THE CHILD-LANGMUIR REGIME FOR ELECTRON TRANSPORT IN A PLASMA INCLUDING A BACKGROUND OF POSITIVE IONS

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Abstract

We consider the Child-Langmuir asymptotics [L1,D1] of a monodimensional electron beam flowing through a background of positve ions of constant density n_D. We extend the results valid for electron beams in the vacuum [L1, D1] to the case of small n_D's. Large n_D's strongly modify the phase portrait of the characteristics of the Vlasov-Poisson equation and yield a different result.

1 Introduction and Main Results

An accurate modelling of charged particle transport requires the coupling of the kinetic equations, (Vlasov, Boltzmann, ...), with the electromagnetic equations, (Poisson, Maxwell, ...). Untill very recently, only the free space problems, in which the boundary data are ignored, were studied. However, the most relevant models require an accurate description of the boundary data. This leads to consider boundary value problems for the Vlasov-Poisson or Vlasov-Maxwell systems. The first existence result for the monodimensional stationary Vlasov-Poisson system has been proved by C. Greengard and P.A. Raviart [Gr1]. Then, F. Poupaud showed the existence of solutions of the boundary-value Problem for the stationary Vlasov-Maxwell, Vlasov-Poisson and Boltzmann-Poisson systems in any dimension [P1, P2, P3]. For the time dependent case, the existence of solutions of the initial boundary-value problem has been shown by Y. Guo [Gu1] and M. Bezart [Be1] in the case of the Vlasov-Maxwell system, and by the author in the case of the Vlasov-Poisson system [Ba2].

In realistic conditions, vacuum diodes operate under very high voltages. This can be analysed mathematically by a scaling of the equations leading to a singular perturbation for the Vlasov-Poisson equation. P. Degond and P.A. Raviart [D1] first introduced this perturbation problem for the one dimensional Vlasov Poisson system and showed that the solution of this problem converges to a reduced solution, referred to as the Child-Langmuir solution. For such solutions the current cannot exceed a limiting value called the Child-Langmuir current and which was introduced by Langmuir and Compton in [L1]. For various extensions of this result, we refer the reader to [D2] , [D3] and [D4]. In this paper we are interested in the Child langmuir asymptotics of a monodimensional electron beam flowing through a background of ions of opposite sign. The system can be written

$$V\frac{\partial F}{\partial X} + \frac{e}{m}\frac{d\Phi}{dX}\frac{\partial F}{\partial V} = 0, \qquad X \in [0,L], \quad V \in \mathbb{R}$$

$$\frac{d^2\Phi}{dX^2} = \frac{e}{\varepsilon_0}(N(X) - N_D), \qquad N(X) = \int_{-\infty}^{+\infty} F(X,V)\,dV$$

$$F(0,V) = G(V), \quad V > 0, \qquad F(L,V) = 0, \qquad V < 0$$

$$\Phi(0) = 0, \qquad \Phi(L) = \Phi_L > 0.$$

where N_D is the density of the background of positive ions. These equations can also model the lowly doped region of an $N^+ - N^- - N^+$ diode, for which we neglect the electron injection at the channel-drain junction ($X = L$), as well as the collisions of the electrons with the cristal lattice. In this case, N_D is the doping profile. Like in [D1], we introduce a small parameter ε:

$$\varepsilon = \frac{V_G}{V_L},$$

where

$$V_G = \frac{\displaystyle\int_0^{+\infty} V G(V)\,dV}{\displaystyle\int_0^{+\infty} G(V)\,dV}, \qquad V_L = \sqrt{\frac{2e\Phi_L}{m}}.$$

Then, we scale the equations by

$$\overline{N} = \frac{\varepsilon_0 \Phi_L}{eL^2}, \qquad \overline{J} = e\overline{N}V_L, \qquad \overline{F} = \frac{\overline{N}}{V_L}$$

We introduce the scaled variables and unknowns

$$X = Lx, \quad V = V_L v, \quad \Phi = \Phi_L \varphi$$

$$N(X) = \overline{N}n(x), \quad N_D = \overline{N}n_D, \quad J(X) = -\overline{J}j(x), \quad F(X,V) = \overline{F}f(x,v).$$

Finally we introduce a dimensionless injection profile:

$$G(V) = \overline{F}g^\varepsilon(v), \qquad g^\varepsilon(v) = \frac{1}{\varepsilon^2}\, g(\frac{v}{\varepsilon}).$$

The physical interpretation of this scaling essentially means that the electrons are injected with a very small average velocity and an infinite density so that the injected current is finite (details can be found in [D1], [D3]). With the new variables and unknowns the dimensionless system is written:

$$v\frac{\partial f^\varepsilon}{\partial x} + \frac{1}{2}\frac{d\varphi^\varepsilon}{dx}\frac{\partial f^\varepsilon}{\partial v} = 0, \tag{1.1}$$

$$\frac{d^2\varphi^\varepsilon}{dx^2} = n^\varepsilon(x) - n_D, \qquad n^\varepsilon(x) = \int_{-\infty}^{+\infty} f^\varepsilon(x,v)\, dv \tag{1.2}$$

$$f^\varepsilon(0,v) = \frac{1}{\varepsilon^2}\, g(\frac{v}{\varepsilon}) = g_\varepsilon(v), \quad \text{for } v > 0, \tag{1.3}$$

$$f^\varepsilon(1,v) = 0 \quad \text{for} \quad v < 0, \tag{1.4}$$

$$\varphi^\varepsilon(0) = 0, \quad \varphi^\varepsilon(1) = 1, \tag{1.5}$$

where n_D is a positive constant.

Since the convexity of the potential given by (1.2) is not guaranteed, the convergence analysis will fairly differs from the previous analyses [D1] - [D4]. Before stating the main results, we introduce some notations and hypotheses.

We assume that g has a compact support and is bounded:

$$supp\ g \subset [0, V_0], \quad 0 \le g \le M.$$

We set:

$$\begin{aligned} j^\varepsilon(x) &= \int_{-\infty}^{+\infty} v f^\varepsilon(x,v)\, dv, \\ k^\varepsilon(x) &= \int_{-\infty}^{+\infty} v^2 f^\varepsilon(x,v)\, dv, \\ h^\varepsilon(x) &= k^\varepsilon(x) - \frac{1}{4}(\frac{d\varphi^\varepsilon}{dx})^2 - \frac{1}{2}n_D\varphi^\varepsilon, \\ j_g &= \int_0^{+\infty} v g(v)\, dv. \end{aligned}$$

where j^ε and k^ε are respectively the total current and kinetic energy, while h^ε is the total (kinetic + potential) energy in scaled variables and j_g the scaled injection current. When the density n_D is small:

$$(SD) \qquad n_D < 12\sqrt{3}, \quad \text{or equivalently} \quad N_D < 12\sqrt{3}\,\frac{\varepsilon_0 \Phi_L}{eL^2}$$

or the current is small:

$$(SC) \qquad j_g < \frac{1}{2} n_D, \quad \text{equivalently} \quad J_{inj} \leq \left(\frac{N_D\, 2eL^2}{\varepsilon_0 \Phi_L} \right) \varepsilon_0 \sqrt{\frac{2e}{m}}\, \frac{\Phi_L^{3/2}}{L^2}$$

where $J_{inj} = e \int_0^\infty V G(V) dV$ is the injected current, we can characterize completely the limit problem; this is stated in

Theorem 1.1 *(i) There exists a limiting value $j_{CL}(n_D)$ such that the elliptic problem:*

$$\begin{cases} \dfrac{d^2\varphi}{dx^2} = \dfrac{j}{\sqrt{\varphi}} - n_D, \\ \varphi(0) = 0, \quad \varphi(1) = 1 \\ \varphi(x) > 0 \quad if \quad x \in \ (0,1]. \end{cases} \tag{1.6}$$

has a unique solution if $j \in [0, j_{CL}(n_D)]$ and no solution if $j > j_{CL}(n_D)$.
(ii) If (SC) or (SD) are satisfied, then the solutions $(f^\varepsilon, \varphi^\varepsilon)$ of the system (1.1) - (1.5) converge as ε tends to zero to (f, φ) where φ is the unique solution of the reduced problem (1.6) and the current j is given by

$$j = \min\left(j_g, j_{CL}(n_D)\right).$$

The distribution function f is a measure solution of the Vlasov equation and is given by

$$f(x,v) = \frac{j}{\sqrt{\varphi(x)}}\, \delta(v - \sqrt{\varphi(x)})$$

where δ is the delta measure. The convergence holds in the $C^1([0,1])$ strong topology for φ^ε and in the weak star topology of bounded measures for f^ε.

In [S1], Shur and Eastman solved the limit problem (1.6) in the case $j = j_{CL}$ or equivalently $\varphi'(0) = 0$. The preceeding theorem gives a mathematical justification of their approach in the high injection regime ($j_{inj} > j_{CL}$). For large n_D's one cannot prove any longer that the potential $\varphi = \lim \varphi^\varepsilon$ satisfies $\varphi(x) > 0$ for $x \in (0,1]$. Therefore, the limit result is different and is given in the following

Theorem 1.2 *Let n_D and j_g be arbitrary positive constants.Then there exists a subsequence of solutions $(f^\varepsilon, \varphi^\varepsilon)$ of the system (1.1) $\cdots$ (1.5) which converges as ε tends to zero to (f, φ) (in the same topologies as in Theorem 1.1), and there exist p real numbers $0 = r_1 < r_2 < \cdots < r_p < 1$ such that:*

$$\varphi(x) = 0 \iff x \in \{r_1, r_2, \cdots, r_p\}.$$

(i) In the case $p = 1$, φ and f are characterized by Theorem 1.1.
(ii) If $p \geq 2$ then there exists a unique sequence $(j^1, j^2, \cdots, j^p)$ such that for $m \in \{1, \cdots, p-1\}$:

$$\begin{cases} \dfrac{d^2\varphi}{dx^2} = \dfrac{j^m}{\sqrt{\varphi}} - n_D, \quad on\ I^m =]r_m, r_{m+1}[\\ \varphi(r_m) = \varphi(r_{m+1}) = \dfrac{d\varphi}{dx}(r_m) = \dfrac{d\varphi}{dx}(r_{m+1}) = 0, \\ \varphi > 0 \quad on\ I^m. \end{cases} \tag{1.7}$$

and

$$\begin{cases} \dfrac{d^2\varphi}{dx^2} = \dfrac{j^p}{\sqrt{\varphi}} - n_D, \quad on\ I^p =]r_p, 1] \\ \varphi(r_p) = \dfrac{d\varphi}{dx}(r_p) = 0, \quad \varphi(1) = 1 \\ \varphi > 0 \quad on\ I^p. \end{cases} \tag{1.8}$$

