ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES

UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 46 00. Fax: 01 69 33 46 46

http://www.cmap.polytechnique.fr/

A large time asymptotics for the solution of the Cauchy problem for the Novikov-Veselov equation at negative energy with non-singular scattering data

Anna Kazeykina

R.I. 728 *October 2011*

A large time asymptotics for the solution of the Cauchy problem for the Novikov-Veselov equation at negative energy with non-singular scattering data

A.V. Kazeykina [1]

Abstract. In the present paper we are concerned with the Novikov–Veselov equation at negative energy, i.e. with the $(2+1)$–dimensional analog of the KdV equation integrable by the method of inverse scattering for the two–dimensional Schrödinger equation at negative energy. We show that the solution of the Cauchy problem for this equation with non–singular scattering data behaves asymptotically as $\frac{\text{const}}{t^{3/4}}$ in the uniform norm at large times t. We also prove that this asymptotics is optimal.

1 Introduction

In the present paper we consider the Novikov–Veselov equation

$$\begin{gathered}\partial_t v = 4\mathrm{Re}(4\partial_z^3 v + \partial_z(vw) - E\partial_z w),\\ \partial_{\bar z} w = -3\partial_z v, \quad v = \bar v, \quad E \in \mathbb{R}, \quad E < 0, \\ v = v(x,t), \quad w = w(x,t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R},\end{gathered} \tag{1.1}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \partial_{\bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

We will say that (v, w) is a rapidly decaying solution of (1.1) if

- $v, w \in C(\mathbb{R}^2 \times \mathbb{R}), \quad v(\cdot, t) \in C^3(\mathbb{R}^3),$ (1.2a)
- $|\partial_x^j v(x,t)| \leqslant \dfrac{q(t)}{(1+|x|)^{2+\varepsilon}}, \ |j| \leqslant 3, \ \text{for some } \varepsilon > 0, \quad w(x,t) \to 0, |x| \to \infty,$ (1.2b)
- (v, w) satisfies (1.1). (1.2c)

Note that if $v(x,t) = v(x_1,t)$, $w(x,t) = w(x_1,t)$, then (1.1) is reduced to the classic KdV equation. In addition, (1.1) is integrable via the inverse scattering method for the two–dimensional Schrödinger equation

$$L\psi = E\psi, \quad L = -\Delta + v(x), \quad x = (x_1, x_2), \quad E = E_{fixed}. \tag{1.3}$$

In this connection, it was shown (see [M], [NV1], [NV2]) that for the Schrödinger operator L from (1.3) there exist appropriate operators A, B (Manakov L–A–B triple) such that (1.1) is equivalent to

$$\frac{\partial(L-E)}{\partial t} = [L-E, A] + B(L-E),$$

where $[\cdot,\cdot]$ is the commutator.

Note that both Kadomtsev–Petviashvili equations can be obtained from (1.1) by considering an appropriate limit $E \to \pm\infty$ (see [ZS], [G2]).

We will consider the Cauchy problem for equation (1.1) with the initial data

$$v(x,0) = v_0(x), \quad w(x,0) = w_0(x). \tag{1.4}$$

[1] CMAP, Ecole Polytechnique, Palaiseau, 91128, France; email: kazeykina@cmap.polytechnique.fr

We will assume that the function $v_0(x)$ satisfies the following conditions

$$\bullet\ v_0 = \bar{v}_0, \tag{1.5a}$$

$$\bullet\ \mathcal{E} = \emptyset,\ \text{where } \mathcal{E} \text{ is the set of zeros of the Fredholm determinant } \Delta \text{ of (2.7) for equation (2.6) with } v(x) = v_0(x), \tag{1.5b}$$

$$\bullet\ v_0 \in \mathcal{S}(\mathbb{R}^2),\ \text{where } \mathcal{S} \text{ denotes the Schwartz class.} \tag{1.5c}$$

As for the function $w_0(x)$, which plays an auxiliary role, we will assume that it is a continuous function decaying at infinity and determined using $\partial_{\bar{z}} w_0(x) = -3\partial_z v_0(x)$ from (1.1).

Condition (1.5b) is equivalent to non–singularity of scattering data for $v_0(x)$. Conditions (1.5) define the class of initial values for which the direct and inverse scattering equations (2.4), (2.9)–(2.11), with time dynamics given by (2.15), are everywhere solvable and the corresponding solution v of (1.1) belongs to $C^\infty(\mathbb{R}^2, \mathbb{R})$. We will call such solution $(v(x,t), w(x,t))$, constructed from $(v_0(x), w_0(x))$ via the inverse scattering method, an "inverse scattering solution" of (1.1).

The main result of this paper consists in the following: we show that for the "inverse scattering solution" $v(x,t)$ of (1.1), (1.4), where $E < 0$ and $v(x,0) = v_0(x)$ satisfies (1.5), the following estimate holds

$$|v(x,t)| \leqslant \frac{\mathrm{const}(v_0)\ln(3+|t|)}{(1+|t|)^{3/4}}, \quad t \in \mathbb{R},\ \text{uniformly on } x \in \mathbb{R}^2. \tag{1.6}$$

We show that this estimate is optimal in the sense that for some initial values $v(x,0)$ and for some lines $x = \omega t$, $\omega \in \mathbb{S}^1$, the exact asymptotics of $v(x,t)$ along these lines is $\frac{\mathrm{const}}{(1+|t|)^{3/4}}$ as $|t| \to \infty$ (where the constant is nonzero). Note that de facto the "inverse scattering solution" is the rapidly decaying solution in the sense of (1.2).

This work is a continuation of the studies on the large time asymptotic behavior of the solution of the Cauchy problem for the Novikov–Veselov equation started in [KN1] for the case of positive energy E. It was shown in [KN1] that if the initial data ($v_0(x)$, $w_0(x)$) satisfy the following conditions:

- ($v_0(x)$, $w_0(x)$) are sufficiently regular and decaying at $|x| \to \infty$,
- $v_0(x)$ is transparent for (1.3) at $E = E_{fixed} > 0$, i.e. its scattering amplitude f is identically zero at fixed energy,
- the additional "scattering data" b for $v_0(x)$ is non–singular,

then the corresponding solution of (1.1),(1.4) can be estimated as

$$|v(x,t)| \leqslant \frac{\mathrm{const} \cdot \ln(3+|t|)}{1+|t|}, \quad t \in \mathbb{R} \text{ uniformly on } x \in \mathbb{R}^2.$$

This estimate implies, in particular, that there are no localized soliton–type traveling waves in the asymptotics of (1.1) with the"transparent" at $E = E_{fixed} > 0$ Cauchy data from the aforementioned class, in contrast with the large time asymptotics for solutions of the KdV equation with reflectionless initial data.

It was shown in [N2] that all soliton–type (traveling wave) solutions of (1.1) with $E > 0$ must have a zero scattering amplitude at fixed energy; in addition it was proved in [N2] that for the equation (1.1) with $E > 0$ no exponentially localized soliton–type solutions exist (even if the scattering data are allowed to have singularities). However, in [G1], [G2] a family of algebraically localized solitons (traveling waves) was constructed de facto (see also [KN2]). We note that for the case $E < 0$, though the absence of exponentially–localized solitons has been proved (see [KN3]), the existence of bounded algebraically localized solitons is still an open question.

Note that studies on the large time asymptotics for solutions of the Cauchy problem for the Kadomtsev–Petviashvili equations were fulfilled in [MST], [HNS], [K].

The proofs provided in the present paper are based on the scheme developed in [KN1], the stationary phase method (see [Fe]). These proofs include, in particular, an analysis of some cubic algebraic equation depending on a complex parameter.

This work was fulfilled in the framework of research carried out under the supervision of R.G. Novikov.

2 Inverse "scattering" transform for the two–dimensional Schrödinger equation at a fixed negative energy

In this section we give a brief description of the inverse "scattering" transform for the two-dimensional Schrödinger equation (1.3) at a fixed negative energy E (see [GN], [N1], [G2]).

First of all, we note that by scaling transform we can reduce the scattering problem with an arbitrary fixed negative energy to the case when $E = -1$. Therefore, in our further reasoning we will assume that $E = -1$.

Let us consider potentials $v(x)$ for the problem (1.3) satisfying the following conditions

$$v = \bar{v}, \quad |v(x)| \leqslant \frac{q}{(1+|x|)^{2+\varepsilon}}, \quad x \in \mathbb{R}^2, \tag{2.1}$$

for some fixed q and $\varepsilon > 0$. Then it is known that for $\lambda \in \mathbb{C}\backslash(0 \cup \mathcal{E})$, where

$$\begin{aligned}&\mathcal{E} \text{ is the set of zeros of the modified Fredholm determinant } \Delta\\ &\text{for equation (2.6),}\end{aligned} \tag{2.2}$$

there exists a unique continuous solution $\psi(z,\lambda)$ of (1.3) with the following asymptotics

$$\psi(z,\lambda) = e^{-\frac{1}{2}(\lambda\bar{z}+z/\lambda)}\mu(z,\lambda), \quad \mu(z,\lambda) = 1 + o(1), \quad |z| \to \infty. \tag{2.3}$$

Here the notation $z = x_1 + ix_2$ is used. These solutions are known as the Faddeev solutions for the Schrodinger equation (1.3), $E = -1$, see for example [Fa], [N1].

The function $\mu(z,\lambda)$ satisfies the following integral equation

$$\mu(z,\lambda) = 1 + \iint\limits_{\zeta\in\mathbb{C}} g(z-\zeta,\lambda)v(\zeta)\mu(\zeta,\lambda)d\mathrm{Re}\zeta d\mathrm{Im}\zeta \tag{2.4}$$

$$g(z,\lambda) = -\left(\frac{1}{2\pi}\right)^2 \iint\limits_{\zeta\in\mathbb{C}} \frac{\exp(i/2(\zeta\bar{z}+\bar{\zeta}z))}{\zeta\bar{\zeta}+i(\lambda\bar{\zeta}+\zeta/\lambda)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{2.5}$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$.

