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HOMOGENIZATION OF A ONE-DIMENSIONAL SPECTRAL PROBLEM FOR A SINGULARLY PERTURBED ELLIPTIC OPERATOR WITH NEUMANN BOUNDARY CONDITIONS

GRÉGOIRE ALLAIRE, YVES CAPDEBOSCQ, AND MARJOLAINE PUEL

ABSTRACT. We study the asymptotic behavior of the first eigenvalue and eigenfunction of a one-dimensional periodic elliptic operator with Neumann boundary conditions. The second order elliptic equation is not self-adjoint and is singularly perturbed since, denoting by ε the period, each derivative is scaled by an ε factor. The main difficulty is that the domain size is not an integer multiple of the period. More precisely, for a domain of size 1 and a given fractional part $0 \leq \delta < 1$, we consider a sequence of periods $\epsilon_n = 1/(n+\delta)$ with $n \in \mathbb{N}$. In other words, the domain contains n entire periodic cells and a fraction δ of a cell cut by the domain boundary. According to the value of the fractional part δ, different asymptotic behaviors are possible: in some cases an homogenized limit is obtained, while in other cases the first eigenfunction is exponentially localized at one of the extreme points of the domain.

1. INTRODUCTION

This paper is devoted to the homogenization of a spectral problem for a singularly perturbed elliptic equation in a one-dimensional periodic medium with Neumann boundary conditions. Without loss of generality we consider a bounded domain $\Omega = (0,1)$ and we denote by $\varepsilon > 0$ its period, or rather the period of the coefficients of the equation posed in Ω. Although we shall sometime use the notations ∇ and div for the gradient and the divergence operators, they simply mean derivation with respect to the single spatial variable. We study the following eigenvalue problem

$$\begin{cases} -\varepsilon^2 \mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) + \varepsilon b\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon + c\left(\frac{x}{\varepsilon}\right)u^\varepsilon = \lambda^\varepsilon \rho\left(\frac{x}{\varepsilon}\right)u^\varepsilon \text{ in } \Omega, \\ a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon = 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

We assume that a, b, c and ρ are continuous periodic functions of period one, defined in the unit cell $Y = [0,1]$. As usual x denotes the macroscopic variable in Ω, while y is the microscopic variable in Y, and they are related by the scaling $y = x/\varepsilon$. We further assume that a and ρ are strictly positive, more precisely there exists a positive constant C such that

$$\forall y \in Y, \quad 0 < C < a(y) < C^{-1}, \quad 0 < C < \rho(y) < C^{-1}.$$

By the Krein-Rutman theorem there exists, at least, a first eigenvalue and eigenvector of (1) that we shall denote by λ^ε and u^ε. Furthermore, λ^ε is real, simple and the smallest in modulus of all other eigenvalues, and u^ε can be chosen to be positive in Ω and is thus unique if it is normalized, say by the choice of $u^\varepsilon(0)$. Since (1) is actually an ordinary differential equation in one space dimension, the eigenfunction u^ε belongs at least to $C^1(\overline{\Omega})$.

We study the asymptotic behavior of the smallest eigenpair $(\lambda^\varepsilon, u^\varepsilon)$, when ε tends to zero. In contrast to the case of Dirichlet boundary conditions, studied in [7], the behavior of the

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first eigencouple depends on the fractional part of $1/\varepsilon$. Furthermore, new asymptotic regimes, corresponding to an exponential localization of the first eigenfunction at one of the extreme points of the domain, are obtained for some values of this fractional part. Nevertheless, for other values of the fractional part we still obtain an homogenized limit as was always the case for Dirichlet boundary conditions. Our main results are Theorems 2.4 and 2.7 below. We therefore choose the sequence $\varepsilon \equiv \varepsilon_n$ to be of the form

$$\varepsilon = \frac{1}{n+\delta}, \tag{2}$$

where n is an integer and $0 \le \delta < 1$ is a constant which is the rescaled size of the fractional part of the extremal periodic cell cut by the right domain boundary. In the sequel, when $\varepsilon \equiv \varepsilon_n$ is said to go to 0, we mean that n goes to infinity with δ fixed.

The special case $\delta = 0$, corresponding to an entire number of cells in the domain, is already known. It already appears in [14] for a similar system of two elliptic equations. In this later case, the proof is a little more involved and uses an exponential change of unknowns together with a viscosity solution approach to the resulting Hamilton-Jacobi equation. In the case of (1) a simpler proof is available for the following proposition.

Proposition 1.1. *Assume that $\delta = 0$ in (2). Let (λ_N, u_N) be the first eigenpair of the following Neumann cell problem*

$$\begin{cases} -\mathrm{div}_y\left(a(y)\nabla_y u_N\right) + b(y)\nabla_y u_N + c(y)u_N = \lambda_N \rho(y) u_N \text{ in } Y, \\ a(0)\nabla_y u_N(0) = a(1)\nabla_y u_N(1) = 0 \\ u_N(0) = 1. \end{cases} \tag{3}$$

Define $\theta_N = \log\left(u_N(1)\right)$. Then, the function $w_N(y) = e^{-\theta_N y} u_N(y)$ is 1-periodic and the first eigenpair of (1) is exactly given by

$$\lambda^\varepsilon = \lambda_N, \qquad u_\varepsilon(x) = e^{\frac{\theta_N x}{\varepsilon}} w_N\left(\frac{x}{\varepsilon}\right).$$

Proof. By the Krein-Rutman theorem u_N is positive, therefore θ_N is well defined, and thus we can define a 1-periodic function $w_N = e^{-\theta_N y} u_N(y)$ on each period. Clearly, $e^{\frac{\theta_N x}{\varepsilon}} w_N\left(\frac{x}{\varepsilon}\right)$ is a positive C^1 solution of (1) for the eigenvalue λ_N. Another application of the Krein-Rutman theorem, which implies that a positive eigenfunction can happen only for the first eigenvalue, yields that λ_N is indeed the smallest eigenvalue λ^ε and then $u_\varepsilon(x) = e^{\frac{\theta_N x}{\varepsilon}} w_N\left(\frac{x}{\varepsilon}\right)$. □

The fact that we can get an explicit and exact formula (in terms of ε) for the solution of (1) is quite special to this case (even though it sometimes happens when $\delta \neq 0$). Nevertheless this example shows that Neumann cell eigenvalue problems are key to the problem, and that the solutions could be of exponential-periodic type.

2. Main results

Before we can state our main results, Theorems 2.4 and 2.7, we need to introduce some notations and auxiliary problems. Since the case $\delta = 0$ is already covered by Proposition 1.1, we assume from now on that $0 < \delta < 1$ in (2). Instead of the single Neumann cell problem (3) there are now two such cell problems to consider, each of them corresponding to one endpoint of the domain Ω. For $t \in [0,1]$ let us introduce the following Neumann cell problem on the shifted cell $(t-1,t)$: we call (u_N^t, λ_N^t), the first eigenpair of

$$(4)\qquad \begin{cases} -\mathrm{div}_y\left(a(y)\nabla_y u_N^t\right) + b(y)\nabla_y u_N^t + c(y)u_N^t = \lambda_N^t \rho(y) u_N^t \text{ in } (t-1,t), \\ \qquad\qquad a(t)\nabla_y u_N^t(t) = a(t-1)\nabla_y u_N^t(t-1) = 0, \end{cases}$$

normalized by $u_N^t(t-1) = 1$. Another application of the Krein-Rutman theorem shows that there exists a first eigenvalue λ_N^t (which is real, simple and the smallest in modulus of all other eigenvalues) and a corresponding eigenvector u_N^t which can be chosen to be positive in Y. Only two values of the parameter t matter: $t = 0$ for the left end point $x = 0$ and $t = \delta$ for the right end point $x = 1$ of Ω.

2.1. **Exponential-periodic cell problems.** We shall recognize (see Lemma 2.2 below) that the auxiliary problem (4) is actually equivalent to the well-known exponential-periodic cell problem (or shifted cell problem) introduced in [2, 6, 7, 14]. These spectral cell problems are key ingredients in the homogenization of (1). Following the lead of [2, 6, 7, 14], for each $\theta \in \mathbb{R}$ we introduce an exponential-periodic cell problem which reads

$$(5)\qquad \begin{cases} -\mathrm{div}_y(a(y)\nabla_y\psi_\theta) + b(y)\nabla_y\psi_\theta + c(y)\psi_\theta = \lambda_\theta\rho(y)\psi_\theta \quad \text{in } Y, \\ y \to e^{-\theta y}\psi_\theta(y) \quad Y\text{-periodic}, \end{cases}$$

together with its associated adjoint problem, with respect to the $L^2(Y)$ scalar product,

$$(6)\qquad \begin{cases} -\mathrm{div}_y(a(y)\nabla_y\psi_\theta^*) - b(y)\nabla_y\psi_\theta^* + (c(y) - \mathrm{div}_y b(y))\psi_\theta^* = \lambda_\theta\rho(y)\psi_\theta^* \quad \text{in } Y, \\ y \to e^{\theta y}\psi_\theta^*(y) \quad Y\text{-periodic}. \end{cases}$$

In the above equations (5) and (6) λ_θ stands for the first eigenvalue and $\psi_\theta, \psi_\theta^*$ for the first eigenfunctions, which exist and are real-valued by virtue, once again, of the Krein-Rutman theorem. It also implies that λ_θ is of algebraic and geometric multiplicity one, that we can impose $\psi > 0$, $\psi^* > 0$ in Y and that there are the only eigenfunctions which are positive. Of course, since (5), (6) and also (4) are just ordinary differential equations, their solutions belong at least to $C^1(Y)$. We choose the following normalization: $\psi_\theta(0) = 1 = \psi_\theta^*(0)$. We recall some properties of these problems, established in [2, 6, 7, 14].

Proposition 2.1. *The following properties hold true.*

- *The map $\theta \to \lambda_\theta$ is strictly concave, and $\lim_{\theta\to\pm\infty}\lambda_\theta = -\infty$.*
- *At the unique θ_∞ such that λ_θ is maximal, the normalized eigenvectors $\psi_\infty \equiv \psi_{\theta_\infty}$ and $\psi_\infty^* \equiv \psi_{\theta_\infty}^*$ satisfy*

$$(7)\qquad a(y)\left(\psi_\infty\nabla_y\psi_\infty^*(y) - \psi_\infty^*\nabla_y\psi_\infty(y)\right) + b(y)\psi_\infty^*(y)\psi_\infty(y) \equiv 0 \text{ in } Y.$$

- *For each $y \in Y$, the map $\theta \to \frac{1}{\psi_\theta(y)}\nabla_y\psi_\theta(y)$ is strictly increasing and one-to-one from $\mathbb{R}$ to $\mathbb{R}$.*
- *The maximizer θ_∞ satisfies*

$$(8)\qquad \theta_\infty = \int_0^1 \frac{b(y)}{2a(y)}dy.$$

Proof. We only prove the last point whose proof is not included in references [2, 6, 7, 14]. By dividing (7) by $\psi_\infty\psi_\infty^*$, we obtain

$$-\theta_\infty + \nabla\log(\tilde{\psi}_\infty^*) - \theta_\infty - \nabla\log(\tilde{\psi}_\infty) + \frac{b}{a} = 0$$

where $\tilde{\psi}_\infty(y) = e^{-\theta_\infty y}\psi_\infty(y)$ and $\tilde{\psi}^*_\infty(y) = e^{\theta_\infty y}\psi^*_\infty(y)$ are Y-periodic functions. Integrating with respect to y, we obtain (8). $\square$

Actually the solution u_N^t of (4) is an exponential periodic function as shown by the following result.

Lemma 2.2. *For each $t \in [0,1]$ there exists $\theta_N^t \in \mathbb{R}$ such that the solution u_N^t of (4) satisfies $u_N^t(y) = e^{\theta_N^t y} w_N^t(y)$ where w_N^t is a 1-periodic function.*

Proof. We define the constant $\theta_N^t = \log\left(u_N^t(t)\right)$. It is then easy to check that the function $w_N^t(y) = e^{-\theta_N^t y} u_N^t(y)$ is 1-periodic. $\square$

Lemma 2.2 shows that the solution $u_N^t(y)$ of (4) coincides with that of (5), $\psi_{\theta_N^t}(y)/\psi_{\theta_N^t}(t-1)$, with the same eigenvalue $\lambda_N^t = \lambda_{\theta_N^t}$. In particular, it allows us to extend the function u_N^t to the whole $\mathbb{R}$ although it is originally defined only in $(t-1,t)$. Depending on the respective positions of θ_N^0 and θ_N^δ with respect to θ_∞, we will exhibit the different behaviors of the sequence u_ε when ε goes to zero.

2.2. **Convergence.** In this subsection, Theorems 2.4 and 2.7 describe completely all possible asymptotic regimes of the spectral problem (1) using the auxiliary spectral problems (4) and (5). However we start with a special case, similar to Proposition 1.1, which is simpler than the general case that will follow. This special case occurs when the solutions u_N^0 and u_N^δ of (4), for $t=0$ and $t=\delta$ respectively, are equal (up to a multiplicative factor).

Proposition 2.3. *If the solutions u_N^0 and u_N^δ of (4) satisfies $u_N^0(y) = \frac{u_N^\delta(y)}{u_N^\delta(-1)}$, then the first eigenpair of (1) is exactly given by*

$$\lambda^\varepsilon = \lambda_N^0, \qquad u_\varepsilon(x) = e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)},$$

where the function $w_N^0(y) = e^{-\theta_N^0 y} u_N^0(y)$, with $\theta_N^0 = \log\left(u_N^0(0)\right)$, is the 1-periodic function defined in Lemma 2.2.

The proof of Proposition 2.3 is given in Proposition 6.1.

When Proposition 2.3 does not apply, i.e., when $u_N^0(y) \neq \frac{u_N^\delta(y)}{u_N^\delta(-1)}$, the asymptotic behavior of u_ε can be of different nature. In some cases, described in Proposition 2.3, the solution of (1) concentrates on the boundaries of the domain.

Theorem 2.4. *The first eigenpair of (1) is localized on one of the end points of Ω in the following two cases.*

- *For $\left(\theta_N^0 < \theta_\infty \text{ and } \theta_N^\delta \le \theta_\infty\right)$ or for $\left(\theta_N^0 < \theta_\infty < \theta_N^\delta \text{ and } \lambda_N^0 < \lambda_N^\delta\right)$, then*

$$\left|\lambda^\varepsilon - \lambda_N^0\right| = \gamma_0 e^{2(\theta_N^0 - \theta_\infty)/\varepsilon}(1 + o(1)),$$

and

$$\left\| u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)} \right\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon} e^{(\theta_N^0 - \theta_\infty)/\varepsilon} \|u^\varepsilon\|_{L^1(\Omega)},$$

where γ_0 is a positive constant defined in Proposition 6.8, independent of ε.