Moreover we have

$$j^p = j = \lim j^\varepsilon$$

$j \leq j^m \leq 2j_g$ for $m \in \{1, \cdots, p-1\}$and the function f is defined by

$$f(x,v) = \frac{j_+^m}{\sqrt{\varphi(x)}}\,\delta(v - \sqrt{\varphi(x)}) + \frac{j_-^m}{\sqrt{\varphi(x)}}\,\delta(v + \sqrt{\varphi(x)}), \qquad (x,v) \in I^m \times \mathbb{R}$$

where j_+^m and j_-^m are defined by

$$j_+^m + j_-^m = j^m, \quad j_+^m - j_-^m = j.$$

Remark 1.3 In the preceeding theorem, the number and the location of the points r_m (where φ vanishes), are unknown. However, the currents j^m are uniquely determined when the points r_m are known. ∎

The results can be understood, by viewing the phase portrait. More precisely, when n_D is small or equivalently, when

$$N_D \leq C\,\frac{\varepsilon_0 \Phi_L}{eL^2},$$

where C is a numerical constant $C \geq 12\sqrt{3}$, the current is saturated at the Child-Langmuir value given by $j_{CL}(n_D)$ in dimensionless variables or by

$$J_{CL}(N_D) = -j_{CL}\left(\frac{N_D\, eL^2}{\varepsilon_0 \Phi_L}\right)\,\varepsilon_0\sqrt{\frac{2e}{m}}\,\frac{\Phi_L^{3/2}}{L^2}$$

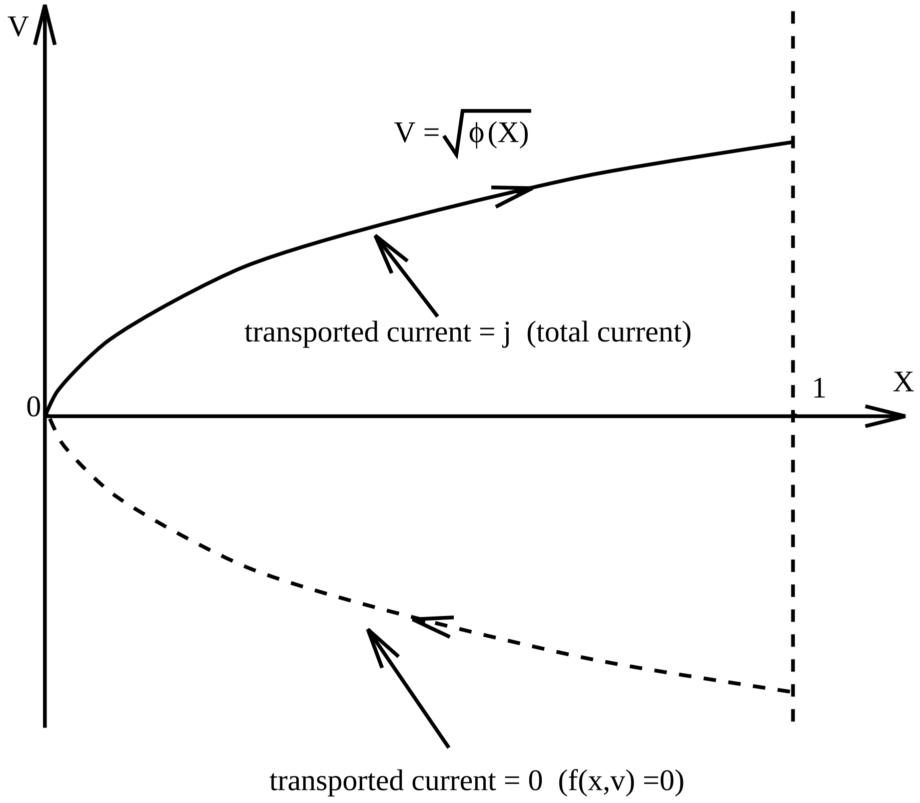

Figure 1: *Phase portrait (small n_D)*

in physical variables. In this case, the limit potential φ is positive on $(0,1]$, and the phase portrait of the characteristics of the Vlasov equation is given like in Fig. 1. Then it follows that $j = \sqrt{\varphi(x)}\, n(x)$ and that φ satisfies (3.11).

When n_D becomes large, we can no more guarantee that the limit potential φ is positive on $(0,1]$. Indeed, φ could vanish and therefore the phase portrait of the characteristics of the Vlasov equation is given in Fig. 2. In this case the density is given by $j = n(x)\sqrt{\varphi(x)}$ for $x \in [x_2, 1]$, and $j^1_+ + j^1_- = n(x)\sqrt{\varphi(x)}$ for $x \in [0, x_1]$, ... The limit potential is then the solution of the problem (1.7)-(1.8).
Since the characteristics passing through x_1 is closed, no information is available to determine the values of j^1_+, $j^1_- \cdots$ and no uniqueness result can be proved.

The outline of the paper is as follows: In section 2 we will start by proving some preliminary estimates. Then, in section 3, we will prove Theorem 1.1. This section is divided into 3 subsections. The first is devoted to the characterization of the reduced problem, the second to the analysis of this reduced problem and the third to the convergence proof. Finally in Section 4 we proceed analogously to prove Theorem 1.2.

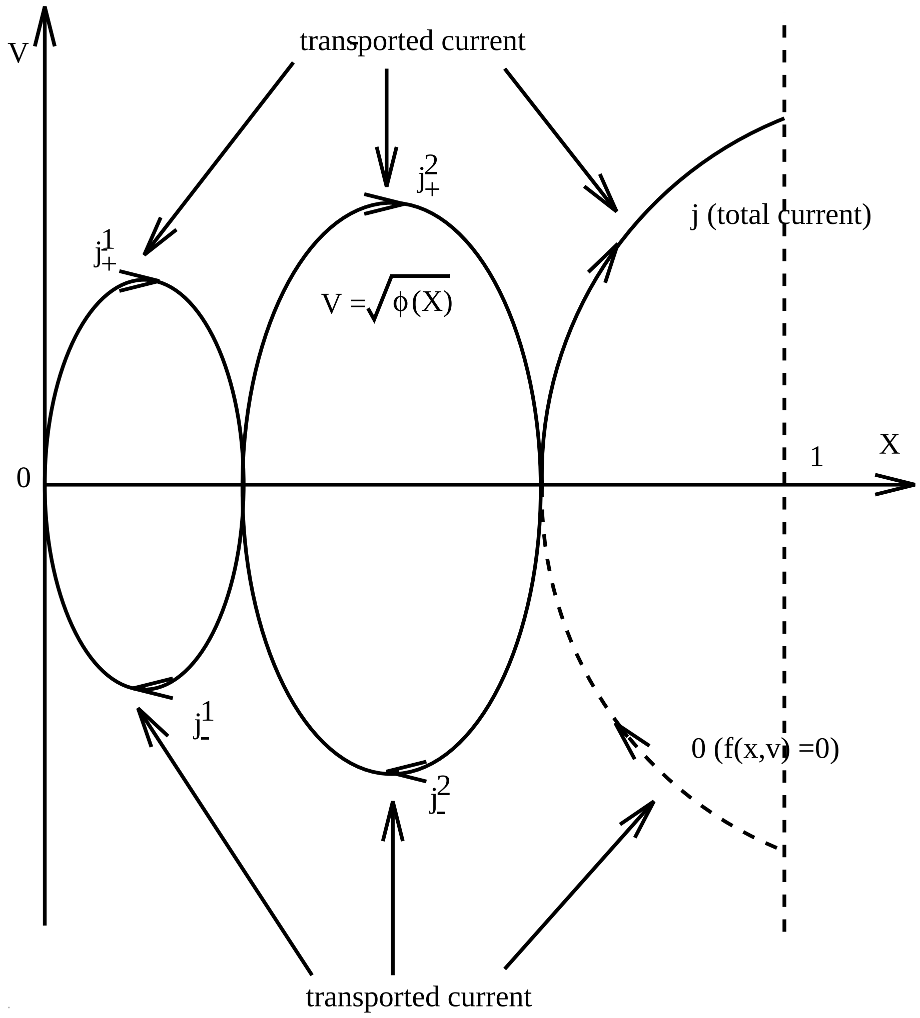

Figure 2: *Phase Portrait (large n_D)*

2 The Singular Perturbation Problem : First Estimates

Now consider the perturbation problem (1.1) - (1.5). F. Poupaud's result [P1] leads to the following theorem:

Theorem 2.1 [P1] *Let ε be fixed and positive. Then the system $(1.1) - (1.5)$ has a solution $(\varphi^\varepsilon, f^\varepsilon)$ such that:*

$$\varphi^\varepsilon \in C^1([0,1]), \quad f^\varepsilon \in L^1 \bigcap L^\infty([0,1] \times \mathbb{R}), \quad n^\varepsilon \in L^\infty([0,1]).$$

Moreover f^ε satisfies the following maximum principle: For every nonnegative function $\theta \in C^1(\mathbb{R})$ such that $g_\varepsilon(v) \leq \theta(v^2)$ on $\mathbb{R}_+$ we have:

$$0 \;\leq\; f^\varepsilon(x,v) \;\leq\; \theta(v^2 - \varphi(x))$$

We now only sketch the differences from [D3]

Lemma 2.2 *The current $j^\varepsilon(x)$ and the energy $h^\varepsilon(x)$ are independent of x.*

Proof: See [D3] Prop 3.2. ∎

Lemma 2.3 *The distribution function satisfies the following properties*

$$supp\, f^\varepsilon \subset \{(x,v) \in [0,1]\times \mathbb{R}, \quad |v^2 - \varphi^\varepsilon| \le V_0^2\varepsilon^2\} \tag{2.1}$$

$$\|f^\varepsilon\|_{L^\infty} \le \frac{M}{\varepsilon^2} \tag{2.2}$$

Proof: The proof proceeds like in [D3] lemma 4.1. ∎

Corollary 2.4 *The following estimates hold for $x \in [0,1]$:*

$$-V_0^2\varepsilon^2 \le \varphi^\varepsilon(x) \le x + \frac{n_D}{2}x(1-x) \tag{2.3}$$

$$0 \le k^\varepsilon(x) \le C\sqrt{\varphi^\varepsilon(x) + V_0^2\varepsilon^2} \tag{2.4}$$

$$\frac{j^\varepsilon}{\sqrt{\varphi^\varepsilon(x) + V_0^2\varepsilon^2}} \le n^\varepsilon(x) \le \frac{C}{\sqrt{\varphi^\varepsilon(x) + 2V_0^2\varepsilon^2}} \tag{2.5}$$

$$|k^\varepsilon(x) - n^\varepsilon(x)\varphi^\varepsilon(x)| \le C\varepsilon \tag{2.6}$$

