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Energy and regularity dependent stability estimates for the Gel'fand inverse problem in multidimensions

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R.I. 751 *April 2012*

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Abstract

We prove new global Hölder-logarithmic stability estimates for the Gel'fand inverse problem at fixed energy in dimension $d \geq 3$. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. Comparisons with preceeding results in this direction are given.

1 Introduction

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \tag{1.1}$$

where

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^d,\ d \geq 2, \\ &\text{with } \partial D \in C^2, \end{aligned} \tag{1.2}$$

$$v \in \mathbb{L}^\infty(D). \tag{1.3}$$

Consider the map $\hat\Phi = \hat\Phi(E)$ such that

$$\hat\Phi(E)(\psi|_{\partial D}) = \frac{\partial\psi}{\partial\nu}|_{\partial D} \tag{1.4}$$

for all sufficiently regular solutions ψ of (1.1) in $\bar D = D \cup \partial D$, where ν is the outward normal to ∂D. Here we assume also that

$$E \text{ is not a Dirichlet eigenvalue for operator } -\Delta + v \text{ in } D. \tag{1.5}$$

The map $\hat\Phi = \hat\Phi(E)$ is called the Dirichlet-to-Neumann map and is considered as boundary measurements.

We consider the following inverse boundary value problem for equation (1.1):

Problem 1.1. Given $\hat\Phi$ for some fixed E, find v.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at fixed energy (see [10], [23]). At zero energy this problem can be considered also as a generalization of the Calderon problem of the electrical impedance tomography (see [6], [23]). Problem 1.1 can be also considered as an example of ill-posed problem: see [18], [4] for an introduction to this theory.

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness results and global reconstruction methods for Problem 1.1 were given for the first time in [23] in dimension $d \geq 3$ and in [5] in dimension $d = 2$.

Global logarithmic stability estimates for Problem 1.1 were given for the first time in [1] in dimension $d \geq 3$ and in [30] in dimension $d = 2$. A principal improvement of the result of [1] was given recently in [29] (for the zero energy case): stability of [29] optimally increases with increasing regularity of v.

For the Calderon problem (of the electrical impedance tomography) in its initial formulation the global uniqueness was firstly proved in [36] for $d \geq 3$ and in [21] for $d = 2$. Global logarithmic stability estimates for this problem were given for the first time in [1] for $d \geq 3$ and [19] for $d = 2$. Principal increasing of global stability of [1], [19] for the regular coefficient case was found in [29] for $d \geq 3$ and [34] for $d = 2$.

In addition, for the case of piecewise constant or piecewise real analytic conductivity the first uniqueness results for the Calderon problem in dimension $d \geq 2$ were given in [7], [16]. Lipschitz stability estimate for the case of piecewise constant conductivity was proved in [2] and additional studies in this direction were fulfilled in [33].

Due to [20] the logarithmic stability results of [1], [19] with their principal effectivization of [29], [34] are optimal (up to the value of the exponent). An extention of the instability estimates of [20] to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was given in [12].

On the other hand, it was found in [25], [26] (see also [28], [31]) that for inverse problems for the Schrödinger equation at fixed energy E in dimension $d \geq 2$ (like Problem 1.1) there is a Hölder stability modulo an error term rapidly decaying as $E \to +\infty$ (at least for the regular coefficient case). In addition, for Problem 1.1 for $d = 3$, global energy dependent stability estimates changing from logarithmic type to Hölder type for high energies were given in [15]. However, there is no efficient stability increasing with respect to increasing coefficient regularity in these results of [15]. An additional study, motivated by [15], [29], was given in [22].

In the present work we give new global Hölder-logarithmic stability estimates for Problem 1.1 in dimension $d \geq 3$ for the regular coefficient case, see Theorem 2.1 and Remark 2.6. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. In particular cases, our new estimates become coherent (although less strong) with respect to results of [29], [26], see Remarks 2.2, 2.3. In general, our new estimates give some synthesis of several important preceeding results.