In terms of $m(z,\lambda) = (1+|z|)^{-(2+\varepsilon)/2}\mu(z,\lambda)$ equation (2.4) takes the form

$$m(z,\lambda) = (1+|z|)^{-(2+\varepsilon)/2} + \iint\limits_{\zeta\in\mathbb{C}} (1+|z|)^{-(2+\varepsilon)/2} g(z-\zeta,\lambda) \frac{v(\zeta)}{(1+|\zeta|)^{-(2+\varepsilon)/2}} m(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{2.6}$$

where $z\in\mathbb{C}$, $\lambda\in\mathbb{C}\backslash 0$. In addition, $A(\cdot,\cdot,\lambda)\in L^2(\mathbb{C}\times\mathbb{C})$, $|\mathrm{Tr}A^2(\lambda)|<\infty$, where $A(z,\zeta,\lambda)$ is the Schwartz kernel of the integral operator $A(\lambda)$ of the integral equation (2.6). Thus, the modified Fredholm determinant for (2.6) can be defined by means of the formula:

$$\ln\Delta(\lambda)=\mathrm{Tr}(\ln(I-A(\lambda))+A(\lambda)) \tag{2.7}$$

(see [GK] for more precise sense of such definition).

Taking the subsequent members in the asymptotic expansion (2.3) for $\psi(z,\lambda)$, we obtain (see [N1]):

$$\psi(z,\lambda)=\exp\left(-\frac{1}{2}\left(\lambda\bar z+\frac{z}{\lambda}\right)\right)\Bigg\{1-2\pi\mathrm{sgn}(1-\lambda\bar\lambda)\times$$
$$\times\left(\frac{i\lambda a(\lambda)}{z-\lambda^2\bar z}+\exp\left(-\frac{1}{2}\left(\left(\frac{1}{\bar\lambda}-\lambda\right)\bar z+\left(\frac{1}{\lambda}-\bar\lambda\right)z\right)\right)\frac{\bar\lambda b(\lambda)}{i(\bar\lambda^2 z-\bar z)}\right)+o\left(\frac{1}{|z|}\right)\Bigg\}, \tag{2.8}$$

$|z|\to\infty$, $\lambda\in\mathbb{C}\backslash(\mathcal{E}\cup 0)$.

The functions $a(\lambda)$, $b(\lambda)$ from (2.8) are called the "scattering" data for the problem (1.3), (2.1) with $E=-1$.

The function $\mu(z,\lambda)$, defined by (2.4), satisfies the following properties:

$$\mu(z,\lambda)\text{ is a continuous function of }\lambda\text{ on }\mathbb{C}\backslash(0\cup\mathcal{E}); \tag{2.9}$$

$$\frac{\partial\mu(z,\lambda)}{\partial\bar\lambda}=r(z,\lambda)\overline{\mu(z,\lambda)}, \tag{2.10a}$$

$$r(z,\lambda)=r(\lambda)\exp\left(\frac{1}{2}\left(\left(\lambda-\frac{1}{\bar\lambda}\right)\bar z-\left(\bar\lambda-\frac{1}{\lambda}\right)z\right)\right), \tag{2.10b}$$

$$r(\lambda)=\frac{\pi\mathrm{sgn}(1-\lambda\bar\lambda)}{\bar\lambda}b(\lambda) \tag{2.10c}$$

for $\lambda\in\mathbb{C}\backslash(0\cup\mathcal{E})$;

$$\mu\to 1,\ \text{as }\lambda\to\infty,\ \lambda\to 0. \tag{2.11}$$

The function $b(\lambda)$ possesses the following symmetries:

$$b\left(-\frac{1}{\bar\lambda}\right)=b(\lambda),\quad b\left(\frac{1}{\bar\lambda}\right)=\overline{b(\lambda)},\quad \lambda\in\mathbb{C}\backslash 0. \tag{2.12}$$

In addition, the following theorem is valid:

Theorem 2.1 (see [GN], [N1], [G2]).

(i) Let v satisfy (1.5). Then the scattering data $b(\lambda)$ for the potential $v(z)$ satisfy properties (2.12) and $b\in\mathcal{S}(\bar D_-)$, where $D_-=\{\lambda\in\mathbb{C}\colon|\lambda|>1\}$, $\bar D_-=D_-\cup\partial D_-$ and $\mathcal{S}$ denotes the Schwartz class.

(ii) Let b be a function on $\mathbb{C}$, such that $b\in\mathcal{S}(\bar D_-)$ and the symmetry properties (2.12) hold. Then the equations of inverse scattering (2.16)–(2.18) are uniquely solvable and the corresponding potential $v(z)$ satisfies the following properties: $v\in C^\infty(\mathbb{C})$, $v=\bar v$, $|v(z)|\to 0$ when $|z|\to\infty$.

Let us denote by T the unit circle on the complex plane:

$$T = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}. \tag{2.13}$$

Then, in addition, it is known that under the assumptions of item (i) of Theorem 2.1 the function b is continuous on $\mathbb{C}$ and its derivative $\partial_\lambda b(\lambda)$ is bounded on $\mathbb{C}$, though discontinuous, in general, on T.

Finally, if $(v(z,t), w(z,t))$ is a solution of equation (1.1) with $E = -1$ in the sense of (1.2), then the dynamics of the "scattering" data is described by the following equations (see [GN])

$$a(\lambda, t) = a(\lambda, 0), \tag{2.14}$$

$$b(\lambda, t) = \exp\left\{\left(\lambda^3 + \frac{1}{\lambda^3} - \bar{\lambda}^3 - \frac{1}{\bar{\lambda}^3}\right) t\right\} b(\lambda, 0). \tag{2.15}$$

The reconstruction of the potential $v(z,t)$ from these "scattering" data is based on the following scheme.

1. Function $\mu(z,\lambda,t)$ is constructed as the solution of the following integral equation

$$\mu(z,\lambda,t) = 1 - \frac{1}{\pi}\iint_{\mathbb{C}} r(z,\zeta,t)\overline{\mu(z,\zeta,t)}\frac{d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta}{\zeta - \lambda}. \tag{2.16}$$

2. Expanding $\mu(z,\lambda,t)$ as $\lambda \to \infty$,

$$\mu(z,\lambda,t) = 1 + \frac{\mu_{-1}(z,t)}{\lambda} + o\left(\frac{1}{|\lambda|}\right), \tag{2.17}$$

   we define $v(z,t)$ as

$$v(z,t) = -2\partial_z \mu_{-1}(z,t). \tag{2.18}$$

3. It can be shown that

$$L\psi = E\psi$$

   where

$$L = -4\partial_z\partial_{\bar{z}} + v(z,t), \quad \overline{v(z,t)} = v(z,t), \quad E = -1$$
$$\psi(z,\lambda,t) = e^{-\frac{1}{2}(\lambda\bar{z} + z/\lambda)}\mu(z,\lambda,t), \quad \lambda \in \mathbb{C}, \quad z \in \mathbb{C}, \quad t \in \mathbb{R}.$$

3 Estimate for the linearized case

Consider

$$\begin{aligned} I(t,z) &= \iint\limits_{\mathbb{C}} f(\zeta)\exp(S(\zeta,z,t))d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \\ J(t,z) &= 3\iint\limits_{\mathbb{C}} \frac{\zeta}{\bar{\zeta}} f(\zeta)\exp(S(\zeta,z,t))d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \end{aligned} \tag{3.1}$$

where $z \in \mathbb{C}$, $t \in \mathbb{R}$, $f(\zeta) \in L^1(\mathbb{C})$, S is defined by

$$S(\lambda,z,t) = \frac{1}{2}\left(\left(\lambda - \frac{1}{\bar{\lambda}}\right)\bar{z} - \left(\bar{\lambda} - \frac{1}{\lambda}\right)z\right) + t\left(\lambda^3 + \frac{1}{\lambda^3} - \bar{\lambda}^3 - \frac{1}{\bar{\lambda}^3}\right). \tag{3.2}$$

We will also assume that $f(\zeta)$ satisfies the following conditions

$$f \in C^{\infty}(\bar{D}_{+}), \quad f \in C^{\infty}(\bar{D}_{-}), \tag{3.3a}$$

$$\partial_{\lambda}^{m} \partial_{\bar{\lambda}}^{n} f(\lambda) = \begin{cases} O\left(|\lambda|^{-\infty}\right) & \text{as } |\lambda| \to \infty, \\ O\left(|\lambda|^{\infty}\right) & \text{as } |\lambda| \to 0, \end{cases} \quad \text{for all } m, n \geqslant 0, \tag{3.3b}$$

where

$$D_{+} = \{\lambda \in \mathbb{C}\colon 0 < |\lambda| \leqslant 1\}, \quad D_{-} = \{\lambda \in \mathbb{C}\colon |\lambda| \geqslant 1\}, \tag{3.4}$$

and $\bar{D}_{+} = D_{+} \cup T$, $\bar{D}_{-} = D_{-} \cup T$ with T defined by (2.13).

Note that if $v(z,t) = I(t,z)$, $w(z,t) = J(t,z)$, where

$$(|\zeta|^{3} + |\zeta|^{-3}) f(\zeta) \in L^{1}(\mathbb{C}) \text{ as a function of } \zeta,$$

and, in addition,

$$\overline{f(\zeta)} = f(-\zeta) \quad \text{and/or} \quad \overline{f(\zeta)} = -|\zeta|^{-4} f\left(\frac{1}{\bar{\zeta}}\right),$$

then v, w satisfy the linearized Novikov–Veselov equation (1.1) with $E = -1$. Besides, if

$$f(\zeta) = \frac{\pi |1 - \zeta\bar{\zeta}|}{2|\zeta|^{2}} b(\zeta), \tag{3.5}$$

where $b(\zeta)$ is the scattering data for the initial functions $(v_0(z), w_0(z))$ of the Cauchy problem (1.1), (1.4), then the integrals $I(t,z)$, $J(t,z)$ of (3.1) represent the approximation of the solution $(v(z,t), w(z,t))$ under the assumption that $\| v \| \ll 1$.

The goal of this section is to give, in particular, a uniform estimate of the large–time behavior of the integral $I(t,z)$ of (3.1).

For this purpose we introduce parameter $u = \frac{z}{t}$ and write the integral I in the following form

$$I(t,u) = \iint\limits_{\mathbb{C}} f(\zeta) \exp(tS(u,\zeta)) d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{3.6}$$

where

$$S(u,\zeta) = \frac{1}{2}\left(\left(\zeta - \frac{1}{\bar{\zeta}}\right)\bar{u} - \left(\bar{\zeta} - \frac{1}{\zeta}\right)u\right) + \left(\zeta^{3} - \bar{\zeta}^{3} + \frac{1}{\zeta^{3}} - \frac{1}{\bar{\zeta}^{3}}\right). \tag{3.7}$$

We will start by studying the properties of the stationary points of the function $S(u,\zeta)$ with respect to ζ. These points satisfy the equation

$$S'_{\zeta} = \frac{\bar{u}}{2} - \frac{u}{2\zeta^{2}} + 3\zeta^{2} - \frac{3}{\zeta^{4}} = 0. \tag{3.8}$$

The degenerate stationary points obey additionally the equation

$$S''_{\zeta\zeta} = \frac{u}{\zeta^{3}} + 6\zeta + \frac{12}{\zeta^{5}} = 0. \tag{3.9}$$

We denote $\xi = \zeta^{2}$ and

$$Q(u,\xi) = \frac{u}{2} - \frac{\bar{u}}{2\xi} + 3\xi - \frac{3}{\xi^{2}}.$$

For each ξ, a root of the function $Q(u,\xi)$, there are two corresponding stationary points of $S(u,\zeta)$, $\zeta = \pm\sqrt{\xi}$.

