

ECOLE POLYTECHNIQUE
CENTRE DE MATHÉMATIQUES APPLIQUÉES
UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tel. 01 69 33 46 00. Fax: 01 69 33 46 46
[http : //www.cmap.polytechnique.fr/](http://www.cmap.polytechnique.fr/)

**Approximate Lipschitz stability
for non-overdetermined inverse scattering
at fixed energy**

R.G. Novikov

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CNRS (UMR 7641), Centre de Mathématiques Appliquées, Ecole Polytechnique,
91128 Palaiseau, France
e-mail: novikov@cmap.polytechnique.fr

Abstract. We prove approximate Lipschitz stability for non-overdetermined inverse scattering at fixed energy with incomplete data in dimension $d \geq 2$. Our estimates are given in uniform norm for coefficient difference and related stability precision efficiently increases with increasing energy and coefficient difference regularity. In addition, our estimates are rather optimal even in the Born approximation.

1. Introduction

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0, \quad (1.1)$$

where

$$v \text{ is real - valued, } v \in L^\infty_\sigma(\mathbb{R}^d) \text{ for some } \sigma > d, \quad (1.2)$$

where

$$\begin{aligned} L^\infty_\sigma(\mathbb{R}^d) &= \{u \in L^\infty(\mathbb{R}^d) : \|u\|_\sigma < +\infty\}, \\ \|u\|_\sigma &= \text{ess sup}_{x \in \mathbb{R}^d} (1 + |x|)^\sigma |u(x)|, \quad \sigma \geq 0. \end{aligned} \quad (1.3)$$

For equation (1.1) we consider the scattering amplitude f on \mathcal{M}_E ,

$$\mathcal{M}_E = \{k \in \mathbb{R}^d, l \in \mathbb{R}^d : k^2 = l^2 = E\}, \quad E > 0. \quad (1.4)$$

For definitions of the scattering amplitude, see formula (1.5) below and, for example, reviews given in [F2], [FM]. The scattering amplitude f arises, in particular, as a coefficient with scattered spherical wave $e^{i|k||x|}/|x|^{(d-1)/2}$ in the asymptotics of the wave solutions $\psi^+(x, k)$ describing scattering of incident plan wave e^{ikx} for equation (1.1):

$$\psi^+(x, k) = e^{ikx} + c(d, |k|) \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k, |k| \frac{x}{|x|}) + o\left(\frac{1}{|x|^{(d-1)/2}}\right) \text{ as } |x| \rightarrow \infty, \quad (1.5)$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $k^2 = E$, $c(d, |k|) = -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}$.

Given v , to determine f one can use, in particular, the Lippmann-Schwinger integral equation

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} + \int_{\mathbb{R}^d} G^+(x - y, k) v(y) \psi^+(y, k) dy, \\ G^+(x, k) &= -\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 - k^2 - i0}, \end{aligned} \quad (1.6)$$

and the formula

$$f(k, l) = \left(\frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} e^{-ily} v(y) \psi^+(y, k) dy, \quad (1.7)$$

where $x, k, l \in \mathbb{R}^d$, $k^2 = l^2 = E$; see, for example, [BS], [F2].

In the present work, in addition to f on \mathcal{M}_E , we consider $f|_{\Gamma_E}$ and $f|_{\Gamma_E^\tau}$, where

$$\begin{aligned} \Gamma_E &= \left\{ k = \frac{p}{2} + \eta_E(p), l = -\frac{p}{2} + \eta_E(p) : p \in \mathcal{B}_{2\sqrt{E}} \right\}, \\ \Gamma_E^\tau &= \left\{ k = \frac{p}{2} + \eta_E(p), l = -\frac{p}{2} + \eta_E(p) : p \in \mathcal{B}_{2\tau\sqrt{E}} \right\}, \end{aligned} \quad (1.8)$$

$$\mathcal{B}_r = \{p \in \mathbb{R}^d : |p| \leq r\}, \quad r > 0, \quad (1.9)$$

where $E > 0$, $\tau \in]0, 1]$, η_E is a piecewise continuous vector-function of $p \in \mathcal{B}_{2\sqrt{E}}$ such that

$$\eta_E(p)p = 0, \quad \frac{p^2}{4} + (\eta_E(p))^2 = E, \quad p \in \mathcal{B}_{2\sqrt{E}}. \quad (1.10)$$

Note that

$$\begin{aligned} \Gamma_E^\tau &\subseteq \Gamma_E \subset \mathcal{M}_E, \\ \dim \mathcal{M}_E &= 2d - 2, \quad \dim \Gamma_E^\tau = \dim \Gamma_E = d, \end{aligned} \quad (1.11)$$

where $E > 0$, $\tau \in]0, 1]$, $d \geq 2$.

