ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES

UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 46 00. Fax: 01 69 33 46 46
http://www.cmap.polytechnique.fr/

Reconstruction of a potential from the impedance boundary map

M.I. Isaev and R.G. Novikov

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Abstract

We give formulas and equations for finding generalized scattering data for the Schrödinger equation in open bounded domain at fixed energy from the impedance boundary map (or Robin-to-Robin map). Combining these results with results of the inverse scattering theory we obtain efficient methods for reconstructing potential from the impedance boundary map.

1 Introduction

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D,\ E \in \mathbb{R}, \tag{1.1}$$

where

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^d,\ d \geq 2, \\ &\text{with } \partial D \in C^2, \end{aligned} \tag{1.2}$$

$$v \in \mathbb{L}^\infty(D), \quad v = \bar{v}. \tag{1.3}$$

Following [11], [17], we consider the impedance boundary map $\hat{M}_\alpha = \hat{M}_{\alpha,v}(E)$ defined by

$$\hat{M}_\alpha[\psi]_\alpha = [\psi]_{\alpha-\pi/2} \tag{1.4}$$

for all sufficiently regular solutions ψ of equation (1.1) in $\bar{D} = D \cup \partial D$, where

$$[\psi]_\alpha = [\psi(x)]_\alpha = \cos\alpha\,\psi(x) - \sin\alpha\,\frac{\partial\psi}{\partial\nu}|_{\partial D}(x), \quad x \in \partial D,\ \alpha \in \mathbb{R} \tag{1.5}$$

and ν is the outward normal to ∂D. Under assumptions (1.2), (1.3), in Lemma 3.2 of [17] it was shown that there is not more than a countable number of $\alpha \in \mathbb{R}$ such that E is an eigenvalue for the operator $-\Delta + v$ in D with the boundary condition

$$\cos\alpha\,\psi|_{\partial D} - \sin\alpha\,\frac{\partial\psi}{\partial\nu}|_{\partial D} = 0. \tag{1.6}$$

Therefore, for any energy level E we can assume that for some fixed $\alpha \in \mathbb{R}$

$$\begin{aligned} &E \text{ is not an eigenvalue for the operator } -\Delta + v \text{ in } D \\ &\text{with boundary condition (1.6)} \end{aligned} \tag{1.7}$$

and, as a corollary, $\hat{M}_\alpha$ can be defined correctly.

We recall that the impedance boundary map $\hat{M}_\alpha$ is reduced to the Dirichlet-to-Neumann(DtN) map if $\alpha = 0$ and is reduced to the Neumann-to-Dirichlet(NtD) map if $\alpha = \pi/2$. The map $\hat{M}_\alpha$ can be called also as the Robin-to-Robin map.

As in [17], we consider the following inverse boundary value problem for equation (1.1).

Problem 1.1. Given $\hat{M}_\alpha$ for some fixed E and α, find v.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at fixed energy (see [10], [25]). At zero energy this problem can be considered also as a generalization of the Calderon problem of the electrical impedance tomography (see [6], [25]).

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness theorems and global reconstruction methods for Problem 1.1 with $\alpha = 0$ were given for the first time in [25] in dimension $d \geq 3$ and in [5] in dimension $d = 2$.

Global stability estimates for Problem 1.1 with $\alpha = 0$ were given for the first time in [1] in dimension $d \geq 3$ and in [34] in dimension $d = 2$. A principal improvement of the result of [1] was given recently in [33] (for the zero energy case). Due to [21] these logarithmic stability results are optimal (up to the value of the exponent). An extention of the instability estimates of [21] to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was given in [16].

Note also that for the Calderon problem (of the electrical impedance tomography) in its initial formulation the global uniqueness was firstly proved in [39] for $d \geq 3$ and in [24] for $d = 2$. In addition, for the case of piecewise constant or piecewise real analytic conductivity the first uniqueness results for the Calderon problem in dimension $d \geq 2$ were given in [7], [18].

It should be noted that in most of previous works on inverse boundary value problems for equation (1.1) at fixed E it was assumed in one way or another that E is not a Dirichlet eigenvalue for the operator $-\Delta + v$ in D, see [1], [21], [25], [33]- [37]. Nevertheless, the results of [5] can be considered as global uniqueness and reconstruction results for Problem 1.1 in dimension $d = 2$ with general α.

Global stability estimates for Problem 1.1 in dimension $d \geq 2$ with general α were recently given in [17].

In the present work we give formulas and equations for finding (generalized) scattering data from the impedance boundary map $\hat{M}_\alpha$ with general α. Combining these results with results of [13], [15], [24], [26]- [28], [30]- [32], we obtain efficient reconstruction methods for Problem 1.1 in multidimensions with general α.

Definitions of (generilized) scattering data are recalled in Section 2. Our main results are presented in Section 3. Proofs of these results are given in Sections 4, 5 and 6.

2 Scattering data

Consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2 \tag{2.1}$$

where

$$(1+|x|)^{d+\varepsilon}v(x) \in \mathbb{L}^\infty(\mathbb{R}^d) \text{ (as a function of } x), \quad \text{for some } \ \varepsilon > 0. \tag{2.2}$$

For equation (2.1) we consider the functions ψ^+ and f of the classical scattering theory and the Faddeev functions ψ, h, ψ_γ, h_γ (see, for example, [3], [8], [9], [12], [15], [22], [26]).

The functions ψ^+ and f can be defined as follows:

$$\psi^+(x,k) = e^{ikx} + \int\limits_{\mathbb{R}^d} G^+(x-y,k)v(y)\psi^+(y,k)dy, \tag{2.3}$$

$$\begin{gathered} G^+(x,k) = -\left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}}{\xi^2 - k^2 - i0}\, d\xi, \\ x, k \in \mathbb{R}^d, \quad k^2 > 0, \end{gathered} \tag{2.4}$$

where (2.3) at fixed k is considered as an equation for ψ^+ in $\mathbb{L}^\infty(\mathbb{R}^d)$;

$$\begin{gathered} f(k,l) = \left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} e^{-ilx}\psi^+(x,k)v(x)dx, \\ k, l \in \mathbb{R}^d, \ k^2 > 0. \end{gathered} \tag{2.5}$$

In addition: $\psi^+(x,k)$ satisfies (2.3) for $E = k^2$ and describes scattering of the plane waves e^{ikx}; $f(k,l)$, $k^2 = l^2$, is the scattering amplitude for equation (2.1) for $E = k^2$. Equation (2.3) is the Lippman-Schwinger integral equation.

The functions ψ and h can be defined as follows:

$$\psi(x,k) = e^{ikx} + \int\limits_{\mathbb{R}^d} G(x-y,k)v(y)\psi(y,k)dy, \tag{2.6}$$

$$\begin{gathered} G(x,k) = -\left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 + 2k\xi}\, e^{ikx}, \\ x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d, \quad \operatorname{Im} k \neq 0, \end{gathered} \tag{2.7}$$

where (2.6) at fixed k is considered as an equation for $\psi = e^{ikx}\mu(x,k)$, $\mu \in \mathbb{L}^\infty(\mathbb{R}^d)$;

$$\begin{gathered} h(k,l) = \left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} e^{-ilx}\psi(x,k)v(x)dx, \\ k, l \in \mathbb{C}^d, \ \operatorname{Im} k = \operatorname{Im} l \neq 0. \end{gathered} \tag{2.8}$$

In addition, $\psi(x,k)$ satisfies (2.1) for $E = k^2$, and ψ, G and h are (nonanalytic) continuations of ψ^+, G^+ and f to the complex domain. In particular, $h(k,l)$ for

$k^2 = l^2$ can be considered as the "scattering" amplitude in the complex domain for equation (2.1) for $E = k^2$. The functions ψ_γ and h_γ are defined as follows:

$$\begin{gathered}\psi_\gamma(x,k) = \psi(x, k + i0\gamma), \qquad h_\gamma(k,l) = h(k + i0\gamma, l + i0\gamma), \\ x, k, l, \gamma \in \mathbb{R}^d,\ \gamma^2 = 1.\end{gathered} \tag{2.9}$$

We recall also that

$$\begin{gathered}\psi^+(x,k) = \psi_{k/|k|}(x,k), \qquad f(k,l) = h_{k/|k|}(k,l), \\ x, k, l \in \mathbb{R}^d,\ |k| > 0.\end{gathered} \tag{2.10}$$

We consider $f(k,l)$ and $h_\gamma(k,l)$, where $k, l, \gamma \in \mathbb{R}^d$, $k^2 = l^2 = E$, $\gamma^2 = 1$, and $h(k,l)$, where $k, l \in \mathbb{C}^d$, $\operatorname{Im} k = \operatorname{Im} l \neq 0$, $k^2 = l^2 = E$, as scattering data S_E for equation (2.1) at fixed $E \in (0, +\infty)$. We consider $h(k,l)$, where $k, l \in \mathbb{C}^d$, $\operatorname{Im} k = \operatorname{Im} l \neq 0$, $k^2 = l^2 = E$, as scattering data S_E for equation (2.1) at fixed $E \in (-\infty, 0]$.

