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New global stability estimates for the Calderón problem in two dimensions

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NEW GLOBAL STABILITY ESTIMATES FOR THE CALDERÓN PROBLEM IN TWO DIMENSIONS

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Abstract. We prove a new global stability estimate for the Gel'fand-Calderón inverse problem on a two-dimensional bounded domain. Specifically, the inverse boundary value problem for the equation $-\Delta\psi+v\,\psi=0$ on D is analysed, where v is a smooth real-valued potential of conductivity type defined on a bounded planar domain D. The main feature of this estimate is that it shows that the more a potential is smooth, the more its reconstruction is stable. Furthermore, the stability is proven to depend exponentially on the smoothness, in a sense to be made precise. As a corollary we obtain a similar estimate for the Calderón problem for the electrical impedance tomography.

1. Introduction

Let $D \subset \mathbb{R}^2$ be a bounded domain equipped with a potential given by a function $v \in L^\infty(D)$. The corresponding Dirichlet-to-Neumann map is the operator $\Phi : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$, defined by

$$\Phi(f) = \left.\frac{\partial u}{\partial \nu}\right|_{\partial D}, \tag{1.1}$$

where $f \in H^{1/2}(\partial D)$, ν is the outer normal of ∂D, and u is the $H^1(D)$-solution of the Dirichlet problem

$$(-\Delta + v)u = 0 \text{ on } D, \quad u|_{\partial D} = f. \tag{1.2}$$

Here we have assumed that

$$0 \text{ is not a Dirichlet eigenvalue for the operator } -\Delta + v \text{ in } D. \tag{1.3}$$

The following inverse boundary value problem arises from this construction:

Problem 1. *Given Φ, find v on D.*

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at zero energy (see [10], [17]) as well

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as a generalization of the Calderón problem for the electrical impedance tomography (see [7], [17]), in two dimensions.

It is convenient to recall how the above problem generalises the inverse conductivity problem proposed by Calderón. In the latter, D is a body equipped with an isotropic conductivity $\sigma(x) \in L^\infty(D)$ (with $\sigma \geq \sigma_{\min} > 0$),

$$v(x) = \frac{\Delta \sigma^{1/2}(x)}{\sigma^{1/2}(x)}, \qquad x \in D, \tag{1.4}$$

$$\Phi = \sigma^{-1/2}\left(\Lambda \sigma^{-1/2} + \frac{\partial \sigma^{1/2}}{\partial \nu}\right), \tag{1.5}$$

where $\sigma^{-1/2}$, $\partial\sigma^{1/2}/\partial\nu$ in (1.5) denote the multiplication operators by the functions $\sigma^{-1/2}|_{\partial D}$, $\partial\sigma^{1/2}/\partial\nu|_{\partial D}$, respectively and Λ is the voltage-to-current map on ∂D, defined as

$$\Lambda f = \sigma \frac{\partial u}{\partial \nu}\bigg|_{\partial D}, \tag{1.6}$$

where $f \in H^{1/2}(\partial D)$, ν is the outer normal of ∂D, and u is the $H^1(D)$-solution of the Dirichlet problem

$$\operatorname{div}(\sigma \nabla u) = 0 \text{ on } D, \quad u|_{\partial D} = f. \tag{1.7}$$

Indeed, the substitution $u = \tilde{u}\sigma^{-1/2}$ in (1.7) yields $(-\Delta + v)\tilde{u} = 0$ in D with v given by (1.4). The following problem is called the Calderón problem:

Problem 2. *Given Λ, find σ on D.*

We remark that Problems 1 and 2 are not overdetermined, in the sense that we consider the reconstruction of a real-valued function of two variables from real-valued inverse problem data dependent on two variables. In addition, the history of inverse problems for the two-dimensional Schrödinger equation at fixed energy goes back to [8].

There are several questions to be answered in these inverse problems: one would like to prove the uniqueness, i.e. the injectivity of the map $v \to \Phi$ (for Problem 1, for example), then the reconstruction of v from Φ and after the stability of the inverse $\Phi \to v$.

In this paper we study interior stability estimates, i.e. (for Problem 1 with a potential of conductivity type, for example) we want to prove that given two Dirichlet-to-Neumann operators Φ_1 and Φ_2, corresponding to potentials v_1 and v_2 on D, we have that

$$\|v_1 - v_2\|_{L^\infty(D)} \leq \omega\left(\|\Phi_1 - \Phi_2\|_{H^{1/2} \to H^{-1/2}}\right),$$

where the function $\omega(t) \to 0$ as fast as possible as $t \to 0$. For Problem 2 similar estimates are considered.

There is a wide literature on the Gel'fand-Calderón inverse problem. In the case of complex-valued potentials the global injectivity of the map $v \to \Phi$ was firstly proved in [17] for $D \subset \mathbb{R}^d$ with $d \geq 3$ and in [6] for $d = 2$ with $v \in L^p$: in particular, these results were obtained by the use of global reconstructions developed in the same papers. A global stability estimate for Problem 1 and 2 for $d \geq 3$ was first found by Alessandrini in [1]; this result was recently improved in [20]. In the two-dimensional case the first global stability estimate for Problem 1 was given in [22].