Proof: (i) The potential φ^ε satisfies

$$\frac{d^2\varphi^\varepsilon}{dx^2} \ge -n_D, \quad \varphi^\varepsilon(0) = 0, \quad \varphi^\varepsilon(1) = 1.$$

Therefore, the maximum principle for elliptic equations leads to the second inequality of (2.3). To prove the first inequality (2.3), we denote by a_ε the point such that

$$\varphi^\varepsilon(a_\varepsilon) = \min_{x\in[0,1]} \varphi^\varepsilon(x).$$

Assume that $\varphi^\varepsilon(a_\varepsilon) < -V_0^2\varepsilon^2$. Then $a_\varepsilon \in]0,1[$ and $\frac{d\varphi^\varepsilon}{dx}(a_\varepsilon) = 0$. Moreover, the continuity of φ^ε implies that $\varphi^\varepsilon < -V_0^2\varepsilon^2$ in a neighbourhood of a_ε. Then by the preceeding lemma, we deduce that n^ε vanishes in this neighborhood. Therefore $\frac{d^2\varphi^\varepsilon}{dx^2} = -n_D < 0$, and this contradicts the fact that a_ε is a minimum. Therefore $\varphi^\varepsilon(a_\varepsilon) \ge -V_0^2\varepsilon^2$.

(ii) The inequality (2.4) is a consequence of the inequalities (2.5) and (2.6).
(iii) The first inequality of (2.5) is a consequence of (2.1). Indeed,

$$\begin{aligned} n^\varepsilon(x) &\geq \int_0^{+\infty} f^\varepsilon(x,v)\,dv = \int_0^{\sqrt{\varphi^\varepsilon(x)+V_0^2\varepsilon^2}} f^\varepsilon(x,v)\,dv \\ &\geq \frac{1}{\sqrt{\varphi^\varepsilon(x)+V_0^2\varepsilon^2}} \int_0^{\sqrt{\varphi^\varepsilon(x)+V_0^2\varepsilon^2}} v f^\varepsilon(x,v)\,dv \\ &= \frac{1}{\sqrt{\varphi^\varepsilon(x)+V_0^2\varepsilon^2}} \int_0^{\infty} v f^\varepsilon(x,v)\,dv \\ &\geq \frac{j^\varepsilon}{\sqrt{\varphi^\varepsilon(x)+V_0^2\varepsilon^2}}. \end{aligned}$$

To prove the second inequality of (2.5) we use (2.2) and get:

$$n^\varepsilon(x) \leq \int_{|v^2-\varphi^\varepsilon(x)|\leq V_0^2\varepsilon^2} \frac{M}{\varepsilon^2}\,dv.$$

If $\varphi^\varepsilon(x) \leq V_0^2\varepsilon^2$, then

$$n^\varepsilon(x) \leq \frac{M}{\varepsilon^2}\int_{-\sqrt{2V_0^2\varepsilon^2}}^{\sqrt{2V_0^2\varepsilon^2}} dv \leq \frac{C}{\varepsilon} \leq \frac{C}{\sqrt{\varphi^\varepsilon(x)+2V_0^2\varepsilon^2}}.$$

Otherwise, $\varphi^\varepsilon(x) \geq V_0^2\varepsilon^2$, and we obtain

$$\begin{aligned} n^\varepsilon(x) &\leq \frac{2M}{\varepsilon^2}\int_{\sqrt{\varphi^\varepsilon(x)-V_0^2\varepsilon^2}}^{\sqrt{\varphi^\varepsilon(x)+V_0^2\varepsilon^2}} dv = \frac{4MV_0^2}{\sqrt{\varphi^\varepsilon(x)-V_0^2\varepsilon^2}+\sqrt{\varphi^\varepsilon(x)+V_0^2\varepsilon^2}} \\ &\leq \frac{C}{\sqrt{\varphi^\varepsilon(x)+2V_0^2\varepsilon^2}}. \end{aligned}$$

(iv) Finally, we prove (2.6). We deduce from (2.1) that.

$$\begin{aligned} |k^\varepsilon(x)-\varphi^\varepsilon(x)n^\varepsilon(x)| &\leq \int_{|v^2-\varphi^\varepsilon(x)|\leq V_0^2\varepsilon^2} |v^2-\varphi^\varepsilon(x)| f^\varepsilon(x,v)\,dv \\ &\leq V_0^2\varepsilon^2 \int_{|v^2-\varphi^\varepsilon(x)|\leq V_0^2\varepsilon^2} f^\varepsilon(x,v)\,dv \\ &\leq V_0^2\varepsilon^2 n^\varepsilon(x). \end{aligned}$$

Then, from (2.3) and (2.5) we have $n^\varepsilon \leq \dfrac{C}{\varepsilon}$ which concludes the proof of (2.6).

∎

Lemma 2.5 *There exists a constant K such that*

$$\frac{d\varphi^\varepsilon}{dx}(0) \;\geq\; -K\,\sqrt{\varepsilon}.$$

Proof: Since

$$\frac{d^2\varphi^\varepsilon}{dx^2} \;\leq\; \frac{C}{\varepsilon},$$

then

$$\varphi^\varepsilon(x) \;\leq\; \frac{C}{2\varepsilon}\,x^2 \;+\; \frac{d\varphi^\varepsilon}{dx}(0)\,x.$$

Using the inequality $\varphi^\varepsilon \;\geq\; -\,V_0^2\varepsilon^2$, we get the following inequality:

$$\frac{C}{2\varepsilon}\,x^2 \;+\; \frac{d\varphi^\varepsilon}{dx}(0)\,x + V_0^2\varepsilon^2 \;\geq\; 0, \quad \forall x \in [0,1]. \tag{2.7}$$

We set

$$P(x) \;=\; \frac{C}{2\varepsilon}\,x^2 \;+\; \frac{d\varphi^\varepsilon}{dx}(0)\,x + V_0^2\varepsilon^2$$

and

$$\Delta \;=\; (\frac{d\varphi^\varepsilon}{dx}(0))^2 - K^2\varepsilon, \qquad K^2 \;=\; 2CV_0^2.$$

Therefore if Lemma 2.5 was not satisfied, the discriminant would be positive and then the polynomial P would have 2 positive roots of $0 < x_1 < x_2 < c'\varepsilon$; hence the inequality (2.7) is not satisfied for $x \in (x_1, x_2)$ which leads to a contradiction.

∎

Lemma 2.6 *There exists a constant C independent of ε such that*

$$\|\varphi^\varepsilon\|_{W^{1,\infty}\cap W^{2,1}} + \|n^\varepsilon\|_{L^1} + |h^\varepsilon| \leq C.$$

Proof: Again the proof proceeds like in [D3] (Prop 4.2 or Cor 4.2). From (2.5) and (2.4) we deduce that

$$\|k^\varepsilon\|_{L^\infty} \;+\; \|\varphi^\varepsilon\|_{L^\infty} \;\leq C.$$

Then, since

$$h^\varepsilon(x) \;=\; k^\varepsilon(x) - \frac{1}{4}(\frac{d\varphi^\varepsilon}{dx})^2 - \frac{1}{2}n_D\varphi^\varepsilon,$$

to prove that $\|\frac{d\varphi^\varepsilon}{dx}\|_{L^\infty}$ is bounded, it is sufficient to prove the boundedness of $\|h^\varepsilon\|_{L^\infty}$. Since h^ε is constant (lemma 2.2) , and since by Lemma 2.5 and the maximum principle we have

$$-K\sqrt{\varepsilon} \;\leq\; \frac{d\varphi^\varepsilon}{dx}(0) \;\leq\; 1 + \frac{n_D}{2},$$

then h^ε is bounded in $L^\infty(0,1)$. This allows to conclude that φ^ε is bounded in $W^{1,\infty}$.

Now we prove the boundedness of $\|n^\varepsilon\|_{L^1}$. Since f^ε is nonnegative, then

$$\|n^\varepsilon\|_{L^1} \;=\; \|f^\varepsilon\|_{L^1} \;=\; \int_0^1 n^\varepsilon(x)\,dx,$$

and using the Poisson equation (1.2), we obtain

$$\|n^\varepsilon\|_{L^1} \;=\; \frac{d\varphi^\varepsilon}{dx}(1) - \frac{d\varphi^\varepsilon}{dx}(0) + n_D.$$

Therefore n^ε is bounded in L^1. By the Poisson equation this implies that $\|\varphi^\varepsilon\|_{W^{2,1}}$ is bounded.

∎

The following proposition is an immediate consequence of the preceding lemma

Proposition 2.7 *There exist some nonnegative functions $\varphi \in W^{1,\infty}(0,1)$, $f \in \mathcal{M}_b([0,1]\times \mathbb{R})$, $n \in \mathcal{M}_b([0,1])$, $k \in L^\infty(0,1)$, a nonnegative constant j and a constant h such that , up to the extraction of a subsequence, the following convergences hold:*

$$\begin{aligned}
\varphi^\varepsilon &\to \varphi && \text{in } C^0[0,1] && (2.8)\\
\frac{d\varphi^\varepsilon}{dx} &\to \frac{d\varphi}{dx} && \text{in } L^p(0,1) \text{ strong} \quad \forall p \in]1,\infty[&& (2.9)\\
\frac{d\varphi^\varepsilon}{dx} &\rightharpoonup \frac{d\varphi}{dx} && \text{in } L^\infty(0,1) \text{ weak } * && (2.10)\\
n^\varepsilon &\rightharpoonup n && \text{in } \mathcal{M}_b([0,1]) \text{ weak } * && (2.11)\\
f^\varepsilon &\rightharpoonup f && \text{in } \mathcal{M}_b([0,1]\times\mathbb{R}) \text{ weak } * && (2.12)\\
k^\varepsilon &\rightharpoonup k && \text{in } L^\infty(0,1) \text{ weak } * && (2.13)\\
h^\varepsilon &\to h && && (2.14)\\
j^\varepsilon &\to j && && (2.15)
\end{aligned}$$

In (2.12), $\mathcal{M}_b([0,1]\times\mathbb{R})$ denotes the set of bounded measures on $[0,1]\times\mathbb{R}$.

