + v$  in  $D$ ). Then  ${}^t v$ , the transpose of  $v$ , also satisfies condition (1.4).*

The proof of Lemma 3.1 is given in section 4.

We can now state and prove a matrix-valued version of Alessandrini's identity (see [1] for the scalar case).

**Lemma 3.2.** *Let  $v_1, v_2 \in C^1(\bar{D}, M_n(\mathbb{C}))$  be two matrix-valued potentials which satisfy (1.4),  $\Phi_1, \Phi_2$  their associated Dirichlet-to-Neumann operators, respectively, and  $u_1, u_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$  two matrix-valued functions such that*

$$(-\Delta + v_1)u_1 = 0, \quad (-\Delta + {}^t v_2)u_2 = 0 \quad \text{on } D,$$

*where  ${}^t A$  stand for the transpose of  $A$ . Then we have the identity*

$$(3.1) \quad \int_{\partial D} {}^t u_2(z) (\Phi_2 - \Phi_1) u_1(z) |dz| = \int_D {}^t u_2(z) (v_2(z) - v_1(z)) u_1(z) d\operatorname{Re}z d\operatorname{Im}z.$$



*Proof.* If  $v \in C^1(\bar{D}, M_n(\mathbb{C}))$  is any matrix-valued potential (which satisfies (1.4)) and  $f_1, f_2 \in C^1(\partial D, M_n(\mathbb{C}))$  then we have

$$(3.2) \quad \int_{\partial D} {}^t f_2 \Phi f_1 |dz| = \int_{\partial D} {}^t (f_1 \Phi^* f_2) |dz|,$$

where  $\Phi$  and  $\Phi^*$  are the Dirichlet-to-Neumann operators associated with  $v$  and  ${}^t v$ , respectively (these operators are well-defined thanks to Lemma 3.1). Indeed, it is sufficient to extend  $f_1$  and  $f_2$  in  $D$  as the solutions of the Dirichlet problems  $(-\Delta + v)\tilde{f}_1 = 0$ ,  $(-\Delta + {}^t v)\tilde{f}_2 = 0$  on  $D$  and  $\tilde{f}_j|_{\partial D} = f_j$ , for  $j = 1, 2$ , so that one obtains

$$\begin{aligned} & \int_{\partial D} ({}^t f_2 \Phi f_1 - {}^t (f_1 \Phi^* f_2)) |dz| \\ &= \int_{\partial D} \left( {}^t f_2 \frac{\partial \tilde{f}_1}{\partial \nu} - {}^t \left( \frac{\partial \tilde{f}_2}{\partial \nu} \right) f_1 \right) |dz| \\ &= \int_D ({}^t \tilde{f}_2 \Delta \tilde{f}_1 - {}^t (\Delta \tilde{f}_2) \tilde{f}_1) d\text{Re}z d\text{Im}z \\ &= \int_D ({}^t \tilde{f}_2 v \tilde{f}_1 - {}^t ({}^t v \tilde{f}_2) \tilde{f}_1) d\text{Re}z d\text{Im}z = 0, \end{aligned}$$

where for the second equality we used the following matrix-valued version of the classical scalar Green's formula:

$$(3.3) \quad \int_{\partial D} \left( {}^t \left( \frac{\partial f}{\partial \nu} \right) g - {}^t f \frac{\partial g}{\partial \nu} \right) |dz| = \int_D ({}^t (\Delta f) g - {}^t f \Delta g) d\text{Re}z d\text{Im}z,$$

for any  $f, g \in C^2(D, M_n(\mathbb{C})) \cap C^1(\bar{D}, M_n(\mathbb{C}))$ .