- *For* $\left(\theta_N^\delta > \theta_\infty \text{ and } \theta_N^0 \geq \theta_\infty\right)$ *or for* $\left(\theta_N^0 < \theta_\infty < \theta_N^\delta \text{ and } \lambda_N^0 > \lambda_N^\delta\right)$*, then*

$$\left|\lambda^\varepsilon - \lambda_N^\delta\right| = \gamma_1 e^{2(\theta_\infty - \theta_N^\delta)/\varepsilon}(1 + o(1)),$$

and

$$\left\| u^\varepsilon(x) - u^\varepsilon(1) \frac{e^{\frac{\theta_N^\delta x}{\varepsilon}} w_N^\delta\left(\frac{x}{\varepsilon}\right)}{e^{\frac{\theta_N^\delta}{\varepsilon}} w_N^\delta(\delta)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{(\theta_\infty - \theta_N^\delta)/\varepsilon} \|u^\varepsilon\|_{L^1(\Omega)},$$

where γ_1 *is a positive constant defined in Proposition 6.8, independent of* ε*.*

The first eigenpair of (1) localizes at one or two end points of Ω *in the following third case.*

- *For* $\left(\theta_N^0 < \theta_\infty < \theta_N^\delta \text{ and } \lambda_N^\delta = \lambda_N^0\right)$*, that is* $\theta_N^\delta - \theta_\infty = \theta_\infty - \theta_N^0 > 0$*, then*

$$\lambda^\varepsilon - \lambda_N^0 = -\gamma_\delta e^{(\theta_N^0 - \theta_\infty)/\varepsilon}(1 + o(1)),$$

and

$$\left\| u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)} - u^\varepsilon(0) c_\delta e^{\frac{\theta_N^0 - \theta_\infty}{\varepsilon}} \frac{\psi_\infty(\delta - 1) e^{\frac{\theta_N^\delta x}{\varepsilon}} w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{(\theta_N^0 - \theta_\infty)/\varepsilon} \|u^\varepsilon\|_{L^1(\Omega)},$$

where $\gamma_\delta > 0$ *and* c_δ *are constants defined in Proposition 6.8, independent of* ε*.*

Remark 2.5. *Throughout this paper,* C *denotes a positive constant independent of* ε*.*

Remark 2.6. *The right hand sides of all estimates in Theorem 2.4 are exponentially small with respect to* ε*. In the two first cases, the eigenfunction* u_ε *is approximately the product of a periodic function and a scaled exponential, which clearly exhibits a localization effect on one and only one end point of* Ω *(at least when* θ_N^0 *and* θ_N^δ*, respectively, are not equal to zero). The precise end point of* Ω *where localization occurs is deduced from the sign of* θ_N^0 *or* θ_N^δ*, respectively. In the third case, the eigenfunction* u_ε *localizes on one endpoint of* Ω *if* $\theta_\infty \neq 0$ *and on the two end points in the special case* $\theta_\infty = 0$*. Indeed, around* $x = 0$*, the ansatz says*

$$u^\varepsilon(x) \approx u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)},$$

whereas around $x = 1$*, we use the following equivalent form of the ansatz*

$$\begin{aligned} & u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)} - u^\varepsilon(0) c_\delta e^{\frac{\theta_N^0 - \theta_\infty}{\varepsilon}} \frac{\psi_\infty(\delta-1) e^{\frac{\theta_N^\delta x}{\varepsilon}} w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)} \\ = \; & u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)} - u^\varepsilon(0) c_\delta e^{\frac{\theta_\infty}{\varepsilon}} e^{\frac{\theta_N^\delta (x-1)}{\varepsilon}} \frac{\psi_\infty(\delta-1) w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)}, \end{aligned}$$

which implies

$$u^\varepsilon(x) \approx u^\varepsilon(0) c_\delta e^{\frac{\theta_\infty}{\varepsilon}} e^{\frac{\theta_N^\delta (x-1)}{\varepsilon}} \frac{\psi_\infty(\delta - 1) w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)}.$$

Therefore, the localisation is determined by the drift factor θ_∞*. If* $\theta_\infty < 0$*, the localization is in* $x = 0$*, and if* $\theta_\infty > 0$ *the localization occurs in* $x = 1$*. In the special case where* $\theta_\infty = 0$ *which includes the self adjoint case (see Proposition 2.1), a double localization occurs, as the solution localizes at both endpoints.*

Proof. It is a consequence of Corollary 6.9 which is expressed in terms of $\phi^\varepsilon(x)$, a factorized solution defined by the relation $u^\varepsilon(x) = \psi_\infty\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)$, of the factorized cell eigenfunctions $\tilde\varphi^t_\theta(y) = e^{-\theta y}\varphi^t_\theta(y)$ where φ^t_θ is the first eigenfunction of (16) and of the factorized Neumann solutions $\phi_t(y)$ given by (17). Introducing the correspondences that, on one hand,

$$\frac{\phi_0\left(\frac{x}{\varepsilon}\right)}{\phi_0(0)} = e^{\frac{\theta_0 x}{\varepsilon}}\frac{\tilde\varphi^0_{\theta_0}\left(\frac{x}{\varepsilon}\right)}{\tilde\varphi^0_{\theta_0}(0)} = \frac{u^0_N\left(\frac{x}{\varepsilon}\right)}{\psi_\infty\left(\frac{x}{\varepsilon}\right)u^0_N(0)} = \frac{e^{\frac{\theta^0_N x}{\varepsilon}}w^0_N\left(\frac{x}{\varepsilon}\right)}{\psi_\infty\left(\frac{x}{\varepsilon}\right)w^0_N(0)}, \tag{9}$$
$$\theta_0 = \theta^0_N - \theta_\infty, \quad u^\varepsilon(0) = \phi^\varepsilon(0), \quad \mu_0 = \lambda^0_N - \lambda_\infty,$$

and on the other hand

$$\frac{\phi_\delta\left(\frac{x}{\varepsilon}\right)}{\phi_\delta\left(\frac{1}{\varepsilon}\right)} = \frac{e^{\frac{\theta_\delta x}{\varepsilon}}\tilde\varphi^\delta_{\theta_\delta}\left(\frac{x}{\varepsilon}\right)}{e^{\frac{\theta_\delta(1)}{\varepsilon}}\tilde\varphi^\delta_{\theta_\delta}(\delta)} = \frac{\psi_\infty\left(\frac{1}{\varepsilon}\right)u^\delta_N\left(\frac{x}{\varepsilon}\right)}{\psi_\infty\left(\frac{x}{\varepsilon}\right)u^\delta_N\left(\frac{1}{\varepsilon}\right)} = \frac{\psi_\infty\left(\frac{1}{\varepsilon}\right)e^{\frac{\theta^\delta_N x}{\varepsilon}}w^\delta_N\left(\frac{x}{\varepsilon}\right)}{\psi_\infty\left(\frac{x}{\varepsilon}\right)e^{\frac{\theta^\delta_N}{\varepsilon}}w^\delta_N(\delta)}, \tag{10}$$
$$\theta_\delta = \theta^\delta_N - \theta_\infty, \quad u^\varepsilon(1) = \psi_\infty\left(\frac{1}{\varepsilon}\right)\phi^\varepsilon(1), \quad \mu_\delta = \lambda^\delta_N - \lambda_\infty,$$

as well as

$$\frac{\phi_\delta\left(\frac{x}{\varepsilon}\right)}{\phi_0(0)} = \frac{e^{\frac{\theta_\delta x}{\varepsilon}}\tilde\varphi^\delta_{\theta_\delta}\left(\frac{x}{\varepsilon}\right)}{\tilde\varphi^0_{\theta_0}(0)} = \frac{\psi_\infty(\delta-1)u^\delta_N\left(\frac{x}{\varepsilon}\right)}{\psi_\infty\left(\frac{x}{\varepsilon}\right)\psi_\infty(-1)u^0_N(0)} = \frac{\psi_\infty(\delta-1)e^{\frac{\theta^\delta_N x}{\varepsilon}}w^\delta_N\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1)\psi_\infty\left(\frac{x}{\varepsilon}\right)w^0_N(0)}, \tag{11}$$

the statements in Theorem 2.4 are equivalent to those in Corollary 6.9. A more precise corrector result is stated in Proposition 6.8. □

The last case, $\theta^0_N \geq \theta_\infty$ and $\theta^\delta_N \leq \theta_\infty$, not covered by Theorem 2.4, corresponds to a homogenization regime. In such a case, the first eigensolution does not localize at the endpoints. Its precise asymptotic form is given by the following result.

Theorem 2.7. *For $\theta^0_N \geq \theta_\infty$ and $\theta^\delta_N \leq \theta_\infty$, the first eigenpair of (1) is of the form*

$$u^\varepsilon(x) \approx \psi_\infty\left(\frac{x}{\varepsilon}\right)u(x) \text{ and } \lambda_\varepsilon = \lambda_\infty + \varepsilon^2(\lambda^*_0 + o(1)),$$

where ψ_∞ is a periodic function and (u, λ^*_0) is the first eigenpair of an homogenized problem

$$\begin{cases} -d^*\Delta u & = & \lambda^*_0 s^* u \text{ in } \Omega, \\ u \in H^1(\Omega) & \text{and either} & u(0) = 0 \text{ or } u(1) = 0, \text{ or both.} \end{cases}$$

where d^* and s^* are positive constants. (See Theorem 4.4 for a more precise statement and for the proof).

It is interesting to notice that, in the case of Dirichlet boundary conditions, Theorem 2.7 gives the only possible asymptotic behavior, for any ε, i.e., for any δ, and in any space dimension (see [7]). Therefore, the case of Neumann boundary conditions is much more sensitive to the precise geometry.

To illustrate our main results, we provide numerical examples of each possible asymptotic behavior described in Theorem 2.4 and 2.7. We will show in the next section that non-selfadjoint problems can be reduced to selfadjoint ones, thus we chose $b(y) = 0$ for our numerical tests. For simplicity we also take $\rho(y) = 1$. Not all possible behavior can be observed with only one pair of coefficient. We use two pairs $(a(y), c_1(y))$ and $(a(y), c_2(y))$, represented in Figure 1 and 2.

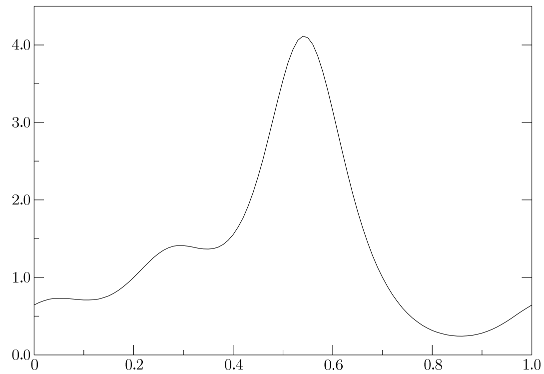

FIGURE 1. The diffusion coefficient a over $Y = (0, 1)$.

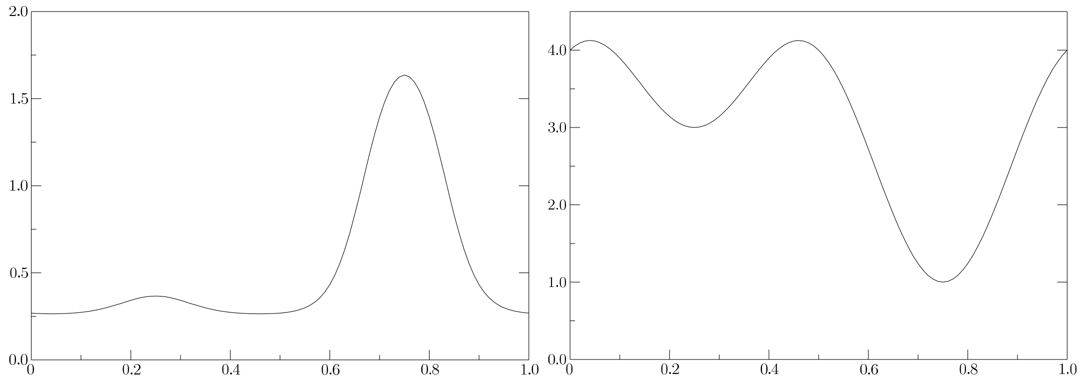

FIGURE 2. The zero-order c_1 (left) and c_2 (right) over $Y = (0, 1)$.

The coefficients (chosen very arbitrarily) are given by

$$\begin{aligned}
\log\left(a\left(y - \frac{3}{10}\right)\right) &= -\sin(2\pi y) - \frac{1}{2}\sin(4\pi y) - \frac{1}{6}\sin(6\pi y) + \frac{1}{4}\sin(8\pi y),\\
\sqrt{c_1(y)} &= \exp\left(-\frac{c_2(y)^2}{4}\right) + \frac{1}{2},\\
c_2(y) &= \sin(2\pi y) + \cos(4\pi y) + 3.
\end{aligned}$$

In all three Figures 3, 4 and 5 we plot the first eigenfunction of (1) for $n = 30$ (dashed line) and $n = 70$ (solid line) to show the trend of convergence as ε goes to zero. Figures 3 and 4 are obtained using the first pair $(a(y), c_1(y))$ and three different values of δ, corresponding to the three configurations identified in Theorem 2.4. In particular the first eigenfunction converges pointwise to zero in the interior of the domain.

Figure 5 was obtained using the second pair $(a(y), c_2(y))$ and $\delta = 0.2$: it illustrates the homogenization effect characterized in Theorem 2.7. In particular the values of the first eigenfunction at the two boundary points converge to zero.

Note that the influence of the δ parameter on the first-order corrector to the eigenvalue of a non singularly perturbed homogenization problem was already observed in [15], [12].

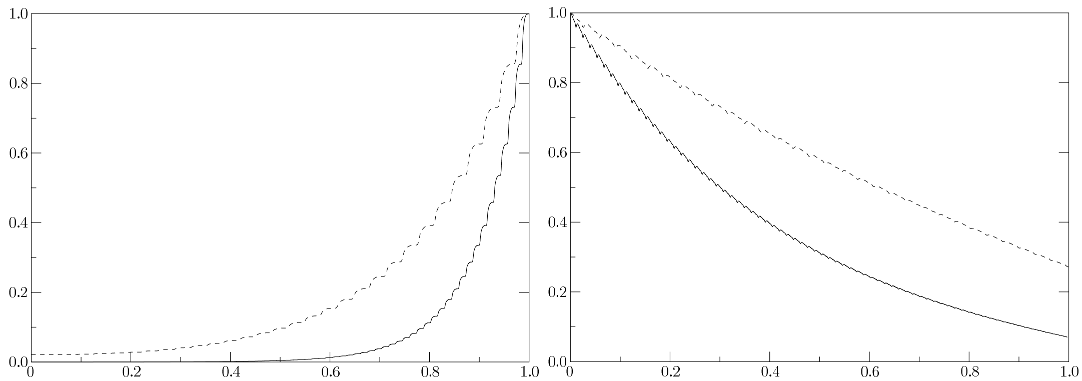

FIGURE 3. Left: concentration at $x = 1$, for $\delta = 0.6$. Right: concentration at $x = 0$, for $\delta = 0.9$ ($n = 30$ dashed line, $n = 70$ solid line).

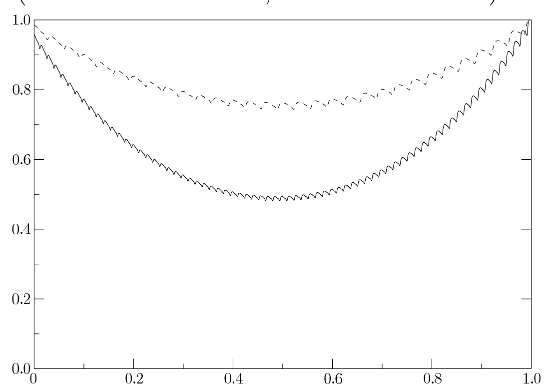

FIGURE 4. Concentration at both end points, for $\delta = 0.2$ ($n = 30$ dashed line, $n = 70$ solid line).

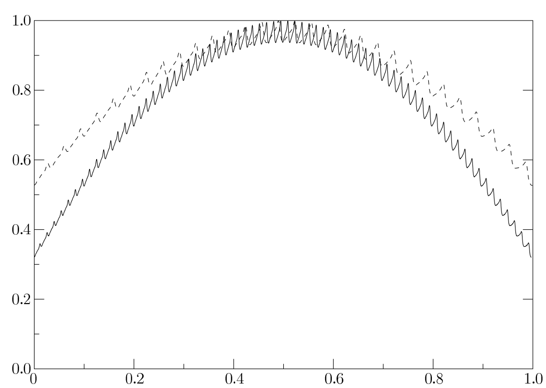

FIGURE 5. The homogenization regime ($n = 30$ dashed line, $n = 70$ solid line).

The purely periodic character of the coefficients in (1) is crucial for our results to hold true. Actually, a completely different behavior can arise if the coefficients depend on the macroscopic variable x too, namely localization inside Ω can appear [4], [5].

The content of our paper is the following. In the next section, by using a factorization principle (in the spirit of [16], [1, 2]) we reduce the original problem (1) to a selfadjoint one. It thus allows us to write a variational characterization of the first eigenvalue. Of course, this "miracle" is possible only in one space dimension. Then, Section 4 adresses the homogenization regime of Theorem 2.7. Section 5 is concerned with the exponential convergence of the eigenvalues in Theorem 2.4. Eventually Section 6 deals with the convergence and localization of the eigenfunctions.