Lemma 2.8

$$n \;=\; \int f\,dv, \qquad j \;=\; \int vf\,dv, \qquad k \;=\; \int v^2 f\,dv,$$

$$h \;=\; k - \frac{1}{4}\Big(\frac{d\varphi}{dx}\Big)^2 - \frac{1}{2} n_D \varphi.$$

Proof: The proof follows from standard arguments by noticing that $v^2 f^\varepsilon$ is bounded in $L^\infty((0,1), L^1(\mathbb{R}))$. ∎

To obtain sharper estimates on the support of f^ε, we define

$$r_\varepsilon \;=\; \sup\{x \in [0,1], \quad \varphi^\varepsilon(x) \;=\; 2V_0^2\varepsilon^2\}. \tag{2.16}$$

Then we have the

Lemma 2.9 *For x in $[r_\varepsilon, 1]$, the support of f satisfies*

$$supp\, f^\varepsilon \ \cap \{[r_\varepsilon, 1] \times \mathbb{R}\} \subset \{(x,v) \in [r_\varepsilon, 1] \times \mathbb{R}, \quad |v - \sqrt{\varphi^\varepsilon}| \leq C\varepsilon\}.$$

Proof: We set

$$E^\varepsilon \ = \Big\{ x \in [0,1], \quad \varphi^\varepsilon(x) \geq 2V_0^2 \varepsilon^2 \Big\} \tag{2.17}$$

and prove that the current $j_-^\varepsilon(x) \ = \ \int_{-\infty}^0 v \ f(x,v)\, dv$ has a vanishing derivative on E^ε (see [D3] lemma 4.3 for details). Then it is constant on every connected component of E^ε. Since $f(1,v)$ vanishes for negative v then j_-^ε vanishes on the connected component of E^ε passing through $x \ = \ 1$. This connected component is $[r_\varepsilon, 1]$. This implies that $f(x,v)$ vanishes for x in $[r_\varepsilon, 1]$ and v negative. Now, combining this with (2.1), we conclude the proof of Lemma 2.9.

∎

As a consequence of this lemma we get

Lemma 2.10 *On $[r_\varepsilon, 1]$ we have:*

$$|k^\varepsilon(x) - j^\varepsilon \sqrt{\varphi^\varepsilon(x)}| \leq C\varepsilon$$

Proof: See [D3] lemma 5.2. for details. ∎

We conclude this section by the following technical lemmas which are specifically needed in the case $n_D > 0$.

Lemma 2.11 *Assume that there exists $r \in]0,1]$ such that $\varphi(r) \ = \ 0$, then:*

$$h \ = \lim_{\varepsilon \to 0} h^\varepsilon \ \equiv \ 0$$

and

$$j \ \geq \ \frac{1}{2} n_D.$$

Proof: Since

$$h^\varepsilon \ = \ k^\varepsilon - \frac{1}{4}(\frac{d\varphi^\varepsilon}{dx})^2 - \frac{1}{2} n_D \varphi^\varepsilon$$

is a constant (with respect to $x \in [0,1]$), and since by (2.4) k^ε is small when φ^ε is small, it is sufficient to exhibit a sequence α^ε such that

$$\lim_{\varepsilon \to 0} \varphi^\varepsilon(\alpha^\varepsilon) \ = \ 0, \qquad \lim_{\varepsilon \to 0} \frac{d\varphi^\varepsilon}{dx}(\alpha^\varepsilon) \ = \ 0. \tag{2.18}$$

Consider $r > 0$ such that $\varphi(r) = 0$, and choose a_ε such that

$$\varphi^\varepsilon(a_\varepsilon) = \sup_{[0,r]} \varphi^\varepsilon$$

Three cases are possible:

(i) Suppose $a_\varepsilon = 0$. Then, $\dfrac{d\varphi^\varepsilon}{dx}(0) \leq 0$ and using Lemma 2.5 we can take $\alpha^\varepsilon = 0$.

(ii) Suppose $a_\varepsilon = r$. There exists $\alpha^\varepsilon \in [0, r]$ such that $\dfrac{d\varphi^\varepsilon}{dx}(\alpha^\varepsilon) = \dfrac{\varphi^\varepsilon(r)}{r}$. Since $\varphi^\varepsilon(\alpha^\varepsilon) \leq \varphi^\varepsilon(r)$ and $\lim \varphi^\varepsilon(r) = 0$, then α^ε satisfies (2.18).

(iii) Finally suppose $a_\varepsilon \in]0, r[$. We choose $\alpha^\varepsilon \in]a_\varepsilon, 1[$ such that

$$\varphi^\varepsilon(\alpha^\varepsilon) = \inf_{[a_\varepsilon,1]} \varphi^\varepsilon.$$

We can choose α^ε in $]a_\varepsilon, 1[$ because $\varphi^\varepsilon(r) < \varphi^\varepsilon(a_\varepsilon)$ and $\varphi^\varepsilon(r) < \varphi^\varepsilon(1) = 1$ for small ε. We have $\dfrac{d\varphi^\varepsilon}{dx}(\alpha^\varepsilon) = 0$, and $\varphi^\varepsilon(\alpha^\varepsilon) \leq \varphi^\varepsilon(r)$ which tends to zero when ε tends to zero.

The only thing left to show is that $j \geq \frac{1}{2} n_D$. From Lemma 2.10 we deduce that $k = j\sqrt{\varphi}$ near $x = 1$. Thus

$$h(x) = j\sqrt{\varphi(x)} - \frac{1}{4}\left(\frac{d\varphi}{dx}(x)\right)^2 - \frac{1}{2} n_D \varphi(x) = 0$$

near $x = 1$. Then, since $\varphi(1) = 1$, the result follows easily.

∎

Lemma 2.12 *Assume that k^ε converges uniformly to a contiuous function k. Then $|\dfrac{d\varphi}{dx}|$ is continuous and $|\dfrac{d\varphi^\varepsilon}{dx}|$ converges uniformly to $|\dfrac{d\varphi}{dx}|$. Furthermore, if the set $\{x \in [0,1], \quad \varphi(x) = 0\}$ is finite, then the denity is given by $n(x) = \dfrac{k(x)}{\varphi(x)}$ and is in $L^1[0,1]$.*

Proof: Since $\dfrac{1}{4}(\dfrac{d\varphi^\varepsilon}{dx})^2 = k^\varepsilon(x) - h^\varepsilon - \dfrac{1}{2} n_D \varphi^\varepsilon$ and h^ε, k^ε and φ^ε converge uniformly, then $(\dfrac{d\varphi^\varepsilon}{dx})^2$ converges uniformly. The limit is obviously $(\dfrac{d\varphi}{dx})^2$.

Furthemore inequality (2.6) implies that $n^\varepsilon \varphi^\varepsilon$ converges uniformly to k. Since n^ε converges weakly in the bounded measure topology and φ^ε strongly in $C^0[0,1]$, we can conclude that $n\varphi = k$. Let $0 = x_1 < x_2 \cdots < x_p$ be the points where φ vanishes. Since n is a bounded positive measure, we can conclude that $\frac{k(x)}{\varphi(x)} \in L^1(0,1)$ and

$$n(x) = \frac{k(x)}{\varphi(x)} + \sum_{i=1}^{p} n_i \delta(x - x_i) \tag{2.19}$$

where $n_1, \cdots, n_p$ are nonnegative constants. It is sufficient to prove that $n_i = 0$.
(i) Suppose first that $p = 1$. To prove that $n_1 = 0$, we suppose the converse is true

$$n_1 > 0. \tag{2.20}$$

Since $|\frac{d\varphi^\varepsilon}{dx}|$ converges uniformly towards $|\frac{d\varphi}{dx}|$, we have:

$$\lim_{\varepsilon\to 0}\lim_{x\to 0} |\frac{d\varphi^\varepsilon}{dx}(x)| = \lim_{x\to 0}\lim_{\varepsilon\to 0} |\frac{d\varphi^\varepsilon}{dx}(x)|. \tag{2.21}$$

On one hand, we have:

$$\lim_{\varepsilon\to 0}\lim_{x\to 0} |\frac{d\varphi^\varepsilon}{dx}(x)| = \lim_{\varepsilon\to 0} |\frac{d\varphi^\varepsilon}{dx}(0)|.$$

As a consequence of Lemma 2.5 we get

$$\lim_{\varepsilon\to 0} |\frac{d\varphi^\varepsilon}{dx}(0)| = \lim_{\varepsilon\to 0} \frac{d\varphi^\varepsilon}{dx}(0) \geq 0. \tag{2.22}$$

On the other hand, we get by the weak convergence of n^ε in $\mathcal{M}_b$ and by (1.2) and (2.19):

$$\lim_{\varepsilon\to 0} \frac{d\varphi^\varepsilon}{dx}(x) = \lim_{\varepsilon\to 0} \frac{d\varphi^\varepsilon}{dx}(0) + n_1 + \int_0^x (\frac{k(\xi)}{\varphi(\xi)} - n_D)\, d\xi.$$

Since n_1 is positive and according to (2.22), we deduce that the right hand side of the above equation is positive for x small enough. Then we get

$$\lim_{x\to 0}\lim_{\varepsilon\to 0} |\frac{d\varphi^\varepsilon}{dx}(x)| = n_1 + \lim_{\varepsilon\to 0} \frac{d\varphi^\varepsilon}{dx}(0). \tag{2.23}$$

Now we combine this equality with (2.22) and using (2.21) we get a contradiction with (2.20).
(ii) In the case $p \geq 2$, we deduce from Lemma 2.11 that $h = 0$, therefore $\frac{d\varphi}{dx}(x_i) = 0$ and $\lim_{\varepsilon\to 0} \frac{d\varphi^\varepsilon}{dx}(x_i) = 0$. We proceed like in the first case and get a contradiction if we suppose that $n_i > 0$.

∎

Lemma 2.13 *Let $Q(x)$ be a $C^1[0,1]$ function and $Q^\varepsilon(x)$ a sequence of $C^1[0,1]$ functions such that:*

$$Q^\varepsilon \to Q \quad in\ C^0[0,1], \qquad |\frac{dQ^\varepsilon}{dx}| \to |\frac{dQ}{dx}| \quad in\ C^0[0,1].$$

Then Q^ε converges to Q in the $C^1[0,1]$ topology.