Identities (3.2) and (3.3) imply

$$\begin{aligned} & \int_{\partial D} {}^t u_2(z) (\Phi_2 - \Phi_1) u_1(z) |dz| \\ &= \int_{\partial D} ({}^t ({}^t u_1(z) \Phi_2^* u_2(z)) - {}^t u_2(z) \Phi_1 u_1(z)) |dz| \\ &= \int_{\partial D} \left( {}^t \left( \frac{\partial u_2(z)}{\partial \nu} \right) u_1(z) - {}^t u_2(z) \frac{\partial u_1(z)}{\partial \nu} \right) |dz| \\ &= \int_D ({}^t (\Delta u_2(z)) u_1(z) - {}^t u_2(z) \Delta u_1(z)) d\text{Re}z d\text{Im}z \\ &= \int_D ({}^t ({}^t v_2(z) u_2(z)) u_1(z) - {}^t u_2(z) v_1(z) u_1(z)) d\text{Re}z d\text{Im}z \\ &= \int_D {}^t u_2(z) (v_2(z) - v_1(z)) u_1(z) d\text{Re}z d\text{Im}z. \quad \square \end{aligned}$$

Now let  $\bar{\mu}_{z_0}$  denote the complex conjugate of  $\mu_{z_0}$  (the solution of (2.4)) for a  $M_n(\mathbb{R})$ -valued potential  $v$  and, more generally, the solution of (2.4) with

$g_{z_0}(z, \zeta, \lambda)$  replaced by  $\overline{g_{z_0}(z, \zeta, \lambda)}$  for a  $M_n(\mathbb{C})$ -valued potential  $v$ . In order to make use of (3.1) we define

$$\begin{aligned} u_1(z) &= \psi_{1,z_0}(z, \lambda) = e^{\lambda(z-z_0)^2} \mu_1(z, \lambda), \\ u_2(z) &= \bar{\psi}_{2,z_0}(z, -\lambda) = e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} \bar{\mu}_2(z, -\lambda), \end{aligned}$$

for  $z_0 \in D$ ,  $\lambda \in C$ ,  $|\lambda| > \rho$  ( $\rho$  is mentioned in section 2), where we set  $\mu_1 = \mu_{1,z_0}$ ,  $\mu_2 = \mu_{2,z_0}$  for simplicity's sake and  $\mu_{1,z_0}$ ,  $\mu_{2,z_0}$  are the solutions of (2.4) with  $v$  replaced by  $v_1$ ,  ${}^t v_2$ , respectively.

Equation (3.1), with the above-defined  $u_1, u_2$ , now reads

$$\begin{aligned} (3.4) \quad & \int_{\partial D} \int_{\partial D} e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} {}^t \bar{\mu}_2(z, -\lambda) (\Phi_2 - \Phi_1)(z, \zeta) e^{\lambda(\zeta-z_0)^2} \mu_1(\zeta, \lambda) |d\zeta| |dz| \\ &= \int_D e_{\lambda,z_0}(z) {}^t \bar{\mu}_2(z, -\lambda) (v_2 - v_1)(z) \mu_1(z, \lambda) d\text{Re}z d\text{Im}z. \end{aligned}$$

with  $e_{\lambda,z_0}(z) = e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2}$  and  $(\Phi_2 - \Phi_1)(z, \zeta)$  is the Schwartz kernel of the operator  $\Phi_2 - \Phi_1$ .

The right side  $I(\lambda)$  of (3.4) can be written as the sum of four integrals, namely

$$\begin{aligned} I_1(\lambda) &= \int_D e_{\lambda,z_0}(z) (v_2 - v_1)(z) d\text{Re}z d\text{Im}z, \\ I_2(\lambda) &= \int_D e_{\lambda,z_0}(z) {}^t (\bar{\mu}_2 - I) (v_2 - v_1)(z) (\mu_1 - I) d\text{Re}z d\text{Im}z, \\ I_3(\lambda) &= \int_D e_{\lambda,z_0}(z) {}^t (\bar{\mu}_2 - I) (v_2 - v_1)(z) d\text{Re}z d\text{Im}z, \\ I_4(\lambda) &= \int_D e_{\lambda,z_0}(z) (v_2 - v_1)(z) (\mu_1 - I) d\text{Re}z d\text{Im}z, \end{aligned}$$

for  $z_0 \in D$ .