We consider the following inverse scattering problems for equation (1.1) under assumptions (1.2):

Problem 1.1. Given f on \mathcal{M}_E at fixed $E > 0$, find v on \mathbb{R}^d (at least approximately).

Problem 1.2. Given f on Γ_E^τ at fixed $E > 0$, $\tau \in]0, 1]$, find v on \mathbb{R}^d (at least approximately).

Using (1.11) one can see that Problem 1.1 is overdetermined for $d \geq 3$, whereas Problem 1.2 is non-overdetermined.

There are many important results on Problem 1.1, see [ABR], [B], [BAR], [E], [ER2], [F1], [G], [HH], [HN], [I], [IN2], [N1]-[N5], [S1], [VW], [W], [WY] and references therein. On the other hand, to our knowledge, Problem 1.2 was not yet considered explicitly in the literature. Concerning known results for some other non-overdetermined multi-dimensional coefficient inverse problems, see [BK], [ER1], [HN], [K], [N6], [S2] and references therein.

Problems 1.1, 1.2 can be also considered as examples of ill-posed problems; see [BK], [LRS] for an introduction to this theory.

In the present work we obtain approximate Lipschitz stability estimates for Problem 1.2 (with $\tau = \tau(E) = \varepsilon E^{(1-d)/(2d)}$ for $E \geq 1$) in dimension $d \geq 2$, see Theorem 2.1 of Section 2. Our estimates are given in uniform norm for coefficient difference and related stability precision efficiently increases with increasing energy and coefficient difference regularity. In addition, at the end of Section 2, we show that our estimates of Theorem 2.1 are rather optimal even for the case of the Born approximation (that is in the linear

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approximation near zero potential). Our new estimates are much different but coherent with respect to results of [N4], [N5] for Problem 1.1.

2. Stability estimates

Let

$$\begin{aligned} W^{n,1}(\mathbb{R}^d) &= \{u : \partial^J u \in L^1(\mathbb{R}^d), |J| \leq n\}, \\ \|u\|_{n,1} &= \max_{|J| \leq n} \|\partial^J u\|_{L^1(\mathbb{R}^d)}, \end{aligned} \quad (2.1)$$

where

$$J \in (\mathbb{N} \cup 0)^d, |J| = \sum_{i=1}^d J_i, \partial^J u(x) = \frac{\partial^{|J|} u(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}, n \in (\mathbb{N} \cup 0).$$

Let $C(\mathcal{M}_E)$ denote continuous functions on \mathcal{M}_E and $C(\Gamma_E)$, $C(\Gamma_E^\tau)$ denote the restrictions of $C(\mathcal{M}_E)$ on Γ_E and Γ_E^τ . Let

$$\begin{aligned} \|f\|_{C(\Gamma_E^\tau)} &= \|f\|_{C(\Gamma_E^\tau),0}, \\ \|f\|_{C(\Gamma_E^\tau),\sigma} &= \sup_{(k,l) \in \Gamma_E^\tau} (1 + |k - l|)^\sigma |f(k, l)|, \end{aligned} \quad (2.2)$$

where $E > 0$, $0 < \tau \leq 1$, $\sigma \geq 0$.