The function $S'_\zeta(u,\zeta)$ can be represented in the following form

$$S'_\zeta(u,\zeta) = \frac{3}{\zeta^4}(\zeta^2 - \zeta_0^2(u))(\zeta^2 - \zeta_1^2(u))(\zeta^2 - \zeta_2^2(u)). \tag{3.10}$$

We will also use hereafter the following notations:

$$\mathcal{U} = \{u = -6(2e^{-i\varphi} + e^{2i\varphi})\colon \varphi \in [0, 2\pi)\}$$

and

$$\mathbb{U} = \{u = re^{i\psi}\colon \psi = \mathrm{Arg}(-6(2e^{-i\varphi} + e^{2i\varphi})),\ 0 \leqslant r \leqslant |6(2e^{i\varphi} + e^{-2i\varphi})|,\ \varphi \in [0, 2\pi)\},$$

the domain limited by the curve $\mathcal{U}$.

Lemma 3.1 (see [KN1]).

1. *If* $u = -18e^{\frac{2\pi ik}{3}}$, $k = 0, 1, 2$, *then*

$$\zeta_0(u) = \zeta_1(u) = \zeta_2(u) = e^{\frac{\pi ik}{3}} \tag{3.11}$$

   and $S(u,\zeta)$ *has two degenerate stationary points, corresponding to a third-order root of the function* $Q(u,\xi)$, $\xi_1 = e^{\frac{2\pi ik}{3}}$.

2. *If* $u \in \mathcal{U}$ (*i.e.* $u = -6(2e^{-i\varphi} + e^{2i\varphi})$) *and* $u \neq -18e^{\frac{2\pi ik}{3}}$, $k = 0, 1, 2$, *then*

$$\zeta_0(u) = \zeta_1(u) = e^{-i\varphi/2}, \quad \zeta_2(u) = e^{i\varphi}. \tag{3.12}$$

   Thus $S(u,\zeta)$ *has two degenerate stationary points, corresponding to a second-order root of the function* $Q(u,\xi)$, $\xi_1 = e^{-i\varphi}$, *and two non–degenerate stationary points corresponding to a first-order root,* $\xi_2 = e^{2i\varphi}$.

3. *If* $u \in \mathrm{int}\mathbb{U} = \mathbb{U}\backslash\partial\mathbb{U}$, *then*

$$\zeta_i(u) = e^{-i\varphi_i} \text{ for some real } \varphi_i, \quad \text{and} \quad \zeta_i(u) \neq \zeta_j(u) \quad \text{for} \quad i \neq j. \tag{3.13}$$

   In this case the stationary points of $S(u,\zeta)$ *are non-degenerate and correspond to the roots of the function* $Q(u,\xi)$ *with absolute values equal to 1.*

4. *If* $u \in \mathbb{C}\backslash\mathbb{U}$, *then*

$$\zeta_0(u) = (1+\omega)e^{-i\varphi/2}, \quad \zeta_1(u) = e^{i\varphi}, \quad \zeta_2(u) = (1+\omega)^{-1}e^{-i\varphi/2} \tag{3.14}$$

   for certain real values φ *and* $\omega > 0$.

   In this case the stationary points of the function $S(u,\zeta)$ *are non-degenerate, and correspond to the roots of the function* $Q(u,\xi)$ *that can be expressed as* $\xi_0 = (1+\tau)e^{-i\varphi}$, $\xi_1 = e^{2i\varphi}$, $\xi_2 = (1+\tau)^{-1}e^{-i\varphi}$, $(1+\tau) = (1+\omega)^2$.

Formula (3.10) and Lemma 3.1 give a complete description of the stationary points of the function $S(u,\zeta)$.

In order to estimate the large–time behavior of the integral having the form

$$I(t,u,\lambda) = \iint\limits_{\mathbb{C}} f(\zeta,\lambda)\exp(tS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta \tag{3.15}$$

(where $S(u,\zeta)$ is an imaginary–valued function) uniformly on $u, \lambda \in \mathbb{C}$, in the present and the following sections we will use the following general scheme.

1. Consider D_ε, the union of disks with a radius of ε and centers in singular points of function $f(\zeta,\lambda)$ and stationary points of $S(u,\zeta)$ with respect to ζ.
2. Represent $I(t,u,\lambda)$ as the sum of integrals over D_ε and $\mathbb{C}\backslash D_\varepsilon$:

$$\begin{aligned} &I(t,u,\lambda) = I_{int} + I_{ext}, \qquad \text{where}\\ &I_{int} = \iint\limits_{D_\varepsilon} f(\zeta,\lambda)\exp(tS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta,\\ &I_{ext} = \iint\limits_{\mathbb{C}\backslash D_\varepsilon} f(\zeta,\lambda)\exp(tS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \end{aligned} \tag{3.16}$$

3. Find an estimate of the form

$$|I_{int}| = O\left(\varepsilon^\alpha\right), \quad \text{as} \quad \varepsilon\to 0 \quad (\alpha \geqslant 1)$$

uniformly on u, λ, t.

4. Integrate I_{ext} by parts using Stokes formula

$$I_{ext} = -\frac{1}{2it}\int\limits_{\partial D_\varepsilon}\frac{f(\zeta,\lambda)\exp(tS(u,\zeta))}{S'_\zeta(u,\zeta)}d\bar\zeta -$$

$$-\frac{1}{2it}\int\limits_{T\backslash D_\varepsilon}\frac{(f_+(\zeta,\lambda)-f_-(\zeta,\lambda))\exp(tS(u,\zeta))}{S'_\zeta(u,\zeta)}d\bar\zeta - \frac{1}{t}\iint\limits_{\mathbb{C}\backslash D_\varepsilon}\frac{f'_\zeta(\zeta,\lambda)\exp(tS(u,\zeta))}{S'_\zeta(u,\zeta)}d\mathrm{Re}\zeta d\mathrm{Im}\zeta +$$

$$+\frac{1}{t}\iint\limits_{\mathbb{C}\backslash D_\varepsilon}\frac{f(\zeta,\lambda)\exp(tS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}d\mathrm{Re}\zeta d\mathrm{Im}\zeta = -\frac{1}{t}(I_1+I_2+I_3-I_4), \tag{3.17}$$

where $f_\pm(\zeta,\lambda) = \lim\limits_{\delta\to 0} f(\zeta(1\mp\delta),\lambda)$, $\zeta\in T$ and T is defined by (2.13).

5. For each I_i find an estimate of the form

$$|I_i| = O\left(\frac{1}{\varepsilon^\beta}\right), \qquad \text{as } \varepsilon\to 0.$$

6. Set $\varepsilon = \dfrac{1}{(1+|t|)^k}$, where $k(\alpha+\beta)=1$, which yields the overall estimate

$$|I(t,u,\lambda)| = O\left(\frac{1}{(1+|t|)^{\frac{\alpha}{\alpha+\beta}}}\right), \quad \text{as} \quad |t|\to\infty.$$

Using this scheme we obtain, in particular, the following result

Lemma 3.2. *Let a function f satisfy assumptions (3.3) and, additionally,*

$$f|_T \equiv 0, \quad T = \{\lambda\in\mathbb{C}\colon |\lambda| = 1\}. \tag{3.18}$$

Then the integral $I(t,u)$ of (3.6) can be estimated

$$|I(t,u)| \leqslant \frac{\mathrm{const}(f)\ln(3+|t|)}{(1+|t|)^{3/4}} \quad \textit{for} \quad t\in\mathbb{R} \tag{3.19}$$

uniformly on $u\in\mathbb{C}$.

Note that condition (3.18) is satisfied if f has the special form (3.5). A detailed proof of Lemma 3.2 is given in Section 6.

4 Estimate for the non–linearized case

In this section we prove estimate (1.6) for the solution $v(x,t)$ of the Cauchy problem for the Novikov–Veselov equation at negative energy with the initial data $v(x,0)$ satisfying properties (1.5).

We proceed from the formulas (2.17), (2.18) for the potential $v(z,t)$ and the integral equation (2.16) for $\mu(z,\lambda,t)$.

We write (2.16) as

$$\mu(z,\lambda,t) = 1 + (A_{z,t}\mu)(z,\lambda,t), \tag{4.1}$$

where

$$(A_{z,t}f)(\lambda) = \partial_{\bar\lambda}^{-1}(r(\lambda)\exp(tS(u,\lambda))\overline{f(\lambda)}) = -\frac{1}{\pi}\iint\limits_{\mathbb{C}} \frac{r(\zeta)\exp(tS(u,\zeta))\overline{f(\zeta)}}{\zeta-\lambda} d\mathrm{Re}\zeta d\mathrm{Im}\zeta$$

and $S(u,\zeta)$ is defined by (3.7), $u = \dfrac{z}{t}$.