2 Stability estimates

In this section we assume for simplicity that

$$v \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d, \ \operatorname{supp} v \subset D, \tag{2.1}$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v : \ \partial^J v \in L^1(\mathbb{R}^d), \ |J| \le m\}, \ m \in \mathbb{N} \cup 0, \tag{2.2}$$

where

$$J \in (\mathbb{N} \cup 0)^d, \ |J| = \sum_{i=1}^{d} J_i, \ \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}. \tag{2.3}$$

Let

$$||v||_{m,1} = \max_{|J| \le m} ||\partial^J v||_{L^1(\mathbb{R}^d)}. \tag{2.4}$$

Let

$$\begin{aligned} &||A|| \text{ denote the norm of an operator} \\ &\qquad A : \mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D). \end{aligned} \tag{2.5}$$

We recall that if v_1, v_2 are potentials satisfying (1.3), (1.5) for some fixed E, then

$$\hat{\Phi}_2(E) - \hat{\Phi}_1(E) \text{ is a compact operator in } \mathbb{L}^\infty(\partial D), \tag{2.6}$$

where $\hat{\Phi}_1$, $\hat{\Phi}_2$ are the DtN maps for v_1, v_2, respectively, see [23], [27]. Note also that (2.1) $\Rightarrow$ (1.3).

Let

$$s_0 = \frac{m-d}{m}, \quad s_1 = \frac{m-d}{d}, \quad s_2 = m - d. \tag{2.7}$$

Theorem 2.1. *Let D satisfy (1.2), where $d \ge 3$. Let v_1, v_2 satisfy (2.1) and (1.5) for some fixed real E. Let $||v_j||_{m,1} \le N$, $j = 1, 2$, for some $N > 0$. Let $\hat{\Phi}_1(E)$ and $\hat{\Phi}_2(E)$ denote the DtN maps for v_1 and v_2, respectively. Then*

$$||v_2 - v_1||_{L^\infty(D)} \le C_1 \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \quad 0 < s \le s_1, \tag{2.8}$$

where $C_1 = C_1(N, D, m, s, E) > 0$, $\delta = ||\hat{\Phi}_2(E) - \hat{\Phi}_1(E)||$ is defined according to (2.5). In addition, for $E \ge 0$, $\tau \in (0, 1)$ and any $s \in [0, s_1]$,

$$||v_2 - v_1||_{L^\infty(D)} \le C_2(1 + \sqrt{E})\delta^\tau + C_3(1 + \sqrt{E})^{s-s_1} \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \tag{2.9}$$

where $C_2 = C_2(N, D, m, \tau) > 0$ and $C_3 = C_3(N, D, m, \tau) > 0$.

Remark 2.1. Estimate (2.8) for $s = s_0$ is a variation of the result of [1] (see also [29], [13]). One can see that estimate (2.8), $s = s_1$, of Theorem 2.1 is more strong (as much as s_1 is greater than s_0) than the aforementioned result going back to [1].

Remark 2.2. Estimate (2.8) for $s = s_2$, $E = 0$, $d = 3$ was proved in [29]. One can see that this estimate of [29] is more strong (as much as s_2 is greater than s_1) than estimate (2.8), $s = s_1$, of Theorem 2.1 for $E = 0$, $d = 3$.

Remark 2.3. Using results of [26] one can obtain estimate (2.9) for $s = 0$, $d = 3$, with s_2 in place of s_1, for sufficiently great E with respect to N. One can see that for this particular case the aforementioned corollary of [26] is more strong (as much as s_2 is greater than s_1) than estimate (2.9) of Theorem 2.1.

Remark 2.4. In a similar way with results of [13], [14], estimates (2.8), (2.9) can be extended to the case when we do not assume that condition (1.5) is fulfiled and consider an appropriate impedance boundary map instead of the Dirichlet-to-Neumann map.

Remark 2.5. Concerning two-dimensional analogs of results of Theorem 2.1, see [25], [31], [34], [35].

Remark 2.6. Actually, in the proof of Theorem 2.1 we obtain the following estimate (see formula (4.19)):

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \le C_4\sqrt{E+\rho^2}\, e^{2\rho L}\delta + C_5(E+\rho^2)^{-s_1/2}, \tag{2.10}$$

where $L = \max_{x\in\partial D}|x|$, $C_4 = C_4(N,D,m) > 0$, $C_5 = C_5(N,D,m) > 0$ and parameter $\rho > 0$ is such that $E+\rho^2$ is sufficiently large: $E+\rho^2 \ge C_6(N,D,m)$. Estimates of Theorem 2.1 follow from estimate (2.10).