Let

$$s_0 = \frac{n-d}{n}, s_1 = \frac{n-d}{d}, s_2 = n-d. \quad (2.3)$$

Theorem 2.1. *Let $v_1, v_2 \in L^\infty_\sigma(\mathbb{R}^d)$ for some $\sigma > d$, $v_1 - v_2 \in W^{n,1}(\mathbb{R}^d)$ for some $n > d$, $\text{supp}(v_1 - v_2) \subset D$, where D is an open bounded domain in \mathbb{R}^d , $d \geq 2$. Let $\|v_j\|_\sigma \leq N_1$, $\|v_1 - v_2\|_{n,1} \leq N_2$, where $\|\cdot\|_\sigma$, $\|\cdot\|_{n,1}$ are defined in (1.3), (2.1). Let f_1, f_2 denote the scattering amplitudes for v_1, v_2 , respectively. Then:*

$$\|v_1 - v_2\|_{L^\infty(D)} \leq C_1 \sqrt{E} \|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)})} + C_2 (\sqrt{E})^{-s_1}, \quad (2.4)$$

$$\|v_1 - v_2\|_{L^\infty(D)} \leq \tilde{C}_1 \|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)}, n_0)} + C_2 (\sqrt{E})^{-s_1}, d < n_0 \leq n, \quad (2.5)$$

where $\|\cdot\|_{C(\Gamma_E^\tau)}$, $\|\cdot\|_{C(\Gamma_E^\tau), n_0}$ are defined by (2.2), $\tau(E) = \varepsilon(\sqrt{E})^{(1-d)/d}$, $\varepsilon = \varepsilon(N_1, D, \sigma)$, $C_1 = C_1(N_1, D, \sigma)$, $C_2 = C_2(N_1, N_2, D, \sigma, n)$, $\tilde{C}_1 = \tilde{C}_1(N_1, D, \sigma, n_0)$, s_1 is defined in (2.3), $E \geq 1$.

In Theorem 2.1, ε , C_1 , C_2 , \tilde{C}_1 denote appropriate positive constants (independent of E). In addition, in particular, $0 < \varepsilon \leq 1$.

Theorem 2.1 is proved in Section 4. There is a considerable similarity between this proof and the proof of recent stability estimates of [IN1].

Note that the old approach to inverse scattering at high energies based on formula (3.3) of Section 3 yields estimates like (2.4), (2.5) with s_0 only instead of s_1 in the error term. In addition, due to (2.3), we have that

$$s_0 \leq 1 \text{ even for } n \rightarrow +\infty, \text{ whereas } s_1 \rightarrow +\infty \text{ for } n \rightarrow +\infty.$$

In Theorem 2.1, we have that $\tau(E) \rightarrow 0$ as $E \rightarrow +\infty$. Therefore, $\Gamma_E^{\tau(E)}$ is a very small part of $\Gamma_E^{\tau_1}$ for any fixed $\tau_1 \in]0, 1]$ for sufficiently high energy E . Therefore, estimates (2.4), (2.5) of Theorem 2.1 can be considered as a stability result for Problem 1.2 with incomplete data.

Let

$$\mathcal{M}_E^\tau = \{(k, l) \in \mathcal{M}_E : k - l \in \mathcal{B}_{2\tau\sqrt{E}}\}, \quad E > 0, \quad \tau \in]0, 1]. \quad (2.6)$$

Let

$$\begin{aligned} \|f\|_{C(\mathcal{M}_E^\tau)} &= \|f\|_{C(\mathcal{M}_E^\tau), 0}, \\ \|f\|_{C(\mathcal{M}_E^\tau), \sigma} &= \sup_{(k, l) \in \mathcal{M}_E^\tau} (1 + |k - l|)^\sigma |f(k, l)|, \end{aligned} \quad (2.7)$$

where $E > 0$, $0 < \tau \leq 1$, $\sigma \geq 0$.

To our knowledge, estimates (2.4), (2.5) are completely new even with $\|f_1 - f_2\|_{C(\mathcal{M}_E^{\tau(E)})}$, $\|f_1 - f_2\|_{C(\mathcal{M}_E^{\tau(E)}, n_0)}$ in place of $\|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)})}$, $\|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)}, n_0)}$ (respectively).

On the other hand, for the case of Problem 1.1 with complete data, estimates (2.4), (2.5) with $\|f_1 - f_2\|_{C(\mathcal{M}_E^1)}$, $\|f_1 - f_2\|_{C(\mathcal{M}_E^1), n_0}$ in place of $\|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)})}$, $\|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)}, n_0)}$ (respectively) are less precise than related results of [N4], [N5] with the error term estimated as $O(E^{-s_2/2})$, $E \rightarrow +\infty$, where s_2 is defined in (2.3).