Equation (4.1) can be also written in the form

$$\mu(z,\lambda,t) = 1 + A_{z,t}\cdot 1 + (A_{z,t}^2\mu)(z,\lambda,t). \tag{4.2}$$

According to the theory of the generalized analytic functions (see [V]), equations (4.1), (4.2) have a unique bounded solution for all z,t. This solution can be written as

$$\mu(z,\lambda,t) = (I - A_{z,t}^2)^{-1}(1 + A_{z,t}\cdot 1). \tag{4.3}$$

Equation (4.3) implies the following formal asymptotic expansion

$$\mu(z,\lambda,t) = (I + A_{z,t}^2 + A_{z,t}^4 + \ldots)(1 + A_{z,t}\cdot 1). \tag{4.4}$$

We also introduce functions $\nu(z,\lambda,t) = \partial_z\mu(z,\lambda,t)$ and $\eta(z,\lambda,t) = \partial_{\bar z}\mu(z,\lambda,t)$. In terms of these functions the potential $v(z,t)$ is obtained by the formula

$$v(z,t) = -2\nu_{-1}(z,t), \tag{4.5}$$

where $\nu_{-1}(z,t)$ is defined by expanding $\nu(z,\lambda,t)$ as $|\lambda|\to\infty$:

$$\nu(z,\lambda,t) = \frac{\nu_{-1}(z,t)}{\lambda} + o\left(\frac{1}{|\lambda|}\right), \quad |\lambda|\to\infty.$$

The pair of function $\nu(z,\lambda,t)$, $\eta(z,\lambda,t)$ satisfy the following system of differential equations:

$$\begin{cases} \dfrac{\partial\nu(z,\lambda,t)}{\partial\bar\lambda} = \partial_z r(z,\lambda,t)\overline{\mu(z,\lambda,t)} + r(z,\lambda,t)\overline{\eta(z,\lambda,t)}, \\ \dfrac{\partial\eta(z,\lambda,t)}{\partial\bar\lambda} = \partial_z r(z,\lambda,t)\overline{\mu(z,\lambda,t)} + r(z,\lambda,t)\overline{\nu(z,\lambda,t)}. \end{cases} \tag{4.6}$$

Equations (4.6) can also be written in the integral form

$$\begin{cases} \nu(z,\lambda,t) = (B_{z,t}\mu)(z,\lambda,t) + (A_{z,t}\eta)(z,\lambda,t), \\ \eta(z,\lambda,t) = (B_{z,t}\mu)(z,\lambda,t) + (A_{z,t}\nu)(z,\lambda,t), \end{cases} \tag{4.7}$$

where operator $B_{z,t}$ is defined

$$(B_{z,t}f)(\lambda) = \partial_{\bar\lambda}^{-1}(\partial_z r(z,\lambda,t)\overline{f(\lambda)}) = -\frac{1}{\pi}\iint\limits_{\mathbb{C}} \frac{\partial_z r(z,\zeta,t)\overline{f(\zeta)}}{\zeta-\lambda} d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \tag{4.8}$$

Thus for the function $\nu(z,\lambda,t)$ we obtain equation

$$\nu = (B_{z,t} + A_{z,t}B_{z,t})\mu + A_{z,t}^2\nu,$$

or the following formal asymptotic expansion

$$\nu = (I + A_{z,t}^2 + A_{z,t}^4 + \ldots)((B_{z,t} + A_{z,t}B_{z,t})(I + A_{z,t}^2 + A_{z,t}^4 + \ldots)(1 + A_{z,t}\cdot 1)). \tag{4.9}$$

We will write this formula in the form $\nu = B_{z,t}\cdot 1 + A_{z,t}B_{z,t}\cdot 1 + R_{z,t}(\lambda)$.

Lemma 4.1. *Let $f(\lambda,z,t)$ be an arbitrary testing function such that*

$$|f| \leqslant \frac{c_f}{(1+|t|)^\delta}, \quad |\partial_\lambda f| \leqslant \frac{c_f}{(1+|t|)^\delta} \quad \forall\lambda\in\mathbb{C}, z\in\mathbb{C}, t\in\mathbb{R}$$

with some positive constant c_f independent of λ, z, t and some $\delta \geqslant 0$. Then:

1. *The following estimates hold for $B_{z,t}\cdot f$:*

$$(B_{z,t}\cdot f)(\lambda) = \frac{\beta_1(z,t)}{\lambda} + o\left(\frac{1}{|\lambda|}\right) \text{ for } \lambda\to\infty,$$

where

$$\beta_1(z,t) = \frac{1}{\pi}\iint_{\mathbb{C}} \partial_z r(z,\zeta,t)\overline{f(\zeta,z,t)}d\mathrm{Re}\zeta d\mathrm{Im}\zeta,$$

and

$$|\beta_1(z,t)| \leqslant \frac{\hat\beta_1 c_f \ln(3+|t|)}{(1+|t|)^{3/4+\delta}}; \tag{4.10}$$

in addition,

$$|(B_{z,t}\cdot f)(\lambda)| \leqslant \frac{\hat\beta_2 c_f \ln(3+|t|)}{(1+|t|)^{1/2+\delta}} \text{ for } \lambda\in T, \tag{4.11}$$

where T is defined by (2.13), and

$$|(B_{z,t}\cdot f)(\lambda)| \leqslant \frac{\hat\beta_3 c_f}{(1+|t|)^{1/4+\delta}} \quad \forall\lambda\in\mathbb{C}. \tag{4.12}$$

2. *The following estimates hold for $A_{z,t}\cdot B_{z,t}\cdot f$:*

$$(A_{z,t}\cdot B_{z,t}\cdot f)(\lambda) = \frac{\alpha_1(z,t)}{\lambda} + o\left(\frac{1}{|\lambda|}\right) \text{ for } \lambda\to\infty,$$

where

$$\alpha_1(z,t) = -\frac{1}{\pi^2}\iint\limits_{\mathbb{C}} d\mathrm{Re}\zeta d\mathrm{Im}\zeta\, r(z,\zeta,t)\iint\limits_{\mathbb{C}} \frac{\partial_z r(z,\eta,t)}{\eta-\zeta}\overline{f(\eta,z,t)}d\mathrm{Re}\eta\, d\mathrm{Im}\eta, \tag{4.13}$$

and

$$|\alpha_1(z,t)| \leqslant \frac{\hat\alpha_1 c_f}{(1+|t|)^{3/4+\delta}}; \tag{4.14}$$

in addition,

$$|(A_{z,t}\cdot B_{z,t}\cdot f)(\lambda)| \leqslant \frac{\hat\alpha_2 c_f}{(1+|t|)^{1/2+\delta}} \quad \forall\lambda\in\mathbb{C}. \tag{4.15}$$

3. The following estimates for $A^n_{z,t}\cdot f$ hold:

$$(A^n_{z,t}\cdot f)(\lambda)=\frac{\gamma_n(z,t)}{\lambda}+o\left(\frac{1}{|\lambda|}\right)\ \textit{for}\ \lambda\to\infty,$$

where

$$\gamma_n(z,t)=\frac{1}{\pi}\iint\limits_{\mathbb{C}} r(z,\zeta,t)\overline{(A^{n-1}_{z,t}\cdot f)(\zeta)}d\mathrm{Re}\zeta d\mathrm{Im}\zeta$$

and

$$|\gamma_n(z,t)|\leqslant\frac{(\hat\gamma_1)^n c_f}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n-1}{2}\rceil+\frac{2}{5}}},\tag{4.16}$$

where $\lceil s\rceil$ denotes the smallest integer following s. In addition,

$$|(A^n_{z,t}\cdot f)(\lambda)|\leqslant\frac{(\hat\gamma_2)^n c_f}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n}{2}\rceil}}\quad\forall\lambda\in\mathbb{C}.\tag{4.17}$$

4. The following estimate holds for $R_{z,t}$:

$$R_{z,t}(\lambda)=\frac{q(z,t)}{\lambda}+o\left(\frac{1}{|\lambda|}\right)\ \textit{for}\ \lambda\to\infty,\tag{4.18}$$

and

$$|q(z,t)|\leqslant\frac{\hat q(c_f)}{(1+|t|)^{9/10}}.\tag{4.19}$$

A detailed proof of Lemma 4.1 is given in Section 6.

From formulas (4.5), (4.9) and Lemma 4.1 follows immediately the following theorem.

Theorem 4.1. *Let $v(x,t)$ be the "inverse scattering solution" of the Cauchy problem for the Novikov–Veselov equation (1.1) with $E=-1$ and the initial data $v(x,0)=v_0(x)$ satisfying (1.5). Then*

$$|\,v(x,t)|\leqslant\frac{\mathrm{const}(v_0)\ln(3+|t|)}{(1+|t|)^{3/4}},\quad x\in\mathbb{R}^2,t\in\mathbb{R}.$$

5 Optimality of estimates (1.6) and (3.19)

In this section we show that estimates (1.6) and (3.19) are optimal in the following sense: there exists such a line $z=\hat u t$ that along this line $I(z,t)$ from (3.19) behaves asymptotically as $\frac{\mathrm{const}}{(1+|t|)^{3/4}}$ with come nonzero constant; there exist such initial data satisfying (1.5) that the corresponding solution $v(z,t)$ behaves asymptotically as $\frac{\mathrm{const}}{(1+|t|)^{3/4}}$ as $|t|\to\infty$, where the constant is nonzero.

5.1 Optimality of the estimate for the linearized case

Let us consider the integral

$$I(t,u)=\iint\limits_{\mathbb{C}} f(\zeta)\exp(tS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta,\tag{5.1}$$

where

$$f(\zeta)=\frac{\pi|1-\zeta\bar\zeta|}{2|\zeta|^2}b(\zeta),\tag{5.2}$$

with $b(\zeta)$ satisfying (3.3), and $S(u,\zeta)$ is defined by (3.7), for $u=\hat{u}=-18$. As shown in Lemma 3.1 for this value of parameter u the phase $S(\hat{u},\zeta)=S(\zeta)$ has two degenerate stationary points $\zeta=\pm1$ of the third order.

To calculate the exact asymptotic behavior of $I(t,\hat{u})$ we will use the classic stationary method as described in [Fe]. First of all, we note that $f(\zeta)$ is continuous, but not continuously differentiable on $\mathbb{C}$. Thus we will consider separately the integrals

$$I_+(t)=\iint\limits_{D_+} f(\zeta)\exp(tS(\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \quad I_-(t)=\iint\limits_{D_-} f(\zeta)\exp(tS(\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta,$$

where D_+ and D_- are defined in (3.4).

Let us introduce the partition of unity $\psi_1(\zeta)+\psi_0(\zeta)+\psi_{-1}(\zeta)\equiv 1$, such that $0\leqslant\psi_i\leqslant 1$, $\psi_i\in C^\infty(\mathbb{C})$, $\psi_{\pm1}(\zeta)\equiv 1$ in some neighborhood of $\zeta=\pm1$, respectively, and $\psi_{\pm1}(\zeta)\equiv 0$ everywhere outside some neighborhood of $\zeta=\pm1$ respectively. Then it is known (see [Fe]) that

$$I_+(t)=\iint\limits_{D_+} f(\zeta)\psi_1(\zeta)\exp(tS(\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta+\iint\limits_{D_+} f(\zeta)\psi_{-1}(\zeta)\exp(tS(\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta+O\left(\frac{1}{|t|}\right)=$$

$$=I_+^+(t)+I_+^-(t)+O\left(\frac{1}{|t|}\right) \text{ as } |t|\to\infty.$$

First we will estimate $I_+^+(t)$. We note that for the phase $S(\zeta)$ the following representation is valid

$$S(\zeta)=P(\zeta)-P(\bar{\zeta}),$$

where $P(\zeta)$ is a holomorphic function defined by

$$P(\zeta)=-9\zeta-\frac{9}{\zeta}+\zeta^3+\frac{1}{\zeta^3}+16;$$

in addition, $P_\zeta(\zeta)=S_\zeta(\zeta)=\frac{3}{\zeta^4}(\zeta-1)^3(\zeta+1)^3$.