Proof: Suppose that $Q^{\varepsilon\prime}$ does not converge uniformly to Q'; by an extraction we can suppose that there exists a sequence x_ε converging to an element $x_0 \in [0,1]$ such that

$$|Q^{\varepsilon\prime}(x_\varepsilon) - Q'(x_\varepsilon)| \geq \alpha \tag{2.24}$$

where α is a positive constant. The uniform convergence of $|Q^{\varepsilon\prime}|$ implies that $Q'(x_0) \neq 0$. Assume for instance that it is positive. Since Q' is continuous, there exists a neighborhood $[x_1, x_2]$ $(x_1 < x_2)$ of x_0 such that $Q'(x) \geq 2\beta$ on $[x_1, x_2]$ where β is a positive constant. For ε small enough, we have

$$\sup_{[x_1,x_2]} ||Q^{\varepsilon\prime}(x)| - |Q'(x)|| \leq \beta,$$

and therefore $Q^{\varepsilon\prime}$ has a constant sign on $[x_1, x_2]$. We deduce from the uniform convergence of Q^ε that $Q^\varepsilon(x_2) > Q^\varepsilon(x_1)$ for ε small enough, and thus $Q^{\varepsilon\prime}$ is positive. This implies that

$$\sup_{[x_1,x_2]} |Q^{\varepsilon\prime}(x) - Q'(x)| \to 0$$

and contradicts (2.24).

∎

3 Small Density n_D or Small Injection Current j_g

In this section we assume one of the following hypotheses:

$$(SD) \quad n_D < 12\sqrt{3} \qquad\qquad (SC) \quad j_g < \frac{n_D}{2}.$$

(SD) means small density and (SC) small current. These assumptions will insure that the set $\{x \in [0,1],\ \varphi(x) = 0\}$ reduces to $\{0\}$ and allow to pass to the limit and to prove theorem 1.1.

3.1 Further estimates

Lemma 3.1 *There exists a constant $K > 0$ independent of ε such that*

$$r_\varepsilon \leq K\varepsilon$$

where r_ε is defined by formula (2.16).

Proof: First, from the definition of r_ε, we get:

$$\varphi^\varepsilon(r_\varepsilon) \ = \ 2V_0^2\varepsilon^2; \qquad \frac{d\varphi^\varepsilon}{dx}(r_\varepsilon) \geq 0.$$

Then, we integrate twice the inequality $\dfrac{d^2\varphi^\varepsilon}{dx^2} \leq \dfrac{C}{\varepsilon}$ on $[x, r_\varepsilon]$ and obtain:

$$(\forall x \in [0, r_\varepsilon]) \quad \varphi^\varepsilon(x) \leq 2V_0^2\varepsilon^2 - \frac{d\varphi^\varepsilon}{dx}(r_\varepsilon)(r_\varepsilon - x) + \frac{C}{2\varepsilon}(r_\varepsilon - x)^2.$$

By noticing that $\varphi^\varepsilon \geq -V_0^2\varepsilon^2$ and setting $\delta = r_\varepsilon - x$, we obtain:

$$(\forall \delta \in [0, r_\varepsilon]) \quad \frac{C}{2\varepsilon}\delta^2 - \frac{d\varphi^\varepsilon}{dx}(r_\varepsilon)\delta + 3V_0^2\varepsilon^2 \;\geq\; 0 \tag{3.1}$$

This second order polynomial has discriminant

$$\Delta \;=\; (\frac{d\varphi^\varepsilon}{dx}(r_\varepsilon))^2 - 6CV_0^2\varepsilon.$$

If $\Delta > 0$, it has two positive solutions:

$$\delta_\pm \;=\; \frac{\varepsilon}{C}\,(\,\frac{d\varphi^\varepsilon}{dx}\,(r_\varepsilon) \pm \sqrt{\Delta}).$$

Notice that $\delta_\pm \leq C\varepsilon$. Thus the inequality (3.1) implies $r_\varepsilon \leq \delta_\pm$ which leads to $r_\varepsilon \leq K\varepsilon$.

We now prove that $\Delta \leq 0$ leads to a contradiction with hypothesis (SC) or (SD). We suppose that $\Delta \leq 0$.
We have

$$h^\varepsilon(x) \;=\; k^\varepsilon(x) - \frac{1}{4}(\frac{d\varphi^\varepsilon}{dx})^2 - \frac{1}{2}n_D\varphi^\varepsilon \;=\; h^\varepsilon(r_\varepsilon),$$

and since $\varphi^\varepsilon(r_\varepsilon) \;=\; 2V_0^2\varepsilon^2$, then by (2.4) we have $k^\varepsilon(r_\varepsilon) \leq\; C\varepsilon$. Besides, from $\Delta \leq 0$, we deduce that $(\dfrac{d\varphi^\varepsilon}{dx}(r_\varepsilon))^2 \leq\; C\varepsilon$, and therefore

$$h^\varepsilon \to 0 \tag{3.2}$$

uniformly. We can extract a subsequence such that $r_\varepsilon \to r$. We get from Lemma 2.10 that $k(x) \;=\; j\sqrt{\varphi}$ on $(r, 1]$. Then taking (3.2) and Lemma 2.10 into account, the following equation holds on $(r, 1]$.

$$j\sqrt{\varphi} - \frac{1}{2}n_D\varphi \;=\; \frac{1}{4}(\frac{d\varphi}{dx})^2 \tag{3.3}$$

First suppose that hypothesis (SC) holds. Then since $\varphi(1) \;=\; 1$ and $j \leq j_g$ we get $(\frac{d\varphi}{dx})^2(1) < 0$ which leads to a contradiction.

Second, suppose that hypothesis (SD) is satisfied.
Since $\varphi^\varepsilon(r_\varepsilon) \;=\; 2V_0^2\varepsilon^2$ the uniform convergence of φ^ε to φ yields $\varphi(r) \;=\; 0$. From $\varphi(1) \;=\; 1$ we have

$$\int_r^1 \varphi'(x)\,dx \;=\; 1.$$

By taking the maximum of the left hand side of (3.3) with respect to φ, *i.e.* for $\varphi = (\frac{j}{n_D})^2$, we deduce that

$$(\varphi')^2 \leq \frac{2j^2}{n_D}, \quad on\ [r, 1].$$

This, combined with the above equation, gives:

$$j \sqrt{\frac{2}{n_D}}\,(1 - r) \geq 1. \tag{3.4}$$

This gives a lower bound for the current j; by considering the interval $[0, r_\varepsilon]$, we will get an upper bound. On $[0, r_\varepsilon]$, φ^ε satisfies:

$$\frac{d^2\varphi^\varepsilon}{dx^2} = n^\varepsilon(x) - n_D,$$

$$\varphi^\varepsilon(0) = 0, \quad \varphi^\varepsilon(r_\varepsilon) = 2V_0^2\varepsilon^2.$$

Let us define the sequence K_p such that

$$K_0 = n_D, \quad K_{p+1} = n_D - \frac{j^\varepsilon}{\sqrt{\frac{K_p r_\varepsilon^2}{8} + 3V_0^2\varepsilon^2}}.$$

Then, for any $p \in \mathbb{N}$, we have:

$$\frac{d^2\varphi^\varepsilon}{dx^2} \geq -K_p \qquad on\ [0, r_\varepsilon].$$

Indeed, proceeding by induction and using the maximum principle, we get:

$$\varphi^\varepsilon(x) \leq \frac{K_p}{2}\, x(r_\varepsilon - x) + 2V_0^2\varepsilon^2 \frac{x}{r_\varepsilon} \leq \frac{K_p r_\varepsilon^2}{8} + 2V_0^2\varepsilon^2,$$

and by using the first inequality of (2.5), we get

$$\frac{d^2\varphi^\varepsilon}{dx^2} \geq -K_{p+1}.$$

The sequence K_p is obviously decreasing and we have :

$$\varphi^\varepsilon(x) \leq \frac{K_p r_\varepsilon^2}{8} + 2V_0^2\varepsilon^2, \quad x \in [0, r_\varepsilon] \tag{3.5}$$

Since $\varphi^\varepsilon \geq -V_0^2\varepsilon^2$, inequality (3.5) shows that K_p is bounded from below. Then it converges to a limit l such that:

$$l = n_D - \frac{j^\varepsilon}{\sqrt{\frac{l r_\varepsilon^2}{8} + 3V_0^2\varepsilon^2}}$$

We set $x = \sqrt{\frac{lr_\varepsilon^2}{8} + 3V_0^2\varepsilon^2}$; then x satisfies:

$$x^3 - (\frac{n_D r_\varepsilon^2}{8} + 3V_0^2\varepsilon^2)\,x + \frac{j^\varepsilon r_\varepsilon^2}{8} = 0.$$

By studying the equation

$$x^3 - ax + b = 0$$

one can see that there is no positive solution unless $b - 2(\frac{a}{3})^{\frac{3}{2}} < 0$. This leads to:

$$j^\varepsilon \leq \frac{16}{(24)^{\frac{3}{2}}} r_\varepsilon n_D^{\frac{3}{2}} [1 + \frac{24}{n_D}\, V_0^2 \frac{\varepsilon^2}{r_\varepsilon^2}]^{\frac{3}{2}}$$

Suppose now that $\limsup_{\varepsilon\to 0} \dfrac{r_\varepsilon}{\varepsilon} = +\infty$. Then passing to the limit into the above inequality leads to

$$j \leq \frac{1}{3\sqrt{6}} r n_D^{\frac{3}{2}} \tag{3.6}$$

This inequality combined with the inequality (3.4) implies that $n_D \geq 12\sqrt{3}$ which contradicts Hypothesis (SD).

∎

The following proposition is an immediate consequence of this lemma.

Proposition 3.2 *We have the following uniform estimates:*
(i) $|\varphi^\varepsilon| \leq C\varepsilon^2$, *for* $x \in [0, r_\varepsilon]$.
(ii) *supp* $f^\varepsilon \subset \{(x,v) \in [0,1] \times \mathbb{R}, \quad |v - \sqrt{|\varphi^\varepsilon|}| \leq C\varepsilon\}$.
(iii) $|k^\varepsilon(x) - j^\varepsilon\sqrt{|\varphi^\varepsilon(x)|}| \leq C\varepsilon$ *for* $x \in [0,1]$.