3. Transformation into a self-adjoint problem

A remarkable feature of this eigenvalue problem is that it can be reformulated, after a suitable change of unknowns, as a self-adjoint problem with compact resolvent. Among the many advantages of working with self-adjoint problems, we shall use in the sequel the fact that the first eigenvalue is characterized as the minimizer of a Rayleigh quotient, and that the normalized eigenvectors span the space $L^2(\Omega)$. This change of unknowns will be made thanks to the exponential-periodic functions introduced in (6), as in [6, 7, 14].

3.1. **Factorization.** To transform the problem into a self-adjoint one, we perform a change of unknown and consider instead of u^ε the function ϕ^ε defined by

$$\phi^\varepsilon(x) = \frac{u^\varepsilon(x)}{\psi_\infty\left(\frac{x}{\varepsilon}\right)} \tag{12}$$

where ψ_∞ is the first cell eigenfunction defined in Proposition 2.1. Because $x \to \psi_\infty\left(\frac{x}{\varepsilon}\right)$ is a solution of the equation (with different boundary conditions) it was proved in [1, 2] that (12) is indeed a change of variable from $H^1(\Omega)$ to $H^1(\Omega)$.

Proposition 3.1. *If u^ε is a solution of the original problem (1), then the function ϕ^ε, defined by (12), is an eigensolution for the following self-adjoint problem*

$$\begin{cases} -\mathrm{div}(d\left(\frac{x}{\varepsilon}\right)\nabla\phi^\varepsilon) = \mu^\varepsilon s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon \ \ \textit{in } \Omega, \\ d\left(\frac{x}{\varepsilon}\right)\nabla\phi^\varepsilon + \frac{1}{\varepsilon}\phi^\varepsilon m\left(\frac{x}{\varepsilon}\right) = 0 \qquad \textit{on } \partial\Omega. \end{cases} \tag{13}$$

The new periodic coefficients are given by

$$d(y) = a(y)\psi_\infty(y)\psi^*_\infty(y), \quad s(y) = \rho(y)\psi_\infty(y)\psi^*_\infty(y), \quad m(y) = d(y)\frac{\nabla\psi_\infty(y)}{\psi_\infty(y)}$$

and the eigenvalues μ_ε are related to the ones of (1) by

$$\mu^\varepsilon = \frac{\lambda^\varepsilon - \lambda_\infty}{\varepsilon^2}.$$

Remark 3.2. *There are other transformations which map a non self-adjoint problem into a self-adjoint one in the theory of Hill's equation (see chapter III in* [11]*).*

Proof. As in [1, 7, 10, 16], replacing $u^\varepsilon(x)$ by $\phi^\varepsilon(x)\psi_\infty\left(\frac{x}{\varepsilon}\right)$ in (1) gives

$$\begin{aligned} -\varepsilon^2\mathrm{div}\left(a\psi_\infty\nabla\phi^\varepsilon\right) - \varepsilon\mathrm{div}\left(a\phi^\varepsilon\nabla_y\psi_\infty\right) + \varepsilon b\psi_\infty\nabla\phi^\varepsilon + b\phi^\varepsilon\nabla_y\psi_\infty + c\psi_\infty\phi^\varepsilon \\ = \lambda^\varepsilon\psi_\infty\phi^\varepsilon. \end{aligned} \tag{14}$$

Using the fact that ψ_∞ is solution of a cell problem, we note that

$$b\phi^\varepsilon\nabla_y\psi_\infty + c\psi_\infty\phi^\varepsilon = \lambda_\infty\psi_\infty\phi^\varepsilon - \mathrm{div}_y\left(a\nabla_y\psi_\infty\right)\phi^\varepsilon.$$

Therefore (14) becomes

$$-\varepsilon^2 \mathrm{div}\,(a\psi_\infty \nabla\phi^\varepsilon) - \varepsilon a \nabla\phi^\varepsilon \nabla_y \psi_\infty + \varepsilon b \psi_\infty \nabla\phi^\varepsilon = (\lambda^\varepsilon - \lambda_\infty)\psi_\infty \phi^\varepsilon.$$

Multiplying this last identity by ψ^*_∞, we obtain

$$-\varepsilon^2 \psi^*_\infty \mathrm{div}\,(a\psi_\infty \nabla\phi^\varepsilon) - \varepsilon a \psi^*_\infty \nabla\phi^\varepsilon \nabla_y \psi_\infty + \varepsilon b \psi^*_\infty \psi_\infty \nabla\phi^\varepsilon = (\lambda^\varepsilon - \lambda_\infty)\psi^*_\infty \psi_\infty \phi^\varepsilon$$

which becomes

$$\begin{aligned} -\varepsilon^2 \mathrm{div}(a\psi^*_\infty \psi_\infty \nabla\phi^\varepsilon) + \varepsilon\left(-a\psi^*_\infty \nabla_y \psi_\infty + a\psi_\infty \nabla_y \psi^*_\infty + b\psi^*_\infty \psi_\infty\right)\nabla\phi^\varepsilon \\ = (\lambda^\varepsilon - \lambda_\infty)\psi^*_\infty \psi_\infty \phi^\varepsilon. \end{aligned}$$

Thanks to (7), the first order term cancels, and we obtain (12). □

Remark 3.3. *Note that because of the regularity and positivity of ψ_∞ and ψ^*_∞ the coefficients d, s and m are continuous and satisfy, for some constant $C > 0$,*

$$C < d(y) < C^{-1},\ C < s(y) < C^{-1} \text{ and } -C < m(y) < C \text{ for all } y \in Y.$$

The coefficients $d(y), s(y), m(y)$ are indeed Y-periodic functions. As $\psi_\infty(y) = \exp(\theta_\infty y) g_\infty(y)$, with g_∞ Y-periodic, and $\psi^*_\infty(y) = \exp(-\theta_\infty y) g^*_\infty(y)$, with g^*_∞ Y-periodic, we have $\psi_\infty \psi^*_\infty = g_\infty g^*_\infty$, and also

$$\frac{\nabla\psi_\infty(y)}{\psi_\infty(y)} = \theta_\infty + \frac{\nabla g_\infty(y)}{g_\infty(y)} = \frac{\nabla\psi_\infty(y+1)}{\psi_\infty(y+1)}.$$

Remark 3.4. *The above factorization principle can actually be applied in any space dimension. However it yields an additional convective term in equation (13) with a periodic velocity which is divergence free and has zero average. It is only in the one-dimensional case that it implies that the velocity is zero. This is the main reason why we restrict ourselves to a one-dimensional setting.*

We have transformed a non-selfadjoint problem into a selfadjoint one, at the cost of changing the Neumann boundary condition into a Fourier or Robin boundary condition. Since we work in one space dimension, we did not write the unit external normal vector in the Fourier boundary condition which thus changes the usual sign convention for the boundary condition at the left end of the interval Ω. Remark that (13) is still singularly perturbed because of the factor ε^{-1} in the boundary condition. Nevertheless, this transformation enables us to characterize the first eigenpair as minimizers of a Rayleigh quotient.

Proposition 3.5. *The first eigenvalue of problem (13) μ^ε is given by*

$$\mu^\varepsilon = \min_{\phi \in H^1(\Omega)} \frac{\displaystyle\int_\Omega d\left(\frac{x}{\varepsilon}\right) |\nabla\phi|^2\, dx + \frac{1}{\varepsilon}\left(m(\delta)\phi^2(1) - m(0)\phi^2(0)\right)}{\displaystyle\int_\Omega s\left(\frac{x}{\varepsilon}\right)\phi^2(x)\, dx}. \tag{15}$$

Furthermore, the minimum in (15) is achieved by any multiple of the first eigenfunction of (13).

The proof of Proposition 3.5 is obvious: simply note that, whatever the signs of $m(0)$ and $m(\delta)$, the boundary terms cause no problems in the coercivity, for fixed ε, of the Rayleigh quotient since, for any small $\kappa > 0$, there exists a constant $C_\kappa > 0$ such that

$$\phi^2(0) \le \kappa \int_\Omega |\nabla\phi|^2\, dx + C_\kappa \int_\Omega \phi^2(x)\, dx \quad \forall\, \phi \in H^1(\Omega).$$

3.2. **Cell Problems.** After the factorization (12) we can again introduce exponential-periodic cell problems, adapted to the new spectral problem (13). For each $\theta \in \mathbb{R}$, define φ_θ^t as the first eigenfunction of

$$\begin{cases} -\mathrm{div}(d(y)\nabla\varphi_\theta^t) = \nu_\theta s(y)\varphi_\theta^t & \text{in } Y, \\ y \to e^{-\theta y}\varphi_\theta^t(y) \quad Y-\text{periodic}, \end{cases} \tag{16}$$

normalized by $\varphi_\theta^t(t-1) = 1$. Since (16) is self-adjoint, there is no need to introduce an adjoint problem. In the periodic case, i.e., $\theta = 0$, the explicit solution of (16) is $\nu_0 = 0$ and $\varphi_0 \equiv 1$.

In the same spirit, we can perform a factorization, similar to (12), for the solution u_N^t of (4) and define

$$\phi_t(y) = \psi_\infty(t-1)\frac{u_N^t(y)}{\psi_\infty(y)} \quad \text{and } \mu_t = \lambda_N^t - \lambda_\infty. \tag{17}$$

Thus ϕ_t is the first eigenfunction of

$$\begin{cases} -\mathrm{div}_y(d(y)\nabla_y\phi_t) = \mu_t s(y)\phi_t & \text{in } (t-1,t), \\ d(t-1)\nabla_y\phi_t(t-1) + m(t-1)\phi_t(t-1) = 0, \\ d(t)\nabla_y\phi_t(t) + m(t)\phi_t(t) = 0, \end{cases} \tag{18}$$

normalized by $\phi_t(t-1) = 1$. Alternatively, (18) can be motivated by a formal study of the influence of the boundary condition in (13). As usual, the simplicity of the first eigenvalue as well as the uniqueness and positivity of the first normalized eigenfunctions of (16) and (18) follows from the Krein-Rutman theorem. The problems (16) and (18) play a role in the final result.

We now show that the eigenvalue problem (18) can be interpreted as an exponential-periodic problem.

Proposition 3.6. *For each $t \in [0,1]$ there exists a unique $\theta_t \in \mathbb{R}$ such that $\varphi_{\theta_t}^t = \phi_t$ and $\nu_{\theta_t} = \mu_t$. The sign of θ_t is the opposite of that of $m(t)$. Furthermore, $\mu_t < 0$ if $m(t) \neq 0$.*

As a consequence, if $m(0) > 0$ then there exists $\theta_0 < 0$ and $C > 0$ such that for all x,

$$0 < C < e^{-\theta_0 x}\phi_0(x) < \frac{1}{C} \text{ and } 0 > -C > \frac{\nabla\phi_0(x)}{\phi_0(x)} > -\frac{1}{C}.$$

If $m(\delta) < 0$ then there exists $\theta_\delta > 0$ and $C > 0$ such that for all x,

$$0 < C < e^{-\theta_\delta x}\phi_\delta(x) < \frac{1}{C} \text{ and } 0 < C < \frac{\nabla\phi_\delta(x)}{\phi_\delta(x)} < \frac{1}{C}.$$

Proof. Recall from Remark 3.3 that d and m are periodic continuous functions. On the same token, $y \to \frac{\nabla\varphi_\theta^t(y)}{\varphi_\theta^t(y)}$ is also Y-periodic. Thanks to Proposition 2.1 (which can also be applied to the spectral problem (16)) we know that there exists a unique θ_t such that $\frac{\nabla\varphi_{\theta_t}^t(t-1)}{\varphi_{\theta_t}^t(t-1)} = -\frac{m(t-1)}{d(t-1)} = -\frac{m(t)}{d(t)}$. Thus, $\varphi_{\theta_t}^t$ satisfies the boundary conditions of (18). Since $\varphi_{\theta_t}^t(t-1) = \phi_t(t-1) = 1$, the uniqueness of the positive normalized first eigenfunction of (18) implies that $\varphi_{\theta_t}^t \equiv \phi_t$.

Finally, note that the maximum of the map $\theta \to \nu_\theta$ is attained at $\theta = 0$, since the maximizer is characterized by (7), which is clearly satisfied for $\varphi_0 = \varphi_0^* = 1$. Therefore, for all $\theta_t \neq 0$, $\mu_t = \nu_{\theta_t} < \nu_0 = 0$.

We have proved that $\phi_0 = \varphi^0_{\theta_0}$ for some θ_0. Note that, thanks to Proposition 2.1, for every $x \in [0,1]$, the map $L(x,\cdot) : \theta_t \to \nabla\varphi^t_{\theta_t}(x)/\varphi^t_{\theta_t}(x)$ is increasing. Since $L(0,0) = 0$ and $L(0,\theta_0) = -m(0)/d(0) < 0$, we conclude that $\theta_0 < 0$. Since $x \to \exp(-\theta_0 x)\phi_0(x)$ is a positive continuous periodic function, it is bounded above and below by positive constants.

Next, notice that, $L(x,0) = 0$ for all $x \in [0,1]$, therefore $L(x,\theta_0) < 0$ since $\theta_0 < 0$. Finally, since $L(\cdot,\theta_0)$ is a negative continuous Y-periodic function, it is therefore bounded above and below by negative constants. The second statement involving θ_δ is proved in a similar way. □

4. The homogenization regime

In this section we show that the assumption $m(0) \le 0 \le m(\delta)$ implies that the spectral problem (13) admits a homogenized limit.

Remark 4.1. *The equality $m(0) = m(\delta) = 0$ is a very special case which is easy to analyze. In this case, the minimum of the Rayleigh quotient (15) is zero, attained by $\phi^\varepsilon = \varphi_0 \equiv 1$, and we deduce that*

$$\lambda_\varepsilon = \lambda_\infty \text{ and } u^\varepsilon(x) = \psi_\infty\left(\frac{x}{\varepsilon}\right).$$

From now on we shall further assume that $m(\delta) \neq m(0)$ since $m(0) = m(\delta)$ together with the assumption $m(0) \le 0 \le m(\delta)$ implies that both term vanish.

Proposition 4.2. *Assume $m(0) \le 0 \le m(\delta)$. The eigenvalue μ_ε satisfies*

$$0 \le \mu_\varepsilon \le \frac{\max(d)}{\min(s)}\pi^2.$$

Proof. Since $H^1_0(\Omega) \subset H^1(\Omega)$,

$$\mu_\varepsilon \le \min_{\phi\in H^1_0(\Omega)} \frac{\displaystyle\int_\Omega d\left(\frac{x}{\varepsilon}\right)|\nabla\phi|^2(x)}{\displaystyle\int_\Omega s\left(\frac{x}{\varepsilon}\right)\phi^2(x)} \le \frac{\max(d)}{\min(s)}\pi^2.$$

When $m(\delta) \ge 0 \ge m(0)$ all terms in the numerator of the Rayleigh quotient (15) are non-negative, and therefore $\mu_\varepsilon \ge 0$. □

This shows that the sequence μ_ε is bounded independently of ε. In this case, following a well-established strategy (see e.g. [1, 2, 3, 13]) we consider the operator S^ε defined as follows

Proposition 4.3. *Assume $m(0) \le 0 \le m(\delta)$, and $m(\delta) \neq m(0)$. Let $S^\varepsilon : L^2(\Omega) \to L^2(\Omega)$ be the self-adjoint operator defined, for $f \in L^2(\Omega)$, by $S^\varepsilon f = w^\varepsilon$ which is the unique solution in $H^1(\Omega)$ of*

$$\int_\Omega d\left(\frac{x}{\varepsilon}\right)\nabla w^\varepsilon \nabla\zeta\, dx + \frac{1}{\varepsilon}\left(m(\delta)w^\varepsilon(1)\zeta(1) - m(0)w^\varepsilon(0)\zeta(0)\right) = \int_\Omega f\zeta dx \tag{19}$$

for all $\zeta \in H^1(\Omega)$. Then, for each $\varepsilon > 0$, S^ε is a compact operator in $L^2(\Omega)$. Furthermore, as ε tends to zero, S^ε converges uniformly to the operator S which to f associates $w \in \mathcal{H}$ given by

$$-d^*\Delta w = f \text{ in } \Omega,$$

where $d^* = \left(\int_Y d^{-1}(y)dy\right)^{-1}$ and $\mathcal{H} = \left\{u \in H^1(\Omega) \text{ s.t. } u(0)m(0) = u(1)m(\delta) = 0\right\}$.