Note that $P(\zeta)$ can be written in the form $P(\zeta)=\rho(\zeta)(\zeta-1)^4$, where $\rho(\zeta)=\frac{\zeta^2+4\zeta+1}{\zeta^3}$ and $\lim\limits_{\zeta\to1}\rho(\zeta)\neq 0$. For the function $\rho(\zeta)$ the expression $(\rho(\zeta))^{1/4}$ can be uniquely defined in some neighborhood of $\zeta=1$. Further, we define the transformation $\zeta\to\eta$:

$$\eta=(\rho(\zeta))^{1/4}(\zeta-1).$$

Since we have that

$$\left.\frac{\partial\eta}{\partial\zeta}\right|_{\zeta=1}=\sqrt[4]{6}\neq 0, \tag{5.3}$$

the inverse transformation $\zeta=\varphi(\eta)$ is defined in some small neighborhood of $\eta=0$. In terms of the new variable η the phase can be represented

$$S(\zeta)=\eta^4-\bar{\eta}^4.$$

Now if we denote $x=\mathrm{Re}\eta$, $y=\mathrm{Im}\eta$, the integral $I_+^+(t)$ becomes

$$I_+^+(t)=\iint\limits_{\Delta_+}\tilde{f}(x+iy)\tilde{\psi}_1(x+iy)\exp(3itxy(x^2-y^2))|\partial_\eta\varphi(x+iy)|^2dxdy,$$

where $\tilde{f} = f \circ \varphi$, $\tilde{\psi}_1 = \psi_1 \circ \varphi$ and $\Delta_+ = \{(x, y) \in \mathbb{R}^2 \colon x < 0\}$, i.e. Δ_+ is the half-plane containing the image of $D_+ \cap B_\varepsilon(1)$ under the transformation $x + iy = \varphi^{-1}(\zeta)$.

The integral $I_+^+(t)$ can be written in the form

$$I_+^+(t) = \int\limits_{-\infty}^{+\infty} dc \exp(3itc) \int\limits_{\gamma_c \cap \Delta_+} \tilde{f}(x+iy)\tilde{\psi}_1(x+iy)|\partial_\eta \varphi(x+iy)|^2 d\omega_S,$$

where $d\omega_S$ is the Gelfand–Leray differential form, defined as

$$dS \wedge d\omega_S = dx \wedge dy \tag{5.4}$$

and in the particular case under study equal to

$$d\omega_S = \frac{-(x^3 - 3xy^2)dx + (3x^2y - y^3)dy}{(x^2 + y^2)^3};$$

γ_c is an oriented contour consisting of points of the set $\{S(x, y) = c\}$ with the orientation chosen so that (5.4) holds.

As $\tilde{\psi}_1(x + iy)$ is equal to zero outside some $B_R(0)$, a disk of radius R centered in the origin, then there exists such $c_* > 0$ that the set $\{S(x, y) = c\}$ lies outside $B_R(0)$ for any $c < -c_*$, $c > c_*$. Thus the integral $I_+^+(t)$ can be written

$$I_+^+(t) = \int\limits_{-c_*}^{c_*} dc \exp(3itc) \int\limits_{\gamma_c \cap \Delta_+} \tilde{f}(x+iy)\tilde{\psi}_1(x+iy)|\partial_\eta \varphi(x+iy)|^2 d\omega_S.$$

Performing the change of variables

$$\begin{cases} x \mapsto c^{1/4}x, \quad y \mapsto c^{1/4}y \quad \text{for } c > 0, \\ x \mapsto (-c)^{1/4}x, \quad y \mapsto (-c)^{1/4}y \quad \text{for } c < 0 \end{cases}$$

yields

$$I_+^+(t) = \int\limits_0^{c_*} \frac{dc \exp(3itc)}{c^{1/2}} F_+(c) + \int\limits_{-c_*}^{0} \frac{dc \exp(3itc)}{(-c)^{1/2}} F_-(c),$$

where

$$F_+(c) = \int\limits_{\gamma_+ \cap \{c^{1/4}(x,y) \in \Delta_+\}} \mathcal{F}(c, x, y) d\omega_S, \quad F_-(c) = \int\limits_{\gamma_- \cap \{(-c)^{1/4}(x,y) \in \Delta_+\}} \mathcal{F}(-c, x, y) d\omega_S, \tag{5.5}$$

γ_+, γ_- are oriented sets consisting of points of the sets $\{S(x, y) = 1\}$, $\{S(x, y) = -1\}$ correspondingly with orientation chosen so that (5.4) holds and

$$\mathcal{F}(c, x, y) = \tilde{f}(c^{1/4}(x+iy))\tilde{\psi}_1(c^{1/4}(x+iy))|\partial_\eta \varphi(c^{1/4}(x+iy))|^2.$$

For any fixed positive c the integrals in (5.5) converge because the set $\{S(x, y) = 1\}$ is separated from zero and, consequently, the denominator does not vanish, and because $\tilde{\psi}_1$ is a function with a bounded support and thus the domains of integration in (5.5) are, in fact, bounded. Besides, since Δ_+ is a conic set, the functions $F_+(c)$ and $F_-(c)$ can be expressed as follows

$$F_+(c) = \int\limits_{\gamma_+ \cap \Delta_+} \mathcal{F}(c, x, y) d\omega_S, \quad F_-(c) = \int\limits_{\gamma_- \cap \Delta_+} \mathcal{F}(-c, x, y) d\omega_S, \tag{5.6}$$

In some neighborhood $\mathcal{U}_0$ of $c^{1/4}(x+iy)=0$ containing the support of $\tilde{\psi}_1$ the function $\tilde{f}(c^{1/4}(x+iy))$ can be represented as

$$\tilde{f}(c^{1/4}(x+iy)) = f(1) + 6^{-1/4}[\partial_\zeta f_{D_+}(1)(x+iy) + \partial_{\bar{\zeta}} f_{D_+}(1)(x-iy)]c^{1/4} + g(c^{1/4}(x+iy)),$$

where $\partial_\zeta f_{D_+}(1) = \lim\limits_{\substack{\zeta\in D_+\\ \zeta\to 1}} \partial_\zeta f$, $\partial_{\bar{\zeta}} f_{D_+}(1) = \lim\limits_{\substack{\zeta\in D_+\\ \zeta\to 1}} \partial_{\bar{\zeta}} f$ and, in addition, g is a function that can be estimated

$$|g(c^{1/4}(x+iy))| \leqslant K|c^{1/4}(x+iy)|^{1+\alpha}, \quad c^{1/4}(x+iy)\in\mathcal{U}_0 \tag{5.7}$$

with some constants $\alpha>0$, $K>0$. Using (5.2), we note that $f(1)=0$, $\partial_\zeta f_{D_+}(1) = \partial_{\bar{\zeta}} f_{D_+}(1) \stackrel{\text{def}}{=} f'_{D_+}(1)$ and thus

$$\tilde{f}(c^{1/4}(x+iy)) = 6^{-1/4} f'_{D_+}(1)xc^{1/4} + g(c^{1/4}(x+iy)), \quad c^{1/4}(x+iy)\in\mathcal{U}_0.$$

Taking into account (5.3) we obtain

$$\mathcal{F}(c,x,y) = \gamma f'_{D_+}(1)xc^{1/4} + \tilde{g}(c^{1/4}(x+iy)), \quad c^{1/4}(x+iy)\in\mathcal{U}_0, \tag{5.8}$$

where $\gamma = 6^{-3/4}$ and $\tilde{g}$ is a function satisfying an estimate similar to (5.7).

It follows then that the functions $F_\pm(c)$ behave asymptotically as

$$F_\pm(c) = \gamma f'_{D_+}(1) J^\pm_{\Delta_+}(\pm c)^{1/4} + R(c), \text{ when } c\to 0,$$

where

$$J^+_{\Delta_+} = \int\limits_{\gamma_+\cap\Delta_+} x\,d\omega_S, \quad J^-_{\Delta_+} = \int\limits_{\gamma_-\cap\Delta_+} x\,d\omega_S$$

and $R(c)$ denotes the remainder.

The integrals $J^\pm_{\Delta_+}$ converge because the set $\{S(x,y)=\pm1\}$ represents a combination of curves which do not pass through zero and converge either to the lines $|y|=|x|$ or to the coordinate axes with velocities $|y|=\frac{1}{|x|^3}$ and $|x|=\frac{1}{|y|^3}$ correspondingly.

The remainder $R(c)$ behaves asymptotically as $o\left(c^{1/4}\right)$ because we can estimate

$$|R(c)| \leqslant \tilde{K}c^{1/4(1+\alpha)} \int\limits_{(\gamma_+\cup\gamma_-)\cap\Delta_+} |x+iy|^{1+\alpha}|d\omega_S|.$$

and the integral converges due to the properties of the set $\{S(x,y)=\pm1\}$ explained above.

Thus $I^+_+(t)$ behaves asymptotically as (see [Fe, Chapter III, §1])

$$I^+_+(t) = \frac{\gamma}{3^{3/4}} f'_{D_+}(1)\Gamma\left(\frac{3}{4}\right)\left[J^+_{\Delta_+}\exp\left(\frac{i\pi 3}{8}\right) + J^-_{\Delta_+}\exp\left(-\frac{i\pi 3}{8}\right)\right]\frac{1}{t^{3/4}} + o\left(\frac{1}{t^{3/4}}\right),$$

where Γ is the Gamma function.