Proof: (i) Since $\dfrac{d^2\varphi^\varepsilon}{dx^2} \geq -n_D$, $\varphi^\varepsilon(0) = 0$ and $\varphi^\varepsilon(r_\varepsilon) = 2V_0^2\varepsilon^2$, then the application of the maximum principle gives

$$\varphi^\varepsilon(x) \leq \frac{n_D}{2} x(r_\varepsilon - x) + 2V_0^2\varepsilon^2 \frac{x}{r_\varepsilon}, \quad \forall x \in [0, r_\varepsilon],$$

and with lemma 3.1, this gives (i).
(ii) The estimate is already shown on $[r_\varepsilon, 1]$ in Lemma 2.9. On $[0, r_\varepsilon]$ it is a direct consequence of (i) and (2.1).
(iii) The proof uses (ii) and is identical to that of Lemma 2.10.

∎

Proposition 3.3 *Under the hypothesis (SD) or (SC) we have:*

$$\varphi > 0 \quad \text{in }]0,1]$$

Proof: Assume that φ vanishes in $]0,1[$, then by Lemma 2.11 we have

$$h(x) \;=\; j\,\sqrt{\varphi(x)} - \frac{1}{4}\,(\frac{d\varphi}{dx})^2 - \frac{1}{2}\,n_D\varphi(x) \;=\; 0. \tag{3.7}$$

Then we proceed like in the proof of Lemma 3.1. We set $r > 0$ such that $\varphi(r) = 0$. The same computation as the one done in the proof of Lemma 3.1 gives (3.4). We define a sequence K_p by

$$K_0 = n_D, \quad K_{p+1} \;=\; n_D \;-\; \frac{j^\varepsilon}{\sqrt{\frac{K_p r^2}{8} \;+\; V_0^2\varepsilon^2 + \varphi^\varepsilon(r)}}.$$

and proceeding like in the proof of Lemma 3.1, we show that

$$j^\varepsilon \le \frac{16}{(24)^{\frac{3}{2}}} r n_D^{\frac{3}{2}} [1 + \frac{8}{n_D}\,\frac{V_0^2\varepsilon^2 + \varphi^\varepsilon(r)}{r^2}]^{\frac{3}{2}}.$$

This leads to (3.6) since $\varphi^\varepsilon(r)$ tends to zero when ε tends to zero.

∎

From Proposition 3.2 (iii), k^ε converges uniformly to $j\sqrt{\varphi}$ in $C^0([0,1])$. In view of Proposition 3.3, Lemma 2.12 applies and yields

Lemma 3.4 (i) *The sequence* $|\frac{d\varphi^\varepsilon}{dx}|$ *converges strongly in* $C^0[0,1]$ *to* $|\frac{d\varphi}{dx}|$.
(ii) $n = \frac{j}{\sqrt{\varphi}}$ *and is in* $L^1(0,1)$.

We conclude this section by the following lemma.

Lemma 3.5 *If* $\frac{d\varphi}{dx}(0) > 0$, *then* $j \;=\; j_g$.

Proof: The proof is similar to that of [D3] lemma 4.3. We set $\varphi'(0) \;=\; 2\beta$. Then by the uniform convergence of $|\frac{d\varphi^\varepsilon}{dx}|$ we deduce that for ε small enough we have $|\frac{d\varphi^\varepsilon}{dx}(0)| \;>\; \beta$; this implies by Lemma 2.5 that $\frac{d\varphi^\varepsilon}{dx}(0) \;>\; \beta$. The above estimate allows to conclude that r_ε defined by the formula (2.16) satisfies

$$r_\varepsilon \le C\varepsilon^2. \tag{3.8}$$

Besides, we define a smooth nonnegative function $\theta_\alpha(v)$ satisfying:

$$\begin{cases} \theta_\alpha(v) = 1 & v \leq -\alpha, \\ \theta_\alpha(v) = 0 & v \geq 0, \\ 0 \leq \theta_\alpha \leq 1 & \frac{d}{dv}(\theta_\alpha(v)) \leq 0. \end{cases} \tag{3.9}$$

The weak formulation of the Vlasov equation yields

$$\begin{aligned} \int_0^1 \int_{-\infty}^{+\infty} \frac{1}{2}\frac{d\varphi^\varepsilon}{dx}\frac{d\theta_\alpha(v)}{dv} f^\varepsilon(x,v)\,dv\,dx &= -\int_{-\infty}^{+\infty} f^\varepsilon(0,v)v\theta_\alpha(v)\,dv \\ &+ \int_{-\infty}^{+\infty} f^\varepsilon(1,v)v\theta_\alpha(v)\,dv. \end{aligned}$$

Since from Lemma 2.9 f^ε vanishes on $\{x \in [r_\varepsilon, 1],\ v \in \mathbb{R}_-\}$ and θ_α vanishes for $v \geq 0$ then the above equation turns to:

$$\int_0^{r_\varepsilon} \int_{-\infty}^{+\infty} \frac{1}{2}\frac{d\varphi^\varepsilon}{dx}\frac{d\theta_\alpha(v)}{dv} f^\varepsilon(x,v)\,dv\,dx = -\int_{-\infty}^{0} f^\varepsilon(0,v)v\theta_\alpha(v)\,dv. \tag{3.10}$$

Since $\dfrac{d\varphi^\varepsilon}{dx}(0) > \beta$ and $\dfrac{d^2\varphi^\varepsilon}{dx^2} \geq -n_D$, we deduce that $\dfrac{d\varphi^\varepsilon}{dx} > 0$ on $[0, r^\varepsilon]$ for ε small enough. Therefore, the left hand side of (3.10) is nonpositive and the right hand is nonnegative. We deduce that both of them vanish. Letting α tend to zero, we get:

$$\int_{-\infty}^{0} vf^\varepsilon(0,v)\,dv = 0.$$

which leads to $j^\varepsilon = j_g$ for small ε.

∎

3.2 Solving the reduced problem

We have shown in the previous section that the solution of the perturbation problem converges as ε goes to zero to a reduced solution satisfying:

$$\begin{cases} \dfrac{d^2\varphi}{dx^2} = \dfrac{j}{\sqrt{\varphi}} - n_D, \\ \varphi(0) = 0, \quad \varphi(1) = 1, \\ \varphi(x) > 0, \quad x \in (0,1]. \end{cases} \tag{3.11}$$

We give the main theorem of this section:

Theorem 3.6 *There exists a limiting current $j_{CL}(n_D)$ such that:*
(i) If $j > j_{CL}$ then the system (3.11) has no solution.
(ii) If $j \leq j_{CL}$, then the system (3.11) has a unique solution.
The limiting current j_{CL} is the unique value of j for which the solution has a vanishing derivative at the point $x = 0$.

To prove this theorem we will consider a scaling of the equation: We set

$$\varphi(x) \;=\; \frac{j^2}{n_D^2}\,\psi(\frac{n_D^{\frac{3}{2}}}{j}\,x). \tag{3.12}$$

The system (3.11) is written in the new variables:

$$\begin{cases} \dfrac{d^2\psi}{dx^2} \;=\; \dfrac{1}{\sqrt{\psi}} - 1, \\ \psi(0) \;=\; 0, \quad \psi(\dfrac{n_D^{\frac{3}{2}}}{j}) \;=\; \dfrac{n_D^2}{j^2} \\ \psi(x) > 0, \quad x \in (0, \dfrac{n_D^{\frac{3}{2}}}{j}]. \end{cases} \tag{3.13}$$

The solution of system (3.13) (if it exists) has a nonnegative derivative at the point $x \;=\; 0$, thus we consider the Cauchy problem (in the same way as done in [D1] or [D4]):

$$\begin{cases} \dfrac{d^2\psi}{dx^2} \;=\; \dfrac{1}{\sqrt{\psi}} - 1, \\ \psi(0) \;=\; 0, \\ \dfrac{d\psi}{dx}(0) \;=\; \beta \geq 0. \end{cases} \tag{3.14}$$

Multiply the equation by $\dfrac{d\psi}{dx}$ and integrate over $[0, x]$. This leads to:

$$(\frac{d\psi}{dx})^2 \;=\; \beta^2 \;+\; 4\sqrt{\psi(x)} \;-\; 2\psi(x). \tag{3.15}$$

Set

$$\psi_{max} \;=\; (1 + \sqrt{1 + \frac{\beta^2}{2}})^2. \tag{3.16}$$

ψ_{max} is the solution of $\beta^2 \;+\; 4\sqrt{\psi} \;-\; 2\psi \;=\; 0$.
Then we can integrate (3.15) for $\psi(x) \leq \psi_{max}$ and get:

$$x \;=\; \int_0^{\psi(x)} \frac{dy}{\sqrt{\beta^2 \;+\; 4\sqrt{y} \;-\; 2y}}\,dy. \tag{3.17}$$

Which defines $\psi(x)$ in a unique way for $x \leq x_\beta$, where

$$x_\beta \;=\; \int_0^{\psi_{max}} \frac{dy}{\sqrt{\beta^2 \;+\; 4\sqrt{y} \;-\; 2y}}\,dy \tag{3.18}$$

From (3.15), the derivative of ψ vanishes at the point $x \;=\; x_\beta$. Besides we have $\dfrac{d^2\psi}{dx^2}(x_\beta) \;=\; \dfrac{1}{\sqrt{\psi_{max}}} \;-\; 1 \;\leq -\dfrac{1}{2}$. Then ψ' becomes negative for $x > x_\beta$, thus using the formula (3.15) we find that the the negative sign must be taken and ψ is symmetric with respect to x_β. Now we can conclude this step of the proof by the following lemma.

Lemma 3.7 *(i) If $\beta > 0$ then there exist a unique solution ψ_β of (3.14) defined on the interval $\mathcal{I}_\beta = [0, 2x_\beta]$ where x_β is given by the formula (3.18). This solution is symmetric with respect to x_β.*
(ii) If $\beta = 0$ then there exists a solution defined by the same formulae as in the case $\beta > 0$. Moreover, this solution can be extended periodically on $\mathbb{R}^+$.

Proof: The derivative $\psi'(2x_\beta) = -\beta$, so it is obvious that the solution can be extended in a periodic way if and only if $\beta = 0$.