Global results for Problem 2 in the two dimensional case have been found much earlier than for Problem 1. In particular, global uniqueness was first proved in [16] for conductivities in the $W^{2,p}(D)$ class ($p > 1$) and after in [2] for L^∞ conductivities. The first global stability result was given in [14], where a logarithmic estimate is obtained for conductivities with two continuous derivatives. This result was improved in [4], where the same kind of estimate is obtained for Hölder continuous conductivities.

The research line delineated above is devoted to prove stability estimates for the least possible regular potentials/conductivities. Here, instead, we focus on the opposite situation, i.e. smooth potentials/conductivities, and try to answer another question: how the stability estimates vary with respect to the smoothness of the potentials/conductivities.

The results, detailed below, also constitute a progress for the case of non-smooth potentials: they indicate stability dependence of the smooth part of a singular potential with respect to boundary value data.

We will assume for simplicity that

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^2, \qquad \partial D \in C^2, \\ &v \in W^{m,1}(\mathbb{R}^2) \text{ for some } m > 2, \qquad \operatorname{supp} v \subset D, \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} &W^{m,1}(\mathbb{R}^2) = \{v \ : \ \partial^J v \in L^1(\mathbb{R}^2),\ |J| \leq m\}, \qquad m \in \mathbb{N} \cup \{0\}, \\ &J \in (\mathbb{N} \cup \{0\})^2, \qquad |J| = J_1 + J_2, \qquad \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \partial x_2^{J_2}}. \end{aligned} \tag{1.9}$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \|\partial^J v\|_{L^1(\mathbb{R}^2)}.$$

The last (strong) hypothesis is that we will consider only potentials of conductivity type, i.e.

$$v = \frac{\Delta \sigma^{1/2}}{\sigma^{1/2}}, \text{ for some } \sigma \in L^{\infty}(D), \text{ with } \sigma \geq \sigma_{\min} > 0. \tag{1.10}$$

The main result is the following.

Theorem 1.1. *Let the conditions* (1.3), (1.8), (1.10) *hold for the potentials* v_1, v_2*, where* D *is fixed, and let* Φ_1 , Φ_2 *be the corresponding Dirichlet-to-Neumann operators. Let* $\|v_j\|_{m,1} \leq N$*,* $j = 1, 2$*, for some* $N > 0$*. Then, for any* $\alpha < m$ *there exists a constant* $C = C(D, N, m, \alpha)$ *such that*

$$\|v_2 - v_1\|_{L^{\infty}(D)} \leq C(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{-\alpha}, \tag{1.11}$$

where $\|\Phi_2 - \Phi_1\| = \|\Phi_2 - \Phi_1\|_{H^{1/2} \to H^{-1/2}}$.

Corollary 1.2. *Let* σ_1, σ_2 *be two isotropic conductivities such that* $\Delta(\sigma_j^{1/2})/\sigma_j^{1/2}$ *satisfies conditions* (1.8)*, where* D *is fixed and* $0 < \sigma_{\min} \leq \sigma_j \leq \sigma_{\max} < +\infty$ *for* $j = 1, 2$ *and some constants* $\sigma_{\min}$ *and* $\sigma_{\max}$*. Let* Λ_1 , Λ_2 *be the corresponding Dirichlet-to-Neumann operators and* $\|\Delta(\sigma_j^{1/2})/\sigma_j^{1/2}\|_{m,1} \leq N$*,* $j = 1, 2$*, for some* $N > 0$*. We suppose, for simplicity, that* $\operatorname{supp}(\sigma_j - 1) \subset D$ *for* $j = 1, 2$*. Then, for any* $\alpha < m$ *there exists a constant* $C = C(D, N, \sigma_{\min}, \sigma_{\max}, m, \alpha)$ *such that*

$$\|\sigma_2 - \sigma_1\|_{L^{\infty}(D)} \leq C(\log(3 + \|\Lambda_2 - \Lambda_1\|^{-1}))^{-\alpha}, \tag{1.12}$$

where $\|\Lambda_2 - \Lambda_1\| = \|\Lambda_2 - \Lambda_1\|_{H^{1/2} \to H^{-1/2}}$.

The main feature of these estimates is that, as $m \to +\infty$, we have $\alpha = \alpha(m) \to +\infty$ (one can take $\alpha(m) = m - 1$). In addition we would like to mention that, under the assumption of Theorem 1.1 and Corollary (1.2), according to instability estimates of Mandache [15] and Isaev [13], our results are almost optimal. Note that in the linear approximation near zero potential Theorem 1.1 (without condition (1.10) but with $\alpha \leq m - 2$) was proved in [21]. In dimension $d \geq 3$ a global stability estimate similar to our result (with respect to dependence on smoothness) was proved in [20].

The proof of Theorem 1.1 relies on the $\bar{\partial}$-techniques introduced by Beals–Coifman [5], Henkin–R. Novikov [12], Grinevich–S. Novikov [11] and developed by R. Novikov [17] and Nachman [16] for solving the Calderón problem in two dimensions.