The proof of Theorem 2.1 and estimate (2.10) is given in Section 4 and is based on results recalled in Section 3. Actually, this proof is technically very similar to the proof of estimate (2.8) for $s = s_0$, see [1], [29], [13]. Possibility of such a proof of estimate (2.8) for $s = s_1$, $E = 0$ was mentioned, in particular, in [32].

3 Faddeev functions

We consider the Faddeev functions G, ψ, h (see [8], [9], [11], [23]):

$$G(x,k) = e^{ikx}g(x,k), \quad g(x,k) = -(2\pi)^{-d}\int\limits_{\mathbb{R}^d}\frac{e^{i\xi x}d\xi}{\xi^2+2k\xi}, \tag{3.1}$$

$$\psi(x,k) = e^{ikx} + \int\limits_{\mathbb{R}^d} G(x-y,k)v(y)\psi(y,k)dy, \tag{3.2}$$

where $x\in\mathbb{R}^d$, $k\in\mathbb{C}^d$, $\operatorname{Im}k \ne 0$, $d\ge 3$,

$$h(k,l) = (2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{-ilx}v(x)\psi(x,k)dx, \tag{3.3}$$

where

$$k,l\in\mathbb{C}^d,\ k^2 = l^2,\ \operatorname{Im}k = \operatorname{Im}l \ne 0. \tag{3.4}$$

One can consider (3.2), (3.3) assuming that

$$\begin{gathered} v \text{ is a sufficiently regular function on } \mathbb{R}^d \\ \text{with suffucient decay at infinity.} \end{gathered} \tag{3.5}$$

For example, in connection with Problem 1.1, one can consider (3.2), (3.3) assuming that

$$v \in \mathbb{L}^\infty(D), \quad v \equiv 0 \text{ on } \mathbb{R} \setminus D. \tag{3.6}$$

We recall that (see [8], [9], [11], [23]):

- The function G satisfies the equation

$$(\Delta + k^2) G(x,k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d; \tag{3.7}$$

- Formula (3.2) at fixed k is considered as an equation for

$$\psi = e^{ikx} \mu(x,k), \tag{3.8}$$

  where μ is sought in $\mathbb{L}^\infty(\mathbb{R}^d)$;

- As a corollary of (3.2), (3.1), (3.7), ψ satisfies (1.1) for $E = k^2$;

- The Faddeev functions G, ψ, h are (non-analytic) continuation to the complex domain of functions of the classical scattering theory for the Schrödinger equation (in particular, h is a generalized "‘scattering"’ amplitude).

In addition, G, ψ, h in their zero energy restriction, that is for $E = 0$, were considered for the first time in [3]. The Faddeev functions G, ψ, h were, actually, rediscovered in [3].

Let

$$\begin{aligned} \Sigma_E &= \left\{ k \in \mathbb{C}^d : k^2 = k_1^2 + \ldots + k_d^2 = E \right\}, \\ \Theta_E &= \left\{ k \in \Sigma_E, \ l \in \Sigma_E : \operatorname{Im} k = \operatorname{Im} l \right\}, \\ |k| &= (|\operatorname{Re} k|^2 + |\operatorname{Im} k|^2)^{1/2}. \end{aligned} \tag{3.9}$$

Under the assumptions of Theorem 2.1, we have that:

$$\mu(x,k) \to 1 \quad \text{as} \quad |k| \to \infty \tag{3.10}$$

and, for any $\sigma > 1$,

$$|\mu(x,k)| \le \sigma \quad \text{for} \quad |k| \ge r_1(N, D, m, \sigma), \tag{3.11}$$

where $x \in \mathbb{R}^d$, $k \in \Sigma_E$;

$$\hat{v}(p) = \lim_{\substack{(k,l) \in \Theta_E, \ k - l = p \\ |\operatorname{Im} k| = |\operatorname{Im} l| \to \infty}} h(k,l) \quad \text{for any } p \in \mathbb{R}^d, \tag{3.12}$$

$$\begin{aligned} |\hat{v}(p) - h(k,l)| \le \frac{c_1(D,m) N^2}{(E + \rho^2)^{1/2}} \quad &\text{for } (k,l) \in \Theta_E, \quad p = k - l, \\ |\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \quad &E + \rho^2 \ge r_2(N, D, m), \\ &p^2 \le 4(E + \rho^2), \end{aligned} \tag{3.13}$$