Since  $(v_2 - v_1)|_{\partial D} = \frac{\partial}{\partial \nu} (v_2 - v_1)|_{\partial D} = 0$ , the first term,  $I_1$ , can be estimated using Lemma 2.2 as

$$(3.5) \quad \left| \frac{2}{\pi} |\lambda| I_1 - (v_2(z_0) - v_1(z_0)) \right| \leq c_3(D, n) \frac{\log(3|\lambda|)}{|\lambda|} \|v_2 - v_1\|_{C^2(\bar{D})},$$

for  $|\lambda| \geq 1$ . The other terms,  $I_2, I_3, I_4$ , satisfy, by Lemmata 2.1 and 2.4,

$$(3.6) \quad \begin{aligned} |I_2| &\leq \left| \int_D e_{\lambda,z_0}(z) {}^t (\overline{g_{z_0,\lambda}} {}^t v_2) (v_2 - v_1)(z) (g_{z_0,\lambda} v_1) d\text{Re}z d\text{Im}z \right| \\ &\quad + O\left(\frac{\log(3|\lambda|)}{|\lambda|^2}\right) c_8(D, N, n), \end{aligned}$$

$$(3.7) \quad \begin{aligned} |I_3| &\leq \left| \int_D e_{\lambda,z_0}(z) {}^t (\overline{g_{z_0,\lambda}} {}^t v_2) (v_2 - v_1)(z) d\text{Re}z d\text{Im}z \right| \\ &\quad + O\left(\frac{\log(3|\lambda|)}{|\lambda|^2}\right) c_9(D, N, n), \end{aligned}$$

$$(3.8) \quad |I_4| \leq \left| \int_D e_{\lambda, z_0}(z) (v_2 - v_1)(z) (g_{z_0, \lambda} v_1) d\operatorname{Re}z d\operatorname{Im}z \right| \\ + O\left(\frac{\log(3|\lambda|)}{|\lambda|^2}\right) c_{10}(D, N, n),$$

where  $N$  is the constant in the statement of Theorem 1.1 and  $|\lambda|$  is sufficiently large, for example for  $\lambda$  such that

$$(3.9) \quad 2n \frac{c_2(D)}{|\lambda|^{\frac{1}{2}}} \leq \frac{1}{2}, \quad |\lambda| \geq 1.$$

Lemmata 2.5, 2.6, applied to (3.6)-(3.8), give us

$$(3.10) \quad |I_2| \leq c_{11}(D, N, n) \frac{(\log(3|\lambda|))^2}{|\lambda|^2},$$

$$(3.11) \quad |I_3| \leq c_{12}(D, N, n) \frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}},$$

$$(3.12) \quad |I_4| \leq c_{13}(D, N, n) \frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}}.$$

The left side  $J(\lambda)$  of (3.4) can be estimated as follows:

$$(3.13) \quad |\lambda| |J(\lambda)| \leq c_{14}(D, n) e^{(2L^2+1)|\lambda|} \|\Phi_2 - \Phi_1\|,$$

for  $\lambda$  which satisfies (3.9), and  $L = \max_{z \in \partial D, z_0 \in D} |z - z_0|$ .

Putting together estimates (3.5)-(3.13) we obtain

$$(3.14) \quad |v_2(z_0) - v_1(z_0)| \leq c_{15}(D, N, n) \frac{(\log(3|\lambda|))^2}{|\lambda|^{3/4}} + \frac{2}{\pi} c_{14}(D, n) e^{(2L^2+1)|\lambda|} \|\Phi_2 - \Phi_1\|$$

for any  $z_0 \in D$ . We call  $\varepsilon = \|\Phi_2 - \Phi_1\|$  and impose  $|\lambda| = \gamma \log(3 + \varepsilon^{-1})$ , where  $0 < \gamma < (2L^2 + 1)^{-1}$  so that (3.14) reads

$$(3.15) \quad |v_2(z_0) - v_1(z_0)| \leq c_{15}(D, N, n) (\gamma \log(3 + \varepsilon^{-1}))^{-\frac{3}{4}} (\log(3\gamma \log(3 + \varepsilon^{-1})))^2 \\ + \frac{2}{\pi} c_{14}(D, n) (3 + \varepsilon^{-1})^{(2L^2+1)\gamma\varepsilon},$$

for every  $z_0 \in D$ , with

$$(3.16) \quad 0 < \varepsilon \leq \varepsilon_1(D, N, \gamma, n),$$

where  $\varepsilon_1$  is sufficiently small or, more precisely, where (3.16) implies that  $|\lambda| = \gamma \log(3 + \varepsilon^{-1})$  satisfies (3.9).