We consider also the sets $\mathcal{E}$, $\mathcal{E}_\gamma$, $\mathcal{E}^+$ defined as follows:

$$\mathcal{E} = \left\{ \begin{array}{c} \zeta \in \mathbb{C}^d \setminus \mathbb{R}^d\ :\ \text{ equation (2.6) for } k = \zeta \text{ is not} \\ \text{uniquely solvable for } \psi = e^{ikx}\mu \text{ with } \mu \in \mathbb{L}^\infty(\mathbb{R}^d) \end{array} \right\}, \tag{2.11a}$$

$$\mathcal{E}_\gamma = \left\{ \begin{array}{c} \zeta \in \mathbb{R}^d \setminus \{0\}\ :\ \text{ equation (2.6) for } k = \zeta + i0\gamma \\ \text{is not uniquely solvable for } \psi = \mathbb{L}^\infty(\mathbb{R}^d) \end{array} \right\}, \quad \gamma \in \mathbb{S}^{d-1}, \tag{2.11b}$$

$$\mathcal{E}^+ = \left\{ \begin{array}{c} \zeta \in \mathbb{R}^d \setminus \{0\}\ :\ \text{ equation (2.6) for } k = \zeta \text{ is not} \\ \text{uniquely solvable for } \psi = \mathbb{L}^\infty(\mathbb{R}^d) \end{array} \right\}. \tag{2.11c}$$

In addition, $\mathcal{E}^+$ is a well-known set of the classical scattering theory for equation (2.1) and $\mathcal{E}^+ = \emptyset$ for real-valued v satisfying (2.2) (see, for example, [3], [22]). Note also that $\mathcal{E}^+$ is spherically symmetric. The sets $\mathcal{E}$, $\mathcal{E}_\gamma$ were considered for the first time in [8], [9]. Concerning the properties of $\mathcal{E}$ and $\mathcal{E}_\gamma$, see [9], [14], [15], [20], [22], [24], [27], [40].

We consider also the functions R, R_γ, R^+ defined as follows:

$$\begin{gathered}R(x,y,k) = G(x-y,k) + \int\limits_{\mathbb{R}^d} G(x-z,k)v(z)R(z,y,k)dz, \\ x, y \in \mathbb{R}^d,\ k \in \mathbb{C}^d,\ \operatorname{Im} k \neq 0,\end{gathered} \tag{2.12}$$

where G is defined by (2.7) and formula (2.12) at fixed y, k is considered as an equation for

$$R(x,y,k) = e^{ik(x-y)}r(x,y,k), \tag{2.13}$$

where r is sought with the properties

$$r(\cdot, y, k) \text{ is continuous on } \mathbb{R}^d \setminus \{y\} \tag{2.14a}$$

$$r(x,y,k) \to 0 \quad \text{as} \quad |x| \to \infty, \tag{2.14b}$$

$$\begin{aligned} r(x,y,k) &= O(|x-y|^{2-d}) \quad \text{as} \quad x \to y \quad \text{for} \quad d \geq 3, \\ r(x,y,k) &= O(|\ln|x-y||) \quad \text{as} \quad x \to y \quad \text{for} \quad d = 2; \end{aligned} \tag{2.14c}$$

$$\begin{aligned} &R_\gamma(x,y,k) = R(x,y,k+i0\gamma), \\ &\qquad\qquad x,y \in \mathbb{R}^d,\ k \in \mathbb{R}^d \setminus \{0\},\ \gamma \in \mathbb{S}^{d-1}; \end{aligned} \tag{2.15}$$

$$\begin{aligned} &R^+(x,y,k) = R_{k/|k|}(x,y,k), \\ &\qquad\qquad x,y \in \mathbb{R}^d,\ k \in \mathbb{R}^d \setminus \{0\}. \end{aligned} \tag{2.16}$$

In addition, the functions $R(x,y,k)$, $R_\gamma(x,y,k)$ and $R^+(x,y,k)$ (for their domains of definition in k and γ) satisfy the following equations:

$$\begin{aligned} &(\Delta_x + E - v(x))R(x,y,k) = \delta(x-y), \\ &(\Delta_y + E - v(y))R(x,y,k) = \delta(x-y), \\ &\qquad\qquad x,y \in \mathbb{R}^d,\ E = k^2. \end{aligned} \tag{2.17}$$

The function $R^+(x,y,k)$ (defined by means of (2.12) for $k \in \mathbb{R}^d \setminus \{0\}$ with G replaced by G^+ of (2.4)) is well-known in the scattering theory for equations (2.1), (2.17) (see, for example, [4]). In particular, this function describes scattering of the spherical waves $G^+(x-y,k)$ generated by a source at y. In addition $R^+(x,y,k)$ is a radial function in k, i.e.

$$R^+(x,y,k) = \mathrm{R}^+(x,y,|k|), \qquad x,y \in \mathbb{R}^d,\ k \in \mathbb{R}^d \setminus \{0\}. \tag{2.18}$$

Apparently, the functions R and R_γ were considered for the first time in [27].

In addition, under the assumption (2.2): equation (2.12) at fixed y and k is uniquely solvable for R with the properties (2.13), (2.14) if and only if $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E})$; equation (2.12) with $k = \zeta + i0\gamma$, $\zeta \in \mathbb{R}^d \setminus \{0\}$, $\gamma \in \mathbb{S}^{d-1}$, at fixed y, ζ and γ is uniquely solvable for R_γ if and only if $\zeta \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}_\gamma)$; equation (2.12) with $k = \zeta + i0\zeta/|\zeta|$, $\zeta \in \mathbb{R}^d \setminus 0$, at fixed y and ζ is uniquely solvable for R^+ if and only if $\zeta \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^+)$.

3 Main results

Let v and v^0 satisfy (1.3), (1.7) for some fixed E and α. Let $M_{\alpha,v}(x,y,E)$, $M_{\alpha,v^0}(x,y,E)$, $x,y \in \partial D$, denote the Schwartz kernels of the impedance boundary maps $\hat{M}_{\alpha,v}$, $\hat{M}_{\alpha,v^0}$, for potentials v and v^0, respectively, where $\hat{M}_{\alpha,v}$, $\hat{M}_{\alpha,v^0}$ are considered as linear integral operators. In addition, we consider v^0 as some known background potential.

Let h, ψ, f, ψ^+, h_γ, ψ_γ, $\mathcal{E}$, $\mathcal{E}^+$, $\mathcal{E}_\gamma$ and h^0, ψ^0, f^0, $\psi^{+,0}$, h^0_γ, ψ^0_γ, $\mathcal{E}^0$, $\mathcal{E}^{+,0}$, $\mathcal{E}^0_\gamma$ denote the functions and sets of (2.3), (2.5), (2.6), (2.8), (2.9), (2.11) for potentials v and v^0, respectively. Here and bellow in this section we always assume that $v \equiv 0$, $v^0 \equiv 0$ on $\mathbb{R}^d \setminus D$.