The Novikov–Nachman method starts with the construction of a special family of solutions $\psi(x, \lambda)$ of equation (1.2), which was originally introduced by Faddeev in [9]. These solutions have an exponential behaviour depending on the complex parameter λ and they are constructed via some function

$\mu(x,\lambda)$ (see (2.5)). One of the most important property of $\mu(x,\lambda)$ is that it satisfies a $\bar\partial$-equation with respect to the variable λ (see equation (2.8)), in which appears the so-called Faddeev generalized scattering amplitude $h(\lambda)$ (defined in (2.6)). On the contrary, if one knows $h(\lambda)$ for every $\lambda \in \mathbb{C}$, it is possible to recover $\mu(x,\lambda)$ via this $\bar\partial$-equation. Starting from these arguments we will prove that the map $h(\lambda) \to v(x)$ satisfies an Hölder condition (Proposition 4.2). This is done in Section 4.

The remaining part of the method relates the scattering amplitude $h(\lambda)$ to the Dirichlet-to-Neumann operator Φ. In the present paper this is done using the Alessandrini identity (see [1]) and an estimate of $h(\lambda)$ for high values of $|\lambda|$ given in [18]. We find that the map $\Phi \to h$ has logarithmic stability in some natural norm (Proposition 3.3). This is explained in Section 3.

The composition of the two above-mentioned maps gives the result of Theorem 1.1, as showed in Section 5.

This work was fulfilled in the framework of researches under the direction of R. G. Novikov.

2. Preliminaries

In this section we recall some definitions and properties of the Faddeev functions, the above-mentioned family of solutions of equation (1.2), which will be used throughout all the paper.

Following [16], we fix some $1 < p < 2$ and define $\psi(x,k)$ to be the solution (when it exists unique) of

$$(-\Delta + v)\psi(x,k) = 0 \text{ in } \mathbb{R}^2, \tag{2.1}$$

with $e^{-ixk}\psi(x,k) - 1 \in W^{1,\tilde p}(\mathbb{R}^2) = \{u \;:\; \partial^J u \in L^{\tilde p}(\mathbb{R}^2),\ |J| \le 1\}$, where $x = (x_1,x_2) \in \mathbb{R}^2$, $k = (k_1,k_2) \in \mathcal{V} \subset \mathbb{C}^2$,

$$\mathcal{V} = \{k \in \mathbb{C}^2 \,:\, k^2 = k_1^2 + k_2^2 = 0\} \tag{2.2}$$

and

$$\frac{1}{\tilde p} = \frac{1}{p} - \frac{1}{2}. \tag{2.3}$$

The variety $\mathcal{V}$ can be written as $\{(\lambda, i\lambda) : \lambda \in \mathbb{C}\} \cup \{(\lambda, -i\lambda) : \lambda \in \mathbb{C}\}$. We henceforth denote $\psi(x,(\lambda,i\lambda))$ by $\psi(x,\lambda)$ and observe that, since v is real-valued, uniqueness for (2.1) yields $\psi(x,(-\bar\lambda, i\bar\lambda)) = \overline{\psi(x,(\lambda,i\lambda))} = \overline{\psi(x,\lambda)}$ so that, for reconstruction and stability purpose, it is sufficient to work on the sheet $k = (\lambda, i\lambda)$.

We now identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$

where $(x_1, x_2) \in \mathbb{R}^2$.

Then we define

$$\psi(z,\lambda) = \psi(x,\lambda), \tag{2.4}$$

$$\mu(z,\lambda) = e^{-iz\lambda}\psi(z,\lambda), \tag{2.5}$$

$$h(\lambda) = \int_D e^{i\bar{z}\bar{\lambda}} v(z)\psi(z,\lambda) d\mathrm{Re} z\, d\mathrm{Im} z, \tag{2.6}$$

for $z, \lambda \in \mathbb{C}$.

Throughout all the paper $c(\alpha, \beta, \ldots)$ is a positive constant depending on parameters $\alpha, \beta, \ldots$

We now restate some fundamental results about Faddeev functions. In the following statement ψ_0 denotes $\sigma^{1/2}$.

Proposition 2.1 (see [16]). *Let $D \subset \mathbb{R}^2$ be an open bounded domain with C^2 boundary, $v \in L^p(\mathbb{R}^2)$, $1 < p < 2$, $\operatorname{supp} v \subset D$, $\|v\|_{L^p(\mathbb{R}^2)} \le N$, be such that there exists a real-valued $\psi_0 \in L^\infty(\mathbb{R}^2)$ with $v = (\Delta\psi_0)/\psi_0$, $\psi_0(x) \ge c_0 > 0$ and $\psi_0 \equiv 1$ outside D. Then, for any $\lambda \in \mathbb{C}$ there is a unique solution $\psi(z,\lambda)$ of (2.1) with $e^{-iz\lambda}\psi(\cdot,\lambda) - 1$ in L^∞. Furthermore, $e^{-iz\lambda}\psi(\cdot,\lambda) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$ ($\tilde{p}$ is defined in (2.3)) and*

$$\|e^{-iz\lambda}\psi(\cdot,\lambda) - 1\|_{W^{s,\tilde{p}}} \le c(p,s)N|\lambda|^{s-1}, \tag{2.7}$$

for $0 \le s \le 1$ and λ sufficiently large.