Proof. This is a classical homogenization result [1, 2, 3, 13], which stems from the following a priori estimate

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^{-1}|m(\delta)|\left(w^\varepsilon(1)\right)^2 + \varepsilon^{-1}|m(0)|\left(w^\varepsilon(0)\right)^2 \le C\|f\|^2_{L^2(\Omega)}.$$

We will therefore only establish this estimate. Choosing w^ε as a test function in (19) we obtain

$$\int_\Omega d\left(\frac{x}{\varepsilon}\right)|\nabla w^\varepsilon|^2 dx + \frac{1}{\varepsilon}\left(m(\delta)\left(w^\varepsilon(1)\right)^2 - m(0)\left(w^\varepsilon(0)\right)^2\right) = \int_\Omega f w^\varepsilon dx.$$

Since each term on the left hand side is non-negative, $d(y) > C > 0$, $m(\delta)$ and $m(0)$ are not both zero, the estimate follows from the Poincaré inequality, for any $\zeta \in H^1(\Omega)$

$$\|\zeta\|^2_{L^2(\Omega)} \le C\left(\alpha|\zeta(0)|^2 + (1-\alpha)|\zeta(1)|^2 + \|\nabla\zeta\|^2_{L^2(\Omega)}\right)$$

where $\alpha = 0$ or 1. □

Theorem 4.4. *Assume $m(0) \le 0 \le m(\delta)$, and $m(\delta) \ne m(0)$. Then*

$$u^\varepsilon(x) = \psi_\infty\left(\frac{x}{\varepsilon}\right)\left(u(x) + r^\varepsilon(x)\right) \text{ and } \lambda_\varepsilon = \lambda_\infty + \varepsilon^2\lambda_0^* + o(\varepsilon^2),$$

where r^ε tends to zero weakly in $H^1(\Omega)$ and (u, λ_0^*) is the first eigenpair of the problem

$$\begin{cases} -d^*\Delta u & = & \lambda_0^* s^* u \text{ in } \Omega, \\ u \in H^1(\Omega) & \text{and} & m(0)u(0) = m(\delta)u(1) = 0, \end{cases}$$

with $s^* = \int_Y s(y)dy$.

Proof. We write (13) as

$$S^\varepsilon\left(\mu^\varepsilon s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon\right) = \phi^\varepsilon$$

Since $\mu^\varepsilon s\left(\frac{x}{\varepsilon}\right)$ is bounded in $L^\infty(\Omega)$, and ϕ^ε is normalized in $L^2(\Omega)$, we can extract a weakly converging subsequence. Since S^ε is compact, ϕ^ε converges strongly in $L^2(\Omega)$ to a limit u. Thus $\mu^\varepsilon s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon$ converges weakly to $\mu s^* u$ in $L^2(\Omega)$. The conclusion follows from Proposition 4.3. □

5. The localization regime: convergence of the eigenvalues

We now turn to the other cases, that is, either $m(0) > 0$ or $m(\delta) < 0$, or both. We shall use two auxiliary cell problems. We introduce p_δ and q_δ as the first normalized eigenfunctions (and l_p, l_q their corresponding first eigenvalues) of the following problems, posed on partial cells,

$$\begin{cases} -\mathrm{div}(d(y)\nabla p_\delta) = l_p s(y) p_\delta \text{ in } (0,\delta) \\ d(0)\nabla p_\delta(0) + m(0)p_\delta(0) = 0, \\ d(\delta)\nabla p_\delta(\delta) + m(\delta)p_\delta(\delta) = 0, \text{ and } p_\delta(0) = 1, \end{cases} \tag{20}$$

and

$$\begin{cases} -\mathrm{div}(d(y)\nabla q_\delta) = l_q s(y) q_\delta \text{ in } (\delta,1) \\ d(\delta)\nabla q_\delta(\delta) + m(\delta)q_\delta(\delta) = 0, \\ d(1)\nabla q_\delta(1) + m(1)q_\delta(1) = 0, \text{ and } q_\delta(\delta) = 1. \end{cases} \tag{21}$$

Note that both p_δ and q_δ are C^1 functions, and satisfy the uniform bounds

$$0 < C < p_\delta < C^{-1} \text{ and } 0 < C < q_\delta < C^{-1}.$$

Proposition 5.1. *The first eigenvalues μ_0, μ_δ of (18) for $t = 0, \delta$ satisfy*

$$\min(l_p, l_q) \leq \mu_0 \leq \max(l_p, l_q)\,, \quad \min(l_p, l_q) \leq \mu_\delta \leq \max(l_p, l_q),$$

and the inequalities are strict except when $l_p = l_q$.

Proof. Define a test function $w(y) = p_\delta(y)$ for $0 \leq y \leq \delta$ and $w(y) = p_\delta(\delta) q_\delta(y)$ for $\delta \leq y \leq 1$. It is easy to see that this function is C^1. We have

$$\begin{aligned}
\mu_0 &\leq \frac{1}{\int_Y s(y) w^2(y)} \left(\int_0^1 d(y)(\nabla w)^2(y) + w^2(1)m(0) - w^2(0)m(0) \right) \\
&= \frac{1}{\int_Y s(y) w^2(y)} \left(\int_0^\delta d(y)(\nabla w)^2(y) + w^2(\delta)m(\delta) - w^2(0)m(0) \right) \\
&+ \frac{1}{\int_Y s(y) w^2(y)} \left(\int_\delta^1 d(y)(\nabla w)^2(y) + w^2(1)m(0) - w^2(\delta)m(\delta) \right) \\
&= \frac{1}{\int_Y s(y) w^2(y)} \left(l_p \int_0^\delta s(y) w^2(y) + l_q \int_\delta^1 s(y) w^2(y) \right) \\
&\leq \max(l_p, l_q).
\end{aligned}$$

Alternatively

$$\begin{aligned}
\mu_0 &= \frac{1}{\int_Y s(y) \phi_0^2(y)} \left(\int_0^1 d(y)(\nabla \phi_0)^2(y) + \phi_0^2(1)m(0) - \phi_0^2(0)m(0) \right) \\
&= \frac{1}{\int_Y s(y) \phi_0^2(y)} \left(\int_0^\delta d(y)(\nabla \phi_0)^2(y) + \phi_0^2(\delta)m(\delta) - \phi_0^2(0)m(0) \right) \\
&+ \frac{1}{\int_Y s(y) \phi_0^2(y)} \left(\int_\delta^1 d(y)(\nabla \phi_0)^2(y) + \phi_0^2(1)m(0) - \phi_0^2(\delta)m(\delta) \right) \\
&\geq \frac{1}{\int_Y s(y) \phi_0^2(y)} \left(l_p \int_0^\delta s(y) \phi_0^2(y) + l_q \int_\delta^1 s(y) \phi_0^2(y) \right) \\
&\geq \min(l_p, l_q).
\end{aligned}$$

Furthermore, the inequalities above show that μ_0 is bounded from above and below by two strictly convex combinations of l_p and l_q. It implies that any inequality becomes an equality if and only if $l_p = l_q$. Indeed, if, for example, $l_p = \mu_0$, the previous inequalities imply $\mu_0 = l_q$, then if an inequality is not strict, we get immediately $l_p = l_q$.

The proof for μ_δ is similar. □

The goal of this section is to prove that $\varepsilon^2 \mu_\varepsilon$ converges to a limit which is either $\min(\mu_0, \mu_\delta)$ or $\max(\mu_0, \mu_\delta)$ depending on the sign of $l_p - l_q$.

Proposition 5.2. *Assume either $m(0) > 0$ or $m(\delta) < 0$, or both. Then, if $l_p \geq l_q$, $\varepsilon^2 \mu_\varepsilon$ is a decreasing sequence converging to a limit L given by*

$$L = \inf_{\varepsilon > 0} \varepsilon^2 \mu_\varepsilon = \max(\mu_0, \mu_\delta),$$

whereas, if $l_q \geq l_p$, then $\varepsilon^2 \mu_\varepsilon$ is an increasing sequence converging to

$$L = \sup_{\varepsilon > 0} \varepsilon^2 \mu_\varepsilon = \min(\mu_0, \mu_\delta).$$

Furthermore,

$$\left|\varepsilon^2\mu_\varepsilon - L\right| \leq C\exp\left(-\frac{C}{\varepsilon}\right).$$

Proposition 5.2 involves four parameters, namely the sign of $m(0)$, the sign of $m(\delta)$, the sign of $l_p - l_q$, and the sign of $\mu_0 - \mu_\delta$. Not all combinations of signs are possible, and in fact the sign of one of the parameters can be determined by the others. We now give a variant of Proposition 5.2, which gives the convergence of the eigenvalues without referring to l_p or l_q.

Proposition 5.3. *If $m(0) > 0$, or $m(\delta) < 0$, or both, then $\varepsilon^2\mu_\varepsilon$ converges monotonically to a limit L, and*

$$|\varepsilon^2\mu_\varepsilon - L| \leq C\exp\left(-\frac{C}{\varepsilon}\right).$$

If $m(0) > 0$ and $m(\delta) \geq 0$, then $L = \mu_0$.
If $m(0) \leq 0$ and $m(\delta) < 0$, then $L = \mu_\delta$.
If both $m(0) > 0$ and $m(\delta) < 0$, then $\varepsilon^2\mu_\varepsilon$ increases monotonically to $\min(\mu_0, \mu_\delta)$.

To prove Proposition 5.2, we rely on several lemmas, that will be proved at the end of this section.

First, we derive an upper bound when $l_q \geq l_p$, and a lower bound when $l_p \geq l_q$.

Lemma 5.4. *Suppose $m(0) > 0$, or $m(\delta) < 0$, or both.*
Then for ε small enough, $\varepsilon^2\mu_\varepsilon < -C < 0$.
If $l_q \geq l_p$, then $\varepsilon^2\mu_\varepsilon \leq \min(\mu_0, \mu_\delta)$.
If $l_p \geq l_q$, then $\varepsilon^2\mu_\varepsilon \geq \max(\mu_0, \mu_\delta)$.

Second, we make use of the dependence on n of the sequence ε. Specifically, in the following lemma we denote $\varepsilon_n = (n+\delta)^{-1}$, and $\mu_n = \mu_{\varepsilon_n}$, for all n. We derive lower and upper bounds for differences between two consecutive terms of the sequence $(\varepsilon_n^2\mu_n)$.

Lemma 5.5. *The following two lower bounds hold:*

$$\varepsilon_{n+1}^2\mu_{n+1} \geq \varepsilon_n^2\mu_n\left(1 - \kappa^1_{\varepsilon_{n+1}}\right) + \kappa^1_{\varepsilon_{n+1}}\mu_\delta, \tag{22}$$

and

$$\varepsilon_{n+1}^2\mu_{n+1} \geq \varepsilon_n^2\mu_n\left(1 - \kappa^0_{\varepsilon_{n+1}}\right) + \kappa^0_{\varepsilon_{n+1}}\mu_0, \tag{23}$$

where

$$0 < \kappa^1_\varepsilon = \frac{\int_{1-\varepsilon}^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx}{\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx} < 1, \ \textit{and}\ 0 < \kappa^0_\varepsilon = \frac{\int_0^\varepsilon s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dy}{\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx} < 1. \tag{24}$$

The following two upper bounds hold:

$$\varepsilon_{n+1}^2\mu_{n+1} \leq \varepsilon_n^2\mu_n\left(1 - \chi^1_{\varepsilon_n}\right) + \mu_\delta\chi^1_{\varepsilon_n}, \tag{25}$$

and

$$\varepsilon_{n+1}^2\mu_{n+1} \leq \varepsilon_n^2\mu_n\left(1 - \chi^0_{\varepsilon_n}\right) + \mu_0\chi^0_{\varepsilon_n}, \tag{26}$$

where

$$0<\chi_\varepsilon^0=\frac{\left(\dfrac{\phi^\varepsilon(0)}{\phi_0(1)}\right)^2\displaystyle\int_0^1 s\,(y)\,\phi_0\,(y)^2\,dy}{\dfrac{1}{\varepsilon}\displaystyle\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx+\left(\dfrac{\phi^\varepsilon(0)}{\phi_0(1)}\right)^2\displaystyle\int_0^1 s\,(y)\,\phi_0\,(y)^2\,dy}<1, \tag{27}$$

and

$$0<\chi_\varepsilon^1=\frac{\left(\dfrac{\phi^\varepsilon(1)}{\phi_\delta(\delta-1)}\right)^2\displaystyle\int_{\delta-1}^\delta s(y)\phi_\delta(y)^2dy}{\dfrac{1}{\varepsilon}\displaystyle\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx+\left(\dfrac{\phi^\varepsilon(1)}{\phi_\delta(\delta-1)}\right)^2\displaystyle\int_{\delta-1}^\delta s(y)\phi_\delta(y)^2dy}<1. \tag{28}$$

Finally, we show that lower bounds on the weights $\kappa_\varepsilon^{0,1}$ and $\chi_\varepsilon^{0,1}$ can be obtained depending on the boundary conditions $m(0)$ and $m(\delta)$.

Lemma 5.6. *The following relations hold*

$$\kappa_\varepsilon^0\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx>\varepsilon C\phi^\varepsilon(0)^2 \ \textit{and}\ \kappa_\varepsilon^1\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx>\varepsilon C\phi^\varepsilon(1)^2.$$

If $m(0)>0$ and $m(\delta)\ge 0$,

$$\phi^\varepsilon(0)^2>\frac{C}{\varepsilon}\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx. \tag{29}$$

As a consequence, $\kappa_\varepsilon^0>C>0$ and $\chi_\varepsilon^0>C>0$.

If $m(0)\le 0$ and $m(\delta)<0$,

$$\phi^\varepsilon(1)^2>\frac{C}{\varepsilon}\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx. \tag{30}$$

As a consequence, $\kappa_\varepsilon^1>C>0$ and $\chi_\varepsilon^1>C>0$.

If $m(0)>0$ and $m(\delta)<0$,

$$\phi^\varepsilon(0)^2+\phi^\varepsilon(1)^2>\frac{C}{\varepsilon}\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx. \tag{31}$$

as a consequence, $\min\left(\kappa_\varepsilon^0,\kappa_\varepsilon^1\right)>C>0$*,* $\min\left(\chi_\varepsilon^0,\chi_\varepsilon^1\right)>C>0$.

We are now in a position to prove Proposition 5.2.

Proof of Proposition 5.2. Suppose $l_p\ge l_q$. Then, Lemma 5.4 shows that $\varepsilon^2\mu_\varepsilon\ge\max(\mu_0,\mu_\delta)$. Using the upper bound $\mu_0\le\max(\mu_0,\mu_\delta)\le\varepsilon_n^2\mu_n$ in (26) yields

$$\varepsilon_{n+1}^2\mu_{n+1}\le\varepsilon_n^2\mu_n,$$

therefore the sequence $\varepsilon^2\mu_\varepsilon$ is decreasing. Now rewrite (26) under the form

$$0\le\varepsilon_{n+1}^2\mu_{n+1}-\mu_0\le\left(1-\chi_{\varepsilon_n}^0\right)\left(\varepsilon_n^2\mu_n-\mu_0\right).$$

This geometric relation implies, for $n\ge 1$, noting that $\mu_{n=0}=l_p$,

$$0\le\varepsilon_n^2\mu_n-\mu_0\le\left(1-\min_{m\le n}\chi_{\varepsilon_m}^0\right)^n(l_p-\mu_0),$$

or in other words,

$$0 \leq \varepsilon^2 \mu_\varepsilon - \mu_0 \leq C e^{-\varepsilon^{-1} \min_{\zeta \geq \varepsilon} \chi^0_\zeta}. \tag{32}$$

Similarly, using (25) instead, we obtain

$$0 \leq \varepsilon^2 \mu_\varepsilon - \mu_\delta \leq C e^{-\varepsilon^{-1} \min_{\zeta \geq \varepsilon} \chi^1_\zeta}. \tag{33}$$

Now, Lemma 5.6 says that when $m(0) > 0$, or $m(\delta) < 0$, or both, then

$$\max(\min_{\varepsilon > 0} \chi^0_\varepsilon, \min_{\varepsilon > 0} \chi^1_\varepsilon) > C > 0.$$

So at least one of inequalities (32) and (33) implies convergence of $\varepsilon^2 \mu_\varepsilon$ to either μ_0 or μ_δ, and since $\varepsilon^2 \mu_\varepsilon > \max(\mu_0, \mu_\delta)$, this in fact shows

$$0 \leq \varepsilon^2 \mu_\varepsilon - \max(\mu_0, \mu_\delta) < C \exp(-C/\varepsilon),$$

as announced.