Let us perform the same procedure for $I^-_+(t)$. In this case we will define $P(\zeta)$ as $P(\zeta) = -9\zeta - \frac{9}{\zeta} + \zeta^3 + \frac{1}{\zeta^3} - 16$ and obtain

$$P(\zeta) = \rho(\zeta)(\zeta+1)^4, \quad \rho(\zeta) = \frac{\zeta^2-4\zeta+1}{\zeta^3}.$$

Since $\rho(-1) = -6$, we will define the following transformation $\zeta \to \eta$ in the neighborhood of $\zeta = -1$: $\eta = (-\rho(\zeta))^{1/4}(\zeta + 1)$ (note that $\frac{\partial \eta}{\partial \zeta}\Big|_{\zeta=-1} = \sqrt[4]{6}$). Then $S(\zeta)$ can be represented $S(\zeta) = -\eta^4 + \bar{\eta}^4$ and integral $I_+^-(t)$ becomes

$$I_+^-(t) = \iint\limits_{\Delta_-} \tilde{f}(x+iy)\tilde{\psi}_{-1}(x+iy)\exp(-3itxy(x^2-y^2))|\partial_\eta \varphi(x+iy)|^2 dxdy,$$

where $\Delta_- = \{(x,y) \in \mathbb{R}^2 \colon x > 0\}$ and the functions $\tilde{f}$, $\tilde{\psi}_{-1}$, φ are defined similarly to the case of $I_+^+(t)$. The integral $I_+^-(t)$ can also be written

$$I_+^-(t) = \int\limits_{-\infty}^{+\infty} dc \exp(-3itc) \int\limits_{\gamma_c \cap \Delta_-} \tilde{f}(x+iy)\tilde{\psi}_1(x+iy)|\partial_\eta \varphi(x+iy)|^2 d\omega_S,$$

where γ_c and $d\omega_S$ are the same as for the case of $I_+^+(t)$. Performing further the same procedure as for the case of $I_+^+(t)$ and taking into account that $J_{\Delta_+}^+ = -J_{\Delta_-}^+$, $J_{\Delta_+}^- = -J_{\Delta_-}^-$, we obtain the following asymptotic expansion for $I_+^-(t)$:

$$I_+^-(t) = -\frac{\gamma}{3^{3/4}} f'_{D_+}(-1)\Gamma\left(\frac{3}{4}\right)\left[J_{\Delta_+}^+ \exp\left(-\frac{i\pi 3}{8}\right) + J_{\Delta_+}^- \exp\left(\frac{i\pi 3}{8}\right)\right]\frac{1}{t^{3/4}} + o\left(\frac{1}{t^{3/4}}\right).$$

Considering the case of $I_-(t)$ we note that in order to get an asymptotic representation for $I_-^+(t)$ and $I_-^-(t)$ we need to replace $D_+ \to D_-$, $\Delta_+ \to \Delta_-$, $\Delta_- \to \Delta_+$ in the formulas for $I_+^+(t)$ and $I_+^-(t)$ correspondingly. Taking into account that $f'_{D_+}(1) = -f'_{D_-}(1)$, $f'_{D_+}(-1) = -f'_{D_-}(-1)$, we obtain

$$I(t) = \frac{C}{t^{3/4}} + o\left(\frac{1}{t^{3/4}}\right),$$

where

$$C = \frac{2\gamma}{3^{3/4}}\Gamma\left(\frac{3}{4}\right)\left(f'_{D_+}(1)\left(J_{\Delta_+}^+ \exp\left(\frac{i\pi 3}{8}\right) + J_{\Delta_+}^- \exp\left(-\frac{i\pi 3}{8}\right)\right) - f'_{D_+}(-1)\left(J_{\Delta_+}^+ \exp\left(-\frac{i\pi 3}{8}\right) + J_{\Delta_+}^- \exp\left(\frac{i\pi 3}{8}\right)\right)\right). \quad (5.9)$$

Thus we have shown that the linear approximation of the solution $v(z,t)$ of (1.1), when $z = -18t$, behaves asymptotically as $\frac{C}{t^{3/4}}$ when $t \to \infty$.

Note that on the set $\gamma_+ \cup \gamma_-$ the differential form $d\omega_S$ is positive. Thus $J_{\Delta_+}^{\pm}$ are some negative constants, and expressions $J_{\Delta_+}^+ \exp\left(\frac{i\pi 3}{8}\right) + J_{\Delta_+}^- \exp\left(-\frac{i\pi 3}{8}\right)$, $J_{\Delta_+}^+ \exp\left(-\frac{i\pi 3}{8}\right) + J_{\Delta_+}^- \exp\left(\frac{i\pi 3}{8}\right)$ do not vanish. On the other hand, from (2.12) and (5.2) it follows that $f'_{D_+}(1)/f'_{D_+}(-1) = b(1)/\overline{b(1)}$. Thus, in the general case the constant C from (5.9) is nonzero. We have proved the optimality of the estimate (1.6) in the linear approximation.

5.2 Optimality of the estimate for the non-linear case

Now we show that for certain initial values $v(z,0)$ the corresponding solution $v(z,t)$ of (1.1) behaves asymptotically as $\frac{c}{t^{3/4}}$ along the line $z = -18t$ for some $c \neq 0$. Let us show that the integral (4.13) with $f \equiv 1$ and $z = -18t$ behaves as $\frac{\text{const}}{t^{3/4}}$.

We can represent $\alpha_1(-18t,t)$ in the form (5.1) with $f(\zeta,t)=r(\zeta,t)\rho(\zeta,t)$, where

$$\rho(\zeta,t)=-\frac{1}{\pi^2}\iint\limits_{\mathbb{C}}\frac{\partial_z r(\eta,z,t)|_{z=-18t}}{\eta-\zeta}d\mathrm{Re}\eta d\mathrm{Re}\zeta.$$

In other terms,

$$f(\zeta)=f(\zeta,t)=\frac{\mathrm{sgn}(1-\zeta\bar{\zeta})}{\bar{\zeta}}b(\zeta)\partial_{\bar{\zeta}}^{-1}\left[\frac{\pi|1-\zeta\bar{\zeta}|}{2|\zeta|^2}b(\zeta)\exp(tS(\zeta))\right].$$

We proceed following the scheme of estimate for $I(t)$ until formula (5.7). Then we represent

$$\tilde{\psi}_1(c^{1/4}(x+iy))|\partial_\eta\varphi(c^{1/4}(x+iy))|^2=\frac{1}{\sqrt{6}}+kc^{1/4}(x+iy)+h(c^{1/4}(x+iy)),$$

where k is some coefficient and $h(c^{1/4}(x+iy))$ satisfies an estimate of type (5.7). Consequently, for $\mathcal{F}(c,x,y)$ we can write

$$\mathcal{F}(c,x,y)=f(1)(6^{-1/2}+kc^{1/4}(x+iy))+6^{-1/4}[\partial_\zeta f_{D_+}(1)(x+iy)+\partial_{\bar{\zeta}}f_{D_+}(1)(x-iy)]c^{1/4}+\tilde{g}(c^{1/4}(x+iy)),$$

where $\tilde{g}(c^{1/4}(x+iy))$ satisfies an estimate of type (5.7). This allows us to obtain in the end

$$\alpha_1(-18t,t)=\frac{l_1(f(\pm1))}{t^{1/2}}\left(1+\frac{\mathrm{const}}{t^{1/4}}\right)+\frac{l_2(f_\zeta(\pm1),f_{\bar{\zeta}}(\pm1))}{t^{3/4}}+o\left(\frac{1}{t^{3/4}}\right),\quad t\to\infty,\tag{5.10}$$

where $l_1(f(\pm1))$ is a linear combination of the limit values of f as ζ tends to 1 and -1 from inside and outside of the unit circle, and $l_2(f_\zeta(\pm1),f_{\bar{\zeta}}(\pm1))$ is a linear combination of the limit values of f_ζ and $f_{\bar{\zeta}}$ as ζ tends to 1 and -1 from inside and outside of the unit circle.

Now let us consider the potential v_θ corresponding to the scattering data θb, where $\theta\in\mathbb{R}$ is some small parameter. In a way similar to which (5.10) was obtained it can be shown that

$$|f(\pm1,t)|\leqslant\frac{c_1\theta^2}{t^{1/2}},\quad |f_\zeta(\pm1,t)|\leqslant c_2\theta^2,\quad |f_{\bar{\zeta}}(\pm1,t)|\leqslant c_2\theta^2$$

for sufficiently large values of t, where c_1, c_2, c_3 are some constants independent of t and θ. When $\theta\to0$ and $t\to\infty$, the linear approximation of v_θ behaves as $O\left(\frac{\theta}{t^{3/4}}\right)$, while the expression $\alpha_1(-18t,t)$ behaves as $O\left(\frac{\theta^2}{t^{3/4}}\right)$ (it can be shown that the member $o\left(\frac{1}{t^{3/4}}\right)$ in (5.10) depends quadratically on θ).

Finally, from (4.9) and Lemma 4.1 it follows that for θ small enough and for $z=-18t$,

$$v_\theta(z,t)=\frac{C_\theta}{t^{3/4}}+o\left(\frac{1}{t^{3/4}}\right),\quad t\to\infty,$$

where C_θ is some nonzero constant. Thus we have shown that the estimate (1.6) is optimal.

6 Proofs of Lemma 3.2 and Lemma 4.1

Proof of Lemma 3.2. The proof follows the scheme described in Section 3 and is carried out separately for four cases depending on the values of the parameter u. In all the reasonings that follow we denote by D_ε the union of disks with the radius ε centered in the stationary points of $S(u,\zeta)$ and we denote by T the unit circle on the complex plane:

$$T=\{\lambda\in\mathbb{C}\colon|\lambda|=1\};\tag{6.1}$$

in addition, const will denote an independent constant and const(f) will denote a constant depending only on function f.

Case 1. $u \in \mathbb{U}$

In this case all the stationary points belong to T and due to assumptions (3.3) and (3.18) of Lemma 3.2 we can estimate

$$|f(\zeta)| \leqslant \mathrm{const}(f)\varepsilon \text{ for } \zeta \in D_\varepsilon. \tag{6.2}$$

Now we estimate the integral I_{int} (as in (3.16)) as follows

$$|I_{int}| = \left| \iint\limits_{D_\varepsilon} f(\zeta)\exp(tS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta \right| \leqslant \mathrm{const}(f)\cdot\varepsilon \iint\limits_{D_\varepsilon} d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \mathrm{const}(f)\varepsilon^3.$$

The estimate for I_{ext} (as in (3.16)) is proved as follows.