The function $\mu(z,\lambda)$ defined in (2.5) satisfies the equation

$$\frac{\partial \mu(z,\lambda)}{\partial \bar{\lambda}} = \frac{1}{4\pi\bar{\lambda}} h(\lambda) e_{-\lambda}(z) \overline{\mu(z,\lambda)}, \qquad z, \lambda \in \mathbb{C}, \tag{2.8}$$

in the $W^{1,\tilde{p}}$ topology, where $h(\lambda)$ is defined in (2.6) and the function $e_{-\lambda}(z)$ is defined as follows:

$$e_\lambda(z) = e^{i(z\lambda + \bar{z}\bar{\lambda})}. \tag{2.9}$$

In addition, the functions $h(\lambda)$ and $\mu(z,\lambda)$ satisfy

$$\left\|\frac{h(\lambda)}{\bar\lambda}\right\|_{L^r(\mathbb{R}^2)} \leq c(r,N), \text{ for all } r\in(\tilde p',\tilde p), \quad \frac{1}{\tilde p}+\frac{1}{\tilde p'}=1, \tag{2.10}$$

$$\inf_{z,\lambda\in\mathbb{C}} |\mu(z,\lambda)| \geq c(D,N) > 0, \tag{2.11}$$

$$\sup_{z\in\mathbb{C}} \|\mu(z,\cdot)-1\|_{L^r(\mathbb{C})} \leq c(r,D,N), \qquad \text{for all } r\in(p',\infty] \tag{2.12}$$

and

$$|h(\lambda)| \leq c(p,D,N)|\lambda|^{\varepsilon}, \tag{2.13}$$

$$\|\mu(\cdot,\lambda)-\psi_0\|_{W^{1,\tilde p}} \leq c(p,D,N)|\lambda|^{\varepsilon}, \tag{2.14}$$

for $\lambda \leq \lambda_0(p,D,N)$ and $0<\varepsilon<\frac{2}{p'}$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Remark. Equation (2.8) means that μ is a generalised analytic function in $\lambda\in\mathbb{C}$ (see [23]). In two-dimensional inverse scattering for the Schrödinger equation, the theory of generalised analytic functions was used for the first time in [11].

We recall that if $v\in W^{m,1}(\mathbb{R}^2)$ with $\operatorname{supp} v\subset D$, then $\|\hat v\|_m < +\infty$, where

$$\hat v(p) = (2\pi)^{-2}\int_{\mathbb{R}^2} e^{ipx}v(x)dx, \qquad p\in\mathbb{C}^2, \tag{2.15}$$

$$\|u\|_m = \sup_{p\in\mathbb{R}^2} |(1+|p|^2)^{m/2}u(p)|, \tag{2.16}$$

for a test function u.

In addition, if $v\in W^{m,1}(\mathbb{R}^2)$ with $\operatorname{supp} v\subset D$ and $m\geq 1$, we have

$$\|v\|_{L^\infty(D)} \leq \operatorname{diam}(D)\|v\|_{m,1}, \tag{2.17}$$

so, in particular, the hypothesis $v\in L^p(\mathbb{R}^2)$, $\operatorname{supp} v\subset D$, in the statement of Proposition 2.1 is satisfied for every $1<p<2$ (since D is bounded).

The following lemma is a variation of a result in [18]:

Lemma 2.2. *Under the assumption* (1.8), *there exists $R=R(m,\|\hat v\|_m)>0$ such that*

$$|h(\lambda)| \leq 8\pi^2\|\hat v\|_m(1+4|\lambda|^2)^{-m/2}, \qquad \text{for } |\lambda|>R. \tag{2.18}$$

Proof. We consider the function $H(k,p)$ defined as

$$H(k,p) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i(p-k)x}v(x)\psi(x,k)dx, \tag{2.19}$$

for $k\in\mathcal{V}$ (where $\mathcal{V}$ is defined in (2.2)), $p\in\mathbb{R}^2$ and $\psi(x,k)$ as defined at the beginning of this section.

We deduce that $h(\lambda) = (2\pi)^2 H(k(\lambda), k(\lambda) + \overline{k(\lambda)})$, for $k(\lambda) = (\lambda, i\lambda)$. By [18, Corollary 1.1] we have

$$|H(k,p)| \leq 2\|\hat{v}\|_m (1+p^2)^{-m/2} \qquad \text{for } |\lambda| > R, \tag{2.20}$$

for $R = R(m, \|\hat{v}\|_m) > 0$ and then the proof follows. □

We restate [3, Lemma 2.6], which will be useful in section 4.

Lemma 2.3 ([3]). *Let $a \in L^{s_1}(\mathbb{R}^2) \cap L^{s_2}(\mathbb{R}^2)$, $1 < s_1 < 2 < s_2 < \infty$ and $b \in L^s(\mathbb{R}^2)$, $1 < s < 2$. Assume u is a function in $L^{\tilde{s}}(\mathbb{R}^2)$, with $\tilde{s}$ defined as in* (2.3)*, which satisfies*

$$\frac{\partial u(\lambda)}{\partial \bar{\lambda}} = a(\lambda)\bar{u}(\lambda) + b(\lambda), \qquad \lambda \in \mathbb{C}. \tag{2.21}$$

Then there exists $c > 0$ such that

$$\|u\|_{L^{\tilde{s}}} \leq c\|b\|_{L^s} \exp(c(\|a\|_{L^{s_1}} + \|a\|_{L^{s_2}})). \tag{2.22}$$

3. From Φ to $h(\lambda)$

Lemma 3.1. *Let the condition* (1.8) *holds. Then we have, for $p > 1$,*

$$\left\| \frac{h(\lambda)}{\bar{\lambda}} \right\|_{L^p(|\lambda|>R)} \leq c(p,m)\|\hat{v}\|_m \frac{1}{R^{m+1-2/p}}, \tag{3.1}$$

where R is as in Lemma 2.2.