In addition, for Problem 1.2 with the scattering amplitude f given on $\Gamma_E^{\tau(E)}$ only, estimates (2.4), (2.5) are rather optimal even for the case of the Born approximation (that is in the linear approximation near zero potential). We recall that, in the Born approximation,

$$f(k, l) \approx \hat{v}(k - l), \quad (k, l) \in \mathcal{M}_E, \quad (2.8)$$

where

$$\hat{v}(p) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \quad (2.9)$$

Let

$$\begin{aligned} \|\hat{v}\|_{C(\mathcal{B}_r)} &= \|\hat{v}\|_{C(\mathcal{B}_r), 0}, \\ \|\hat{v}\|_{C(\mathcal{B}_r), \sigma} &= \sup_{p \in \mathcal{B}_r} (1 + |p|)^\sigma |\hat{v}(p)| \quad r > 0, \quad \sigma \geq 0. \end{aligned} \quad (2.10)$$

Born approximation analogs of (2.4), (2.5) can be written as

$$\begin{aligned} \|v_1 - v_2\|_{L^\infty(D)} &\leq c_1(d) \varepsilon^d \sqrt{E} \|\hat{v}_1 - \hat{v}_2\|_{C(\mathcal{B}_{2\varepsilon(\sqrt{E})^{1/d})}} + \\ &c_2(d, n) N_2 \varepsilon^{-(n-d)} (\sqrt{E})^{-s_1}, \end{aligned} \quad (2.11)$$

$$\begin{aligned} \|v_1 - v_2\|_{L^\infty(D)} &\leq \tilde{c}_1(d, n_0) \|\hat{v}_1 - \hat{v}_2\|_{C(\mathcal{B}_{2\varepsilon(\sqrt{E})^{1/d}), n_0}} + \\ &c_2(d, n) N_2 \varepsilon^{-(n-d)} (\sqrt{E})^{-s_1}, \end{aligned} \quad (2.12)$$

where s_1 , n , n_0 , d , N_2 are the same that in (2.3)-(2.5), $0 < \varepsilon < 1$, $E \geq 1$.

3. Some results of direct scattering

We recall that, under assumptions (1.2), the Lippmann-Schwinger integral equation (1.6) is uniquely solvable for $\psi^+(\cdot, k) \in L^\infty(\mathbb{R}^d)$ for fixed $k \in \mathbb{R}^d \setminus \{0\}$; see [BS], [F2] and references therein.

We recall that the following estimate holds:

$$\begin{aligned} \|\Lambda^{-s}G^+(k)\Lambda^{-s}\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} &= O(|k|^{-1}), \\ \text{as } |k| \rightarrow \infty, k \in \mathbb{R}^d, \text{ for } s > 1/2, \end{aligned} \quad (3.1)$$

where $G^+(k)$ denotes the integral operator with the Schwartz kernel $G^+(x-y, k)$ of (1.6), Λ denotes the multiplication operator by the function $(1+|x|^2)^{1/2}$; see [E], [J] and references therein.

As a corollary of (1.6), (3.1), under assumptions (1.2), we have that

$$\|\Lambda^{-\sigma/2}\psi^+(\cdot, k) - \Lambda^{-\sigma/2}\psi_0^+(\cdot, k)\|_{L^2(\mathbb{R}^d)} \leq a_1(d, \sigma)\|v\|_\sigma|k|^{-1} \quad (3.2)$$

for $|k| \geq \rho_1(d, \sigma)\|v\|_\sigma$, $k \in \mathbb{R}^d$, where $\psi_0^+(x, k) = e^{ikx}$.

As a corollary of (1.7), (3.2), under assumptions (1.2), we have that

$$|f(k, l) - \hat{v}(k-l)| \leq a_2(d, \sigma)(\|v\|_\sigma)^2|k|^{-1} \quad (3.3)$$

for $k, l \in \mathbb{R}^d$, $|k| = |l| \geq \rho_1(d, \sigma)\|v\|_\sigma$, where \hat{v} is defined by (2.9).