where

$$\hat{v}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \tag{3.14}$$

Results of the type (3.10), (3.11) go back to [3]. For more information concerning (3.11) see estimate (4.11) of [13]. Results of the type (3.12), (3.13) (with less precise right-hand side in (3.13)) go back to [11]. Estimate (3.13) follows, for example, from formulas (3.2), (3.3) and the estimate

$$\begin{gathered} \|\Lambda^{-s} g(k) \Lambda^{-s}\|_{\mathbb{L}^2(\mathbb{R}^d)\to\mathbb{L}^2(\mathbb{R}^d)} = O(|k|^{-1}) \\ \text{as} \ \ |k| \to \infty, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d, \end{gathered} \tag{3.15}$$

for $s > 1/2$, where $g(k)$ denotes the integral operator with the Schwartz kernel $g(x-y, k)$ and Λ denotes the multiplication operator by the function $(1+|x|^2)^{1/2}$. Estimate (3.15) was formulated, first, in [17] for $d \geq 3$. Concerning proof of (3.15), see [37].

In addition, we have that:

$$\begin{aligned} h_2(k,l) - h_1(k,l) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \psi_1(x,-l)(v_2(x) - v_1(x))\psi_2(x,k)dx \\ \text{for } (k,l) \in \Theta_E, \ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \neq 0, \\ \text{and } v_1,\, v_2 \text{ satisfying (3.5)}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} h_2(k,l) - h_1(k,l) = (2\pi)^{-d} \int\limits_{\partial D} \psi_1(x,-l) \left[\left(\hat{\Phi}_2 - \hat{\Phi}_1\right) \psi_2(\cdot,k)\right](x)dx \\ \text{for } (k,l) \in \Theta_E, \ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \neq 0, \\ \text{and } v_1,\, v_2 \text{ satisfying (1.5), (3.6)}, \end{aligned} \tag{3.17}$$

and, under assumtions of Theorem 2.1,

$$\begin{aligned} |\hat{v}_1(p) - \hat{v}_2(p) - h_1(k,l) + h_2(k,l)| \leq \frac{c_2(D,m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E+\rho^2)^{1/2}} \\ \text{for } (k,l) \in \Theta_E, \quad p = k - l, \quad |\mathrm{Im}\, k| = |\mathrm{Im}\, l| = \rho, \\ E + \rho^2 \geq r_3(N,D,m), \quad p^2 \leq 4(E+\rho^2), \end{aligned} \tag{3.18}$$

where h_j, ψ_j denote h and ψ of (3.3) and (3.2) for $v = v_j$, and $\hat{\Phi}_j$ denotes the Dirichlet-to-Neumann map for $v = v_j$, where $j = 1, 2$.

Formulas (3.16), (3.17) were given in [24], [27]. Estimate (3.18) follows from (3.2), (3.15), (3.16) in a similar way as estimate (3.13) follows from (3.2), (3.3), (3.15).

4 Proof of Theorem 2.1

Let

$$\begin{gathered} \mathbb{L}^\infty_\mu(\mathbb{R}^d) = \{u \in \mathbb{L}^\infty(\mathbb{R}^d) : \|u\|_\mu < +\infty\}, \\ \|u\|_\mu = \operatorname{ess}\sup_{p\in\mathbb{R}^d} (1+|p|)^\mu |u(p)|, \quad \mu > 0. \end{gathered} \tag{4.1}$$

Note that

$$\begin{gathered} w \in \mathbb{W}^{m,1}(\mathbb{R}^d) \Longrightarrow \hat{w} \in \mathbb{L}^\infty_\mu(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d), \\ \|\hat{w}\|_\mu \le c_3(m,d)\|w\|_{m,1} \quad \text{for} \quad \mu = m, \end{gathered} \tag{4.2}$$

where $\mathbb{W}^{m,1}$, $\mathbb{L}^\infty_\mu$ are the spaces of (2.2), (4.1),

$$\hat{w}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} w(x) dx, \quad p \in \mathbb{R}^d. \tag{4.3}$$