Proof. It's a corollary of Lemma 2.2. Indeed we have

$$\left\| \frac{h(\lambda)}{\bar{\lambda}} \right\|^p_{L^p(|\lambda|>R)} \leq c\|\hat{v}\|^p_m \int_{r>R} r^{1-mp-p} dr = \frac{c(p,m)\|\hat{v}\|^p_m}{R^{(m+1)p-2}}.$$ □

Lemma 3.2. *Let $D \subset \{x \in \mathbb{R}^2 : |x| \leq l\}$, v_1, v_2 be two potentials satisfying* (1.3), (1.8), (1.10)*, let Φ_1, Φ_2 the corresponding Dirichlet-to-Neumann operator and h_1, h_2 the corresponding generalised scattering amplitude. Let $\|v_j\|_{m,1} \leq N$, $j = 1, 2$. Then we have*

$$|h_2(\lambda) - h_1(\lambda)| \leq c(D,N) e^{2l|\lambda|} \|\Phi_2 - \Phi_1\|_{H^{1/2} \to H^{-1/2}}, \quad \lambda \in \mathbb{C}. \tag{3.2}$$

Proof. We have the following identity:

$$h_2(\lambda) - h_1(\lambda) = \int_{\partial D} \overline{\psi_1(z,\lambda)} (\Phi_2 - \Phi_1) \psi_2(z,\lambda) |dz|, \tag{3.3}$$

where $\psi_i(z,\lambda)$ are the Faddeev functions associated to the potential v_i, $i = 1, 2$. This identity is a particular case of the one in [19, Theorem 1]: we refer to that paper for a proof.

From this identity we have:

(3.4)

$$|h_2(\lambda) - h_1(\lambda)| \le \|\psi_1(\cdot,\lambda)\|_{H^{1/2}(\partial D)} \|\Phi_2 - \Phi_1\|_{H^{1/2}\to H^{-1/2}} \|\psi_2(\cdot,\lambda)\|_{H^{1/2}(\partial D)}.$$

Now take $\tilde{p} > 2$ and use the trace theorem to get

$$\begin{aligned}\|\psi_j(\cdot,\lambda)\|_{H^{1/2}(\partial D)} &\le C\|\psi_j(\cdot,\lambda)\|_{W^{1,\tilde{p}}(D)} \le Ce^{l|\lambda|}\|e^{-iz\lambda}\psi_i(\cdot,\lambda)\|_{W^{1,\tilde{p}}(D)} \\ &\le Ce^{l|\lambda|}\left(\|e^{-iz\lambda}\psi_i(\cdot,\lambda) - 1\|_{W^{1,\tilde{p}}(D)} + \|1\|_{W^{1,\tilde{p}}(D)}\right), \qquad i = 1,2,\end{aligned}$$

which from (2.7) and (2.12) is bounded by $C(D,N)e^{l|\lambda|}$. These estimates together with (3.4) give (3.2). □

The main result of this section is the following:

Proposition 3.3. *Let v_1, v_2 be two potentials satisfying* (1.3), (1.8), (1.10), *let Φ_1, Φ_2 the corresponding Dirichlet-to-Neumann operator and h_1, h_2 the corresponding generalised scattering amplitude. Let $0 < \varepsilon < 1$, $1 < p < \frac{2}{1-\varepsilon}$ and $\|v_j\|_{m,1} \le N$, $j = 1,2$. Then for every $\alpha < m + 1 - 2/p$ there exists a constant $c = c(D,N,m,p,\alpha)$ such that*

$$\left\|\frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}}\right\|_{L^p(\mathbb{C})} \le c\log(3 + \|\Phi_2 - \Phi_1\|^{-1}_{H^{1/2}\to H^{-1/2}})^{-\alpha}. \tag{3.5}$$

Proof. Let choose $a, b > 0$, a close to 0 and b big to be determined and let

$$\delta = \|\Phi_2 - \Phi_1\|_{H^{1/2}\to H^{-1/2}}. \tag{3.6}$$

We split down the left term of (3.5) as follows:

$$\begin{aligned}\left\|\frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}}\right\|_{L^p(\mathbb{C})} \le \left\|\frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}}\right\|_{L^p(|\lambda|<a)} + \left\|\frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}}\right\|_{L^p(a<|\lambda|<b)} \\ + \left\|\frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}}\right\|_{L^p(|\lambda|>b)}.\end{aligned}$$

From (2.13) we obtain

$$\begin{aligned}\left\|\frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}}\right\|_{L^p(|\lambda|<a)} &\le c(D,N,p)\left(\int_{|\lambda|<a} |\lambda|^{(\varepsilon-1)p} d\mathrm{Re}\lambda\, d\mathrm{Im}\lambda\right)^{\frac{1}{p}} \\ &= c(D,N,p)a^{\varepsilon-1+2/p}.\end{aligned} \tag{3.7}$$

