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Absence of traveling wave solutions of conductivity type for the Novikov-Veselov equation at zero energy

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Abstract. We prove that the Novikov-Veselov equation (an analog of KdV in dimension $2+1$) at zero energy does not have sufficiently localized soliton solutions of conductivity type.

1 Introduction

In this note we are concerned with the Novikov-Veselov equation at zero energy

$$\begin{aligned}
&\partial_t v = 4\mathrm{Re}(4\partial_z^3 v + \partial_z(vw)),\\
&\partial_{\bar z} w = -3\partial_z v, \quad v = \bar v,\\
&v = v(x,t), \quad w = w(x,t), \quad x = (x_1,x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R},
\end{aligned} \tag{1}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \partial_{\bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Definition 1. *A pair (v,w) is a sufficiently localized solution of equation (1) if*

- $v, w \in C(\mathbb{R}^2 \times \mathbb{R})$, $v(\cdot,t) \in C^3(\mathbb{R}^3)$,
- $|\partial_x^j v(x,t)| \leqslant \dfrac{q(t)}{(1+|x|)^{2+\varepsilon}}$, $|j| \leqslant 3$, *for some* $\varepsilon > 0$, $w(x,t) \to 0, |x| \to \infty$,
- (v,w) *satisfies (1).*

Definition 2. *A solution (v,w) of (1) is a soliton (a traveling wave) if $v(x,t) = V(x-ct)$, $c \in \mathbb{R}^2$.*

Equation (1) is an analog of the classic KdV equation. When $v = v(x_1,t)$, $w = w(x_1,t)$, then equation (1) is reduced to KdV. Besides, equation (1) is integrable via the scattering transform for the 2–dimensional Schrödinger equation

$$\begin{aligned}
&L\psi = 0,\\
&L = -\Delta + v(x,t), \quad \Delta = 4\partial_z\partial_{\bar z}, \quad x \in \mathbb{R}^2.
\end{aligned} \tag{2}$$

Equation (1) is contained implicitly in [M] as an equation possessing the following representation

$$\frac{\partial(L-E)}{\partial t} = [L-E, A] + B(L-E), \tag{3}$$

where L is defined in (2), A and B are suitable differential operators of the third and zero order respectively and $[\cdot,\cdot]$ denotes the commutator. In the explicit form equation (1) was written in [NV1], [NV2], where it was also studied in the periodic setting. For the rapidly decaying potentials the studies of equation (1) and the scattering problem for (2) were carried out in [BLMP], [GN] [T], [LMS]. In [LMS] the relation with the Calderón conductivity problem was discussed in detail.

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Definition 3. *A potential* $v \in L^p(\mathbb{R}^2)$, $1 < p < 2$, *is of conductivity type if* $v = \gamma^{-1/2}\Delta\gamma^{1/2}$ *for some real-valued positive* $\gamma \in L^\infty(\mathbb{R}^2)$, *such that* $\gamma \geqslant \delta_0 > 0$ *and* $\nabla\gamma^{1/2} \in L^p(\mathbb{R}^2)$.

The potentials of conductivity type arise naturally when the Calderón conductivity problem is studied in the setting of the boundary value problem for the 2-dimensional Schrödinger equation at zero energy (see [Nov1], [N], [LMS]); in addition, in [N] it was shown that for this type of potentials the scattering data for (2) are well-defined everywhere.

The main result of the present note consists in the following: there are no solitons of conductivity type for equation (1). The proof is based on the ideas proposed in [Nov2].

This work was fulfilled in the framework of research carried out under the supervision of R.G. Novikov.

2 Scattering data for the 2-dimensional Schrödinger equation at zero energy with a potential of conductivity type

Consider the Schrödinger equation (2) on the plane with the potential $v(z)$, $z = x_1 + ix_2$, satisfying

$$\begin{aligned} &v(z) = \overline{v(z)}, \quad v(z) \in L^\infty(\mathbb{C}), \\ &|v(z)| < q(1+|z|)^{-2-\varepsilon} \text{ for some } q > 0,\ \varepsilon > 0. \end{aligned} \tag{4}$$

For $k \in \mathbb{C}$ we consider solutions $\psi(z,k)$ of (2) having the following asymptotics

$$\psi(z,k) = e^{ikz}\mu(z,k), \quad \mu(z,k) = 1 + o(1), \text{ as } |z| \to \infty, \tag{5}$$

i.e. Faddeev's exponentially growing solutions for the two-dimensional Schrödinger equation (2) at zero energy, see [F], [GN], [Nov1].