Theorem 3.1. *Let D satisfy* (1.2) *and potentials v, v^0 satisfy (1.3), (1.7) for some fixed E and α. Then:*

$$\begin{aligned}
&h(k,l)-h^0(k,l)=\\
&=\left(\frac{1}{2\pi}\right)^d\int\limits_{\partial D}\int\limits_{\partial D}[\psi^0(x,-l)]_\alpha\left(M_{\alpha,v}-M_{\alpha,v_0}\right)(x,y,E)[\psi(y,k)]_\alpha dx\,dy, \qquad (3.1)\\
&\qquad k,l\in\mathbb{C}^d\setminus(\mathcal{E}\cup\mathcal{E}^0),\ k^2=l^2=E,\ Im\,k=Im\,l\neq 0,
\end{aligned}$$

$$\begin{aligned}
&[\psi(x,k)]_\alpha=[\psi^0(x,k)]_\alpha+\int\limits_{\partial D}A_\alpha(x,y,k)[\psi(y,k)]_\alpha dy,\\
&x\in\partial D,\ k\in\mathbb{C}^d\setminus(\mathcal{E}\cup\mathcal{E}^0),\ Im\,k\neq 0,\ k^2=E
\end{aligned} \qquad (3.2)$$

where

$$A_\alpha(x,y,k)=\lim_{\varepsilon\to+0}\int\limits_{\partial D}D_{\alpha,\varepsilon}R^0(x,\xi,k)\left(M_{\alpha,v}-M_{\alpha,v^0}\right)(\xi,y,E)d\xi, \qquad (3.3)$$

$$\begin{aligned}
&D_{\alpha,\varepsilon}R^0(x,\xi,k)=[[R^0(x+\varepsilon\nu_x,\xi,k)]_{\xi,\alpha}]_{x,\alpha}=\\
&=\left(\cos^2\alpha-\sin\alpha\cos\alpha\left(\frac{\partial}{\partial\nu_x}+\frac{\partial}{\partial\nu_\xi}\right)+\sin^2\alpha\frac{\partial^2}{\partial\nu_x\partial\nu_\xi}\right)R^0(x+\varepsilon\nu_x,\xi,k),\\
&\qquad x,\xi,y\in\partial D,
\end{aligned} \qquad (3.4)$$

where R^0 denotes the Green function of (2.12) *for potential v^0, ν_x is the outward normal to ∂D at x. In addition, formulas completely similar to* (3.1) - (3.4) *are also valid for the classical scattering functions f, ψ^+, f^0, $\psi^{+,0}$ and sets $\mathcal{E}^+$, $\mathcal{E}^{+,0}$ of* (2.3), (2.5), (2.11c) *for v and v^0, respectively, but with $R^{+,0}$ in place of R^0 in* (3.3), (3.4), *where $R^{+,0}$ denotes the Green function of* (2.16) *for potential v^0.*

Note that formula of the type (3.1) for h_γ is not completely similar to (3.1). To present this formula for h_γ we consider also $\psi_\gamma(x,k,l)$ defined as follows:

$$\begin{aligned}
&\psi_\gamma(x,k,l)=e^{ilx}+\int\limits_{\mathbb{R}^d}G_\gamma(x-y,k)v(y)\psi_\gamma(y,k,l)dy,\\
&G_\gamma(x,k)=G(x,k+i0\gamma),\\
&\gamma\in\mathbb{S}^{d-1},\ x,k,l\in\mathbb{R}^d,k^2=l^2>0,
\end{aligned} \qquad (3.5)$$

where (3.5) at fixed γ, k, l is considered as an equation for $\psi_\gamma(\cdot,k,l)$ in $\mathbb{L}^\infty(\mathbb{R}^d)$, G is defined by (2.7).

Proposition 3.1. *Let the asssumptions of Theorem 3.1 hold. Let $\psi_\gamma(x,k)$ correspond to v according to* (2.9) *and $\psi^0_{-\gamma}(\cdot,k,l)$ correspond to v^0 according to*

(3.5). *Then*

$$h_\gamma(k,l) - h^0_\gamma(k,l) =$$
$$= \left(\frac{1}{2\pi}\right)^d \int\limits_{\partial D}\int\limits_{\partial D} [\psi^0_{-\gamma}(x,-k,-l)]_\alpha \left(M_{\alpha,v} - M_{\alpha,v^0}\right)(x,y,E)[\psi_\gamma(y,k)]_\alpha dx\, dy,$$
$$\gamma \in \mathbb{S}^{d-1},\ k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}_\gamma \cup \mathcal{E}^0_\gamma),\ l \in \mathbb{R}^d,\ k^2 = l^2 = E. \tag{3.6}$$

In addition, formulas completely similar to (3.2) - (3.4) *are also valid for the functions* $\psi_\gamma(x,k)$, $\psi^0_\gamma(x,k)$ *and sets* $\mathcal{E}_\gamma$, $\mathcal{E}^0_\gamma$ *of* (2.9), (2.11b) *for* v *and* v^0, *respectively, but with* R^0_γ *in place of* R^0 *in* (3.3), (3.4), *where* R^0_γ *denotes the Green function of* (2.15) *for potential* v^0.

Note that (3.2) is considered as a linear integral equation for finding $[\psi(x,k)]_\alpha$, $x \in \partial D$, at fixed k, from $\hat{M}_{\alpha,v} - \hat{M}_{\alpha,v^0}$ and $[\psi^0(x,k)]_\alpha$, whereas (3.1) is considered as an explicit formula for finding h from h^0, $\hat{M}_{\alpha,v} - \hat{M}_{\alpha,v^0}$, $[\psi^0(x,k)]_\alpha$ and $[\psi(x,k)]_\alpha$. In addition, we use similar interpretation for similar formulas for ψ^+, f and for ψ_γ, h_γ, mentioned in Theorem 3.1 and Proposition 3.1.

Under the assumptions of Theorem 3.1, the following propositions are valid:

Proposition 3.2. *Equation* (3.2) *for* $[\psi(x,k)]_\alpha$ *at fixed* $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$ *is a Fredholm linear integral equation of the second kind in the space of bounded functions on* ∂D. *In addition, the same is also valid for the equation for* $[\psi^+(x,k)]_\alpha$ *at fixed* $k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^{+,0})$, *mentioned in Theorem 3.1, and for the equation for* $[\psi_\gamma(x,k)]_\alpha$ *at fixed* $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^0_\gamma)$, *mentioned in Proposition 3.1.*

Proposition 3.3. *For* $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$ *equation* (3.2) *is uniquely solvable in the space of bounded functions on* ∂D *if and only if* $k \notin \mathcal{E}$. *In addition, the aforementioned equations for* $[\psi^+(x,k)]_\alpha$, $k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^{+,0})$, *and* $[\psi_\gamma(x,k)]_\alpha$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^0_\gamma)$, *are uniquely solvable in the space of bounded functions on* ∂D *if and only if* $k \notin \mathcal{E}^+$ *and* $k \notin \mathcal{E}_\gamma$, *respectively.*

Proposition 3.4. *Let* $\phi_\alpha(x,y)$ *be the solution of the Dirichlet boundary value problem at fixed* $y \in \partial D$, $\lambda \in \mathbb{C}$:

$$\begin{aligned} &-\Delta_x \phi_\alpha(x,y) = \lambda \phi_\alpha(x,y), \qquad x \in D,\\ &\phi_\alpha(x,y) = \left(M_{\alpha,v} - M_{\alpha,v^0}\right)(x,y,E), \quad x \in \partial D, \end{aligned} \tag{3.7}$$

where we assume that λ *is not a Dirichlet eigenvalue for* $-\Delta$ *in* D. *Then*

$$\begin{aligned} A_\alpha(x,y,k) = \lim_{\varepsilon \to +0} \int\limits_{\partial D} [R^0(x + \varepsilon\nu_x, \xi, k)]_{x,\alpha} [\phi_\alpha(\xi,y)]_{\xi,\alpha} d\xi -\\ - \sin\alpha \int\limits_{D} [R^0(x,\xi,k)]_{x,\alpha} (v^0(\xi) - E + \lambda)\phi_\alpha(\xi,y) d\xi, \end{aligned} \tag{3.8}$$
$$x, y \in \partial D,$$

where

$$[R^0(x+\varepsilon\nu_x,\xi,k)]_{x,\alpha} = \left(\cos\alpha - \sin\alpha\frac{\partial}{\partial\nu_x}\right) R^0(x+\varepsilon\nu_x,\xi,k), \quad x\in\partial D,\ \xi\in\bar{D}, \tag{3.9}$$

$$\begin{aligned}[\phi_\alpha(\xi,y)]_{\xi,\alpha} &= \left(\cos\alpha - \sin\alpha\frac{\partial}{\partial\nu_\xi}\right)\phi_\alpha(\xi,y) = \\ &= \cos\alpha\,\phi_\alpha(\xi,y) - \sin\alpha\left(\hat{\Phi}(\lambda)\phi_\alpha(\cdot,y)\right)(\xi), \quad \xi,y\in\partial D,\end{aligned} \tag{3.10}$$

where A_α *is defined in* (3.3), $\hat{\Phi}(\lambda) = \hat{M}_{0,0}(\lambda)$ *is the Dirichlet-to-Neumann map for* (3.7)*. In addition, formulas completely similar to* (3.8) *are also valid for the kernels* A^+_α *(but with* R^+_0 *in place of* R_0*) and* $A_{\alpha,\gamma}$ *(but with* R^0_γ *in place of* R^0*), arising in the equations for* $[\psi^+]_\alpha$ *and* $[\psi_\gamma]_\alpha$*, mentioned in Theorem 3.1 and Proposition 3.1.*

Note that, for the case when $\sin\alpha = 0$, formula (3.8) coincides with (3.3). However, for $\sin\alpha \neq 0$, formula (3.8) does not contain $\partial^2 R^0/\partial\nu_x\partial\nu_\xi$ in contrast with (3.3) and is more convenient than (3.8) in this sense.