Suppose now $l_q > l_p$. Then, Lemma 5.4 shows that $\varepsilon^2 \mu_\varepsilon < \min(\mu_0, \mu_\delta)$. Using the upper bound $\mu_0 \geq \min(\mu_0, \mu_\delta) > \varepsilon_n^2 \mu_n$ in (23) yields

$$\varepsilon_{n+1}^2 \mu_{n+1} > \varepsilon_n^2 \mu_n,$$

therefore the sequence $\varepsilon^2 \mu_\varepsilon$ is increasing. Now rewrite (23) under the form

$$0 > \varepsilon_{n+1}^2 \mu_{n+1} - \mu_0 \geq \left(1 - \kappa^0_{\varepsilon_n}\right) \left(\varepsilon_n^2 \mu_n - \mu_0\right).$$

As above this geometric relation implies

$$0 < \mu_0 - \varepsilon^2 \mu_\varepsilon \leq C e^{-\varepsilon^{-1} \min_{\zeta \geq \varepsilon} \kappa^0_\zeta}. \tag{34}$$

Similarly, using (22) instead of (23), we obtain

$$0 < \mu_\delta - \varepsilon^2 \mu_\varepsilon \leq C e^{-\varepsilon^{-1} \min_{\zeta \geq \varepsilon} \kappa^1_\zeta}. \tag{35}$$

And, again, Lemma 5.6 says that when $m(0) > 0$, or $m(\delta) < 0$, or both, at least one of the two terms $\min_{\varepsilon > 0} \kappa^0_\varepsilon$ and $\min_{\varepsilon > 0} \kappa^1_\varepsilon$ is positive. So at least one of inequalities (34) and (35) implies

$$0 < \min(\mu_0, \mu_\delta) - \varepsilon^2 \mu_\varepsilon < C \exp(-C/\varepsilon),$$

as announced. □

We now turn to the proof of the different Lemmas.

Proof of Lemma 5.5. Let us prove the two lower bounds (22) and (23). Take two successive small positive parameters $\varepsilon_{n+1} < \varepsilon_n$. Let us denote by $\phi^{n+1} = \phi^{\varepsilon_{n+1}}$ the first eigenfunction of (13) or the minimizer of (15). We make the change of variables $y = x/\varepsilon_{n+1}$ and we define

$\tilde{\phi}^{n+1}(y) = \phi^{n+1}(\varepsilon_{n+1}y)$. Recalling that $\varepsilon_{n+1} = (n+1+\delta)^{-1}$, we get

$$\varepsilon_{n+1}^2\mu_{n+1} =$$
$$= \varepsilon_{n+1}^2 \frac{\int_\Omega d(\frac{x}{\varepsilon_{n+1}})\left(\nabla\phi^{n+1}\right)^2(x)dx + \frac{1}{\varepsilon_{n+1}}\left(m(\delta)\phi^{n+1}(1)^2 - m(0)\phi^{n+1}(0)^2\right)}{\int_\Omega s(\frac{x}{\varepsilon_{n+1}})\phi^{n+1}(x)^2dx}$$
$$= \frac{\int_0^{\varepsilon_{n+1}^{-1}} d(y)\left(\nabla\tilde{\phi}^{n+1}\right)^2(y)dy + m(\delta)\tilde{\phi}^{n+1}(\varepsilon_{n+1}^{-1})^2 - m(0)\tilde{\phi}^{n+1}(0)^2}{\int_0^{\varepsilon_{n+1}^{-1}} s(y)\tilde{\phi}^{n+1}(y)^2dy}$$
$$= \frac{\int_0^{n+\delta} d(y)\left(\nabla\tilde{\phi}^{n+1}\right)^2(y)dy + m(\delta)\tilde{\phi}^{n+1}(n+\delta)^2 - m(0)\tilde{\phi}^{n+1}(0)^2}{\int_0^{n+1+\delta} s(y)\tilde{\phi}^{n+1}(y)^2dy}$$
$$+ \frac{\int_{n+\delta}^{n+1+\delta} d(y)\left(\nabla\tilde{\phi}^{n+1}\right)^2(y)dy + m(\delta)\left(\tilde{\phi}^{n+1}(n+1+\delta)^2 - \tilde{\phi}^{n+1}(n+\delta)^2\right)}{\int_0^{n+1+\delta} s(y)\tilde{\phi}^{n+1}(y)^2dy}.$$

From the minimizing properties of μ_n, we get

$$\frac{\int_0^{n+\delta} d(y)\left(\nabla\tilde{\phi}^{n+1}\right)^2(y)dy + m(\delta)\tilde{\phi}^{n+1}(n+\delta)^2 - m(0)\tilde{\phi}^{n+1}(0)^2}{\int_0^{n+1+\delta} s(y)\tilde{\phi}^{n+1}(y)^2dy}$$
$$\geq \varepsilon_n^2\mu_n \frac{\int_0^{n+\delta} s(y)\tilde{\phi}^{n+1}(y)^2dy}{\int_0^{n+1+\delta} s(y)\tilde{\phi}^{n+1}(y)^2dy}.$$

On the other hand, the segment $[n+\delta, n+1+\delta]$ is a translation of $[\delta-1,\delta]$ and from the minimizing property of μ_δ we deduce

$$\frac{\int_{n+\delta}^{n+1+\delta} d\left(\nabla\tilde{\phi}^{n+1}\right)^2dy + m(\delta)\left(\tilde{\phi}^{n+1}(n+1+\delta)^2 - \tilde{\phi}^{n+1}(n+\delta)^2\right)}{\int_0^{n+1+\delta} s\tilde{\phi}^{n+1}(y)^2dy}$$
$$\geq \mu_\delta \frac{\int_{n+\delta}^{n+1+\delta} s\tilde{\phi}^{n+1}(y)^2dy}{\int_0^{n+1+\delta} s\tilde{\phi}^{n+1}(y)^2dy}.$$

Thus we obtain the lower bound (22),

$$\varepsilon_{n+1}^2\mu_{n+1} \geq \varepsilon_n^2\mu_n(1-\kappa_{\varepsilon_{n+1}}^1) + \kappa_{\varepsilon_{n+1}}^1\mu_\delta,$$

where $\kappa_{\varepsilon_{n+1}}^1$ is defined by (24). By a symmetric argument, exchanging the two endpoints, we obtain in a similar way (23).

Let us now turn to the upper bounds. Since $\varepsilon_{n+1} < \varepsilon_n$, we define a test function

$$w^{n+1} = \begin{cases} \phi^n\left(\frac{\varepsilon_n}{\varepsilon_{n+1}}x\right) & \text{on } [0, \varepsilon_{n+1}/\varepsilon_n], \\ \frac{\phi^n(1)}{\phi_\delta(\delta-1)}\phi_\delta\left(\frac{x}{\varepsilon_{n+1}} + \delta - 1 - \frac{1}{\varepsilon_n}\right) & \text{on } [\varepsilon_{n+1}/\varepsilon_n, 1], \end{cases}$$

which is clearly continuous on Ω (it is even $C^1(\Omega)$ by further inspection). Taking w^{n+1} as a test function in the Rayleigh quotient for μ_{n+1}, and arguing as above, we deduce (25), namely,

$$\varepsilon_{n+1}^2\mu_{n+1} \leq \varepsilon_n^2\mu_n(1-\chi_{\varepsilon_n}^1) + \mu_\delta\chi_{\varepsilon_n}^1,$$

where

$$\chi^1_{\varepsilon_n} = \frac{\frac{|\phi^n(1)|^2}{|\phi_\delta(\delta-1)|^2} \int\limits_{\frac{\varepsilon_{n+1}}{\varepsilon_n}}^{1} s\left(\frac{x}{\varepsilon_{n+1}}\right) \phi^\varepsilon_\delta(x)^2 dx}{\int\limits_0^{\frac{\varepsilon_{n+1}}{\varepsilon_n}} s\left(\frac{x}{\varepsilon_{n+1}}\right) \phi^n\left(\frac{\varepsilon_n x}{\varepsilon_{n+1}}\right)^2 dx + \frac{|\phi^n(1)|^2}{|\phi_\delta(\delta-1)|^2} \int\limits_{\frac{\varepsilon_{n+1}}{\varepsilon_n}}^{1} s\left(\frac{x}{\varepsilon_{n+1}}\right) \phi^\varepsilon_\delta(x)^2 dx}$$

with $\phi^\varepsilon_\delta(x) = \phi_\delta\left(\frac{x}{\varepsilon_{n+1}} + \delta - 1 - \frac{1}{\varepsilon_n}\right)$. By the change of variables $y = x/\varepsilon_{n+1}$, we obtain that $\chi^1_{\varepsilon_n}$ is indeed given by (28).

To prove the other upper bound (26), the argument is similar, using in this case the test function

$$w^{n+1} = \begin{cases} \frac{\phi^n(0)}{\phi_0\left(\frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n}\right)} \phi_0\left(\frac{x}{\varepsilon_{n+1}}\right) & \text{on } [0, 1 - \varepsilon_{n+1}/\varepsilon_n], \\ \phi^n\left(\frac{\varepsilon_n}{\varepsilon_{n+1}} x + 1 - \frac{\varepsilon_n}{\varepsilon_{n+1}}\right) & \text{on } [1 - \varepsilon_{n+1}/\varepsilon_n, 1]. \end{cases}$$

□

Proof of Lemma 5.6. If either $m(\delta) < 0$ or $m(0) > 0$, or both, Lemma 5.4 shows that $\mu_\varepsilon < -\varepsilon^{-2} C < 0$. Integrating directly (13) we obtain, for $t \in (0,1)$,

$$d\left(\frac{t}{\varepsilon}\right) \nabla\phi^\varepsilon(t) + \frac{1}{\varepsilon} m(0)\phi^\varepsilon(0) = -\mu_\varepsilon \int\limits_0^t s\left(\frac{x}{\varepsilon}\right) \phi^\varepsilon dx$$

Dividing by $d\left(\frac{t}{\varepsilon}\right)$ and integrating again

$$\begin{aligned} &\phi^\varepsilon(t) - \phi^\varepsilon(0) + \frac{1}{\varepsilon}\left(\int\limits_0^t d^{-1}\left(\frac{\tau}{\varepsilon}\right) d\tau\right) m(0)\phi^\varepsilon(0) \\ &\qquad = -\mu_\varepsilon \int\limits_0^t d^{-1}\left(\frac{u}{\varepsilon}\right) \int\limits_0^u s\left(\frac{x}{\varepsilon}\right) \phi^\varepsilon dx du. \end{aligned} \tag{36}$$

The right-hand-side of (36) is positive because $\mu_\varepsilon < -\varepsilon^{-2} C < 0$ and $\phi^\varepsilon > 0$. If $m(0) \leq 0$, this implies that, for $0 \leq t \leq 1$,

$$\phi^\varepsilon(t) \geq \phi^\varepsilon(0).$$

On the other hand, if $m(0) > 0$, we write

$$\phi^\varepsilon(t) \geq \phi^\varepsilon(0)\left(1 - \frac{1}{\varepsilon}\left(\int\limits_0^t d^{-1}\left(\frac{u}{\varepsilon}\right) du\right) m(0)\right) \geq \phi^\varepsilon(0)\left(1 - \frac{t}{\varepsilon}\frac{m(0)}{\min d}\right),$$

which implies, for $0 \leq t \leq \frac{\varepsilon}{2}\min(\frac{m(0)}{\min d}, 1)$, that

$$\phi^\varepsilon(t) \geq \frac{1}{2}\phi^\varepsilon(0).$$

Consequently, in either case

$$\kappa^0_\varepsilon \int\limits_0^1 s\left(\frac{x}{\varepsilon}\right) \phi^\varepsilon(x)^2 dx = \int\limits_0^\varepsilon s\left(\frac{x}{\varepsilon}\right) \phi^\varepsilon(x)^2 dx \geq \varepsilon \frac{\min(s)}{4} \phi^\varepsilon(0)^2.$$

The proof of $\int_{1-\varepsilon}^{1} s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2 dx \geq C\varepsilon\phi^\varepsilon(1)^2$ is similar.

Let us now prove the lower bounds (29-31). The variational formulation of (13) with ϕ^ε as a test function yields

$$-\int_0^1 d\left(\frac{x}{\varepsilon}\right)(\nabla\phi^\varepsilon)^2 dx - \frac{1}{\varepsilon}m(\delta)\phi^\varepsilon(1)^2 + \frac{1}{\varepsilon}m(0)\phi^\varepsilon(0)^2 = -\mu_\varepsilon\int_0^1 s\left(\frac{x}{\varepsilon}\right)(\phi^\varepsilon)^2 dx.$$

Since the first term is negative and $\mu_\varepsilon < -\varepsilon^{-2}C < 0$, we deduce

$$\begin{aligned} \max\left(-m(\delta)\phi^\varepsilon(1)^2, m(0)\phi^\varepsilon(0)^2\right) &\geq -\frac{\varepsilon\mu_\varepsilon}{2}\int_0^1 s\left(\frac{x}{\varepsilon}\right)(\phi^\varepsilon)^2 dx \\ &\geq \frac{C}{\varepsilon}\int_0^1 s\left(\frac{x}{\varepsilon}\right)(\phi^\varepsilon)^2 dx. \end{aligned} \tag{37}$$

If $m(\delta) \geq 0$, and $m(0) > 0$ the maximum is $m(0)\phi^\varepsilon(0)^2$, which proves (29). Conversely, if $m(\delta) < 0$, and $m(0) \leq 0$, the maximum is $-m(\delta)\phi^\varepsilon(1)^2$, which proves (30). If $m(\delta) < 0$, and $m(0) > 0$, the maximum is attained by at least one of the points, or both, which proves (31). Finally, notice that for $i = 0, 1$,

$$\chi_\varepsilon^i = \left(\frac{\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2 dx}{\varepsilon\phi^\varepsilon(i)^2}c_i + 1\right)^{-1},$$

where c_i is a positive constant, therefore the bound (37) implies the desired lower bound on $\min(\chi_\varepsilon^0, \chi_\varepsilon^1) > C > 0$.

Finally, note that

$$\chi_\varepsilon^i < C\frac{\varepsilon\phi^\varepsilon(i)^2}{\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2 dx} \leq C\kappa_\varepsilon^i,$$

therefore $\min(\chi_\varepsilon^0, \chi_\varepsilon^1) > C > 0$ implies $\min(\kappa_\varepsilon^0, \kappa_\varepsilon^1) > C > 0$. □

Lemma 5.4 will be a consequence of the following Lemma.

Lemma 5.7. *There exist two parameters $0 < \tau_\varepsilon^0 < 1$ and $0 < \tilde{\kappa}_\varepsilon^1 < \kappa_\varepsilon^1 < 1$ such that*

$$\mu_0\left(1 - \tilde{\kappa}_\varepsilon^1\right) + l_p\tilde{\kappa}_\varepsilon^1 \leq \varepsilon^2\mu_\varepsilon \leq \mu_0\left(1 - \tau_\varepsilon^0\right) + l_p\tau_\varepsilon^0. \tag{38}$$

Similarly, there exist two parameters $0 < \tau_\varepsilon^\delta < 1$ and $0 < \tilde{\kappa}_\varepsilon^0 < \kappa_\varepsilon^0 < 1$ such that

$$\mu_\delta\left(1 - \kappa_\varepsilon^0\right) + l_p\kappa_\varepsilon^0 \leq \varepsilon^2\mu_\varepsilon \leq \mu_\delta\left(1 - \tau_\varepsilon^\delta\right) + l_p\tau_\varepsilon^\delta. \tag{39}$$

This allows to prove Lemma 5.4.

Proof of Lemma 5.4. Proposition 5.1 implies that $\min(l_p, l_q) \leq \mu_0, \mu_\delta \leq \max(l_p, l_q)$.