From Lemma 3.2 and (3.6) we get

$$\left\|\frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}}\right\|_{L^p(a<|\lambda|<b)} \le c(D,N)\left(\frac{\delta}{a^{1-2/p}} + \delta b^{1/p} e^{(2l+1)b}\right). \tag{3.8}$$

From Lemma 3.1

$$\left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(|\lambda|>b)} \leq \frac{c(N)}{b^{m+1-2/p}}. \tag{3.9}$$

We now define, for $0 < \alpha < m + 1 - \frac{2}{p}$,

$$a = \log(3 + \delta^{-1})^{-\frac{\alpha}{\varepsilon - 1 + 2/p}}, \qquad b = \log(3 + \delta^{-1})^{\frac{\alpha}{m+1-2/p}}, \tag{3.10}$$

in order to have (3.7) and (3.9) of the order $\log(3 + \delta^{-1})^{-\alpha}$. We also choose $\delta_\alpha < 1$ such that for every $\delta \leq \delta_\alpha$, a is sufficiently small in order to have (2.13) (which yields (3.7)), $b \geq R$ (with R as in Lemma 2.2) and also

$$\frac{\delta}{a^{1-2/p}} = \delta \log(3 + \delta^{-1})^{\left(\frac{\alpha}{\varepsilon - 1 + 2/p}\right)(1-2/p)} < \log(3 + \delta^{-1})^{-\alpha}. \tag{3.11}$$

Thus we obtain

$$\begin{aligned} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C})} &\leq \frac{c(D, N, p)}{\log(3 + \delta^{-1})^\alpha} \\ &\quad + c(D, N)\delta \log(3 + \delta^{-1})^{\frac{\alpha}{p(m+1-2/p)}} e^{(2l+1)\log(3+\delta^{-1})^{\frac{\alpha}{m+1-2/p}}}, \end{aligned} \tag{3.12}$$

for $\delta \leq \delta_\alpha$. As

$$\delta \log(3 + \delta^{-1})^{\frac{\alpha}{p(m+1-2/p)}} e^{(2l+1)\log(3+\delta^{-1})^{\frac{\alpha}{m+1-2/p}}} \to 0 \text{ for } \delta \to 0$$

more rapidly than the other term, we obtain that

$$\left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C})} \leq \frac{c(D, N, m, p, \alpha)}{\log(3 + \delta^{-1})^\alpha}, \tag{3.13}$$

for $0 < \alpha < m + 1 - \frac{2}{p}$, $\delta \leq \delta_\alpha$.

Estimate (3.13) for general δ (with modified constant) follows from (3.13) for $\delta \leq \delta_\alpha$ and the property (2.10) of the scattering amplitude. This completes the proof of Proposition 3.3. $\square$

4. From $h(\lambda)$ to $v(x)$

Lemma 4.1. *Let v_1, v_2 be two potentials satisfying* (1.3), (1.8), (1.10), *with $\|v_j\|_{m,1} \leq N$, h_1, h_2 the corresponding scattering amplitude and $\mu_1(z, \lambda), \mu_2(z, \lambda)$ the corresponding Faddeev functions. Let $1 < s < 2$, and $\tilde{s}$ be as in* (2.3). *Then*

$$\sup_{z \in \mathbb{C}} \|\mu_2(z, \cdot) - \mu_1(z, \cdot)\|_{L^{\tilde{s}}(\mathbb{C})} \leq c(D, N, s) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})}, \tag{4.1}$$

$$\sup_{z \in \mathbb{C}} \|v_2\mu_2(z, \cdot) - v_1\mu_1(z, \cdot)\|_{L^{\tilde{s}}(\mathbb{C})} \leq c(D, N, s) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})}. \tag{4.2}$$

Proof. Let

$$\nu(z,\lambda) = \mu_2(z,\lambda) - \mu_1(z,\lambda), \tag{4.3}$$

$$\tau(z,\lambda) = v_2(z)\mu_2(z,\lambda) - v_1(z)\mu_1(z,\lambda). \tag{4.4}$$

From the $\bar\partial$-equation (2.8) (and the fact that v_1 and v_2 are real-valued) we deduce that ν, τ satisfy the following non-homogeneous $\bar\partial$-equations:

$$\frac{\partial}{\partial\bar\lambda}\nu(z,\lambda) = \frac{e_{-\lambda}(z)}{4\pi}\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\overline{\mu_2(z,\lambda)} + \frac{e_{-\lambda}(z)}{4\pi}\frac{h_1(\lambda)}{\bar\lambda}\overline{\nu(z,\lambda)}, \tag{4.5}$$

$$\frac{\partial}{\partial\bar\lambda}\tau(z,\lambda) = \frac{e_{-\lambda}(z)}{4\pi}\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\overline{v_2\mu_2(z,\lambda)} + \frac{e_{-\lambda}(z)}{4\pi}\frac{h_1(\lambda)}{\bar\lambda}\overline{\tau(z,\lambda)}, \tag{4.6}$$

for $\lambda \in \mathbb{C}$, where $e_{-\lambda}(z)$ is defined in (2.9).