Theorem 3.1, Propositions 3.1 - 3.4 and the reconstruction results from generalized scattering data (see [12], [13], [15], [26]- [28], [30]- [32], [36]) imply the following corollary:

Corollary 3.1. To reconstruct a potential v in the domain D from its impedance boundary map $\hat{M}_{\alpha,v}(E)$ at fixed E and α one can use the following schema:

1. $v^0 \to \{S^0_E\}, \{R^0\}, \{[\psi^0]_\alpha\}, \hat{M}_{\alpha,v^0}$ via direct problem methods,
2. $\{R^0\}, \hat{M}_{\alpha,v^0}, \hat{M}_{\alpha,v} \to \{A_\alpha\}$ as described in Theorem 3.1 and Propositions 3.1, 3.4,
3. $\{A_\alpha\}, \{[\psi^0]_\alpha\} \to \{[\psi]_\alpha\}$ as described in Theorem 3.1 and Proposition 3.1,
4. $\{S^0_E\}, \{[\psi^0]_\alpha\}, \{[\psi]_\alpha\}, \hat{M}_{\alpha,v^0}, \hat{M}_{\alpha,v} \to \{S_E\}$ as described in Theorem 3.1 and Proposition 3.1,
5. $\{S_E\} \to v$ as described in [12], [13], [15], [26]- [28], [30]- [32], [36],

where $\{S^0_E\}$ and $\{S_E\}$ denote some appropriate part of h^0, f^0, h^0_γ and h, f, h_γ, respectively, $\{[\psi^0]_\alpha\}$ and $\{[\psi]_\alpha\}$ denote some appropriate part of $[\psi^0]_\alpha$, $[\psi^{+,0}]_\alpha$, $[\psi^0_\gamma]_\alpha$ and $[\psi]_\alpha$, $[\psi^+]_\alpha$, $[\psi_\gamma]_\alpha$, respectively, $\{R^0\}$, $\{A_\alpha\}$ denote some appropriate part of R^0, $R^{+,0}$, R^0_γ, A_α, A^+_α, $A_{\alpha,\gamma}$.

Remark 3.1. For the case when $v^0 \equiv 0$, $\sin\alpha = 0$, Theorem 3.1, Propositions 3.1 - 3.3 and Corollary 3.1 (with available references at that time at step 5) were obtained in [25] (see also [23], [24]). Note that basic results of [25] were presented already in the survey given in [15]. For the case when $\sin\alpha = 0$ Theorem 3.1, Propositions 3.1 - 3.3 and Corollary 3.1 (with available references at that time at step 5) were obtained in [29].

Remark 3.2. The results of Theorem 3.1, Propositions 3.1 - 3.4 and Corollary 3.1 remain valid for complex-valued v, v^0 and complex E, α, under the condition that (1.7) holds for both v and v^0.

Remark 3.3. Under the assumptions of Theorem 3.1, the following formula holds:

$$\hat{M}_{\alpha,v}(E) - \hat{M}_{\alpha,v^0}(E) = (D_\alpha \mathrm{R}^{+,0}(E))^{-1} - (D_\alpha \mathrm{R}^{+}(E))^{-1}, \tag{3.11}$$

$$\begin{aligned} D_\alpha \mathrm{R}^{+}(E)u(x) &= \lim_{\varepsilon\to+0} \int\limits_{\partial D} D_{\alpha,\varepsilon}\mathrm{R}^{+}(x,y,\sqrt{E})u(y)dy, \\ D_\alpha \mathrm{R}^{+,0}(E)u(x) &= \lim_{\varepsilon\to+0} \int\limits_{\partial D} D_{\alpha,\varepsilon}\mathrm{R}^{+,0}(x,y,\sqrt{E})u(y)dy, \\ &\qquad x \in \partial D, \end{aligned} \tag{3.12}$$

where $D_{\alpha,\varepsilon}$ is defined as in (3.4), $\mathrm{R}^{+}(x,y,\sqrt{E})$, $\mathrm{R}^{+,0}(x,y,\sqrt{E})$, $\sqrt{E} > 0$, are the Green functions of (2.16) written as in (2.18) for potentials v, v^0, respectively, u is the test function. For the case when $\sin\alpha = 0$, $v^0 \equiv 0$, $d \geq 3$, formula (3.11) was given in [23]. Using techniques developed in [17] and in the present work, we obtain (3.11) in the general case.

4 Proofs of Theorem 3.1 and Propositions 3.1, 3.2, 3.4

In this section we will use formulas and equations for impedance boundary map from [17]. These results are presented in detail in Subsection 4.1. Proofs of Theorem 3.1 and Propositions 3.1, 3.2, 3.4 are given in Subsections 4.2, 4.3.

4.1 Preliminaries

Let $G_{\alpha,v}(x,y,E)$ be the Green function for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6) under assumptions (1.2), (1.3) and (1.7). We recall that (see formulas (3.12), (3.13) of [17]):

$$G_{\alpha,v}(x,y,E) = G_{\alpha,v}(y,x,E), \quad x,y \in \bar{D}, \tag{4.1}$$

and, for $\sin\alpha \neq 0$,

$$M_{\alpha,v}(x,y,E) = \frac{1}{\sin^2\alpha} G_{\alpha,v}(x,y,E) - \frac{\cos\alpha}{\sin\alpha}\delta_{\partial D}(x-y), \quad x,y \in \partial D, \tag{4.2}$$

where $M_\alpha(x,y,E)$ and $\delta_{\partial D}(x-y)$ denote the Schwartz kernels of the impedance boundary map $\hat{M}_{\alpha,v}(E)$ and the identity operator $\hat{I}$ on ∂D, respectively, where $\hat{M}_\alpha$ and $\hat{I}$ are considered as linear integral operators.

We recall also that (see, for example, formula (3.16) of [17]):

$$\psi(x) = \frac{1}{\sin\alpha} \int\limits_{\partial D} (\cos\alpha\, \psi(\xi) - \sin\alpha \frac{\partial}{\partial\nu}\psi(\xi)) G_{\alpha,v}(x,\xi,E) d\xi, \quad x \in D, \tag{4.3}$$

for all sufficiently regular solutions ψ of equation (1.1) in $\bar{D}$ and $\sin\alpha \neq 0$.

We will use the following properties of the Green function $G_\alpha(x,y,E)$:

$$G_{\alpha,v}(x,y,E) \text{ is continuous in } x,y \in \bar{D},\ x \neq y, \tag{4.4}$$

$$\begin{aligned} |G_{\alpha,v}(x,y,E)| &\leq c_1(|x-y|^{2-d}), \quad x,y \in \bar{D}, \text{ for } d \geq 3, \\ |G_{\alpha,v}(x,y,E)| &\leq c_1(|\ln|x-y||), \quad x,y \in \bar{D}, \text{ for } d = 2, \end{aligned} \tag{4.5}$$

where $c_1 = c_1(D,E,v,\alpha) > 0$.

Actually, properties (4.4), (4.5) are well-known for $\sin\alpha = 0$ (the case of the Direchlet boundary condition) and for $\cos\alpha = 0$ (the case of the Neumann boundary condition). Properties (4.4), (4.5) with $d \geq 3$, $\frac{\cos\alpha}{\sin\alpha} < 0$, $v \equiv 0$ and $E = 0$ were proven in [19]. For $d = 2$ see also [2]. In Section 6 we give proofs of (4.4), (4.5) for the case of general α, v and E.