As  $(3 + \varepsilon^{-1})^{(2L^2+1)\gamma} \varepsilon \rightarrow 0$  for  $\varepsilon \rightarrow 0$  more rapidly than the other term, we obtain that

$$(3.17) \quad \|v_2 - v_1\|_{L^\infty(D)} \leq c_{16}(D, N, \gamma, n) \frac{(\log(3 \log(3 + \|\Phi_2 - \Phi_1\|^{-1})))^2}{(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{\frac{3}{4}}}$$

for any  $\varepsilon = \|\Phi_2 - \Phi_1\| \leq \varepsilon_1(D, N, \gamma, n)$ .

Estimate (3.17) for general  $\varepsilon$  (with modified  $c_{16}$ ) follows from (3.17) for  $\varepsilon \leq \varepsilon_1(D, N, \gamma, n)$  and the assumption that  $\|v_j\|_{L^\infty(D)} \leq N$ ,  $j = 1, 2$ . This completes the proof of Theorem 1.1.  $\square$

#### 4. PROOFS OF LEMMATA 2.5, 2.6, 3.1.

*Proof of Lemma 2.5.* We decompose the operator  $g_{z_0, \lambda}$ , defined in (2.12), as the product  $\frac{1}{4}T_{z_0, \lambda}\bar{T}_{z_0, \lambda}$ , where

$$(4.1) \quad T_{z_0, \lambda}u(z) = \frac{1}{\pi} \int_D \frac{e^{-\lambda(\zeta - z_0)^2 + \bar{\lambda}(\bar{\zeta} - \bar{z}_0)^2}}{z - \zeta} u(\zeta) d\operatorname{Re}\zeta d\operatorname{Im}\zeta,$$

$$(4.2) \quad \bar{T}_{z_0, \lambda}u(z) = \frac{1}{\pi} \int_D \frac{e^{\lambda(\zeta - z_0)^2 - \bar{\lambda}(\bar{\zeta} - \bar{z}_0)^2}}{\bar{z} - \bar{\zeta}} u(\zeta) d\operatorname{Re}\zeta d\operatorname{Im}\zeta,$$

for  $z_0, \lambda \in \mathbb{C}$ . From the proof of [13, Lemma 3.1] we have the estimate

$$(4.3) \quad \|\bar{T}_{z_0, \lambda}u\|_{C(\bar{D})} \leq \frac{\eta_1(D)}{|\lambda|^{1/2}} \|u\|_{C(\bar{D})} + \eta_2(D) \frac{\log(3|\lambda|)}{|\lambda|} \left\| \frac{\partial u}{\partial \bar{z}} \right\|_{C(\bar{D})},$$

for  $u \in C_z^1(\bar{D})$ ,  $z_0 \in D$ ,  $|\lambda| \geq 1$ . As the kernels of  $T_{z_0, \lambda}$  and  $\bar{T}_{z_0, \lambda}$  are conjugates of each other we deduce immediately that

$$(4.4) \quad \|T_{z_0, \lambda}u\|_{C(\bar{D})} \leq \frac{\eta_1(D)}{|\lambda|^{1/2}} \|u\|_{C(\bar{D})} + \eta_2(D) \frac{\log(3|\lambda|)}{|\lambda|} \left\| \frac{\partial u}{\partial z} \right\|_{C(\bar{D})}, \quad |\lambda| \geq 1,$$