We recall also that, under assumptions (1.2) for $v = v_j$, $j = 1, 2$, the following formula holds:

$$\begin{aligned} f_2(k, l) - f_1(k, l) &= \\ \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \psi_1^+(x, -l)(v_2(x) - v_1(x))\psi_2^+(x, k)dx, \quad k, l \in \mathbb{R}^d, k^2 = l^2 > 0, \end{aligned} \quad (3.4)$$

where f_j, ψ_j^+ denote f and ψ^+ for $v = v_j$, $j = 1, 2$; see [S2].

In addition, in the proof of Theorem 2.1 we use, in particular, the following lemma:

Lemma 3.1. *Let $v = v_j$ satisfy (1.2), $\|v_j\|_\sigma \leq N$, where $j = 1, 2$. Let $\text{supp}(v_1 - v_2) \subset D$, where D is an open bounded domain in \mathbb{R}^d . Then the following estimate holds:*

$$\begin{aligned} |(f_2(k, l) - f_1(k, l)) - (\hat{v}_2(k-l) - \hat{v}_1(k-l))| &\leq \\ a_3(D, \sigma)N\|v_2 - v_1\|_{L^\infty(D)}|k|^{-1} \end{aligned} \quad (3.5)$$

for $k, l \in \mathbb{R}^d$, $|k| = |l| \geq \rho_1(d, \sigma)N$.

Lemma 3.1 follows from formula (3.4), estimate (3.2) and the property that $\inf_{x \subset D} (1+|x|^2)^{-\sigma/4} > 0$.

4. Proof of Theorem 2.1

We have that

$$\|v_2 - v_1\|_{L^\infty(D)} \leq \sup_{x \in D} \left| \int_{\mathbb{R}^d} e^{-ipx} (\hat{v}_2(p) - \hat{v}_1(p)) dp \right| \leq \quad (4.1)$$

$$I_1(\kappa) + I_2(\kappa) \quad \text{for any } \kappa > 0,$$

where

$$I_1(\kappa) = \int_{|p| \leq \kappa} |\hat{v}_2(p) - \hat{v}_1(p)| dp, \quad (4.2)$$

$$I_2(\kappa) = \int_{|p| \geq \kappa} |\hat{v}_2(p) - \hat{v}_1(p)| dp.$$

Due to Lemma 3.1, we have that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \leq |f_2(k_E(p), l_E(p)) - f_1(k_E(p), l_E(p))| + a_3(D, \sigma) N_1 \|v_2 - v_1\|_{L^\infty(D)} (\sqrt{E})^{-1} \quad (4.3)$$

for $p \in \mathcal{B}_{2\sqrt{E}}$, $\sqrt{E} \geq \rho_1(d, \sigma) N_1$, where

$$k_E(p) = \frac{p}{2} + \eta_E(p), \quad l_E(p) = -\frac{p}{2} + \eta_E(p), \quad (4.4)$$

where η_E is the function of (1.8), (1.10).

Using (4.2), (4.3), (2.2), we obtain that

$$I_1(2\tau\sqrt{E}) \leq |\mathcal{B}_1| (2\tau\sqrt{E})^d \left(\|f_2 - f_1\|_{C(\Gamma_E^\tau)} + \frac{a_3(D, \sigma) N_1 \|v_2 - v_1\|_{L^\infty(D)}}{\sqrt{E}} \right), \quad (4.5)$$

$$I_2(2\tau\sqrt{E}) \leq \frac{|\mathbb{S}^{d-1}|}{n_0 - d} \|f_2 - f_1\|_{C(\Gamma_E^\tau), n_0} + |\mathcal{B}_1| a_3(D, \sigma) N_1 \frac{(2\tau\sqrt{E})^d}{\sqrt{E}} \|v_2 - v_1\|_{L^\infty(D)} \quad (4.6)$$

for $\sqrt{E} \geq \rho_1(d, \sigma) N_1$, $\tau \in]0, 1]$, where $|\mathcal{B}_1|$ and $|\mathbb{S}^{d-1}|$ denote standard Euclidean volumes of \mathcal{B}_1 and \mathbb{S}^{d-1} (respectively), n_0 is the number of (2.5).