∎

The formula (3.17) can be integrated analytically and gives:

$$\begin{aligned} x &= \beta - \sqrt{\beta^2 + 4\sqrt{\psi(x)} - 2\psi(x)} - \sqrt{2}\arcsin\left[\frac{1-\sqrt{\psi(x)}}{\sqrt{1+\frac{\beta^2}{2}}}\right] \\ &\quad + \sqrt{2}\arcsin\left[\frac{1}{\sqrt{1+\frac{\beta^2}{2}}}\right]. \end{aligned} \tag{3.19}$$

From this formula we can deduce the value of x_β.

$$x_\beta = \frac{\pi}{\sqrt{2}} + \sqrt{2}\arcsin\left[\frac{1}{\sqrt{1+\frac{\beta^2}{2}}}\right] + \beta. \tag{3.20}$$

We can differentiate this equality and get

$$\frac{dx_\beta}{d\beta} = \frac{\beta^2}{2+\beta^2},$$

then x_β is an increasing function of β. From this we deduce the following lemma.

Lemma 3.8 *Let $\beta_1 < \beta_2$ be two nonegative real numbers; then the respective solutions ψ_1 and ψ_2 of (3.14) with β_1 and β_2 as initial data satisfy:*

$$\psi_1 < \psi_2 \qquad on \quad (0, 2x_{\beta_1}].$$

Proof: It is sufficient to prove the result for $x \in (0, x_{\beta_1}]$, because thanks to the symmetry of the functions ψ_1 and ψ_2 with respect to x_{β_1} and x_{β_2} it can immediately be extended to $(0, 2x_{\beta_1}]$.

From formula (3.17), we get for $x \in (0, x_{\beta_1}]$:

$$\int_0^{\psi_1(x)} \frac{dy}{\sqrt{\beta_1^2 \,+\, 4\sqrt{y} \,-\, 2y}}\, dy \;=\; \int_0^{\psi_2(x)} \frac{dy}{\sqrt{\beta_2^2 \,+\, 4\sqrt{y} \,-\, 2y}}\, dy.$$

Then, since $\beta_1 < \beta_2$, the above equality implies that $\psi_1 < \psi_2$.

■

The current j is determined by $\psi_\beta(\frac{n_D^{3/2}}{j}) \;=\; (\frac{n_D}{j})^2$. We set

$$X \;=\; \frac{n_D^{3/2}}{j}. \tag{3.21}$$

The above equation is then written

$$\psi_\beta(X) \;=\; \frac{X^2}{n_D}. \tag{3.22}$$

According to the last equality of (3.13), ψ does not vanish in $(0, \frac{n_D^{3/2}}{j}]$, therefore X has to be in $[0, 2x_\beta]$.

Lemma 3.9 *For every $\beta \geq 0$, the equation (3.22) has a unique solution $X(\beta)$ in $[0, 2x_\beta]$. Moreover, $X(\beta)$ is an increasing continuous function such that:*

$$\lim_{\beta \to +\infty} X(\beta) \;=\; +\infty.$$

Proof: We set

$$F_\beta(x) \;=\; \psi_\beta(x) - \frac{x^2}{n_D}. \tag{3.23}$$

We have

$$F_\beta(0) \;=0, \quad F'_\beta(0) \;=\; \beta, \quad F''_\beta(x) \;=\; \frac{1}{\sqrt{\psi_\beta(x)}} - 1 - \frac{2}{n_D}$$

Since $\sqrt{\psi_{max}} \geq 2$, then the equation $F''_\beta \;=\; 0$ has two solutions $y_1 < x_\beta < y_2$ such that

$$\sqrt{\psi_\beta(y_i)} = \frac{1}{1 + \frac{2}{n_D}}, \quad i = 1, 2.$$

We deduce that F'_β vanishes at a unique point z_β; it is positive on $[0, z_\beta]$ and negative on $[z_\beta, 1]$. From this, we deduce the existence and uniqueness of $X(\beta)$ such that

$$F_\beta(x) > 0, \;\text{ if } x < X(\beta), \quad F_\beta(x) < 0, \;\text{ if } x > X(\beta). \tag{3.24}$$

The function $X(\beta)$ is obviously continuous. It is increasing because in view of Lemma 3.8 we have:

$$(\beta_1 < \beta_2) \rightarrow (\psi_1 < \psi_2).$$

Then $F_{\beta_2}(X(\beta_1)) > F_{\beta_1}(X(\beta_1)) = 0$. We deduce from (3.24) that $X(\beta_1) < X(\beta_2)$. To prove that $\lim\limits_{\beta \to \infty} X(\beta) = +\infty$ we suppose that the converse is true. Thus, since $X(\beta)$ is increasing, there exists a positive constant C such that

$$\lim_{\beta \to \infty} X(\beta) = C.$$

From (3.19) and (3.23), we deduce that $X(\beta)$ solves the equation

$$\begin{aligned} X(\beta) &= \beta - \sqrt{\beta^2 + \frac{4X(\beta)}{\sqrt{n_D}} - \frac{2X(\beta)^2}{n_D}} - \sqrt{2}\arcsin\left[\frac{1 - \frac{X(\beta)}{\sqrt{n_D}}}{\sqrt{1 + \frac{\beta^2}{2}}}\right] \\ &\quad + \sqrt{2}\arcsin\left[\frac{1}{\sqrt{1 + \frac{\beta^2}{2}}}\right]. \end{aligned}$$

The last two terms of the right hand side vanish when β goes to infinity. The difference between the two first terms vanishes too. Then we find that the limit C of $X(\beta)$ is equal to zero, which is obviously impossible because $X(\beta)$ is increasing.

∎

End of the proof of Theorem 3.6. We have a one to one monotonically increasing map

$$X : \beta \in [0, +\infty[:\rightarrow X(\beta) \in [X(0), \infty).$$

We define j_{CL} by

$$X(0) = \frac{n_D^{3/2}}{j_{CL}}.$$

Therefore, if $j \in (0, j_{CL}]$ and , we define $X = \frac{n_D^{3/2}}{j}$ and we have $X \in [X(0), +\infty[$. There exists a unique β such that $X = X(\beta)$ which insures the existence and uniqueness of a solution of the reduced problem (3.11). For $j > j_{CL}$, we have no solution because $X < X(0)$.

∎

3.3 Passing to the limit

Proposition 3.10 *We assume that (SC) or (SD) are satisfied.*
(i) *The distribution function $f(x,v)$ is a solution of the Vlasov equation and is written*

$$f(x,v) = \frac{j}{\sqrt{\varphi(x)}}\,\delta(v-\sqrt{\varphi(x)}). \tag{3.25}$$

(ii) *The current j satisfies $j=\min(j_g,j_{CL})$.*
(iii) *We have the following convergences:*

$$n^\varepsilon \to n \qquad L^1[0,1] \text{ strong and } L^\infty[\alpha,1] \text{ strong } (\forall\alpha>0) \tag{3.26}$$

$$\varphi^\varepsilon \to \varphi \qquad W^{2,1}[0,1] \text{ strong and } W^{2,\infty}[\alpha,1] \text{ strong } (\forall\alpha>0). \tag{3.27}$$

Proof: (i) It is obvious from the support properties of f that there exists $n_1(x)\in\mathcal{M}_b([0,1])$ such that $f(x,v) = n_1(x)\,\delta(v-\sqrt{\varphi(x)})$. Then integrating over v, one can find that $n_1(x)=n(x)$ and in view of Lemma 3.4 we get (3.25).
(ii) The proof is identical to the one given in [D4]. We know from Theorem 3.6 and (2.2) that $j\le\min(j_g,j_{CL})$. If $\frac{d\varphi}{dx}(0)>0$ then $j=j_g$ (see Lemma 3.5). If $\frac{d\varphi}{dx}(0)=0$ then $j=j_{CL}$ (see Theorem 3.6 (ii)) . This yields $j=\min(j_g,j_{CL})$.
(iii) The limit solution φ is in $C^1([0,1])$ (because $n\in L^1[0,1]$). Therefore we can apply Lemma 2.13 and deduce that φ^ε converges towards φ in $C^1([0,1])$. The convergence (3.27) is a direct consequence of (3.26). The strong L^1 convergence of n^ε is a consequence of the local L^∞ convergence:

$$\|n^\varepsilon-n\|_{L^1} = \int_0^\alpha |n^\varepsilon(x)-n(x)|\,dx + \int_\alpha^1 |n^\varepsilon(x)-n(x)|\,dx.$$

Let α small enough such that $\frac{d^2\varphi^\varepsilon}{dx^2}$ and $\frac{d^2\varphi}{dx^2}$ are positive on $(0,\alpha]$. The first integral of the right hand side can be estimated as follows:

$$\begin{aligned}\int_0^\alpha |n^\varepsilon(x)-n(x)|\,dx &= \int_0^\alpha |\frac{d^2\varphi^\varepsilon}{dx^2}-\frac{d^2\varphi}{dx^2}|\,dx\\ &\le \int_0^\alpha \frac{d^2\varphi^\varepsilon}{dx^2}+\frac{d^2\varphi}{dx^2}\,dx\\ &\le \frac{d\varphi^\varepsilon}{dx}(\alpha)-\frac{d\varphi^\varepsilon}{dx}(0) + \frac{d\varphi}{dx}(\alpha)-\frac{d\varphi}{dx}(0)\end{aligned}$$

From the $C^1([0,1])$ convergence of φ^ε to φ we deduce that the right hand side of the above inequality can be chosen arbitrarily small with respect to ε by taking α small enough. Since the second integral goes to zero for every fixed α because of the local L^∞ convergence, the above estimate yields the L^1 convergence of n^ε towards n.

The only thing left to show is the strong convergence of n^ε in $L^\infty[\alpha,1]$. Since φ^ε is positive on $[\alpha,1]$, we deduce from (2.6) and Proposition 3.2 (iii) that :

$$\left| n^\varepsilon(x) - \frac{j^\varepsilon}{\sqrt{\varphi^\varepsilon(x)}} \right| \le \frac{C\varepsilon}{\varphi^\varepsilon(x)}, \qquad x \in [\alpha,1].$$

Then for ε small enough, the above estimate becomes:

$$\left| n^\varepsilon(x) - \frac{j^\varepsilon}{\sqrt{\varphi^\varepsilon(x)}} \right| \le C(\alpha)\,\varepsilon,$$

and then we get $n^\varepsilon \to \dfrac{j}{\sqrt{\varphi}} = n$ in $L^\infty[\alpha,1]$.