for  $u \in C_z^1(\bar{D})$ . Combining the two estimates we obtain

$$\begin{aligned} \|g_{\lambda, z_0}u\|_{C(\bar{D})} &= \frac{1}{4} \|T_{z_0, \lambda}\bar{T}_{z_0, \lambda}u\|_{C(\bar{D})} \\ &\leq \frac{1}{4} \left( \eta_1(D) \frac{\|\bar{T}_{z_0, \lambda}u\|_{C(\bar{D})}}{|\lambda|^{1/2}} + \eta_2(D) \frac{\log(3|\lambda|)}{|\lambda|} \left\| \frac{\partial \bar{T}_{z_0, \lambda}u}{\partial z} \right\|_{C(\bar{D})} \right) \\ &\leq \eta_3(D) \left( \frac{\|u\|_{C(\bar{D})}}{|\lambda|} + \frac{\log(3|\lambda|)}{|\lambda|^{3/2}} \left\| \frac{\partial u}{\partial \bar{z}} \right\|_{C(\bar{D})} + \frac{\log(3|\lambda|)}{|\lambda|} \|u\|_{C(\bar{D})} \right) \\ &\leq \eta_4(D) \frac{\log(3|\lambda|)}{|\lambda|} \|u\|_{C_z^1(\bar{D})}, \quad |\lambda| \geq 1, \end{aligned}$$

where we use the fact that  $\left\| \frac{\partial \bar{T}_{z_0, \lambda}u}{\partial z} \right\|_{C(\bar{D})} = \|u\|_{C(D)}$ .  $\square$

*Proof of Lemma 2.6.* For  $0 < \varepsilon \leq 1$ ,  $z_0 \in D$ , let  $B_{z_0, \varepsilon} = \{z \in \mathbb{C} : |z - z_0| \leq \varepsilon\}$ . We write  $W(u, v)(\lambda) = W^1(\lambda) + W^2(\lambda)$ , where

$$\begin{aligned} W^1(\lambda) &= \int_{D \cap B_{z_0, \varepsilon}} e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} u(z) g_{z_0, \lambda} v(z) d\operatorname{Re}z d\operatorname{Im}z, \\ W^2(\lambda) &= \int_{D \setminus B_{z_0, \varepsilon}} e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} u(z) g_{z_0, \lambda} v(z) d\operatorname{Re}z d\operatorname{Im}z. \end{aligned}$$

The first term,  $W^1$ , can be estimated as follows:

$$(4.5) \quad |W^1(\lambda)| \leq \sigma_1(D, n) \|u\|_{C(\bar{D})} \|v\|_{C_{\bar{z}}^1(\bar{D})} \frac{\varepsilon^2 \log(3|\lambda|)}{|\lambda|}, \quad |\lambda| \geq 1,$$

where we use estimates (2.16) and (2.22).

For the second term,  $W^2$ , we proceed using integration by parts, in order to obtain

$$\begin{aligned} W^2(\lambda) &= \frac{1}{4i\bar{\lambda}} \int_{\partial(D \setminus B_{z_0, \varepsilon})} e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} \frac{u(z) g_{z_0, \lambda} v(z)}{\bar{z} - \bar{z}_0} dz \\ &\quad - \frac{1}{2\bar{\lambda}} \int_{D \setminus B_{z_0, \varepsilon}} e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} \frac{\partial}{\partial \bar{z}} \left( \frac{u(z) g_{z_0, \lambda} v(z)}{\bar{z} - \bar{z}_0} \right) d\operatorname{Re}z d\operatorname{Im}z. \end{aligned}$$

This implies that

$$(4.6) \quad |W^2(\lambda)| \leq \frac{1}{4|\lambda|} \int_{\partial(D \setminus B_{z_0, \varepsilon})} \frac{\|u(z) g_{z_0, \lambda} v(z)\|_{C(\bar{D})}}{|\bar{z} - \bar{z}_0|} |dz| \\ + \frac{1}{2|\lambda|} \left| \int_{D \setminus B_{z_0, \varepsilon}} e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} \frac{\partial}{\partial \bar{z}} \left( \frac{u(z) g_{z_0, \lambda} v(z)}{\bar{z} - \bar{z}_0} \right) d\operatorname{Re}z d\operatorname{Im}z \right|,$$

for  $\lambda \neq 0$ . Again by estimates (2.16) and (2.22) we obtain

$$(4.7) \quad |W^2(\lambda)| \leq \sigma_2(D, n) \|u\|_{C_{\bar{z}}^1(\bar{D})} \|v\|_{C_{\bar{z}}^1(\bar{D})} \frac{\log(3\varepsilon^{-1}) \log(3|\lambda|)}{|\lambda|^2} \\ + \frac{1}{8|\lambda|} \left| \int_{D \setminus B_{z_0, \varepsilon}} u(z) \frac{\bar{T}_{z_0, \lambda} v(z)}{\bar{z} - \bar{z}_0} d\operatorname{Re}z d\operatorname{Im}z \right|, \quad |\lambda| \geq 1,$$