The assumptions that $v_1 - v_2 \in W^{n,1}(\mathbb{R}^d)$, $\|v_1 - v_2\|_{n,1} \leq N_2$ for some $n > d$, imply that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \leq a_4(n, d) N_2 (1 + |p|)^{-n}, \quad p \in \mathbb{R}^d. \quad (4.7)$$

Using (4.2), (4.7) we obtain that

$$I_2(2\tau\sqrt{E}) \leq \frac{|\mathbb{S}^{d-1}| a_4(n, d) N_2}{n - d} \frac{1}{(2\tau\sqrt{E})^{n-d}}, \quad (4.8)$$

$\sqrt{E} > 0$, $\tau \in]0, 1]$.

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Let

$$\begin{aligned}\tau(E) &= \varepsilon(\sqrt{E})^{(1-d)/d}, \\ \varepsilon &= \min\left(\frac{1}{2}\left(\frac{1}{2|\mathcal{B}_1|a_3(D, \sigma)N_1}\right)^{1/d}, 1\right).\end{aligned}\tag{4.9}$$

Due to (4.9), we have, in particular, that

$$\begin{aligned}|\mathcal{B}_1|a_3(D, \sigma)N_1(2\tau(E)\sqrt{E})^d(\sqrt{E})^{-1} &\leq \frac{1}{2}, \\ \tau(E) &\leq 1, \quad E \geq 1.\end{aligned}\tag{4.10}$$

Using (4.1), (4.5), (4.6), (4.8), (4.10), we obtain that

$$\begin{aligned}\|v_2 - v_1\|_{L^\infty(D)} &\leq \frac{\sqrt{E}}{a_3(D, \sigma)N_1}\|f_2 - f_1\|_{C(\Gamma_E^{\tau(E)})} + \\ \frac{1}{2}\|v_2 - v_1\|_{L^\infty(D)} &+ \frac{|\mathbb{S}^{d-1}|a_4(n, d)N_2}{(n-d)(2\varepsilon)^{n-d}}\frac{1}{(\sqrt{E})^{(n-d)/d}},\end{aligned}\tag{4.11}$$

$$\begin{aligned}\|v_2 - v_1\|_{L^\infty(D)} &\leq \frac{|\mathbb{S}^{d-1}|}{n_0 - d}\|f_2 - f_1\|_{C(\Gamma_E^{\tau(E)}, n_0)} + \\ \frac{1}{2}\|v_2 - v_1\|_{L^\infty(D)} &+ \frac{|\mathbb{S}^{d-1}|a_4(n, d)N_2}{(n-d)(2\varepsilon)^{n-d}}\frac{1}{(\sqrt{E})^{(n-d)/d}}\end{aligned}\tag{4.12}$$

for $\sqrt{E} \geq \max(\rho_1(d, \sigma)N_1, 1)$.

Estimates (2.4), (2.5) with

$$C_1 = \frac{2}{a_3(D, \sigma)N_1}, \quad C_2 = \frac{2|\mathbb{S}^{d-1}|a_4(n, d)N_2}{(n-d)(2\varepsilon)^{n-d}}, \quad \tilde{C}_1 = \frac{2|\mathbb{S}^{d-1}|}{n_0 - d},\tag{4.13}$$

for $\sqrt{E} \geq \max(\rho_1(d, \sigma)N_1, 1)$, follow from (4.11), (4.12).

Using also that $\|v_2 - v_1\|_{L^\infty(D)} \leq 2N_1$ we obtain estimates (2.4), (2.5) with

$$\begin{aligned}C_1 &= \frac{2}{a_3(D, \sigma)N_1}, \quad \tilde{C}_1 = \frac{2|\mathbb{S}^{d-1}|}{n_0 - d}, \\ C_2 &= \max\left(\frac{2|\mathbb{S}^{d-1}|a_4(n, d)N_2}{(n-d)(2\varepsilon)^{n-d}}, 2N_1(\max(\rho_1(d, \sigma)N_1, 1))^{(n-d)/d}\right)\end{aligned}\tag{4.14}$$

for $E \geq 1$.

This completes the proof of Theorem 2.1.

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