In addition, under assumptions of Theorem 3.1, the following identity holds (see formula (3.9) of [17]):

$$\int\limits_D (v - v^0)\psi\psi^0 dx = \int\limits_{\partial D} [\psi]_\alpha \left(\hat{M}_{\alpha,v} - \hat{M}_{\alpha,v^0}\right) [\psi^0]_\alpha dx \tag{4.6}$$

for all sufficiently regular solutions ψ, ψ^0 of equation (1.1) in $\bar{D}$ for potentials v, v^0, respectively, where $[\psi]_\alpha$, $[\psi^0]_\alpha$ are defined according to (1.5).

Identity (4.6) for $\sin\alpha = 0$ is reduced to the Alessandrini identity (Lemma 1 of [1]).

We will use also that:

$$\begin{aligned} \|\hat{R}(k)u\|_{C^{1+\delta}(\Omega)} &\leq c_2(D,\Omega,v,k,\delta)\|u\|_{\mathbb{L}^\infty(D)}, \\ \hat{R}(k)u(x) &= \int\limits_D R(x,y,k)u(y)dy, \quad x \in \Omega, \\ & \qquad k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}), \end{aligned} \tag{4.7a}$$

$$\begin{aligned} \|\hat{R}_\gamma(k)u\|_{C^{1+\delta}(\Omega)} &\leq c_3(D,\Omega,v,k,\gamma,\delta)\|u\|_{\mathbb{L}^\infty(D)}, \\ \hat{R}_\gamma(k)u(x) &= \int\limits_D R_\gamma(x,y,k)u(y)dy, \quad x \in \Omega, \\ & \gamma \in \mathbb{S}^{d-1},\ k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}_\gamma), \end{aligned} \tag{4.7b}$$

for $u \in \mathbb{L}^\infty(D)$, $\delta \in [0,1)$, where Ω is such an open bounded domain in $\mathbb{R}^d$ that $\bar{D} \in \Omega$ and $C^{1+\delta}$ denotes C^1 with the first derivatives belonging to the Hölder space C^δ.

We will use also the Green formula:

$$\int_{\partial D} \left(\phi_1 \frac{\partial \phi_2}{\partial \nu} - \phi_2 \frac{\partial \phi_1}{\partial \nu} \right) dx = \int_{D} (\phi_1 \Delta \phi_2 - \phi_2 \Delta \phi_1)\, dx, \tag{4.8}$$

where ϕ_1 and ϕ_2 are arbitrary sufficiently regular functions in $\bar{D}$.

4.2 Proof of Theorem 3.1 and Proposition 3.1

For the case when $\sin\alpha = 0$, Theorem 3.1 and Proposition 3.1 were proved in [29]. In this subsection we generalize the proof of [29] to the case $\sin\alpha \neq 0$. We proceed from the following formulas and equations (being valid under assumption (2.2) on v^0 and v):

$$\begin{gathered} h(k,l) - h^0(k,l) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \psi^0(x,-l)(v(x) - v^0(x))\psi(x,k)dx, \\ k,l \in \mathbb{C}^d \setminus (\mathcal{E}^0 \cup \mathcal{E}),\ k^2 = l^2,\ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \neq 0, \end{gathered} \tag{4.9}$$

$$\begin{gathered} \psi(x,k) = \psi^0(x,k) + \int_{\mathbb{R}^d} R^0(x,y,k)(v(y) - v^0(y))\psi(y,k)dy, \\ x \in \mathbb{R}^d,\ k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0), \end{gathered} \tag{4.10}$$

where (4.10) at fixed k is considered as an equation for $\psi = e^{ikx}\mu(x,k)$ with $\mu \in \mathbb{L}^\infty(\mathbb{R}^d)$;

$$\begin{gathered} f(k,l) - f^0(k,l) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \psi^{+,0}(x,-l)(v(x) - v^0(x))\psi^+(x,k)dx, \\ k,l \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^{+,0} \cup \mathcal{E}^+),\ k^2 = l^2, \end{gathered} \tag{4.11}$$

$$\begin{gathered} \psi^+(x,k) = \psi^{+,0}(x,k) + \int_{\mathbb{R}^d} R^{+,0}(x,y,k)(v(y) - v^0(y))\psi^+(y,k)dy, \\ x \in \mathbb{R}^d,\ k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^{+,0}), \end{gathered} \tag{4.12}$$

where (4.12) at fixed k is an equation for $\psi^+ \in \mathbb{L}^\infty(\mathbb{R}^d)$;

$$\begin{gathered} h_\gamma(k,l) - h_\gamma^0(k,l) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \psi^0_{-\gamma}(x,-k,-l)(v(x) - v^0(x))\psi_\gamma(x,k)dx, \\ \gamma \in \mathbb{S}^{d-1}, k \in \mathbb{R}^d \setminus (\mathcal{E}^0_\gamma \cup \mathcal{E}_\gamma),\ l \in \mathbb{R}^d,\ k^2 = l^2, \end{gathered} \tag{4.13}$$

$$\psi_\gamma(x,k) = \psi^0_\gamma(x,k) + \int\limits_{\mathbb{R}^d} R^0_\gamma(x,y,k)(v(y)-v^0(y))\psi_\gamma(y,k)dy,$$
$$x\in\mathbb{R}^d,\ \gamma\in\mathbb{S}^{d-1},\ k\in\mathbb{R}^d\setminus(\{0\}\cup\mathcal{E}^0_\gamma), \tag{4.14}$$

where (4.14) at fixed γ and k is considered as an equation for $\psi_\gamma\in\mathbb{L}^\infty(\mathbb{R}^d)$.

We recall that ψ^+, f, ψ, h, ψ_γ, h_γ were defined in Sections 2, 3 by means of (2.3) - (2.9), (3.5). Equation (4.12) is well-known in the classical scattering theory for the Schrödinger equation (2.1). Formula (4.11) was given, in particular, in [38]. To our knowledge formula and equations (4.9), (4.10), (4.14) were given for the first time in [27], whereas formula (4.13) was given for the first time in [29].

In addition, under assumption (2.2) on v^0 and v:

$$\begin{gathered}\text{equation (4.10) at fixed } k\in\mathbb{C}^d\setminus(\mathbb{R}^d\cup\mathcal{E}^0) \text{ is uniquely solvable}\\ \text{for } \psi=e^{ikx}\mu(x,k) \text{ with } \mu\in\mathbb{L}^\infty(\mathbb{R}^d) \text{ if and only if } k\notin\mathcal{E};\end{gathered} \tag{4.15a}$$

$$\begin{gathered}\text{equation (4.12) at fixed } k\in\mathbb{R}^d\setminus(\{0\}\cup\mathcal{E}^{+,0}) \text{ is uniquely}\\ \text{solvable for } \psi^+\in\mathbb{L}^\infty(\mathbb{R}^d) \text{ if and only if } k\notin\mathcal{E}^+;\end{gathered} \tag{4.15b}$$

$$\begin{gathered}\text{equation (4.14) at fixed } \gamma\in\mathbb{S}^{d-1} \text{ and } k\in\mathbb{R}^d\setminus(\{0\}\cup\mathcal{E}^+_\gamma)\\ \text{is uniquely solvable for } \psi_\gamma\in\mathbb{L}^\infty(\mathbb{R}^d) \text{ if and only if } k\notin\mathcal{E}_\gamma.\end{gathered} \tag{4.15c}$$

Let us prove Theorem 3.1 for the case of the Faddeev functions ψ, h. The proof of Theorem 3.1 for the case of ψ^+, f and the proof of Proposition 3.1 are similar.

Note that formula (3.1) follows directly from (4.6) and (4.9).