where we used the fact that  $\frac{\partial}{\partial \bar{z}} g_{z_0, \lambda} v(z) = \frac{1}{4} e^{-\lambda(z-z_0)^2 + \bar{\lambda}(\bar{z}-\bar{z}_0)^2} \bar{T}_{z_0, \lambda} v(z)$ , with  $\bar{T}_{z_0, \lambda}$  defined in (4.2).

The last term in (4.7) can be estimated independently of  $\varepsilon$  by

$$(4.8) \quad \sigma_3(D, n) \|u\|_{C(\bar{D})} \|v\|_{C_{\bar{z}}^1(\bar{D})} \frac{\log(3|\lambda|)}{|\lambda|^{1+3/4}}.$$

This is a consequence of (4.3) and of the estimate

$$(4.9) \quad |\bar{T}_{z_0, \lambda} u(z)| \leq \frac{\log(3|\lambda|)(1 + |z - z_0|) \tau_1(D)}{|\lambda| |z - z_0|^2} \|u\|_{C_{\bar{z}}^1(\bar{D})}, \quad |\lambda| \geq 1,$$

for  $u \in C_{\bar{z}}^1(\bar{D})$ ,  $z, z_0 \in D$  (a proof of (4.9) can be found in the proof of [13, Lemma 3.1]).

Indeed, for  $0 < \delta \leq \frac{1}{2}$  we have

$$\begin{aligned}
& \left| \int_D u(z) \frac{\bar{T}_{z_0, \lambda} v(z)}{\bar{z} - \bar{z}_0} d\text{Re}z d\text{Im}z \right| \\
& \leq \int_{B_{z_0, \delta} \cap D} |u(z)| \frac{|\bar{T}_{z_0, \lambda} v(z)|}{|z - z_0|} d\text{Re}z d\text{Im}z + \int_{D \setminus B_{z_0, \delta}} |u(z)| \frac{|\bar{T}_{z_0, \lambda} v(z)|}{|z - z_0|} d\text{Re}z d\text{Im}z \\
& \leq \|u\|_{C(\bar{D})} \|v\|_{C_{\bar{z}}^1(\bar{D})} \frac{\tau_2(D, n)}{|\lambda|^{1/2}} \int_{B_{z_0, \delta} \cap D} \frac{d\text{Re}z d\text{Im}z}{|z - z_0|} \\
& \quad + \|u\|_{C(\bar{D})} \|v\|_{C_{\bar{z}}^1(\bar{D})} \frac{\log(3|\lambda|)}{|\lambda|} \tau_3(D, n) \int_{D \setminus B_{z_0, \delta}} \frac{d\text{Re}z d\text{Im}z}{|z - z_0|^3} \\
& \leq 2\pi \|u\|_{C(\bar{D})} \|v\|_{C_{\bar{z}}^1(\bar{D})} \tau_2(D, n) \frac{\delta}{|\lambda|^{1/2}} + \|u\|_{C(\bar{D})} \|v\|_{C_{\bar{z}}^1(\bar{D})} \tau_4(D, n) \frac{\log(3|\lambda|)}{|\lambda|\delta},
\end{aligned}$$

for  $|\lambda| \geq 1$ . Putting  $\delta = \frac{1}{2}|\lambda|^{-1/4}$  in the last inequality gives (4.8).

Finally, defining  $\varepsilon = |\lambda|^{-1/2}$  in (4.7), (4.5) and using (4.8), we obtain the main estimate (2.24), which thus finishes the proof of Lemma 2.6.  $\square$

*Proof of Lemma 3.1.* Take  $u \in H^1(D, M_n(\mathbb{C}))$  such that  $(-\Delta + {}^t v)u = 0$  on  $D$  and  $u|_{\partial D} = 0$ . We want to prove that  $u \equiv 0$  on  $D$ .