Using the inverse Fourier transform formula

$$w(x) = \int\limits_{\mathbb{R}^d} e^{-ipx} \hat{w}(p) dp, \quad x \in \mathbb{R}^d, \tag{4.4}$$

we have that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \le \sup_{x \in \bar{D}} | \int\limits_{\mathbb{R}^d} e^{-ipx} \left(\hat{v}_2(p) - \hat{v}_1(p)\right) dp| \le \\ \le I_1(r) + I_2(r) \quad \text{for any} \quad r > 0, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} I_1(r) &= \int\limits_{|p| \le r} |\hat{v}_2(p) - \hat{v}_1(p)| dp, \\ I_2(r) &= \int\limits_{|p| \ge r} |\hat{v}_2(p) - \hat{v}_1(p)| dp. \end{aligned} \tag{4.6}$$

Using (4.2), we obtain that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \le 2c_3(m,d) N (1 + |p|)^{-m}, \quad p \in \mathbb{R}^d. \tag{4.7}$$

Due to (3.18), we have that

$$\begin{aligned} |\hat{v}_2(p) - \hat{v}_1(p)| \le |h_2(k,l) - h_1(k,l)| + \frac{c_2(D,m) N \|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E + \rho^2)^{1/2}}, \\ \text{for } (k,l) \in \Theta_E, \quad p = k - l, \quad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \\ E + \rho^2 \ge r_3(N, D, m), \quad p^2 \le 4(E + \rho^2). \end{aligned} \tag{4.8}$$

Let

$$\begin{gathered} c_4 = (2\pi)^{-d} \int\limits_{\partial D} dx, \quad L = \max_{x \in \partial D} |x|, \\ \delta = \|\hat{\Phi}_2(E) - \hat{\Phi}_1(E)\|, \end{gathered} \tag{4.9}$$

where $\|\hat{\Phi}_2(E) - \hat{\Phi}_1(E)\|$ is defined according to (2.5).

Due to (3.17), we have that

$$|h_2(k,l) - h_1(k,l)| \le c_4 \|\psi_1(\cdot,-l)\|_{\mathbb{L}^\infty(\partial D)}\, \delta\, \|\psi_2(\cdot,k)\|_{\mathbb{L}^\infty(\partial D)}, \qquad (4.10)$$
$$(k,l) \in \Theta_E,\ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \ne 0.$$

Using (3.11), we find that

$$\|\psi(\cdot,k)\|_{\mathbb{L}^\infty(\partial D)} \le \sigma\, \exp\Big(|\mathrm{Im}\, k| L\Big), \qquad (4.11)$$
$$k \in \Sigma_E,\ |k| \ge r_1(N,D,m,\sigma).$$

Here and bellow in this section the constant σ is the same that in (3.11).

Combining (4.10) and (4.11), we obtain that

$$|h_2(k,l) - h_1(k,l)| \le c_4 \sigma^2 e^{2\rho L}\delta, \quad \text{for } (k,l) \in \Theta_E, \qquad (4.12)$$
$$\rho = |\mathrm{Im}\, k| = |\mathrm{Im}\, l|,$$
$$E + \rho^2 \ge r_1^2(N,D,m,\sigma).$$

Using (4.8), (4.12), we get that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \le c_4 \sigma^2 e^{2\rho L}\delta + \frac{c_2(D,m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E+\rho^2)^{1/2}}, \qquad (4.13)$$
$$p \in \mathbb{R}^d,\ p^2 \le 4(E+\rho^2),\ E+\rho^2 \ge \max\{r_1^2, r_3\}.$$

Let

$$\varepsilon = \left(\frac{1}{2c_2(D,m)Nc_5}\right)^{1/d}, \quad c_5 = \int\limits_{p\in\mathbb{R}^d, |p|\le 1} dp, \qquad (4.14)$$

and $r_4(N,D,m,\sigma) > 0$ be such that

$$E + \rho^2 \ge r_4(N,D,m,\sigma) \Longrightarrow \begin{cases} E+\rho^2 \ge r_1^2(N,D,m,\sigma), \\ E+\rho^2 \ge r_3(N,D,m), \\ \Big(\varepsilon(E+\rho^2)^{\frac{1}{2d}}\Big)^2 \le 4(E+\rho^2). \end{cases} \qquad (4.15)$$

Let

$$c_6 = \int\limits_{p\in\mathbb{R}^d, |p|=1} dp. \qquad (4.16)$$

Using (4.6), (4.13), we get that

$$I_1(r) \le c_5 r^d \Big(c_4\sigma^2 e^{2\rho L}\delta + \frac{c_2(D,m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E+\rho^2)^{1/2}}\Big), \qquad (4.17)$$
$$r > 0,\ r^2 \le 4(E+\rho^2),$$
$$E + \rho^2 \ge r_4(N,D,m,\sigma).$$