Using (2.17) and applying (4.6) for equation (4.10), we get that

$$\begin{aligned}&\psi(x,k)-\psi^0(x,k) =\\ &= \int\limits_{\partial D}\int\limits_{\partial D}[R^0(x,\xi,k)]_{\xi,\alpha}\left(M_{\alpha,v}-M_{\alpha,v^0}\right)(\xi,y,E)[\psi(y,k)]_\alpha d\xi dy,\\ &\qquad x\in\mathbb{R}^d\setminus\bar{D},\end{aligned} \tag{4.16}$$

where
$$[R^0(x,\xi,k)]_{\xi,\alpha} = \left(\cos\alpha-\sin\alpha\frac{\partial}{\partial\nu_\xi}\right)R^0(x,\xi,k). \tag{4.17}$$

Equation (3.2) follows from formula (4.16), definition (1.5) and the property that

$$\lim_{\varepsilon\to+0}\left(\cos\alpha-\sin\alpha\frac{\partial}{\partial\nu_x}\right)u(x+\varepsilon\nu_x) = [u(x)]_\alpha,\quad x\in\partial D, \tag{4.18}$$

for $u(x)=\psi(x,k)-\psi^0(x,k)$.

4.3 Proofs of Propositions 3.2 and 3.4

In this subsection we prove Propositions 3.2, 3.4 for the case of equation (3.2) for $[\psi]_\alpha$. The proofs of Propositions 3.2 and 3.4 for the cases of ψ^+ and ψ_γ are absolutely similar.

Proof of Proposition 3.2. The proof of Proposition 3.2 for the case of $\sin\alpha = 0$ was given in [29]. Let us assume that $\sin\alpha \neq 0$.

Using (4.2), we find that

$$\left(M_{\alpha,v} - M_{\alpha,v^0}\right)(\xi, y, E) = \frac{1}{\sin^2\alpha}\left(G_{\alpha,v} - G_{\alpha,v^0}\right)(\xi, y, E), \quad \xi, y \in \partial D. \tag{4.19}$$

Using (2.17), (4.1), (4.8) and the impedance boundary condition (1.6) for $G_{\alpha,v}$, G_{α,v^0}, we get that

$$\begin{aligned}
&\int\limits_{\partial D} [R^0(x,\xi,k)]_{\alpha,\xi}(G_{\alpha,v} - G_{\alpha,v^0})(\xi, y, E)d\xi = \\
&\quad = \int\limits_{\partial D}\Bigg([R^0(x,\xi,k)]_{\alpha,\xi}\left(G_{\alpha,v} - G_{\alpha,v^0}\right)(\xi,y,E)d\xi - \\
&\qquad\qquad - R^0(x,\xi,k)[\left(G_{\alpha,v} - G_{\alpha,v^0}\right)(\xi,y,E)]_{\alpha,\xi}\Bigg)d\xi = \\
&\quad = \sin\alpha\int\limits_{D}\Bigg(R^0(x,\xi,k)\Delta_\xi\left(G_{\alpha,v} - G_{\alpha,v^0}\right)(\xi,y,E)d\xi - \\
&\qquad\qquad - \left(G_{\alpha,v} - G_{\alpha,v^0}\right)(\xi,y,E)\Delta_\xi R^0(x,\xi,k)\Bigg)d\xi = \\
&\quad = \sin\alpha\int\limits_{D} R^0(x,\xi,k)\left(v(\xi) - v^0(\xi)\right)G_{\alpha,v}(\xi,y,E)d\xi, \\
&\qquad\qquad\qquad\qquad x \in \mathbb{R}^d \setminus \bar{D},\ y \in \partial D.
\end{aligned} \tag{4.20}$$

Combining (4.16), (4.19) and (4.20), we obtain that

$$A_\alpha(x,y,k) = \lim_{\varepsilon\to+0}\left(\cos\alpha - \sin\alpha\frac{\partial}{\partial\nu_x}\right)B_\alpha(x+\varepsilon\nu_x, y, k), \quad x, y \in \partial D, \tag{4.21}$$

where

$$\begin{aligned}
B_\alpha(x,y,k) &= \int\limits_{\partial D}[R^0(x,\xi,k)]_{\alpha,\xi}(M_{\alpha,v} - M_{\alpha,v^0})(\xi,y,E)d\xi = \\
&= \frac{1}{\sin\alpha}\int\limits_{D} R^0(x,\xi,k)\left(v(\xi) - v^0(\xi)\right)G_{\alpha,v}(\xi,y,E)d\xi, \\
&\qquad\qquad x \in \mathbb{R}^d \setminus \bar{D},\ y \in \partial D.
\end{aligned} \tag{4.22}$$

Thus, we have that the limit in (4.21) (and, hence, in (3.3)) is well defined and

$$A_\alpha(x,y,k) = \frac{1}{\sin\alpha}\int\limits_D [R^0(x,\xi,k)]_{x,\alpha}\left(v(\xi)-v^0(\xi)\right)G_{\alpha,v}(\xi,y,E)d\xi, \qquad x,y\in\partial D. \tag{4.23}$$

Let $\hat{A}_\alpha(k)$ denote the linear integral operator on ∂D with the Schwartz kernel $A_\alpha(x,y,k)$ of (3.3), (4.23). Using (4.5), (4.7), (4.23), we obtain that

$$\begin{aligned} &\hat{A}_\alpha(k):\ \mathbb{L}^\infty(\partial D)\to C^\delta(\partial D)\\ &\text{is a bounded linear operator.}\end{aligned} \tag{4.24}$$

As a corollary of (4.24), $\hat{A}_\alpha(k)$ is a compact operator in $\mathbb{L}^\infty(D)$. ∎

Proof of Proposition 3.4. Using (2.17), (3.7) and (4.8), we get that

$$\begin{aligned}\int\limits_{\partial D}\left(\phi_\alpha(\xi,y)\frac{\partial}{\partial\nu_\xi}R^0(x,\xi,k)-R^0(x,\xi,k)\frac{\partial}{\partial\nu_\xi}\phi_\alpha(\xi,y)\right)d\xi &=\\ =\int\limits_D\left(\phi_\alpha(\xi,y)\Delta_\xi R^0(x,\xi,k)-R^0(x,\xi,k)\Delta_\xi\phi_\alpha(\xi,y)\right)d\xi &=\\ =\int\limits_D R^0(x,\xi,k)(v^0(\xi)-E+\lambda)\phi_\alpha(\xi,y)d\xi&,\\ x\in\mathbb{R}^d\setminus\bar{D},\ y\in\partial D&.\end{aligned} \tag{4.25}$$

Combining (3.7), (4.22) and (4.25), we find that

$$\begin{aligned}B_\alpha(x,y,k) &= \int\limits_{\partial D}[R^0(x,\xi,k)]_{\xi,\alpha}\phi_\alpha(\xi,y)d\xi =\\ &= \int\limits_{\partial D}R^0(x,\xi,k)[\phi_\alpha(\xi,y)]_{\xi,\alpha}d\xi-\sin\alpha\int\limits_D R^0(x,\xi,k)(v^0(\xi)-E+\lambda)\phi_\alpha(\xi,y)d\xi,\\ &\qquad x\in\mathbb{R}^d\setminus\bar{D},\ y\in\partial D.\end{aligned} \tag{4.26}$$

Combining (4.21) and (4.26), we obtain (3.8).

Formula (3.10) follows from (3.7) and the definition of $\hat{\Phi}$.

∎

5 Proof of Proposition 3.3

For the case when $\sin\alpha=0$, Proposition 3.3 was proved in [29]. In this section we prove Proposition 3.3 for $\sin\alpha\neq 0$. We will prove Proposition 3.3 for the case of equation (3.2) for $[\psi]_\alpha$. The proofs for the cases of ψ^+ and ψ_γ are similar.

According to (4.15), to prove Proposition 3.3 (for the case of ψ) it is sufficient to show that equation (3.2) (at fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$) is uniquely solvable in the space of bounded functions on ∂D if and only if equation (4.10) is uniquely solvable for $\psi = e^{ikx}\mu(x,k)$ with $\mu \in \mathbb{L}^\infty(\mathbb{R}^d)$.

Let equation (4.10) have several solutions. Then, repeating the proof of Theorem 3.1 separately for each solution, we find that $[\psi]_\alpha$ on ∂D for each of these solutions satisfies equation (3.2). Thus, using also (1.7) we obtain that equation (3.2) has at least as many solutions as equation (4.10).

To prove the converse (and thereby to prove Proposition 3.3) it remains to show that any solution $[\psi]_\alpha$ of (3.2) can be continued to a continuos solution of (4.10).