We note that the function $S'_\zeta(u,\zeta)$ can be estimated as

$$\begin{aligned} &|S'_\zeta(u,\zeta)| \geqslant \mathrm{const}\frac{\varepsilon_0^3}{|\zeta|^4} \quad \text{for } \zeta \in \mathbb{C}\backslash D_{\varepsilon_0}, \quad \text{and} \\ &|S'_\zeta(u,\zeta)| \geqslant \mathrm{const}\frac{\rho^3}{|\zeta|^4} \quad \text{for } \zeta \in \partial D_\rho, \quad \varepsilon \leqslant \rho \leqslant \varepsilon_0. \end{aligned} \tag{6.3}$$

Similarly, we can estimate

$$\begin{aligned} &\left|\frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \mathrm{const}\frac{|\zeta|^4}{\varepsilon_0^4} \quad \text{for } \zeta \in \mathbb{C}\backslash D_{\varepsilon_0}, \quad \text{and} \\ &\left|\frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \mathrm{const}\frac{|\zeta|^4}{\rho_0^4} \quad \text{for } \zeta \in \partial D_\rho, \quad \varepsilon \leqslant \rho \leqslant \varepsilon_0. \end{aligned} \tag{6.4}$$

Thus we obtain the following estimate for I_1 from (3.17)

$$|I_1| \leqslant \frac{1}{2}\int\limits_{\partial D_\varepsilon} \frac{|f(\zeta)|}{|S'_\zeta(u,\zeta)|}|d\bar\zeta| \leqslant \mathrm{const}\frac{\varepsilon}{\varepsilon^3}\int\limits_{\partial D_\varepsilon} |\zeta|^4|d\bar\zeta| \leqslant \mathrm{const}(f)\frac{\varepsilon}{\varepsilon^2}(1+\varepsilon)^4 \leqslant \frac{\mathrm{const}(f)}{\varepsilon}.$$

Due to assumption (3.18) of Lemma 3.2 the integral I_2 from (3.17) is equivalent to zero.

When estimating I_3 and I_4 from (3.17) we fix some independent $\varepsilon_0 > 0$ and integrate separately over $D_{\varepsilon_0}\backslash D_\varepsilon$ and $\mathbb{C}\backslash D_{\varepsilon_0}$:

$$\begin{aligned} |I_3| \leqslant \iint\limits_{D_{\varepsilon_0}\backslash D_\varepsilon} \left|\frac{f'_\zeta(\zeta)\exp(tS(u,\zeta))}{S'_\zeta(u,\zeta)}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}} \left|\frac{f'_\zeta(\zeta)\exp(tS(u,\zeta))}{S'_\zeta(u,\zeta)}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \\ \leqslant \mathrm{const}(f)\int\limits_\varepsilon^{\varepsilon_0} \frac{\rho}{\rho^3}d\rho + \mathrm{const}\iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}} |f'_\zeta(\zeta)||\zeta^4| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \frac{\mathrm{const}(f)}{\varepsilon}, \end{aligned}$$

$$|I_4| \leqslant \iint\limits_{D_{\varepsilon_0}\setminus D_\varepsilon} \left| \frac{f(\zeta)\exp(tS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2} \right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta +$$

$$+ \iint\limits_{\mathbb{C}\setminus D_{\varepsilon_0}} \left| \frac{f(\zeta)\exp(tS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2} \right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant$$

$$\leqslant \mathrm{const}(f) \int\limits_{\varepsilon}^{\varepsilon_0} \frac{\rho^2}{\rho^4} d\rho + \mathrm{const} \iint\limits_{\mathbb{C}\setminus D_{\varepsilon_0}} |f(\zeta)||\zeta^3| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \frac{\mathrm{const}(f)}{\varepsilon}.$$

Setting finally $\varepsilon = \frac{1}{(1+|t|)^{1/4}}$ yields

$$I(t,u) \leqslant \frac{\mathrm{const}(f)}{(1+|t|)^{3/4}}$$

uniformly on $u \in \mathbb{U}$.

Case 2. $u \in \mathbb{C}\setminus\mathbb{U}$ and ω from (3.14) satisfies $\omega_0 < \frac{\omega}{1+\omega} < 1-\omega_1$ for some fixed independent positive constants ω_0 and ω_1 (i.e. the roots ζ_0, ζ_2 from (3.14) are separated from T, defined by (6.1), and the root ζ_2 is separated from the origin)

In this case the we can estimate

$$|S'_\zeta(u,\zeta)| \geqslant \frac{\mathrm{const}\rho}{|\zeta|^4} \text{ for } \zeta \in \partial D_\rho,$$

$$\left| \frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2} \right| \leqslant \frac{\mathrm{const}|\zeta|^4}{\rho^2} \text{ for } \zeta \in \partial D_\rho.$$

Using these estimates and proceeding as in case 1, we obtain

$$|I_{int}| \leqslant \mathrm{const}(f)\varepsilon^2, \quad |I_1| \leqslant \mathrm{const}(f), \quad I_2 \equiv 0, \quad |I_3| \leqslant \mathrm{const}(f), \quad |I_4| \leqslant \mathrm{const}(f) \ln \frac{1}{\varepsilon}.$$

Setting $\varepsilon = \frac{1}{1+|t|}$, we obtain that

$$I(t,u) \leqslant \mathrm{const}(f) \frac{\ln(3+|t|)}{1+|t|}$$

uniformly for the considered values of the parameter u.

Case 3. $u \in \mathbb{C}\setminus\mathbb{U}$ and $\frac{\omega}{1+\omega} < \omega_0$ (i.e. the roots ζ_0 and ζ_2 from (3.14) lie in some neighborhood of T from (6.1))

Lemma 6.1. *For any $t \geqslant t_0$ with some fixed $t_0 > 0$ and any $\omega > 0$ one of the following conditions holds*

(a) $0 < \omega < \frac{2}{(1+|t|)^{1/4}}$;

(b) $\omega > \frac{1}{(1+|t|)^{1/8}}$;

(c) $\exists n$: $\frac{1}{(1+|t|)^{\gamma_{n+1}/(2+2\gamma_{n+1})}} < \omega < \frac{2}{(1+|t|)^{\gamma_n/(2+2\gamma_{n+1})}}$, *where* $\gamma_{n+1} = \frac{2}{3}\gamma_n + \frac{1}{3}$, $\gamma_1 = \frac{1}{3}$.

Proof. We note that

$$\begin{aligned}
&\frac{\gamma_{n+1}}{2+2\gamma_{n+1}} \to \frac{1}{4}, \quad n\to\infty;\\
&\frac{\gamma_n}{2+2\gamma_{n+1}} < \frac{\gamma_n}{2+2\gamma_n};\\
&\frac{\gamma_1}{2+2\gamma_2} < \frac{1}{8}.
\end{aligned}$$

Thus the intervals from the cases (a), (b), (c) $\forall n \in \mathbb{N}$ cover the whole range $0<\omega<+\infty$. $\square$

We will prove the result separately for three different cases depending on the value of parameter ω

(a) $0<\omega<2\varepsilon = \frac{2}{(1+|t|)^{1/4}}$

In this case estimates (6.2), (6.3), (6.4) hold and so the reasoning of the case 1 can be carried out to obtain that

$$I(t,u) \leqslant \frac{\mathrm{const}(f)}{(1+|t|)^{3/4}}$$

uniformly for the considered values of the parameter u satisfying

$$0<\omega<\frac{2}{(1+|t|)^{1/4}}. \tag{6.5}$$

(b) $\omega > \varepsilon^{1/3} = \frac{1}{(1+|t|)^{1/8}}$

In this case we estimate $|I_{int}| \leqslant \mathrm{const}(f)\varepsilon^2$.

Further, we note that the derivative of the phase is estimated as

$$|S'_\zeta(u,\zeta)| \geqslant \frac{\mathrm{const}\,\varepsilon\omega^2}{|\zeta|^4} \text{ for } \zeta\in\partial D_\varepsilon. \tag{6.6}$$

Thus for I_1 we obtain $|I_1| \leqslant \mathrm{const}(f)\frac{1}{\varepsilon^{2/3}}$.

In order to estimate the integral I_3 we use the following estimate of the derivative S'_ζ for $\zeta\in\partial D_\rho$ when $\varepsilon\leqslant\rho\leqslant\varepsilon_0$:

$$\begin{cases} |S'_\zeta(u,\zeta)| \geqslant \frac{\mathrm{const}\rho\omega^2}{|\zeta|^4}, \text{ if } \rho<\omega,\\ |S'_\zeta(u,\zeta)| \geqslant \frac{\mathrm{const}\rho^3}{|\zeta|^4}, \text{ if } \rho>\omega. \end{cases} \tag{6.7}$$

It allows to derive $|I_3| \leqslant \mathrm{const}(f)\frac{1}{\varepsilon^{2/3}}$.

Finally, we proceed to the study of the integral I_4. We use the following estimates

$$\left|\frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \frac{\mathrm{const}|\zeta|^4}{\rho^2\omega^2}$$

and

$$\begin{cases} |f(\zeta)| \leqslant \mathrm{const}(f)\,\omega, \text{ if } \rho<\omega,\\ |f(\zeta)| \leqslant \mathrm{const}(f)\,\rho, \text{ if } \rho>\omega. \end{cases} \tag{6.8}$$

After integration we obtain the estimate $|I_4| \leqslant \mathrm{const}(f)\frac{1}{\varepsilon^{2/3}}$.

Setting finally $\varepsilon = \frac{1}{(1+|t|)^{3/8}}$, we obtain

$$I(t,u) \leqslant \frac{\mathrm{const}(f)}{(1+|t|)^{3/4}}$$

uniformly for the considered values of the parameter u satisfying

$$\omega > \frac{1}{(1+|t|)^{1/8}}. \tag{6.9}$$

(c) $\varepsilon^{\gamma_{n+1}} < \omega < 2\varepsilon^{\gamma_n}$, where $\varepsilon = \frac{1}{(1+|t|)^{1/(2+2\gamma_{n+1})}}$ and $\gamma_{n+1} = \frac{2}{3}\gamma_n + \frac{1}{3}$, $\gamma_1 = \frac{1}{3}$ (note that $\gamma_n \to 1$)

We proceed similarly to the case (b). Evidently, I_{int} can be estimated $|I_{int}| \leqslant \mathrm{const}(f)\varepsilon^{2+\gamma_n}$. Employing the estimate (6.6) we obtain $|I_1| \leqslant \mathrm{const}(f)\frac{\varepsilon^{\gamma_n}}{\varepsilon^{2\gamma_{n+1}}}$.

Using (6.7) in order to estimate I_3 we obtain $|I_3| \leqslant \mathrm{const}(f)\frac{1}{\varepsilon^{\gamma_{n+1}}}$.