If $l_p \leq l_q$, then the upper bound in (38) shows that $\varepsilon^2\mu_\varepsilon \leq \mu_0$, whereas the upper bound in (39) shows that $\varepsilon^2\mu_\varepsilon \leq \mu_\delta$. Thus, $\varepsilon^2\mu_\varepsilon \leq \min(\mu_0, \mu_\delta) < 0$ by virtue of Proposition 3.6.

Symmetrically if $l_p \geq l_q$ using the lower bounds in (38) and (39) we obtain $\varepsilon^2\mu_\varepsilon \geq \max(\mu_0, \mu_\delta)$.

Finally, let us show that $\varepsilon^2\mu_\varepsilon < -C < 0$ for ε small enough. Suppose $m(0) > 0$. Choosing as a test function $\exp(-\alpha x/\varepsilon)$ with $\alpha > 0$, in the Rayleigh quotient (15) defining μ_ε, we obtain

$$\begin{aligned} \mu_\varepsilon &\leq \frac{\dfrac{\alpha^2}{\varepsilon^2}\displaystyle\int_0^1 d\left(\frac{x}{\varepsilon}\right)\exp(-2\alpha x/\varepsilon)dx - \frac{1}{\varepsilon}m(0) + \frac{1}{\varepsilon}m(\delta)\exp(-2\alpha/\varepsilon)}{\int_0^1 s\left(\frac{x}{\varepsilon}\right)\exp(-2\alpha x/\varepsilon)dx}, \\ &\leq \frac{1}{\varepsilon^2}\frac{\alpha\max(d)/2 - m(0) + m(\delta)\exp(-2\alpha/\varepsilon)}{\min(s)(1-\exp(-2\alpha/\varepsilon))}, \end{aligned}$$

Pick for example $\alpha = m(0)/\max(d)$, to obtain $\mu_\varepsilon \leq \frac{-m(0)}{2\varepsilon^2\min(s)}(1 + C\exp(-C/\varepsilon))$, which shows that $\varepsilon^2\mu_\varepsilon < -C < 0$ for ε small enough. The argument is similar for $m(\delta) < 0$, choosing instead a test function $\exp(-\alpha(1-x)/\varepsilon)$ with $\alpha > 0$. $\square$

Proof of Lemma 5.7. Let us focus on the proof of the first bound (38). To obtain an upper bound, we construct a continuous (actually C^1) test function for the Rayleigh quotient (15) as follows. Recall that $\varepsilon^{-1} = n + \delta$, so that $\varepsilon^{-1} - 1 < n < \varepsilon^{-1}$ and $n\varepsilon \leq x \leq 1 \Leftrightarrow 0 \leq (x - n\varepsilon)\varepsilon^{-1} \leq \delta$. We define w^ε as

$$w^\varepsilon(x) = \begin{cases} \phi_0\left(\frac{x}{\varepsilon}\right) & \text{for } 0 \leq x \leq n\varepsilon, \\ \phi_0(n)p_\delta\left(\frac{x-n\varepsilon}{\varepsilon}\right) & \text{for } n\varepsilon \leq x \leq 1. \end{cases}$$

Recall that, by virtue of Proposition 3.6, ϕ_0 is equal to an exponential-periodic function φ_{θ_0} and thus is defined everywhere in $\mathbb{R}$ and not only on the interval $(0,1)$. By construction, w^ε

is continuous and we can use it as a test function in (15) to obtain

$$
\begin{aligned}
\mu^\varepsilon &\leq \frac{\displaystyle\int_0^{n\varepsilon} d\left(\frac{x}{\varepsilon}\right)|\nabla w^\varepsilon|^2\,dx + \frac{1}{\varepsilon}\left(m(0)\,|w^\varepsilon(n\varepsilon)|^2 - m(0)\,|w^\varepsilon(0|^2\right)}{\displaystyle\int_0^1 s\left(\frac{x}{\varepsilon}\right) w^\varepsilon(x)^2 dx} \\
&+ \frac{\displaystyle\int_{n\varepsilon}^{1} d\left(\frac{x}{\varepsilon}\right)|\nabla w^\varepsilon|^2\,dx + \frac{1}{\varepsilon}\left(m(\delta)\,|w^\varepsilon(1)|^2 - m(0)\,|w^\varepsilon(n\varepsilon)|^2\right)}{\displaystyle\int_0^1 s\left(\frac{x}{\varepsilon}\right) w^\varepsilon(x)^2 dx} \\
&\leq \frac{\displaystyle\varepsilon^{-2}\mu_0 \int_0^{n\varepsilon} s\left(\frac{x}{\varepsilon}\right)\phi_0\left(\frac{x}{\varepsilon}\right)^2 dx + \varepsilon^{-2} l_p \phi_0(n)^2 \int_{n\varepsilon}^1 s\left(\frac{x}{\varepsilon}\right) p_\delta\left(\frac{x-n\varepsilon}{\varepsilon}\right)^2 dx}{\displaystyle\int_0^1 s\left(\frac{x}{\varepsilon}\right) w^\varepsilon(x)^2 dx} \\
&\leq \varepsilon^{-2}\mu_0\left(1-\tau_\varepsilon^0\right) + \varepsilon^{-2} l_p \tau_\varepsilon^0,
\end{aligned}
$$

where, using the change of variables $y = (x - n\varepsilon)/\varepsilon$, we defined

$$
\tau_\varepsilon^0 = \phi_0(n)^2 \frac{\varepsilon \int_0^\delta s(y) p_\delta(y)^2 dy}{\int_0^1 s\left(\frac{x}{\varepsilon}\right) w^\varepsilon(x)^2 dx}.
$$

Let us now turn to the lower bound in (38). The idea is to get a lower bound in the Rayleigh quotient (15), using the fact that μ_0 and l_p are themselves given as minima of Rayleigh quotients. In (38) the coefficient $\tilde{\kappa}_\varepsilon^1$ is going to be defined by

$$
\tilde{\kappa}_\varepsilon^1 = \frac{\int_{1-\delta\varepsilon}^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2 dx}{\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2 dx}. \tag{40}
$$

Indeed,

$$\begin{aligned}
&\int_0^1 d\left(\frac{x}{\varepsilon}\right)|\nabla\phi^\varepsilon|^2 dx + \frac{1}{\varepsilon}\Big(m(\delta)\,(\phi^\varepsilon(1))^2 - m(0)\,(\phi^\varepsilon(\,0))^2\Big)\\
= &\int_0^{n\varepsilon} d\left(\frac{x}{\varepsilon}\right)|\nabla\phi^\varepsilon|^2 dx + \frac{1}{\varepsilon}\Big(m(0)\,(\phi^\varepsilon(n\varepsilon))^2 - m(0)\,(\phi^\varepsilon(0))^2\Big)\\
+ &\int_{n\varepsilon}^{n\varepsilon+\delta\varepsilon} d\left(\frac{x}{\varepsilon}\right)|\nabla\phi^\varepsilon|^2 dx + \frac{1}{\varepsilon}\Big(m(\delta)\,(\phi^\varepsilon(n\varepsilon+\delta\varepsilon))^2 - m(0)\,(\phi^\varepsilon(n\varepsilon))^2\Big)\\
\geq\ &\varepsilon^{-2}\mu_0\int_0^{n\varepsilon} s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2 dx + \varepsilon^{-2} l_p \int_{n\varepsilon}^{n\varepsilon+\delta\varepsilon} s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2 dx,
\end{aligned}$$

thanks to the minimizing properties of μ_0 and l_p. So, altogether,

$$\begin{aligned}
\mu_\varepsilon &= \frac{\displaystyle\int_0^1 d\left(\frac{x}{\varepsilon}\right)|\nabla\phi^\varepsilon|^2 dx + \frac{1}{\varepsilon}\Big(m(\delta)\,(\phi^\varepsilon)^2(1) - m(0)\,(\phi^\varepsilon)^2(0)\Big)}{\displaystyle\int_0^1 s\left(\frac{x}{\varepsilon}\right)(\phi^\varepsilon)^2(x)dx}\\
&\geq \varepsilon^{-2}((1-\tilde\kappa_\varepsilon^1)\mu_0 + \tilde\kappa_\varepsilon^1 l_p).
\end{aligned}$$

The proof of the inequalities (39), involving μ_δ, is similar. We use instead

$$w^\varepsilon(x) = \begin{cases} p_\delta\left(\frac{x}{\varepsilon}\right) & \text{for } 0\leq x\leq \delta\varepsilon,\\ p_\delta(\delta)\phi_\delta\left(\frac{x-\delta\varepsilon}{\varepsilon}\right) & \text{for } \delta\varepsilon\leq x\leq 1.\end{cases}$$

□

Proof of Proposition 5.3. The fact that the convergence is exponential in all cases is already established in Proposition 5.2. When $m(0)>0$ and $m(\delta)\geq 0$, let us check that the limit of μ_ε is always μ_0. In the course of the proof of Proposition 5.2, we have established (32) and (34) which prove that the limit is μ_0 if either $\min_{\varepsilon>0}\kappa_\varepsilon^0$ or $\min_{\varepsilon>0}\chi_\varepsilon^0$ is positive. Lemma 5.6 provides such a result when $m(0)>0$ and $m(\delta)\geq 0$.

The case $m(0)\leq 0$ and $m(\delta)<0$ is handled by similar arguments using (33) and (35).

If $m(0)>0$ and $m(\delta)<0$, we have

$$l_q = \min_{\phi\in H^1(\delta,1)} \frac{\int_\delta^1 d(y)|\nabla\phi|^2 dy + \big(m(0)\phi(1)^2 - m(\delta)\phi(\delta)^2\big)}{\int_\delta^1 s(y)\phi^2 dy} \geq 0.$$

From Proposition 5.1, $\min(l_p,l_q)\leq\mu_0<0$, therefore $l_p<0<l_q$. Then Proposition 5.2 shows that $\varepsilon^2\mu_\varepsilon$ is an increasing sequence converging to $\min(\mu_0,\mu_\delta)$. □

6. The localization regime: a corrector result

In this section, we show that, in the self adjoint case, the first eigenfunction must localize at one of the end-points when, either $m(0) > 0$ or $m(\delta) < 0$, or both. More precisely, if $\mu_0 \neq \mu_\delta$, then localization occurs at only one end point. On the other hand, if $\mu_0 = \mu_\delta$, then two cases can happen: when $m(0)m(\delta) < 0$ localization takes place at both endpoints, while, when $m(0)m(\delta) > 0$ the first eigenfunction can be computed exactly and localization occurs at only one end point.

We start with this last case which is peculiar because it is equivalent to $\phi_0 = \phi_\delta$ – up to a renormalization.

Proposition 6.1. *If $\mu_0 = \mu_\delta$ and $m(\delta)m(0) > 0$, then $\phi_0 = \frac{\phi_\delta}{\phi_\delta(-1)}$, and we have the exact relation*

$$\mu_\varepsilon = \mu_0, \text{ and } u^\varepsilon(x) = \psi_\infty\left(\frac{x}{\varepsilon}\right)\frac{\phi_0\left(\frac{x}{\varepsilon}\right)}{\phi_0(0)}.$$

Conversely, if $\phi_0 = \frac{\phi_\delta}{\phi_\delta(-1)}$ then $\mu_0 = \mu_\delta$ and $m(\delta)m(0) > 0$.

Remark 6.2. *Proposition 6.1 is very similar to Proposition 2.3 when the two Neumann eigenfunctions coincide $u_N^0 = \frac{u_N^\delta}{u_N^\delta(-1)}$.*

Proof. Recall that, in view of Proposition 3.6, ϕ_0 and ϕ_δ are exponential-periodic functions, namely $\phi_0 = \varphi^0_{\theta_0}$ and $\phi_\delta = \varphi^\delta_{\theta_\delta}$. Since $\mu_0 = \mu_\delta$ they are also solutions of the same equation,

$$-\mathrm{div}_y\left(d(y)\nabla_y\phi\right) = \mu_0 s(y)\phi \text{ in } \mathbb{R}.$$

If $m(\delta)$ and $m(0)$ have the same sign, then the exponent θ_0 and θ_δ have the same sign too. But the maps $\theta \to \varphi^t_\theta$ and $\theta \to \nu_\theta$, where $(\nu_\theta, \varphi_\theta)$ is the solution of the spectral problem (16) are one-to-one when restricted to $\theta \in \mathbb{R}^+$ or $\theta \in \mathbb{R}^-$. Thus, it implies that $\phi_0 = \frac{\phi_\delta}{\phi_\delta(-1)}$. This implies in turn that $x \to \phi_0\left(\frac{x}{\varepsilon}\right)$ is positive, and satisfies both $\nabla\phi_0(0) = -\frac{1}{\varepsilon}m(0)\phi_0(0)$ and $\nabla\phi_0(1/\varepsilon) = -\frac{1}{\varepsilon}m(\delta)\phi_0(1/\varepsilon)$, i.e., it is the first eigensolution of problem (13) and then is equal to ϕ^ε after a renormalization. □

To handle the other cases, we shall now make full use of the one-dimensional nature of the problem. Notice that problem (13) can be viewed as a linear second order ordinary differential equation, thus ϕ^ε is a combination of any two other linear independent solutions of (13) with different boundary conditions.

We first need the following lemmas.

Lemma 6.3. *Assume $m(0) > 0$ or $m(\delta) < 0$, or both. Then, there exists $\theta_\varepsilon \neq 0$ such that $\mu_\varepsilon = \nu_{\theta_\varepsilon} = \nu_{-\theta_\varepsilon}$ where ν_θ is the first eigenvalue of (16).*

Proof. According to Lemma 5.4 we have $\mu_\varepsilon < 0$ since either $m(0) > 0$ or $m(\delta) < 0$. Proposition 2.1, applied to the selfadjoint case (16), tells us that $(\max_\theta \nu_\theta) = \nu_0 = 0$ and thus the range of ν_θ is $\mathbb{R}^-$. Therefore, there exists $\theta_\varepsilon \neq 0$ such that $\mu_\varepsilon = \nu_{\theta_\varepsilon} = \nu_{-\theta_\varepsilon}$. □

Lemma 6.4. *Suppose $m(0) > 0$ or $m(\delta) < 0$, or both. Then,*

$$\phi^\varepsilon(0) + \phi^\varepsilon(1) \leq \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)}.$$

Remark 6.5. *Note that in the case of constant coefficients, $\phi_0(\cdot/\varepsilon)$ would be the form $\exp(-B\cdot/\varepsilon)$, and $\|\exp(-B\cdot/\varepsilon)\|_{L^1(\Omega)} \leq \varepsilon/B$, so in this sense this estimate is sharp.*

Proof. Integrating by part (13) against $D_\varepsilon(x) = \int_0^x d\left(\frac{z}{\varepsilon}\right)^{-1} dz$ shows that

$$\begin{aligned}\varepsilon\mu_\varepsilon \int_0^1 s\left(\frac{x}{\varepsilon}\right) D_\varepsilon(x)\phi^\varepsilon dx &= m(\delta)\phi^\varepsilon(1)D_\varepsilon(1) + \varepsilon\int_0^1 \nabla\phi^\varepsilon dx \\ &= m(\delta)\phi^\varepsilon(1)D_\varepsilon(1) + \varepsilon(\phi^\varepsilon(1) - \phi^\varepsilon(0)),\end{aligned}$$

since the left hand side is negative and since $-C' < \varepsilon^2\mu_\varepsilon < -C$ by Proposition 5.2, we obtain

$$0 \le (-m(\delta)\int_0^1 d\left(\frac{x}{\varepsilon}\right)^{-1} dx - \varepsilon)\phi^\varepsilon(1) + \varepsilon\phi^\varepsilon(0)) \le \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)},$$

thus, if $m(\delta) < 0$, $\phi^\varepsilon(1) \le \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)}$. Symmetrically, integrating by part (13) against $D_\varepsilon(x) = \int_x^1 d\left(\frac{z}{\varepsilon}\right)^{-1} dz$ we obtain

$$\varepsilon\mu_\varepsilon \int_0^1 s\left(\frac{x}{\varepsilon}\right) D_\varepsilon(x)\phi^\varepsilon dx = -m(0)\phi^\varepsilon(0)D_\varepsilon(0) - \varepsilon\int_0^1 \nabla\phi^\varepsilon dx,$$

so, if $m(0) > 0$, we deduce $\phi^\varepsilon(0) \le \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)}$. Therefore, when either $m(0) > 0$ or $m(\delta) < 0$, or both, we obtain

$$\phi^\varepsilon(0) + \phi^\varepsilon(1) \le \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)},$$