Let ψ be the solution of (1.1) with the impedance boundary data $[\psi]_\alpha$, satisfying (3.2). Let

$$\psi_1(x) = \psi^0(x,k) + \int\limits_D R^0(x,y,k)(v(y) - v^0(y))\psi(y)dy, \quad x \in \mathbb{R}^d. \tag{5.1}$$

Using (4.7), we obtain that

$$\psi_1 \text{ defined by (5.1) belongs to } C^{1+\delta}(\mathbb{R}^d),\ \delta \in [0,1). \tag{5.2}$$

We have that

$$(-\Delta + v^0(x) - E)\psi(x) = (v^0(x) - v(x))\psi(x), \quad x \in D, \tag{5.3}$$

$$\begin{aligned}(-\Delta + v^0(x) - E)\psi_1(x) = \int\limits_D -\delta(x-y)(v(y) - v^0(y))\psi(y)dy = \\ = (v^0(x) - v(x))\psi(x), \quad x \in D.\end{aligned} \tag{5.4}$$

Combining (4.6) and (4.22), we get that

$$\int\limits_D R^0(x,y,k)(v(y) - v^0(y))\psi(y)dy = \int\limits_{\partial D} B_\alpha(x,y,k)[\psi(y)]_\alpha dy, \quad x \in \mathbb{R}^d \setminus \bar{D}. \tag{5.5}$$

Using (3.2), (4.21), (5.2), (5.5), we find that

$$[\psi_1(x)]_\alpha = [\psi^0(x,k)]_\alpha + \int\limits_{\partial D} A_\alpha(x,y,k)[\psi(y)]_\alpha dy = [\psi(x)]_\alpha, \quad x \in \partial D. \tag{5.6}$$

Using (5.3), (5.4) and (5.6), we obtain that

$$\begin{aligned}(-\Delta + v^0(x) - E)(\psi_1(x) - \psi(x)) = 0, \quad x \in D, \\ [\psi_1(x) - \psi(x)]_\alpha = 0, \quad x \in \partial D.\end{aligned} \tag{5.7}$$

Since v^0 satisfies (1.7), we get that

$$\psi_1(x) = \psi(x), \quad x \in \bar{D}. \tag{5.8}$$

Combining (5.1), (5.2) and (5.8), we find that ψ_1 is a continuos solution of (4.10).

6 Proofs of properties (4.4), (4.5)

As it was mentioned in Subsection 4.1, properties (4.4), (4.5) are well-known for $\cos\alpha = 0$ (the case of the Neumann boundary condition). To extend these properties to the case of general α, v, E, we use the following schema:

1. $G_{\alpha_1,v} \to G_{\alpha_2,v}$ by means of Lemma 6.1 given bellow (with $\sin\alpha_1 \neq 0$ and $\sin\alpha_2 \neq 0$),

2. $G_{\alpha,v_1} \to G_{\alpha,v_2}$ by means of Lemma 6.2 given bellow.

The proofs of steps 1, 2 are based on the theory of Fredholm linear integral equations of the second kind.

Starting from (4.4), (4.5) for $\cos\alpha = 0$ and combining steps 1, 2 and the property

$$G_{\alpha,v}(\cdot,\cdot,E) = G_{\alpha,v-E}(\cdot,\cdot,0), \tag{6.1}$$

we obtain these properties for the case when $\sin\alpha \neq 0$.

As it was already mentioned in Section 4, properties (4.4), (4.5) are well-known for $\sin\alpha = 0$ (the case of the Dirichlet boundary condition).

Lemma 6.1. *Let D satisfy* (1.2) *and potential v satisfy (1.3), (1.7) for some fixed E and for $\alpha = \alpha_1$, $\alpha = \alpha_2$ simultaneously, where* $\sin\alpha_1 \neq 0$ *and* $\sin\alpha_2 \neq 0$. *Let G_j denote the Green function $G_{\alpha_j,v}$, $j = 1,2$. Let G_1 satisfy:*

$$G_1(x,y,E) \ \text{ is continuous in } \ x,y \in \bar{D},\ x \neq y, \tag{6.2}$$

$$\begin{aligned} |G_1(x,y,E)| &\le a_1|x-y|^{2-d} \quad \text{for } \ d \ge 3,\\ |G_1(x,y,E)| &\le a_1|\ln|x-y|| \quad \text{for } \ d = 2,\\ & \qquad\qquad x,y \in \bar{D}. \end{aligned} \tag{6.3}$$

Then:

$$G_2(x,y,E) \ \text{ is continuous in } \ x,y \in \bar{D},\ x \neq y, \tag{6.4}$$

$$\begin{aligned} |G_2(x,y,E)| &\le a_2|x-y|^{2-d} \quad \text{for } \ d \ge 3,\\ |G_2(x,y,E)| &\le a_2|\ln|x-y|| \quad \text{for } \ d = 2,\\ & \qquad\qquad x,y \in \bar{D}, \end{aligned} \tag{6.5}$$

where $a_2 = a_2(D,E,a_1,v,\alpha_1,\alpha_2) > 0$.

Proof of Lemma 6.1. First, we derive formally some formulas and equations relating the Green functions G_1 and G_2. Then, proceeding from these formulas and equations, we obtain, in particular, estimates (6.4), (6.5).

Consider $W = G_2 - G_1$. Using definitions of G_1, G_2 and formula (4.3), we find that:

$$(-\Delta_x + v(x) - E)W(x,y) = 0, \quad x,y \in D, \tag{6.6}$$

$$
\begin{aligned}
\Big(\cos\alpha_2\, W(x,y) - \sin\alpha_2\, \frac{\partial W}{\partial \nu_x}(x,y)\Big)\Big|_{x\in\partial D} &= \\
= -\Big(\cos\alpha_2\, G_1(x,y,E) - \sin\alpha_2\, \frac{\partial G_1}{\partial \nu_x}(x,y,E)\Big)\Big|_{x\in\partial D} &= \\
= -\Big(\cos\alpha_2\, G_1(x,y,E) - \sin\alpha_2\, \frac{\cos\alpha_1}{\sin\alpha_1} G_1(x,y,E)\Big)\Big|_{x\in\partial D} &= \\
= \frac{\sin(\alpha_2-\alpha_1)}{\sin\alpha_1} G_1(x,y,E)\Big|_{x\in\partial D}, \quad y\in D&,
\end{aligned}
\tag{6.7}
$$

$$
W(x,y) = \frac{1}{\sin\alpha_1}\int\limits_{\partial D}\Big(\cos\alpha_1 W(\xi,y) - \sin\alpha_1 \frac{\partial W}{\partial \nu_\xi}(\xi,y)\Big) G_1(\xi,x,E)d\xi, \quad x,y\in D. \tag{6.8}
$$

Using (6.7) and (6.8), we find the following linear integral equation for $W(\cdot,y)$ on ∂D:

$$
W(\cdot,y) = W_0(\cdot,y) + \hat K_1 W(\cdot,y), \quad y\in D, \tag{6.9}
$$

where

$$
W_0(x,y) = \frac{\sin(\alpha_2-\alpha_1)}{\sin\alpha_2}\int\limits_{\partial D} G_1(\xi,x,E)G_1(\xi,y,E)d\xi, \tag{6.10}
$$

$$
\hat K_1 u(x) = \frac{\sin(\alpha_2-\alpha_1)}{\sin\alpha_2\sin\alpha_1}\int\limits_{\partial D} G_1(\xi,x,E)u(\xi)d\xi, \quad x\in\partial D,\ y\in D,\ u \text{ is a test function.} \tag{6.11}
$$

In addition, for

$$
\delta_n W = W - \sum_{j=1}^{n}(\hat K_1)^{j-1}W_0 \tag{6.12}
$$

equation (6.9) takes the form

$$
\delta_n W = (\hat K_1)^n W_0 + \hat K_1 \delta_n W. \tag{6.13}
$$

Our analysis based on (6.6)-(6.13) is given bellow.
Using (6.2), (6.3), we obtain that

$$
(\hat K_1)^n W_0 \in C(\partial D\times \bar D) \text{ for sufficiently great } n \text{ with respect to } d, \tag{6.14}
$$

$$
\hat K_1 \text{ is a compact operator in } C(\partial D). \tag{6.15}
$$

Let us show that the homogeneous equation

$$
u = \hat K_1 u, \quad u\in C(\partial D), \tag{6.16}
$$

has only trivial solution $u\equiv 0$.