Finally, to estimate I_4 we use (6.8) and

$$\left|\frac{S''_{\zeta\zeta}}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \begin{cases} \frac{\mathrm{const}|\zeta|^4}{\rho^2\omega^2}, & \rho < \omega, \\ \frac{\mathrm{const}|\zeta|^4}{\rho^3\omega}, & \rho > \omega \end{cases}$$

to obtain $|I_4| \leqslant \mathrm{const}(f)\frac{\varepsilon^{\gamma_n}\ln(1/\varepsilon)}{\varepsilon^{2\gamma_{n+1}}}$.

Setting $\varepsilon = \frac{1}{(1+|t|)^{1/(2+2\gamma_{n+1})}}$ yields

$$|I(t,u)| \leqslant \frac{\mathrm{const}(f)\ln(3+|t|)}{(1+|t|)^{3/4}}$$

uniformly for the considered values of the parameter u satisfying

$$\frac{1}{(1+|t|)^{\gamma_{n+1}/(2+2\gamma_{n+1})}} < \omega < \frac{2}{(1+|t|)^{\gamma_n/(2+2\gamma_{n+1})}}. \tag{6.10}$$

Finally, from Lemma 6.1 it follows that we have proved the required estimate uniformly on the values of parameter $u \in \mathbb{C}\backslash\mathbb{U}$ such that $\frac{\omega}{1+\omega} < \omega_0$.

Case 4. $u \in \mathbb{C}\backslash\mathbb{U}$ and $\frac{\omega}{1+\omega} > 1-\omega_1$ (i.e. the roots ζ_2, $-\zeta_2$ lie in the ω_1–neighborhood of the origin)

This case is treated similarly to the previous one. We denote $\tilde{\omega} = \frac{1}{1+\omega}$. Then we use estimates

$$|f(\zeta)| \leqslant c(f)|\tilde{\omega}+\rho|,$$
$$|S'_\zeta(u,\zeta)| \geqslant \frac{\mathrm{const}\rho^2}{|\zeta|^4} \text{ or } |S'_\zeta(u,\zeta)| \geqslant \frac{\mathrm{const}\rho\tilde{\omega}}{|\zeta|^4},$$
$$\left|\frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \frac{\mathrm{const}|\zeta|^4}{\rho^3} \text{ or } \left|\frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \frac{\mathrm{const}|\zeta|^4}{\rho^2\tilde{\omega}},$$

which hold for $\zeta \in D_\rho$, to obtain the necessary estimates. □

Proof of Lemma 4.1. 1. The proof of inequality (4.10) repeats the proof of Lemma 3.2. The proof of inequality (4.11) also follows the scheme of the proof of Lemma 3.2. In this case we take D_ε to be the union of disks of the radius ε with centers in the stationary points of $S(u,\zeta)$ and in the point λ.

For the case when $\lambda \notin T$, where T is defined by (6.1), an estimate weaker than (4.11) can be obtained via a simplified reasoning. Indeed, I_{int}, as in (3.16), can be estimated $|I_{int}| \leqslant O\left(\frac{\varepsilon}{(1+|t|)^{\delta}}\right)$. Using estimates (6.3), (6.4) and

$$|\zeta - \lambda| \geqslant \rho \text{ for } \zeta \in \partial D_{\rho} \tag{6.11}$$

we obtain that $|I_{ext}| \leqslant O\left(\frac{1}{(1+|t|)^{1+\delta}\varepsilon^3}\right)$. Setting $\varepsilon = \frac{1}{(1+|t|)^{1/4}}$ we get the estimate (4.12).

2. In order to obtain estimates (4.14), (4.15) we proceed according to the scheme outlined in Section 3. In this case the integral I_2 does not annul. On the other hand, when the variable of integration belongs to T, the estimate (4.11) on the integrand of I_2 is stronger than the estimate (4.12) for the general case. Thus we obtain for $\alpha_1(z,t)$

$$|I_{int}| \leqslant O\left(\frac{\varepsilon^2}{(1+|t|)^{\delta+\frac{1}{4}}}\right), \quad |I_1| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{4}}\,\varepsilon^2}\right), \quad |I_2| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{4}}\,\varepsilon^3}\right),$$

$$|I_3| \leqslant O\left(\frac{1}{(1+|t|)^{\delta}\,\varepsilon}\right), \quad |I_4| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{4}}\,\varepsilon^2}\right).$$

Setting $\varepsilon = \frac{1}{(1+|t|)^{1/4}}$ yields the required estimate. The estimate (4.15) is obtained similarly.

3. We will give the scheme of the proof for estimate (4.16). The estimate (4.17) is obtained similarly.

   We will prove (4.16) by induction. Suppose that (4.16) holds for all $n = 1, 2, \ldots, N$. Then following the scheme of Section 3 and taking into account that $\partial_{\lambda}\overline{(A^n_{z,t} \cdot f)(\lambda)} = \overline{(A^{n-1}_{z,t} \cdot f)(\lambda)}$, we obtain for $n = N+1$:

$$|I_{int}| \leqslant O\left(\frac{\varepsilon}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n-1}{2}\rceil}}\right), \quad |I_1| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n-1}{2}\rceil}\varepsilon^3}\right),$$

$$|I_2| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n-1}{2}\rceil}\varepsilon^4}\right), \quad |I_3| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n-2}{2}\rceil}\varepsilon^3}\right),$$

$$|I_4| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n-1}{2}\rceil}\varepsilon^3}\right).$$

   Setting $\varepsilon = \frac{1}{(1+|t|)^{1/5}}$ we obtain the required estimate.

4. We represent $R_{z,t}(\lambda)$ as the sum of the following members

$$\begin{aligned} R_{z,t}(\lambda) &= B(A + A^2 + A^3 + \ldots) \cdot 1 + AB(A + A^2 + A^3 + \ldots) \cdot 1 + \\ &+ (A + A^2 + A^3 + \ldots)AB(I + A + A^2 + \ldots) \cdot 1 = R^1_{z,t}(\lambda) + R^2_{z,t}(\lambda) + R^3_{z,t}(\lambda). \end{aligned}$$

   The convergence of the series at sufficiently large times follows from the estimate (4.17). Now let

$$R^i_{z,t}(\lambda) = \frac{q_i(z,t)}{\lambda} + o\left(\frac{1}{|\lambda|}\right), \text{ as } \lambda \to \infty.$$

   From (4.10) and (4.17) it follows that $|q_1(z,t)| \leqslant \frac{\hat{q}_1(c_f)\ln(3+|t|)}{(1+|t|)^{3/4+1/5}}$. From (4.14) and (4.17) we obtain that $|q_2(z,t)| \leqslant \frac{\hat{q}_2(c_f)}{(1+|t|)^{3/4+1/5}}$. Finally, from (4.15), (4.16) and (4.17) it follows that $|q_3(z,t)| \leqslant \frac{\hat{q}_3(c_f)}{(1+|t|)^{1/2+2/5}}$. This yields the required estimate. □

References

[Fa] Faddeev L.D.: Growing solutions of the Schrödinger equation. Dokl. Akad. Nauk SSSR. 165(3), 514-517 (1965), translation in Sov. Phys. Dokl. 10, 1033-1035 (1966)

[Fe] Fedoryuk M.V.: Asymptotics: integrals and series. Mathematical Reference Library, Nauka, Moscow (1987) (in Russian)

[GK] Gohberg I.C., Krein M.G.: Introduction to the theory of linear nonselfadjoint operators. Moscow: Nauka. (1965)

[G1] Grinevich P.G.: Rational solitons of the Veselov–Novikov equation are reflectionless potentials at fixed energy. TMF. 69(2), 307-310 (1986), translation in Theor. Math. Phys. 69, 1170-1172 (1986)

[G2] Grinevich P.G.: Scattering transformation at fixed non-zero energy for the two-dimensional Schrödinger operator with potential decaying at infinity. Russ. Math. Surv. 55(6), 1015–1083 (2000)

[GN] Grinevich P.G., Novikov S.P.: Two-dimensional «inverse scattering problem» for negative energies and generalized-analytic functions. I. Energies below the ground state. Funct. Anal. Appl. 22(1), 19–27 (1988)

[HNS] Hayashi, N., Naumkin, P.I., Saut, J.-C.: Asymptotics for large time of global solutions to the generalized Kadomtsev-Petviashvili equation. Commun. Math. Phys. 201(3), 577–590 (1999)

[KN1] Kazeykina A.V., Novikov R.G.: A large time asymptotics for transparent potentials for the Novikov–Veselov equation at positive energy. J. Nonlinear Math. Phys. 18(3), 377–400 (2011)

[KN2] Kazeykina A.V., Novikov R.G.: Large time asymptotics for the Grinevich-Zakharov potentials. Bulletin des Sciences Mathématiques. 135, 374–382 (2011)

[KN3] Kazeykina A.V., Novikov R.G.: Absence of exponentially localized solitons for the Novikov–Veselov equation at negative energy. Nonlinearity. 24, 1821-1830 (2011)

[K] Kiselev, O.M.: Asymptotics of a solution of the Kadomtsev–Petviashvili–2 equation. Tr. Inst. Mat. Mekh. 7(1), 105–134 (2001), translation in Proc. Inst. Math. Mech. suppl.1, S107–S139 (2001)

[M] Manakov S.V.: The inverse scattering method and two-dimensional evolution equations. Uspekhi Mat. Nauk. 31(5), 245–246 (1976) (in Russian)

[N1] Novikov R.G.: The inverse scattering problem on a fixed energy level for the two-dimensional Schrödinger operator. J. Funkt. Anal. and Appl. 103, 409–463 (1992)

[N2] Novikov R.G.: Absence of exponentially localized solitons for the Novikov–Veselov equation at positive energy. Physics Letters A. 375, 1233–1235 (2011)

[NV1] Novikov S.P., Veselov A.P.: Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formula and evolutions equations. Dokl. Akad. Nauk SSSR. 279, 20–24 (1984), translation in Sov. Math. Dokl. 30, 588–591 (1984)

[NV2] Novikov S.P., Veselov A.P.: Finite-zone, two-dimensional Schrödinger operators. Potential operators. Dokl. Akad. Nauk SSSR. 279, 784–788 (1984), translation in Sov. Math. Dokl. 30, 705–708 (1984)

[MST] Manakov, S.V., Santini, P.M., Takchtadzhyan, L.A.: An asymptotic behavior of the solutions of the Kadomtsev-Petviashvili equations. Phys. Lett. A. 75, 451–454 (1980)

[V] Vekua I.N.: Generalized analytic functions. Oxford: Pergamon Press (1962)

[ZS] Zakharov V.E., Shulman E.I.: Integrability of nonlinear systems and perturbation theory // What is integrability? Berlin: Springer–Verlag. 185–250 (1991)