By our hypothesis, for any  $f \in C^1(\partial D, M_n(\mathbb{C}))$  there exists a unique  $\tilde{f} \in H^1(D, M_n(\mathbb{C}))$  such that  $(-\Delta + v)\tilde{f} = 0$  on  $D$  and  $\tilde{f}|_{\partial D} = f$ . Thus we have, using Green's formula (3.3),

$$\begin{aligned}
\int_{\partial D} {}^t \left( \frac{\partial u}{\partial \nu} \right) f |dz| &= \int_D \left( {}^t(\Delta u) \tilde{f} - {}^t u \Delta \tilde{f} \right) d\text{Re}z d\text{Im}z \\
&= \int_D \left( {}^t({}^t v u) \tilde{f} - {}^t u v \tilde{f} \right) d\text{Re}z d\text{Im}z = 0,
\end{aligned}$$

which yields  $\frac{\partial u}{\partial \nu}|_{\partial D} = 0$ . Now consider the following straightforward generalization of Green's formula (3.3),

$$(4.10) \quad \int_{\partial D} \left( {}^t \left( \frac{\partial f}{\partial \nu} \right) g - {}^t f \frac{\partial g}{\partial \nu} \right) |dz| = \int_D \left( {}^t((\Delta - {}^t v)f) g - {}^t f ((\Delta - v)g) \right) d\text{Re}z d\text{Im}z,$$

which holds (weakly) for any  $f, g \in H^1(D, M_n(\mathbb{C}))$ . If we put  $f = u$  we obtain

$$(4.11) \quad \int_D {}^t u (-\Delta + v)g d\text{Re}z d\text{Im}z = 0,$$

for any  $g \in H^1(D, M_n(\mathbb{C}))$ . By Fredholm alternative (see [6, Sec. 6.2]), for each  $h \in L^2(D, M_n(\mathbb{C}))$  there exists a unique  $g \in H_0^1(D, M_n(\mathbb{C})) = \{g \in$

$H^1(D, M_n(\mathbb{C})) : g|_{\partial D} = 0$  such that  $(-\Delta + v)g = h$ . This yields  $u \equiv 0$  on  $D$  and thus Lemma 3.1 is proved.  $\square$

## 5. AN EXTENSIONS OF THEOREM 1.1

As an extension of Theorem 1.1 to the case where we do not assume that  $v_1|_{\partial D} = v_2|_{\partial D}$  and  $\frac{\partial}{\partial \nu} v_1|_{\partial D} = \frac{\partial}{\partial \nu} v_2|_{\partial D}$ , we give the following proposition:

**Proposition 5.1.** *Let  $D \subset \mathbb{R}^2$  be an open bounded domain with a  $C^2$  boundary,  $v_1, v_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$  two matrix-valued potentials which satisfy (1.4), with  $\|v_j\|_{C^2(\bar{D})} \leq N$  for  $j = 1, 2$ , and  $\Phi_1, \Phi_2$  the corresponding Dirichlet-to-Neumann operators. Then, for any  $0 < \alpha < \frac{1}{5}$ , there exists a constant  $C = C(D, N, n, \alpha)$  such that*

$$(5.1) \quad \|v_2 - v_1\|_{L^\infty(D)} \leq C (\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))^{-\alpha},$$

where, for an operator  $A$  which acts on  $L^\infty(\partial D, M_n(\mathbb{C}))$  with kernel  $A(x, y)$ ,  $\|A\|_1$  is the norm defined as  $\|A\|_1 = \sup_{x, y \in \partial D} |A(x, y)| (\log(3 + |x - y|^{-1}))^{-1}$  and  $|A(x, y)| = \max_{1 \leq i, j \leq n} |A_{i,j}(x, y)|$ .

The only properties of  $\|\cdot\|_1$  we will use are the following:

- i)  $\|A\|_{L^\infty(\partial D) \rightarrow L^\infty(\partial D)} \leq \text{const}(D, n) \|A\|_1$ ;
- ii) In a similar way as in formula (4.9) of [11] one can deduce

$$\|v\|_{L^\infty(\partial D)} \leq \text{const}(n) \|\Phi_v - \Phi_0\|_1,$$

for a matrix-valued potential  $v$ ,  $\Phi_v$  its associated Dirichlet-to-Neumann operator and  $\Phi_0$  the Dirichlet-to-Neumann operator of the 0 potential.