Using the fact that the potential v satisfy (1.7) for $\alpha = \alpha_1$, we define ψ by

$$\begin{aligned} (-\Delta + v(x) - E)\psi(x) = 0, \quad x \in D, \\ \cos\alpha_1 \psi|_{\partial D} - \sin\alpha_1 \frac{\partial \psi}{\partial \nu}|_{\partial D} = u. \end{aligned} \tag{6.17}$$

Due to (4.3), we have that

$$\psi(x) = \frac{1}{\sin\alpha_1} \int\limits_{\partial D} (\cos\alpha_1 \psi(\xi) - \sin\alpha_1 \frac{\partial \psi}{\partial \nu}(\xi)) G_1(\xi, x, E) d\xi, \quad x \in D. \tag{6.18}$$

Using (6.16), (6.18), we find that

$$\frac{\sin(\alpha_2 - \alpha_1)}{\sin\alpha_2} \psi(x) = \hat{K}_1 u(x) = u(x), \quad x \in \partial D. \tag{6.19}$$

Therefore, we have that

$$\cos\alpha_1 \psi(x) - \sin\alpha_1 \frac{\partial \psi}{\partial \nu}(x) = \frac{\sin(\alpha_2 - \alpha_1)}{\sin\alpha_2} \psi(x), \quad x \in \partial D. \tag{6.20}$$

Since $\sin\alpha_1 \neq 0$ and $\sin\alpha_2 \neq 0$, using (6.20), we obtain that

$$\cos\alpha_2 \psi(x) - \sin\alpha_2 \frac{\partial \psi}{\partial \nu}(x) = 0 \tag{6.21}$$

Taking into account the fact that the potential v satisfy (1.7) for $\alpha = \alpha_2$, we get that $\psi \equiv 0$ and $u \equiv 0$.

Proceeding from

$$\begin{aligned} F = W(x, y) \ \text{ and } \ F' = \frac{\cos\alpha_2}{\sin\alpha_2} W(x, y) - \frac{\sin(\alpha_2 - \alpha_1)}{\sin\alpha_1 \sin\alpha_2} G_1(x, y, E), \\ x \in \partial D,\ y \in \bar{D}, \end{aligned} \tag{6.22}$$

found from (6.9), (6.13) and (6.7) (with F' substituted in place of $\partial W/\partial \nu_x$), we consider

$$\begin{aligned} W(x, y) = \frac{1}{\sin\alpha_1} \int\limits_{\partial D} \Big(\cos\alpha_1 F(\xi, y) - \sin\alpha_1 F'(\xi, y) \Big) G_1(\xi, x, E) d\xi, \\ x, y \in \bar{D}. \end{aligned} \tag{6.23}$$

Using (6.9) and properties of G_1 (including formula (4.3)), we subsequently obtain that

$$\lim_{\varepsilon \to +0} W(x - \varepsilon \nu_x, y) = F(x, y), \quad x \in \partial D,\ y \in \bar{D}, \tag{6.24}$$

$$W \text{ satisfies (6.6)}, \tag{6.25}$$

$$\lim_{\varepsilon \to +0} \frac{\partial}{\partial \nu_x} W(x - \varepsilon \nu_x, y) = F'(x, y), \quad x \in \partial D,\ y \in \bar{D}. \tag{6.26}$$

From (6.2), (6.3), (6.10)-(6.16), (6.24)-(6.26) it follows that G_2 defined as $G_2 = G_1 + W$ is the Green function for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6) for $\alpha = \alpha_2$ and that G_2 satisfies (6.4), (6.5). ■

Lemma 6.2. *Let D satisfy* (1.2) *and potentials v_1, v_2 satisfy (1.3), (1.7) for some fixed E and α. Let G_j denote the Green function G_{α,v_j}, $j = 1, 2$. Let G_1 satisfy:*

$$G_1(x, y, E) \ \text{ is continuous in } \ x, y \in \bar{D}, \ x \neq y, \tag{6.27}$$

$$\begin{aligned} |G_1(x, y, E)| \le a_3 |x - y|^{2-d} \quad &\text{for} \ \ d \ge 3, \\ |G_1(x, y, E)| \le a_3 |\ln|x - y|| \quad &\text{for} \ \ d = 2, \\ &x, y \in \bar{D}. \end{aligned} \tag{6.28}$$

Then:

$$G_2(x, y, E) \ \text{ is continuous in } \ x, y \in \bar{D}, \ x \neq y, \tag{6.29}$$

$$\begin{aligned} |G_2(x, y, E)| \le a_4 |x - y|^{2-d} \quad &\text{for} \ \ d \ge 3, \\ |G_2(x, y, E)| \le a_4 |\ln|x - y|| \quad &\text{for} \ \ d = 2, \\ &x, y \in \bar{D}, \end{aligned} \tag{6.30}$$

where $a_4 = a_4(D, E, a_3, v_1, v_2, \alpha) > 0$.

Proof of Lemma 6.2. First, we derive formally some formulas and equations relating the Green functions G_1 and G_2. Then, proceeding from these formulas and equations, we obtain, in particular, estimates (6.29), (6.30).

Using (4.1), the impedance boundary condition for G_1, G_2, we find that

$$\begin{aligned} G_1(x, y, E) &= \int\limits_D G_1(x, \xi, E)\Big(\Delta_\xi - v_2(\xi) + E\Big) G_2(\xi, y, E)\, d\xi, \\ G_2(x, y, E) &= \int\limits_D G_2(\xi, y, E)\Big(\Delta_\xi - v_1(\xi) + E\Big) G_1(x, \xi, E)\, d\xi, \\ \int\limits_{\partial D} \bigg(G_1(x, \xi, E) &\frac{\partial G_2}{\partial \nu_\xi}(\xi, y, E) - G_2(\xi, y, E) \frac{\partial G_1}{\partial \nu_\xi}(x, \xi, E) \bigg)\, d\xi = 0, \\ &x, y \in D. \end{aligned} \tag{6.31}$$

Combining (6.31) with (4.8), we get that

$$G_2(\cdot, y, E) - G_1(\cdot, y, E) = \hat{K}_2 G_2(\cdot, y, E), \quad y \in D, \tag{6.32}$$

where

$$\hat{K}_2 u(x) = \int\limits_D \left(v_2(\xi) - v_1(\xi)\right) G_1(x, \xi, E) u(\xi) d\xi. \tag{6.33}$$

In addition, for

$$\delta_n G = G_2 - \sum_{j=1}^{n} (\hat{K}_2)^{j-1} G_1 \tag{6.34}$$

equation (6.32) takes the form

$$\delta_n G = (\hat{K}_2)^n G_1 + \hat{K}_2 \delta_n G. \tag{6.35}$$

Our analysis based on (6.31)-(6.35) is given bellow.
Using (6.27), (6.28), we find that

$$(\hat{K}_2)^n G_1 \in C(\bar{D} \times \bar{D}) \text{ for sufficiently great } n \text{ with respect to } d, \tag{6.36}$$

$$\hat{K}_2 \text{ is a compact operator in } C(\bar{D}). \tag{6.37}$$

Let us show that the homogeneous equation

$$u = \hat{K}_2 u, \quad u \in C(\bar{D}), \tag{6.38}$$

has only trivial solution $u \equiv 0$. Using (6.33), (6.38) and properties of the Green function G_1, we find that

$$\begin{aligned} (-\Delta + v_1(x) - E)u(x) = \int\limits_D -\delta(x-\xi)\,(v_2(\xi) - v_1(\xi))\,u(\xi)d\xi = \\ = (v_1 - v_2)u(x), \quad x \in D, \\ \cos\alpha\, u(x) - \sin\alpha \frac{\partial u}{\partial \nu}(x) = 0, \quad x \in \partial D. \end{aligned} \tag{6.39}$$

Using (6.27), (6.28), we find that $u \in C(\bar{D})$. Taking into account the fact that the potential v_2 satisfy (1.7), we get that $u \equiv 0$.

Proceeding from (6.27), (6.28), (6.36), (6.37) it follows that G_2 found from (6.32), (6.35) is the Green function for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6) for $v = v_2$ and that G_2 satisfies (6.29), (6.30). ■

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M.I. Isaev,
Centre de Mathématiques Appliquées,
Ecole Polytechnique, 91128 Palaiseau, France
e-mail: `isaev.m.i@gmail.com`

R.G. Novikov,
Centre de Mathématiques Appliquées,
Ecole Polytechnique, 91128 Palaiseau, France
e-mail: `novikov@cmap.polytechnique.fr`