□

Lemma 6.6. *The first eigencouple $(\nu_\theta, \varphi^t_\theta)$ of (16) is real analytic as function of $\theta \in \mathbb{R}$ with values in $\mathbb{R} \times L^2(Y)$. If the sequence θ_ε, defined in Lemma 6.3, converges to a limit θ_t, then the eigenfunction $\varphi^t_{\theta_\varepsilon}$ can be expanded as follows*

$$||\varphi^t_{\theta_\varepsilon}(y) - \varphi^t_{\theta_t}(y) - (\theta_\varepsilon - \theta_t)v_{\theta_t}(y)||_{L^\infty(Y)} = O((\theta_\varepsilon - \theta_t)^2) \tag{41}$$

where the function $v_{\theta_t} \in L^2(Y)$ is defined by (44), and

$$d(t-1)\nabla v_{\theta_t}(t-1) + m(t-1)v_{\theta_t}(t-1) \neq 0. \tag{42}$$

Remark 6.7. *Recall that, according to Proposition 3.6, $\phi_t = \varphi^t_{\theta_t}$.*

Proof. The analyticity property is well-known by changing the unknown φ^t_θ into $\tilde{\varphi}^t_\theta = e^{-\theta y}\varphi^t_\theta$ which is a 1-periodic function, defined in a space independent of θ, satisfying an elliptic equation with coefficients that depend quadratically on θ. The variational formulation for $\tilde{\varphi}^t_\theta$ is

$$\int_Y d(y)(\nabla\tilde{\varphi}^t_\theta + \theta\tilde{\varphi}^t_\theta)(\nabla\tilde{\phi} - \theta\tilde{\phi}) = \nu_\theta \int_Y s(y)\tilde{\varphi}^t_\theta\tilde{\phi}, \tag{43}$$

for any 1-periodic test function $\tilde{\phi} \in H^1(Y)$. We conclude using Kato's Theorem [9] to prove the analyticity of the eigenvector $\tilde{\varphi}^t_\theta$. Since $\varphi^t_\theta = e^{\theta y}\tilde{\varphi}^t_\theta$, (41) holds in the L^∞ norm by Sobolev embedding. To characterize the function v_{θ_t} we differentiate (43) with respect to θ and obtain for the value θ_t

$$\int_Y d(y)(\nabla\tilde{v}_{\theta_t}+\theta_t\tilde{v}_{\theta_t})(\nabla\tilde{\phi}-\theta_t\tilde{\phi})+\int_Y [d(y)\tilde{\varphi}^t_{\theta_t}(\nabla\tilde{\phi}-\theta_t\tilde{\phi})-d(y)(\nabla\tilde{\varphi}^t_{\theta_t}+\theta_t\tilde{\varphi}^t_{\theta_t})\tilde{\phi}]$$
$$=\frac{d\nu}{d\theta}(\theta_t)\int_Y s(y)\tilde{\varphi}^t_{\theta_t}\tilde{\phi}+\nu_{\theta_t}\int_Y s(y)\tilde{v}_{\theta_t}\tilde{\phi}.$$

Introducing the test function $\phi=e^{-\theta_t y}\tilde{\phi}$ and defining $v_{\theta_t}=e^{-\theta_t y}\tilde{v}_{\theta_t}$ we deduce

$$\int_Y d(y)\nabla v_{\theta_t}\nabla\phi+\int_Y [d(y)\varphi^t_{\theta_t}\nabla\phi-d(y)\nabla\varphi^t_{\theta_t}\phi]=\frac{d\nu}{d\theta}(\theta_t)\int_Y s(y)\varphi^t_{\theta_t}\phi+\nu_{\theta_t}\int_Y s(y)v_{\theta_t}\phi. \tag{44}$$

To prove (42), we argue by contradiction. Assume $d(t-1)\nabla v_{\theta_t}(t-1)+m(t-1)v_{\theta_t}(t-1)=0$. Since $v_{\theta_t}(t-1)=0$, it implies that $\nabla v_{\theta_t}(t-1)=0$. As a consequence, the 1-periodic function $\tilde{v}_{\theta_t}=e^{-\theta_t y}v_{\theta_t}$ satisfies the following boundary conditions

$$\tilde{v}_{\theta_t}(t-1)=\tilde{v}_{\theta_t}(t)=0 \quad \text{and} \quad \nabla\tilde{v}_{\theta_t}(t-1)=\nabla\tilde{v}_{\theta_t}(t)=0.$$

Returning back to the function v_{θ_t} we deduce

$$v_{\theta_t}(t-1)=v_{\theta_t}(t)=0 \quad \text{and} \quad \nabla v_{\theta_t}(t-1)=\nabla v_{\theta_t}(t)=0.$$

In other words, v_{θ_t} is solution of the over-determined boundary value problem

$$\begin{cases} -\mathrm{div}_y\,(d(y)\nabla_y v_{\theta_t})-\nu_{\theta_t}s(y)v_{\theta_t}=\mathrm{div}_y(d(y)\varphi^t_{\theta_t})+d(y)\nabla_y\varphi^t_{\theta_t}+\frac{d\nu}{d\theta}(\theta_t)s(y)\varphi^t_{\theta_t} \\ v_{\theta_t}(t-1)=v_{\theta_t}(t)=0 \\ \nabla_y v_{\theta_t}(t-1)=\nabla_y v_{\theta_t}(t)=0 \end{cases}$$

Multiplying the above equation by $\varphi^t_{\theta_t}$, integrating two times by parts (without any boundary contribution) and using the spectral equation satisfied by $\varphi^t_{\theta_t}$, we deduce

$$\frac{d\nu}{d\theta}(\theta_t)\int_Y s(y)|\varphi^t_{\theta_t}|^2=0, \qquad \text{that is,} \qquad \frac{d\nu}{d\theta}(\theta_t)=0,$$

which leads to a contradiction since $\theta\to\nu(\theta)$ is strictly concave and the only root of $\frac{d\nu}{d\theta}(\theta)=0$ is $\theta=0$. □

We are now in a position to evaluate how close the solution ϕ^ε is to a linear combination of $\varphi_{\pm\theta_\varepsilon}$. Recall that Proposition 5.3 implies that the only possible limits of the sequence θ_ε is θ_0 or θ_δ.

Proposition 6.8. *Suppose $m(0)>0$, or $m(\delta)<0$, or both.*

(1) *If $\mu_0\neq\mu_\delta$ and $\theta_\varepsilon\to\theta_0$, we have*

$$\left|\phi^\varepsilon(x)-\left(\frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right)-\frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}e^{2\theta_\varepsilon n}K(\delta)\varphi^0_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right)\right)\right|\leq\frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)}e^{2\theta_0/\varepsilon}$$

and

$$|\mu_\varepsilon-\mu_0|=\gamma_0 e^{2\theta_0/\varepsilon}(1+o(1)), \tag{45}$$

with

$$\gamma_0 = \left| \left(\frac{K(\delta)}{k(0)} \right) \frac{d(0)\nabla\varphi^0_{-\theta_0}(0)\phi_0(0) + m(0)\varphi^0_{-\theta_0}(0)\phi_0(0)}{\int\limits_Y s(y)\phi_0\varphi^0_{-\theta_0}} \right|. \tag{46}$$

(2) *If* $\mu_0 \neq \mu_\delta$ *and* $\theta_\varepsilon \to \theta_\delta$, *we have*

$$\left| \phi^\varepsilon(x) - \left(\frac{\phi^\varepsilon(1)}{\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)} \varphi^\delta_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(1)}{\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)} K(0)\varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) \right) \right| \leq \frac{C}{\varepsilon} \|\phi_\varepsilon\|_{L^1(\Omega)} e^{-2\theta_\delta/\varepsilon},$$

and

$$|\mu_\varepsilon - \mu_\delta| = \gamma_1 e^{-2\theta_\delta/\varepsilon}(1 + o(1)). \tag{47}$$

with

$$\gamma_1 = \left| \left(\frac{K(0)}{k(\delta)} \right) \frac{d(\delta)\nabla\varphi^\delta_{-\theta_\delta}(\delta)\phi_\delta(\delta) + m(\delta)\varphi^\delta_{-\theta_\delta}(\delta)\phi_\delta(\delta)}{\int\limits_Y s(y)\phi_\delta\varphi^\delta_{-\theta_\delta}} \right|. \tag{48}$$

(3) *If* $\mu_0 = \mu_\delta$, *we have*

$$\left| \phi^\varepsilon(x) - \left(\frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)} \varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)} c_\delta e^{\theta_\varepsilon n} \varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) \right) \right| \leq \frac{C}{\varepsilon} \|\phi_\varepsilon\|_{L^1(\Omega)} e^{\theta_0/\varepsilon},$$

with

$$c_\delta = -\sqrt{-\left(\frac{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta)} \right) \left(\frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)} \right)} \tag{49}$$

and

$$|\mu_\varepsilon - \mu_0| = \gamma_\delta e^{\theta_0/\varepsilon}(1 + o(1)), \tag{50}$$

with

$$\gamma_\delta = \left| \sqrt{-\frac{k(\delta)}{k(0)}} \frac{d(0)\nabla\phi_\delta(0)\phi_0(0) + m(0)\phi_\delta(0)\phi_0(0)}{\int\limits_Y s(y)\phi_0\phi_\delta} \right|. \tag{51}$$

We used the following notations

$$K(\delta) = \frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(\delta)\nabla\varphi^0_{-\theta_0}(\delta) + m(\delta)\varphi^0_{-\theta_0}(\delta)},\ K(0) = \frac{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}{d(0)\nabla\varphi^\delta_{-\theta_\delta}(0) + m(0)\varphi^\delta_{-\theta_\delta}(0)},$$

and

$$k(0) = \frac{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}{d(0)\nabla\varphi_{-\theta_0}(0) + m(0)\varphi_{-\theta_0}(0)} \quad k(\delta) = \frac{d(\delta)\nabla\varphi^\delta_{-\theta_\delta}(\delta) + m(\delta)\varphi^\delta_{-\theta_\delta}(\delta)}{d(\delta)\nabla v_{-\theta_\delta}(\delta) + m(\delta)v_{-\theta_\delta}(\delta)}.$$

Proposition 6.8 provides a detailed description of the first order correctors for the first eigenpair. The following corollary limit the results of Proposition 6.8 to the leading order term. This highlights the main trend of the first eigenvectors, at the cost of an exponentially small loss of accuracy. The case when a double localization occurs is a limit case when zero and first order terms are of the same strength. In that case, characterizing the main trend means calculating first order correctors.

Corollary 6.9. *Suppose $\theta_0 > 0$, or $\theta_\delta < 0$, or both. Let $\tilde{\varphi}^t_\theta$ be the positive, bounded and Y-periodic function given by $\tilde{\varphi}^t_\theta = e^{-\theta y}\varphi^t_\theta$ where φ^t_θ is the first eigenfunction of (16).*

(1) *If $\mu_0 \neq \mu_\delta$, the first eigenvector localize in one of the endpoints. Indeed when $\theta_0 < 0$ and either $\theta_\delta \leq 0$, or $\theta_\delta > 0$ and $\mu_0 < \mu_\delta$, we have $L = \mu_0$,*

$$\left\| \phi^\varepsilon(x) - \phi^\varepsilon(0) e^{\frac{\theta_0 x}{\varepsilon}} \frac{\tilde{\varphi}^0_{\theta_0}\left(\frac{x}{\varepsilon}\right)}{\tilde{\varphi}^0_{\theta_0}(0)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{\theta_0/\varepsilon} \|\phi^\varepsilon\|_{L^1(\Omega)}$$

and $|\mu_\varepsilon - \mu_0| = \gamma_0 e^{2\theta_0/\varepsilon}(1 + o(1))$ where γ_0 is defined by (46).

Alternatively when $\theta_\delta > 0$ and either $\theta_0 \geq 0$, or $\theta_0 < 0$ and $\mu_\delta < \mu_0$, we have $L = \mu_\delta$,

$$\left\| \phi^\varepsilon(x) - \phi^\varepsilon(1) e^{\frac{\theta_\delta (x-1)}{\varepsilon}} \frac{\tilde{\varphi}^\delta_{\theta_\delta}\left(\frac{x}{\varepsilon}\right)}{\tilde{\varphi}^\delta_{\theta_\delta}(\delta)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{-\theta_\delta/\varepsilon} \|\phi^\varepsilon\|_{L^1(\Omega)}$$

and $|\mu_\varepsilon - \mu_\delta| = \gamma_1 e^{-2\theta_\delta/\varepsilon}(1 + o(1))$ where γ_1 is defined by (48).

(2) *If $\mu_0 = \mu_\delta$, then the eigenvector could mix both boundary layers. We obtain*

$$\begin{aligned} & \left\| \phi^\varepsilon(x) - \frac{\phi^\varepsilon(0)}{\tilde{\varphi}^0_{\theta_0}(0)} e^{\frac{\theta_0 x}{\varepsilon}} \tilde{\varphi}^0_{\theta_0}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(0)}{\tilde{\varphi}^0_{\theta_0}(0)} c_\delta e^{\frac{\theta_0}{\varepsilon}} e^{\frac{\theta_\delta x}{\varepsilon}} \tilde{\varphi}^\delta_{\theta_\delta}\left(\frac{x}{\varepsilon}\right) \right\|_{L^\infty(\Omega)} \\ = & \left\| \phi^\varepsilon(x) - \frac{\phi^\varepsilon(0)}{\tilde{\varphi}^0_{\theta_0}(0)} e^{\frac{\theta_0 x}{\varepsilon}} \tilde{\varphi}^0_{\theta_0}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(0)}{\tilde{\varphi}^0_{\theta_0}(0)} c_\delta e^{\frac{\theta_\delta (x-1)}{\varepsilon}} \tilde{\varphi}^\delta_{\theta_\delta}\left(\frac{x}{\varepsilon}\right) \right\|_{L^\infty(\Omega)} \\ \leq & \; C \frac{e^{\theta_0/\varepsilon}}{\varepsilon} \|\phi^\varepsilon\|_{L^1(\Omega)} \end{aligned}$$

and $|\mu_\varepsilon - \mu_0| = \gamma_\delta e^{\theta_0/\varepsilon}(1 + o(1))$ where γ_δ is defined by (51) and c_δ is defined by (49). Note that in this last case $\theta_0 = -\theta_\delta < 0$.

Proof of Corollary 6.9. To prove this corollary starting from Proposition 6.8, we notice that, when by Proposition 5.3, if $\theta_0 < 0$ and $\theta_\delta \leq 0$, or if $\theta_0 < 0$, $\theta_\delta > 0$ and $\mu_0 < \mu_\delta$, we have $L = \mu_0$, θ_ε tends to $\theta_0 < 0$ and that $\theta_\varepsilon - \theta_0 = O(e^{2\theta_0/\varepsilon})$. This implies that

$$\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) = \phi_0\left(\frac{x}{\varepsilon}\right) + O(e^{2\theta_0/\varepsilon}), \quad {\varphi^0}^{-1}_{\theta_\varepsilon} = \phi_0^{-1} + O(e^{\theta_0/\varepsilon})$$

and $e^{2\theta_\varepsilon n} K(\delta) \varphi^0_{-\theta_\varepsilon}\left(\dfrac{x}{\varepsilon}\right) = O(e^{\theta_0/\varepsilon})$. Since

$$\frac{\phi_0\left(\frac{x}{\varepsilon}\right)}{\phi_0(0)} = e^{\frac{\theta_0 x}{\varepsilon}} \frac{\tilde{\varphi}^0_{\theta_0}\left(\frac{x}{\varepsilon}\right)}{\tilde{\varphi}^0_{\theta_0}(0)}, \tag{52}$$

we have proved the first estimate.