It was shown that if v satisfies (4) and is of conductivity type, then $\forall k \in \mathbb{C}\backslash 0$ there exists a unique continuous solution of (1) satisfying (5) (see [N]). Thus the scattering data b for the potential v of conductivity type are well-defined and continuous:

$$b(k) = \iint\limits_{\mathbb{C}} e^{i(ky+\bar{k}\bar{y})}v(y)\mu(y,k)d\mathrm{Re}y d\mathrm{Im}y, \quad k \in \mathbb{C}\backslash 0. \tag{6}$$

In addition (see [N]), the function $\mu(z,k)$ from (5) satisfies the following $\bar\partial$-equation

$$\frac{\partial\mu(z,k)}{\partial\bar{k}} = \frac{1}{4\pi\bar{k}}e^{-i(kz+\bar{k}\bar{z})}b(k)\overline{\mu(z,k)}, \quad z \in \mathbb{C}, \quad k \in \mathbb{C}\backslash 0 \tag{7}$$

and the following limit properties:

$$\mu(z,k) \to 1, \text{ as } |k| \to \infty, \tag{8}$$

$$\mu(z,k) \text{ is bounded in the neighborhood of } k = 0. \tag{9}$$

The following lemma describes the scattering data corresponding to a shifted potential.

Lemma 1. *Let* $v(z)$ *be a potential satisfying (4) with the scattering data* $b(k)$. *The scattering data* $b_y(k)$ *for the potential* $v_y(z) = v(z-y)$ *are related to* $b(k)$ *by the following formula*

$$b_y(k) = e^{i(ky+\bar{k}\bar{y})}b(k), \quad k \in \mathbb{C}\backslash 0, \quad y \in \mathbb{C}. \tag{10}$$

Proof. We note that $\psi(z-y,k)$ satisfies (1) with $v_y(z)$ and has the asymptotics $\psi(z-y,k) = e^{ik(z-y)}(1+o(1))$ as $|z| \to \infty$. Thus $\psi_y(z,k) = e^{iky}\psi(z-y,k)$ and $\mu_y(z,k) = \mu(z-y,k)$. Finally, we have

$$b_y(k) = \iint_{\mathbb{C}} e^{i(k\zeta+\bar{k}\bar{\zeta})} v_y(\zeta)\mu_y(\zeta,k)d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$

$$= \iint_{\mathbb{C}} e^{i(k\zeta+\bar{k}\bar{\zeta})} v(\zeta-y)\mu(\zeta-y,k)d\mathrm{Re}\zeta d\mathrm{Im}\zeta = e^{i(ky+\bar{k}\bar{y})}b(k).$$

□

As for the time dynamics of the scattering data, in [BLMP], [GN] it was shown that if the solution (v,w) of (1) exists and the scattering data for this solution are well-defined, then the time evolution of these scattering data is described as follows:

$$b(k,t) = e^{i(k^3+\bar{k}^3)t}b(k,0), \quad k \in \mathbb{C}\backslash 0, \quad t \in \mathbb{R}. \tag{11}$$

3 Absence of solitons of conductivity type

Theorem 1. *Let (v,w) be a sufficiently localized traveling wave solution of (1) of conductivity type. Then $v \equiv 0$, $w \equiv 0$.*

Scheme of proof. From (10), (11), continuity of $b(k)$ on $\mathbb{C}\backslash 0$ and the fact that the functions k, $\bar{k}$, k^3, $\bar{k}^3$, 1 are linearly independent in the neighborhood of any point, it follows that $b \equiv 0$. Equation (7) implies that in this case the function $\mu(z,k)$ is holomorphic on k, $k \in \mathbb{C}\backslash 0$. Using properties (8) and (9) we apply Liouville theorem to obtain that $\mu \equiv 1$. Then $\psi(z,k) = e^{ikz}$ and from (2) it follows that $v \equiv 0$. □

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