By Lemma 2.3 and (2.10) we obtain

$$\begin{aligned}
\|\nu(z,\cdot)\|_{L^{\tilde s}} &\le c(D,N)\left\|\overline{\mu_2(z,\lambda)}\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^s(\mathbb{C})}\\
&\le c(D,N)\sup_{z\in\mathbb{C}}\|\mu_2(z,\cdot)\|_{L^\infty}\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^s(\mathbb{C})}\\
&\le c(D,N)\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^s(\mathbb{C})},
\end{aligned}$$

where we used the property (2.12) of $\mu_2(z,\lambda)$. With the same arguments (along with (2.17)) we also obtain

$$\begin{aligned}
\|\tau(z,\cdot)\|_{L^{\tilde s}} &\le c(D,N)\left\|\overline{v_2\mu_2(z,\lambda)}\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^s(\mathbb{C})}\\
&\le c(D,N)\sup_{z\in\mathbb{C}}\|v_2(z)\mu_2(z,\cdot)\|_{L^\infty}\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^s(\mathbb{C})}\\
&\le c(D,N)\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^s(\mathbb{C})},
\end{aligned}$$

which ends the proof. □

The main result of this section is the following proposition.

Proposition 4.2. *Let v_1, v_2 be two potentials satisfying* (1.3), (1.8), (1.10), *with $\|v_j\|_{m,1} \le N$, and let h_1, h_2 be the corresponding scattering amplitude. Let p, p' such that $1 < p < 2 < p' < \infty$, $1/p + 1/p' = 1$. Then*

$$\|v_2 - v_1\|_{L^\infty(D)} \le c(D,N,p)\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(\mathbb{C})\cap L^{p'}(\mathbb{C})}. \tag{4.7}$$

Proof. We write

$$v_2(z) - v_1(z) = \frac{1}{\mu_2(z,0)}\big(v_2\mu_2(z,0) - v_1\mu_1(z,0) - v_1(z)[\mu_2(z,0)-\mu_1(z,0)]\big),$$

that yields

$$|v_2(z)-v_1(z)| \le \frac{1}{|\mu_2(z,0)|}\big(|v_2\mu_2(z,0)-v_1\mu_1(z,0)|+|v_1(z)||\mu_2(z,0)-\mu_1(z,0)|\big). \tag{4.8}$$

We claim that

$$\|v_2\mu_2(\cdot,0)-v_1\mu_1(\cdot,0)\|_{L^\infty(D)} \le c(D,N,p)\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(\mathbb{C})\cap L^{p'}(\mathbb{C})}, \tag{4.9}$$

$$\|\mu_2(\cdot,0)-\mu_1(\cdot,0)\|_{L^\infty(D)} \le c(D,N,p)\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(\mathbb{C})\cap L^{p'}(\mathbb{C})}, \tag{4.10}$$

for $1 < p < 2 < p' < \infty$, $1/p+1/p' = 1$. Suppose (4.9), (4.10) already proved; then estimate (4.7) follows from (4.8), (4.9), (4.10), property (2.11) and (2.17).

Before proving (4.9), (4.10), we would like to recall that if $v \in W^{m,1}(\mathbb{R}^2)$, $m \ge 1$, with $\operatorname{supp} v \subset D$ then $v \in L^p(D)$ for $p \in [1,\infty]$; in particular, from Proposition 2.1, this yields $h(\lambda)/\bar\lambda \in L^p(\mathbb{C})$, for $1 < p < \infty$.

Now, in order to prove (4.9), (4.10) we write as before

$$\nu(z,\lambda) = \mu_2(z,\lambda)-\mu_1(z,\lambda), \tag{4.11}$$

$$\tau(z,\lambda) = v_2(z)\mu_2(z,\lambda)-v_1(z)\mu_1(z,\lambda), \tag{4.12}$$

which satisfy the non-homogeneous $\bar\partial$-equations (4.5) and (4.6), respectively. From these equations we obtain

$$\begin{aligned}|\nu(z,0)| &= \frac{1}{\pi}\left|\int_{\mathbb{C}}\frac{e_{-\lambda}(z)}{4\pi\lambda}\frac{h_1(\lambda)}{\bar\lambda}\overline{\nu(z,\lambda)}d\mathrm{Re}\lambda\, d\mathrm{Im}\lambda\right.\\ &\quad\left.+\int_{\mathbb{C}}\frac{e_{-\lambda}(z)}{4\pi\lambda}\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\overline{\mu_2(z,\lambda)}d\mathrm{Re}\lambda\, d\mathrm{Im}\lambda\right|\\ &\le \frac{1}{4\pi^2}\sup_{z\in\mathbb{C}}\|\nu(z,\cdot)\|_{L^r}\left\|\frac{h_1(\lambda)}{\lambda\bar\lambda}\right\|_{L^{r'}}\\ &\quad+\frac{1}{4\pi^2}\sup_{z\in\mathbb{C}}\|\mu_2(z,\cdot)\|_{L^\infty}\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\lambda\bar\lambda}\right\|_{L^1}\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}|\tau(z,0)| &= \frac{1}{\pi}\left|\int_{\mathbb{C}}\frac{e_{-\lambda}(z)}{4\pi\lambda}\frac{h_1(\lambda)}{\bar\lambda}\overline{\tau(z,\lambda)}d\mathrm{Re}\lambda\, d\mathrm{Im}\lambda\right.\\ &\quad\left.+\int_{\mathbb{C}}\frac{e_{-\lambda}(z)}{4\pi\lambda}\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\overline{v_2\mu_2(z,\lambda)}d\mathrm{Re}\lambda\, d\mathrm{Im}\lambda\right|\\ &\le \frac{1}{4\pi^2}\sup_{z\in\mathbb{C}}\|\tau(z,\cdot)\|_{L^r}\left\|\frac{h_1(\lambda)}{\lambda\bar\lambda}\right\|_{L^{r'}}\\ &\quad+\frac{\|v_2\|_{L^\infty(D)}}{4\pi^2}\sup_{z\in\mathbb{C}}\|\mu_2(z,\cdot)\|_{L^\infty}\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\lambda\bar\lambda}\right\|_{L^1},\end{aligned} \tag{4.14}$$