We recall a lemma from [13], which generalizes Lemma 2.2 to the case of potentials without boundary conditions. We then define  $(\partial D)_\delta = \{z \in \mathbb{C} : \text{dist}(z, \partial D) < \delta\}$ .

**Lemma 5.2.** *For  $v \in C^2(\bar{D})$  we have that*

$$(5.2) \quad \left| v(z_0) - \frac{2}{\pi} |\lambda| h_{z_0}^{(0)}(\lambda) \right| \leq \kappa_1(D, n) \delta^{-4} \frac{\log(3|\lambda|)}{|\lambda|} \|v\|_{C^2(\bar{D})} \\ + \kappa_2(D, n) \log(3 + \delta^{-1}) \|v\|_{C(\partial D)},$$

for  $z_0 \in D \setminus (\partial D)_\delta$ ,  $0 < \delta < 1$ ,  $\lambda \in \mathbb{C}$ ,  $|\lambda| \geq 1$ .

The proof of Lemma 5.2 for the scalar case can be found in [13] and its generalization to the matrix-valued case is straightforward.

*Proof of Proposition 5.1.* Fix  $0 < \alpha < \frac{1}{5}$  and  $0 < \delta < 1$ . We then have the following chain of inequalities

$$\begin{aligned}
& \|v_2 - v_1\|_{L^\infty(D)} \\
&= \max(\|v_2 - v_1\|_{L^\infty(D \cap (\partial D)_\delta)}, \|v_2 - v_1\|_{L^\infty(D \setminus (\partial D)_\delta)}) \\
&\leq C_1 \max\left(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{\log(3 \log(3 + \|\Phi_2 - \Phi_1\|^{-1}))}{\delta^4 \log(3 + \|\Phi_2 - \Phi_1\|^{-1})}\right. \\
&\quad \left. + \log\left(3 + \frac{1}{\delta}\right)\|\Phi_2 - \Phi_1\|_1 + \frac{(\log(3 \log(3 + \|\Phi_2 - \Phi_1\|^{-1})))^2}{(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{\frac{3}{4}}}\right) \\
&\leq C_2 \max\left(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{1}{\delta^4} (\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))^{-5\alpha}\right. \\
&\quad \left. + \log\left(3 + \frac{1}{\delta}\right)\|\Phi_2 - \Phi_1\|_1 + \frac{(\log(3 \log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})))^2}{(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))^{\frac{3}{4}}}\right),
\end{aligned}$$

where we followed the outline of the proof of Theorem 1.1 with the following modifications: we made use of Lemma 5.2 instead of Lemma 2.2 and we also used i)-ii); note that  $C_1 = C_1(D, N, n)$  and  $C_2 = C_2(D, N, n, \alpha)$ .

Putting  $\delta = (\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))^{-\alpha}$  we obtain the desired inequality

$$(5.3) \quad \|v_2 - v_1\|_{L^\infty(D)} \leq C_3 (\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))^{-\alpha},$$

with  $C_3 = C_3(D, N, n, \alpha)$ ,  $\|\Phi_2 - \Phi_1\|_1 = \varepsilon \leq \varepsilon_1(D, N, n, \alpha)$  with  $\varepsilon_1$  sufficiently small or, more precisely when  $\delta_1 = (\log(3 + \varepsilon_1^{-1}))^{-\alpha}$  satisfies:

$$\delta_1 < 1, \quad \varepsilon_1 \leq 2N\delta_1, \quad \log\left(3 + \frac{1}{\delta_1}\right)\varepsilon_1 \leq \delta_1.$$

Estimate (5.3) for general  $\varepsilon$  (with modified  $C_3$ ) follows from (5.3) for  $\varepsilon \leq \varepsilon_1(D, N, n, \alpha)$  and the assumption that  $\|v_j\|_{L^\infty(\bar{D})} \leq N$  for  $j = 1, 2$ . This completes the proof of Proposition 5.1.  $\square$

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