Using (4.6), (4.7), we find that, for any $r>0$,

$$I_2(r)\le 2c_3(m,d)Nc_6\int\limits_r^{+\infty}\frac{dt}{t^{m-d+1}}\le\frac{2c_3(m,D)Nc_6}{m-d}\frac{1}{r^{m-d}}. \tag{4.18}$$

Combining (4.5), (4.17), (4.18) for $r=\varepsilon(E+\rho^2)^{\frac{1}{2d}}$ and (4.15), we get that

$$\begin{aligned}\|v_1-v_2\|_{\mathbb{L}^\infty(D)}\le c_7(N,D,m,\sigma)\sqrt{E+\rho^2}\,e^{2\rho L}\delta+&\\ +c_8(N,D,m)(E+\rho^2)^{-\frac{m-d}{2d}}+\frac{1}{2}\|v_1-v_2\|_{\mathbb{L}^\infty(D)},&\\ E+\rho^2\ge r_4(N,D,m,\sigma).&\end{aligned} \tag{4.19}$$

Let $\tau'\in(0,1)$ and

$$\beta=\frac{1-\tau'}{2L},\quad \rho=\beta\ln\left(3+\delta^{-1}\right), \tag{4.20}$$

where δ is so small that $E+\rho^2\ge r_4(N,D,m,\sigma)$. Then due to (4.19), we have that

$$\begin{aligned}&\frac{1}{2}\|v_1-v_2\|_{\mathbb{L}^\infty(D)}\le\\ &\quad\le c_7(N,D,m,\sigma)\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{1/2}\left(3+\delta^{-1}\right)^{2\beta L}\delta+\\ &\quad\quad+c_8(N,D,m)\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{-\frac{m-d}{2d}}=\\ &\quad=c_7(N,D,m,\sigma)\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{1/2}(1+3\delta)^{1-\tau'}\delta^{\tau'}+\\ &\quad\quad+c_8(N,D,m)\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{-\frac{m-d}{2d}},\end{aligned} \tag{4.21}$$

where τ',β and δ are the same as in (4.20).

Using (4.21), we obtain that

$$\|v_1-v_2\|_{\mathbb{L}^\infty(D)}\le c_9(N,D,E,m,\sigma,\tau')\left(\ln\left(3+\delta^{-1}\right)\right)^{-\frac{m-d}{d}} \tag{4.22}$$

for $\delta=\|\hat\Phi_2-\hat\Phi_1\|\le\delta_1(N,D,E,m,\sigma,\tau')$, where δ_1 is a sufficiently small positive constant. Estimate (4.22) in the general case (with modified c_9) follows from (4.22) for $\delta\le\delta_1(N,D,E,m,\sigma,\tau')$ and the property that

$$\|v_j\|_{\mathbb{L}^\infty(D)}\le c_{10}(D,m)N. \tag{4.23}$$

This completes the proof of (2.8).

If $E\ge0$ then there is a constant $\delta_2=\delta_2(N,D,m,\sigma,\tau')>0$ such that

$$\delta\in(0,\delta_2)\Longrightarrow\begin{cases}E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\ge r_4(N,D,m,\sigma),\\ E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\le\left((1+\sqrt{E})\beta\ln\left(3+\delta^{-1}\right)\right)^2,\\ \beta\ln\left(3+\delta^{-1}\right)\ge1,\end{cases} \tag{4.24}$$

where β is the same as in (4.20). Combining (4.21), (4.24), we obtain that for $s \in [0, (m-d)/d]$, $\tau \in (0, \tau')$ and $\delta \in (0, \delta_2)$ the following estimate holds:

$$||v_2 - v_1||_{L^\infty(D)} \le c_{11}(1+\sqrt{E})\delta^\tau + c_{12}(1+\sqrt{E})^{s-\frac{m-d}{d}} \left(\ln\left(3+\delta^{-1}\right)\right)^{-s}, \quad (4.25)$$

where constants $c_{11}, c_{12} > 0$ depend only on N, D, m, σ, τ' and τ.

Estimate (4.25) in the general case (with modified c_{11} and c_{12}) follows from (4.25) for $\delta \le \delta_2(N, D, m, \sigma, \tau')$ and (4.23).

This completes the proof of (2.9)

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