where $1/r + 1/r' = 1$, $1 < r' < 2 < r < \infty$. The number $s = 2r/(r+2)$ can be chosen $s < 2$ and as close to 2 as wanted, by taking r big enough.

Then

$$\left\| \frac{h_1(\lambda)}{\lambda\bar{\lambda}} \right\|_{L^{r'}(|\lambda|<R)} \le \left\| \frac{h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p} \left\| \frac{1}{\lambda} \right\|_{L^q(|\lambda|<R)} \le c(N,r), \tag{4.15}$$

where we have chosen $p > 2$ such that $\left\| h_1(\lambda)/\bar{\lambda} \right\|_{L^p} \le c(N,p)$ from (2.10) and also, since $1/q = 1/r' - 1/p = 1 - 1/r - 1/p$, q can be chosen less than 2 by taking r big enough depending on p. With the same choice of p, q we also obtain

$$\left\| \frac{h_1(\lambda)}{\lambda\bar{\lambda}} \right\|_{L^{r'}(|\lambda|>R)} \le \left\| \frac{h_1(\lambda)}{\bar{\lambda}} \right\|_{L^q} \left\| \frac{1}{\lambda} \right\|_{L^p(|\lambda|>R)} \le c(N,r). \tag{4.16}$$

From Lemma 4.1 with $r = \tilde{s}$ we get

$$\sup_{z\in\mathbb{C}} \|\nu(z,\cdot)\|_{L^r} \le c(D,N,r) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})}, \tag{4.17}$$

$$\sup_{z\in\mathbb{C}} \|\tau(z,\cdot)\|_{L^r} \le c(D,N,r) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})}, \tag{4.18}$$

and from (2.12)

$$\sup_{z,\lambda\in\mathbb{C}} |\mu_2(z,\lambda)| \le c(D,N). \tag{4.19}$$

Finally

$$\begin{aligned} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\lambda\bar{\lambda}} \right\|_{L^1} \le{}& \left\| \frac{1}{\lambda} \right\|_{L^p(|\lambda|>R)} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^{p'}} \\ &+ \left\| \frac{1}{\lambda} \right\|_{L^{p'}(|\lambda|<R)} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p}, \end{aligned} \tag{4.20}$$

by taking $p' = s$ and p such that $1/p + 1/p' = 1$.

Now (4.9) and (4.10) follow from (4.11)–(4.20); this finishes the proof of Proposition 4.2. □

5. Proof of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. Fix $\alpha < m$ and take p such that

$$\max\left(1, \frac{2}{m - \alpha + 1}\right) < p < 2.$$

From Proposition 4.2 we have

$$\|v_2 - v_1\|_{L^\infty(D)} \le c(D,N,p) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C})\cap L^{p'}(\mathbb{C})}, \tag{5.1}$$

where $1/p + 1/p' = 1$. From Proposition 3.3

$$\left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C}) \cap L^{p'}(\mathbb{C})} \leq c(D, N, p) \log(3 + \|\Phi_2 - \Phi_1\|^{-1}_{H^{1/2} \to H^{-1/2}})^{-\alpha}, \tag{5.2}$$

as $\alpha < m + 1 - \frac{2}{p}$. Theorem 1.1 is proved. □

Proof of Corollary 1.2. We first extend σ on the whole plane by putting $\sigma(x) = 1$ for $x \in \mathbb{R}^2 \setminus D$ (this extension is smooth by our hypothesis on σ). Now since $\sigma_j|_{\partial D} = 1$ and $\frac{\partial \sigma_j}{\partial \nu}|_{\partial D} = 0$ for $j = 1, 2$, from (1.5) we deduce that

$$\Phi_j = \Lambda_j, \qquad j = 1, 2. \tag{5.3}$$

In addition, from (2.14) we get

$$\lim_{\lambda \to 0} \mu_j(z, \lambda) = \sigma_j^{1/2}(z), \qquad j = 1, 2; \tag{5.4}$$

thus we obtain, using the fact that σ_j is bounded from above and below, for $j = 1, 2$,

$$\begin{aligned} \|\sigma_2 - \sigma_1\|_{L^\infty(D)} &\leq c(N) \|\sigma_2^{1/2} - \sigma_1^{1/2}\|_{L^\infty(D)} \\ &= c(N) \|\mu_2(\cdot, 0) - \mu_1(\cdot, 0)\|_{L^\infty(D)}. \end{aligned} \tag{5.5}$$

Now the proof follows by repeating the proof of Theorem 1.1, using (5.5), (4.10) and (5.3). □

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