In a same way, when either $\theta_0 \geq 0$ and $\theta_\delta > 0$, or $\theta_0 < 0$, $\theta_\delta > 0$ and $\mu_\delta < \mu_0$, by Proposition 5.3, $L = \mu_\delta$, θ_ε tends to $\theta_\delta > 0$, and $\theta_\varepsilon - \theta_\delta = O(e^{-2\theta_\delta/\varepsilon})$. We then obtain

$$\varphi^\delta_{\theta_\varepsilon} = \phi_\delta + O(e^{-2\theta_\delta/\varepsilon}), \quad {\varphi^\delta}^{-1}_{\theta_\varepsilon} = \phi_\delta^{-1} + O(e^{-\theta_\delta/\varepsilon})$$

and $\dfrac{K(0)}{\varphi^{\delta}_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)}\varphi^{\delta}_{-\theta_\varepsilon}\left(\dfrac{x}{\varepsilon}\right) = O(e^{-\theta_\delta/\varepsilon})$. As before, we write

$$\frac{\phi_\delta\left(\frac{x}{\varepsilon}\right)}{\phi_\delta\left(\frac{1}{\varepsilon}\right)} = e^{\frac{\theta_\delta(x-1)}{\varepsilon}}\frac{\tilde{\varphi}^{\delta}_{\theta_\delta}\left(\frac{x}{\varepsilon}\right)}{\tilde{\varphi}^{\delta}_{\theta_\delta}\left(\frac{1}{\varepsilon}\right)} = e^{\frac{\theta_\delta(x-1)}{\varepsilon}}\frac{\tilde{\varphi}^{\delta}_{\theta_\delta}\left(\frac{x}{\varepsilon}\right)}{\tilde{\varphi}^{\delta}_{\theta_\delta}(\delta)}. \tag{53}$$

Finally, when $\mu_0 = \mu_\delta$, $m(0) > 0$ and $m(\delta) < 0$, θ_ε tends to $\theta_0 < 0$ and $\theta_\varepsilon - \theta_0 = O(e^{\theta_0/\varepsilon})$. This implies that $\exp(\theta_\varepsilon n) = \exp(\theta_0 n)(1 + o(1))$, and therefore that

$$\varphi^{0}_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) = \phi_0\left(\frac{x}{\varepsilon}\right) + O(e^{\theta_0/\varepsilon}) \quad \text{and} \quad \varphi^{\delta}_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) = \phi_\delta\left(\frac{x}{\varepsilon}\right) + O(e^{\theta_0/\varepsilon}).$$

Together with the observation that

$$\varphi^{0}_{\theta_\varepsilon}(0) = \phi_0(0)(1 + O(e^{2\theta_0/\varepsilon})),$$

this shows that

$$\frac{1}{\varphi^{0}_{\theta_\varepsilon}(0)} = \frac{1}{\phi_0(0)} + O(e^{2\theta_0/\varepsilon}).$$

This allows us to conclude. □

Proof of Proposition 6.8. Since $\varphi^{t_1}_{+\theta_\varepsilon}(y)$ and $\varphi^{t_2}_{-\theta_\varepsilon}(y)$ are linearly independent solutions of (13), we have

$$\phi^\varepsilon = \alpha^\varepsilon \varphi^{t_1}_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \beta^\varepsilon \varphi^{t_2}_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right).$$

Inserting the boundary conditions of problem (13), the existence of of a non trivial pair $(\alpha^\varepsilon, \beta^\varepsilon)$ implies

$$\begin{vmatrix} d(0)\nabla\varphi^{t_1}_{\theta_\varepsilon}(0) + m(0)\varphi^{t_1}_{\theta_\varepsilon}(0) & d(0)\nabla\varphi^{t_2}_{-\theta_\varepsilon}(0) + m(0)\varphi^{t_2}_{-\theta_\varepsilon}(0) \\ \left(d(\delta)\nabla\varphi^{t_1}_{\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_1}_{\theta_\varepsilon}(\delta)\right)\frac{\varphi^{t_1}_{\theta_\varepsilon}(\varepsilon^{-1})}{\varphi^{t_1}_{\theta_\varepsilon}(\delta)} & \left(d(\delta)\nabla\varphi^{t_2}_{-\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_2}_{-\theta_\varepsilon}(\delta)\right)\frac{\varphi^{t_2}_{-\theta_\varepsilon}(\varepsilon^{-1})}{\varphi^{t_2}_{-\theta_\varepsilon}(\delta)} \end{vmatrix} = 0.$$

This identity can also be written as

$$\begin{aligned} \frac{d(0)\nabla\varphi^{t_1}_{\theta_\varepsilon}(0) + m(0)\varphi^{t_1}_{\theta_\varepsilon}(0)}{d(0)\nabla\varphi^{t_2}_{-\theta_\varepsilon}(0) + m(0)\varphi^{t_2}_{-\theta_\varepsilon}(0)} &= \\ \frac{d(\delta)\nabla\varphi^{t_1}_{\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_1}_{\theta_\varepsilon}(\delta)}{d(\delta)\nabla\varphi^{t_2}_{-\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_2}_{-\theta_\varepsilon}(\delta)}\frac{\varphi^{t_1}_{\theta_\varepsilon}(\varepsilon^{-1})}{\varphi^{t_1}_{\theta_\varepsilon}(\delta)}\frac{\varphi^{t_2}_{-\theta_\varepsilon}(\delta)}{\varphi^{t_2}_{-\theta_\varepsilon}(\varepsilon^{-1})} &= \\ \frac{d(\delta)\nabla\varphi^{t_1}_{\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_1}_{\theta_\varepsilon}(\delta)}{d(\delta)\nabla\varphi^{t_2}_{-\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_2}_{-\theta_\varepsilon}(\delta)}e^{2\theta_\varepsilon n}&, \end{aligned} \tag{54}$$

the second relation being a consequence of the relation

$$\frac{\varphi^{t_1}_{\theta_\varepsilon}(\varepsilon^{-1})}{\varphi^{t_1}_{\theta_\varepsilon}(\delta)}\frac{\varphi^{t_2}_{-\theta_\varepsilon}(\delta)}{\varphi^{t_2}_{-\theta_\varepsilon}(\varepsilon^{-1})} = \frac{\varphi^{t_1}_{\theta_\varepsilon}(n+\delta)}{\varphi^{t_1}_{\theta_\varepsilon}(\delta)}\frac{\varphi^{t_2}_{-\theta_\varepsilon}(\delta)}{\varphi^{t_2}_{-\theta_\varepsilon}(n+\delta)} = e^{2\theta_\varepsilon n}.$$

At $x = 0$, we obtain the following additional relation

$$\alpha^\varepsilon \varphi^{t_1}_{\theta_\varepsilon}(0) + \beta^\varepsilon \varphi^{t_2}_{-\theta_\varepsilon}(0) = \phi^\varepsilon(0).$$

The key point of the proof will be the computation of α^ε and β^ε.

We will now consider three cases. In the first one, θ_ε tends to θ_0, with $\theta_0 < 0$, and $\mu_0 \neq \mu_\delta$. In the second one, θ_ε tends to θ_δ, with $\theta_\delta > 0$, and $\mu_0 \neq \mu_\delta$. Finally, we will consider the limit case when θ_ε tends to θ_0, with $\theta_0 < 0$, and $\mu_0 = \mu_\delta$. Proposition 5.3 shows that these are the only possible cases when concentration occurs.

Case 1. Assume that $\mu_0 \neq \mu_\delta$ and θ_ε tends to θ_0, with $\theta_0 < 0$. This implies that $\varphi^0_{\theta_\varepsilon}$ tends to ϕ_0. Define $\eta := \theta_\varepsilon - \theta_0$. Thanks to Lemma 6.6, the following first order expansions in η hold

$$\begin{aligned} \varphi^0_{\theta_\varepsilon}(y) &= \phi_0(y) + \eta v_{\theta_0}(y) + O(\eta^2) \\ \varphi^0_{-\theta_\varepsilon}(y) &= \varphi_{-\theta_0}(y) + O(\eta). \end{aligned}$$

Inserting this ansatz in (54), we obtain

$$\frac{\eta\left(d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)\right) + O(\eta^2)}{d(0)\nabla\varphi^0_{-\theta_0}(0) + m(0)\varphi^0_{-\theta_0}(0) + O(\eta)} = \frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta) + O(\eta)}{d(\delta)\nabla\varphi^0_{-\theta_0}(\delta) + m(\delta)\varphi^0_{-\theta_0}(\delta) + O(\eta)} e^{2\theta_\varepsilon n}.$$

Note that $d(\delta)\nabla\varphi^0_{-\theta_0}(\delta) + m(\delta)\varphi^0_{-\theta_0}(\delta) \neq 0$, as this would imply $\mu_0 = \mu_\delta$, which we assume does not hold. Thanks to Lemma 6.6 we know that, $d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0) \neq 0$, therefore we can write

$$\eta = \frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(\delta)\nabla\varphi^0_{-\theta_0}(\delta) + m(\delta)\varphi^0_{-\theta_0}(\delta)} \frac{d(0)\nabla\varphi^0_{-\theta_0}(0) + m(0)\varphi^0_{-\theta_0}(0)}{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)} e^{2\theta_\varepsilon n} + o(e^{2\theta_\varepsilon n}). \tag{55}$$

This provides a first order correction (in exponential terms) for θ_ε. This value of η allows us to compute α^ε and β^ε, namely

$$\begin{aligned} \beta^\varepsilon &= -\eta\alpha^\varepsilon \frac{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}{d(0)\nabla\varphi^0_{-\theta_0}(0) + m(0)\varphi^0_{-\theta_0}(0)} + O(\alpha^\varepsilon\eta^2) \\ \alpha^\varepsilon &= \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)} + \phi^\varepsilon(0)O(\eta). \end{aligned}$$

Turning now to the solution ϕ^ε, we have obtained

$$\begin{aligned} \phi^\varepsilon &= \alpha^\varepsilon\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \beta^\varepsilon\varphi^0_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right), \\ &= \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)} \frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(\delta)\nabla\varphi^0_{-\theta_0}(\delta) + m(\delta)\varphi^0_{-\theta_0}(\delta)} e^{2\theta_\varepsilon n}\varphi^0_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) \\ &+ \phi^\varepsilon(0)O\left(e^{2\theta_\varepsilon n}\right). \end{aligned}$$

Using Lemma 6.4, the proof of the asymptotic formula for the eigenvector is complete. Let us now turn to the eigenvalue. Testing (44) against $\phi_\theta = \varphi^0_{-\theta_0}$, we obtain

$$\int_Y \left(d(y)\phi_0\nabla\varphi^0_{-\theta_0} - d(y)\nabla\phi_0\varphi^0_{-\theta_0}\right)dy = \frac{d\mu}{d\theta}(\theta_0)\int_Y s(y)\phi_0\varphi^0_{-\theta_0}dy. \tag{56}$$

Note that the wronskian $d\phi_0\nabla\varphi^0_{-\theta_0} - d\nabla\phi_0\varphi^0_{-\theta_0}$ is a constant, therefore

$$\int_Y d(y)\left(\phi_0\nabla\varphi^0_{-\theta_0} - \nabla\phi_0\varphi^0_{-\theta_0}\right)dy = d(0)\nabla\varphi^0_{-\theta_0}(0) + m(0)\varphi^0_{-\theta_0}(0).$$

Thanks to Lemma 6.6, ν_θ is analytic, with $\nu_{\theta_0} = \mu_0$ and $\nu_{\theta_\varepsilon} = \mu_\varepsilon$. We write

$$\mu_\varepsilon = \mu_0 + \eta\frac{d\mu}{d\theta}(\theta_0) + O(\eta^2).$$

and inserting (55) and (56) we obtain

$$\mu_\varepsilon = \mu_0 + \frac{K(\delta)}{k(0)} \frac{d(0)\nabla\varphi^0_{-\theta_0}(0) + m(0)\varphi^0_{-\theta_0}(0)}{\int\limits_Y s(y)\phi_0\varphi^0_{-\theta_0}} e^{2\theta_0/\varepsilon}(1+o(1))$$

which is (45).

Case 2. If $\mu_0 \neq \mu_\delta$ and θ_ε tends to θ_δ, then $\varphi^\delta_{\theta_\varepsilon}$ tends to ϕ_δ. The same strategy and similar arguments shows that

$$\eta = \frac{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}{d(0)\nabla\varphi^\delta_{-\theta_\delta}(0) + m(0)\varphi^\delta_{-\theta_\delta}(0)} \frac{d(\delta)\nabla\varphi^\delta_{-\theta_\delta}(\delta) + m(\delta)\varphi^\delta_{-\theta_\delta}(\delta)}{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta)} e^{-2\theta_\varepsilon n} + o(e^{-2\theta_\varepsilon n}),$$

and, in turn, using $\phi^\varepsilon(1) = \alpha^\varepsilon\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right) + \beta^\varepsilon\varphi^\delta_{-\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)$ we obtain,

$$\beta^\varepsilon = -\eta\alpha^\varepsilon e^{2\theta_\varepsilon n} \frac{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta)}{d(\delta)\nabla\varphi^\delta_{-\theta_\delta}(\delta) + m(\delta)\varphi^\delta_{-\theta_\delta}(\delta)} + O(e^{2\theta_\varepsilon n}\alpha^\varepsilon\eta^2)$$

and

$$\alpha^\varepsilon = \frac{\phi^\varepsilon(1)}{\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)} + \phi^\varepsilon(1)O(\eta).$$

This implies

$$\begin{aligned} \phi^\varepsilon &= \frac{\phi^\varepsilon(1)}{\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)}\varphi^\delta_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(1)}{\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)} \frac{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}{d(0)\nabla\varphi^\delta_{-\theta_\delta}(0) + m(0)\varphi^\delta_{-\theta_\delta}(0)} \varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) \\ &+ \phi^\varepsilon(0)O\left(e^{-2\theta_\varepsilon n}\right), \end{aligned}$$

which is the announced result. The proof of (47) follows that of the first case.

Case 3. If $\mu_0 = \mu_\delta$, then $\phi_\delta = \varphi^\delta_{-\theta_0}$ and we can rewrite the expansion as follows

$$\begin{aligned} \varphi^0_{\theta_\varepsilon} &= \phi_0 + \eta v_{\theta_0} + O(\eta^2) \\ \varphi^\delta_{-\theta_\varepsilon} &= \phi_\delta - \eta v_{\theta_\delta} + O(\eta^2). \end{aligned}$$

In this case $\varphi^0_{\theta_0} = \phi_0$ satisfies the boundary condition at 0 whereas $\varphi^\delta_{-\theta_0} = \phi_\delta$ satisfies the boundary conditions at δ, and equation (54) shows that

$$\frac{\eta^2\left(d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0) + O(\eta)\right)}{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0) + O(\eta)} = -\frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta) + O(\eta)}{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta) + O(\eta)} e^{2\theta_\varepsilon n}.$$

Thus

$$\eta = \sqrt{-\frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta)} \frac{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}} e^{\theta_\varepsilon n} + o(e^{\theta_\varepsilon n}).$$

Following the same steps as in the first case, we obtain

$$\phi^\varepsilon(x) = \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}\eta \frac{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)} \varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \phi^\varepsilon(0)O(e^{\theta_\varepsilon n}),$$

and finally

$$\phi^\varepsilon(x) \;=\; \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)} c_\delta e^{\theta_\varepsilon n}\varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \phi^\varepsilon(0)O(e^{\theta_\varepsilon n}),$$

with

$$
\begin{aligned}
c_\delta &= -\sqrt{-\frac{d(\delta)\nabla\phi_0(\delta)+m(\delta)\phi_0(\delta)}{d(\delta)\nabla v_{\theta_\delta}(\delta)+m(\delta)v_{\theta_\delta}(\delta)}\frac{d(0)\nabla\phi_\delta(0)+m(0)\phi_\delta(0)}{d(0)\nabla v_{\theta_0}(0)+m(0)v_{\theta_0}(0)}} \\
&\quad\times\frac{d(0)\nabla v_{\theta_0}(0)+m(0)v_{\theta_0}(0)}{d(0)\nabla\phi_\delta(0)+m(0)\phi_\delta(0)} \\
&= \sqrt{-\frac{d(0)\nabla v_{\theta_0}(0)+m(0)v_{\theta_0}(0)}{d(\delta)\nabla v_{\theta_\delta}(\delta)+m(\delta)v_{\theta_\delta}(\delta)}\frac{d(\delta)\nabla\phi_0(\delta)+m(\delta)\phi_0(\delta)}{d(0)\nabla\phi_\delta(0)+m(0)\phi_\delta(0)}}
\end{aligned}
$$

as claimed. The proof of (50) follows that of the first case. □

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