∎

4 Large Density n_D and Large Injection Current j_g

4.1 Further estimates

If none of the hypotheses (SC) or (SD) are satisfied, the limit solution may have zeros inside the interval $(0,1)$. As a consequence, lemma 3.1 may not be satisfied any longer, then we begin by showing the following lemma:

Lemma 4.1 *There exist p real numbers $0 = r_1 < r_2 < \cdots < r_p < 1$ such that*

$$E_0 = \{x \in [0,1], \quad \varphi(x) = 0\} = \{r_1, r_2, \cdots, r_p\}.$$

Proof: Let $r_1 < r_2$ two real numbers such that $\varphi(r_1) = \varphi(r_2) = 0$. From Lemma 2.11 we deduce that $j \ge \frac{1}{2}\, n_D$. We will prove that $r_2 - r_1 > C$ where C is a positive constant. Like in the proof of Lemma 3.1 we build a sequence K_p such that

$$\frac{d^2\varphi^\varepsilon}{dx^2} \ge -K_p, \quad x \in [r_1, r_2].$$

We have

$$\varphi^\varepsilon(x) \le \frac{K_p(r_2-r_1)^2}{8} + \varphi^\varepsilon(r_1), \quad x \in [r_1, r_2] \tag{4.1}$$

$$K_{p+1} = n_D - \frac{j^\varepsilon}{\sqrt{\frac{K_p(r_2-r_1)^2}{8} + 4V_0^2\varepsilon^2 + \varphi^\varepsilon(r_1)}}.$$

The limit l^ε of K_p satisfies

$$l^\varepsilon \,=\, n_D \,-\, \frac{j^\varepsilon}{\sqrt{\frac{l^\varepsilon (r_2-r_1)^2}{8} \,+\, 4V_0^2\varepsilon^2 \,+\, \varphi^\varepsilon(r_1)}}.$$

Now we pass to the limit when ε tends to zero and we get:

$$l \,=\, n_D \,-\, \frac{j}{\sqrt{\frac{l(r_2-r_1)^2}{8}}}, \tag{4.2}$$

$$\varphi \leq \frac{l(r_2-r_1)^2}{8}, \quad x \in [r_1, r_2]. \tag{4.3}$$

From these inequalities, we deduce an analogous estimate to (3.6)

$$j \leq \frac{1}{3\sqrt{6}}(r_2-r_1)n_D^{\frac{3}{2}}$$

which implies

$$r_2 - r_1 \,\geq \sqrt{\frac{27}{2n_D}}$$

since $j \geq \frac{1}{2}\,n_D$.

This indicates that the number of roots of φ is finite and is bounded by $\sqrt{\dfrac{2n_D}{27}}$.

∎

Let $r_1, \cdots r_p$ the points where φ vanishes. Since on (r_m, r_{m+1}), the function φ is positive, then, for small ε there exists a constant $\alpha(\varepsilon)$ tending to zero as ε tends to zero, such that

$$\varphi^\varepsilon(x) \,\geq 2V_0^2\varepsilon^2, \quad \forall\, x \in \cup_{m=1}^p I_\varepsilon^m$$

where we denoted

$$I_\varepsilon^m \,=\, [r_m + \alpha(\varepsilon), r_{m+1} - \alpha(\varepsilon)], \quad m = 1, \cdots, p-1,$$

and

$$I_\varepsilon^p \,=\, [r_p + \alpha(\varepsilon), 1].$$

Then from the proof of Lemma 2.9 the current $j_-^\varepsilon(x) \,=\int_{-\infty}^0 v f^\varepsilon(x,v)\,dv$ is constant on each range I_ε^m. Since the total current $j \,=\int_{\mathbb{R}} v f^\varepsilon\,dv$ is constant everywhere, we deduce that $\int_{\mathbb{R}} |v|\, f^\varepsilon\,dv$ is constant on each interval I_ε^m. We set

$$j_\varepsilon^m \,=\, \int_{\mathbb{R}} |v|\, f^\varepsilon(x,v)\,dv, \qquad x \in I_\varepsilon^m.$$

Remark 4.2 (i) Notice that by Lemma 2.9 we have $j_\varepsilon^p \,=\, j^\varepsilon$.
(ii) Since $f^\varepsilon(x,v) \,\leq\, g^\varepsilon(\sqrt{v^2 - \varphi^\varepsilon(x)})$ if $v^2 \,\geq\, \sqrt{\varphi^\varepsilon(x)}$ and $f^\varepsilon(x,v) \,=\, 0$ otherwise, then $\int_{\mathbb{R}} |v|\, f^\varepsilon(x,v)\,dv \leq 2j_g$ when $\varphi^\varepsilon(x) \geq 0$. ∎

We have the following lemma which is a generalization of Lemma 2.10:

Lemma 4.3 *There exists a constant C such that*

$$\left|k^\varepsilon(x) - j_\varepsilon^m\sqrt{\varphi^\varepsilon(x)}\right| \leq C\varepsilon, \quad x \in I_\varepsilon^m, \ m = 1, \cdots, p.$$

Proof: On every interval I_ε^m, we write

$$k^\varepsilon(x) - j_\varepsilon^m \sqrt{\varphi^\varepsilon(x)} \, = \, \int_{\mathbb{R}} |v| \, (|v| \, - \, \sqrt{\varphi(x)}) \, f^\varepsilon(x,v) \, dv.$$

From (2.1) we deduce that $supp\, f^\varepsilon \, \subset \, \{||v| - \sqrt{\varphi^\varepsilon}| \, \leq \, V_0 \, \varepsilon\}$.
Thus, since $\int_{\mathbb{R}} |v| \, f^\varepsilon(x,v) \, dv \, \leq 2 j_g$, we obtain

$$\left|k^\varepsilon(x) - j_\varepsilon^m\sqrt{\varphi^\varepsilon(x)}\right| \, \leq \, 2 \, j_g \, V_0 \, \varepsilon.$$

∎

As a consequence of this lemma we have the following convergence result.

Proposition 4.4 *There exist p positive constants $j^1, \cdots, j^p$ such that the kinetic energy k^ε converges in the L^∞ norm to the continuous function k defined by*

$$k(x) \, = \, j^m \sqrt{\varphi(x)} \qquad x \, \in [r_m, r_{m+1}], \quad m \, = 1, \cdots, p \tag{4.4}$$

where $r_{p+1} \, = \, 1$ and $j^p \, = \, j$.

Proof: Since the sequences j_ε^m are bounded, then, after extraction of a subsequence we obtain $j_\varepsilon^m \, \to \, j^m$.
On one hand, we deduce from the definition 4.4 of k and from lemma 4.3 that

$$|k^\varepsilon(x) - k(x)| \, \leq C\varepsilon \, + \, |j_\varepsilon^m\sqrt{\varphi^\varepsilon(x)} - j^m\sqrt{\varphi(x)}|, \quad x \in \, I_\varepsilon^m. \tag{4.5}$$

On ther hand, for $x \, \in [r_m - \alpha(\varepsilon), r_m + \alpha(\varepsilon)]$ we have

$$|\varphi^\varepsilon(x)| \, \leq |\varphi^\varepsilon(r_m)| \, + \, \alpha(\varepsilon) \, \sup |\frac{d\varphi^\varepsilon}{dx}|.$$

Since $\varphi(r_m) \, = 0$, $|\frac{d\varphi^\varepsilon}{dx}|$ is bounded in L^∞ (see lemma 2.6), and $\alpha(\varepsilon) \, \to \, 0$, then we deduce that

$$\sup_{x \in [0,1] \setminus \cup I_\varepsilon^m} |\varphi^\varepsilon(x)| + |\varphi(x)| \, = \, o(1).$$

Therefore using (2.4) we deduce that

$$\sup_{x \in [0,1] \setminus \cup I_\varepsilon^m} |k^\varepsilon(x)| + |k(x)| \, = \, o(1). \tag{4.6}$$

This combined with (4.5) ends the proof of Proposition 4.4.

∎

We apply Lemma 2.12 and conclude that

Lemma 4.5 $\left|\dfrac{d\varphi^\varepsilon}{dx}\right| \to \left|\dfrac{d\varphi}{dx}\right|$ *in the* $C^0(0,1)$ *topology.*

The function n *is an* $L^1[0,1]$ *function and satisfies:*

$$n(x) = \frac{j^m}{\sqrt{\varphi(x)}}, \quad x \in I^m = (r_m, r_{m+1}),\ m = 1, \cdots, p.$$

Proof:

Remark 4.6 In the case $p = 1$, Lemma 3.5 is still valid and the conclusion is the same as in the previous section (The limit solution and the convergence spaces are still characterized by proposition 3.10): indeed, to prove Lemma 3.5, the key point is that $r_\varepsilon \le C\varepsilon^2$, which is still true in this case ∎

The limit problem is characterized by the points r_m and the currents j^m. We were not able to determine the number p and the location of the points r_m. However, if the points r_m are known, then the limit problem is entirely characterized. This is shown in the following section.

4.2 The reduced problem

From now on, we suppose that we are in the case $p \ge 2$. Then the total energy h is identically equal to zero, and the derivative of φ vanishes at the points $r_1, \cdots, r_p$.Then, φ satisfies:

$$\begin{cases} \dfrac{d^2\varphi}{dx^2} = \dfrac{j^m}{\sqrt{\varphi}} - n_D, \quad on\ I^m = (r_m, r_{m+1}) \\ \varphi(r_m) = \varphi(r_{m+1}) = \dfrac{d\varphi}{dx}(r_m) = \dfrac{d\varphi}{dx}(r_{m+1}) = 0, \\ \varphi > 0 \quad on\ I^m. \end{cases} \tag{4.7}$$

for $m = 1, 2, \cdots, p-1$, and

$$\begin{cases} \dfrac{d^2\varphi}{dx^2} = \dfrac{j^p}{\sqrt{\varphi}} - n_D, \quad on\ I^p = (r_p, 1] \\ \varphi(r_p) = \dfrac{d\varphi}{dx}(r_p) = 0, \quad \varphi(1) = 1 \\ \varphi > 0 \quad on\ I^p. \end{cases} \tag{4.8}$$

Proposition 4.7 *For* r_m *and* r_{m+1} *fixed, the system (4.7) has a unique solution* (j^m, φ). *It is given by:*

$$j^m = \frac{n_D^{3/2}}{2\sqrt{2}\,\pi}(r_{m+1} - r_m), \tag{4.9}$$

and

$$\varphi(x) = \left(\frac{j^m}{n_D}\right)^2 \psi_0\left(\frac{n_D^{3/2}}{j^m}(x - r_m)\right), \tag{4.10}$$

where ψ_0 *is the unique solution of (3.14) with* $\beta = 0$.

Proof: Set

$$\varphi(x) \;=\; \left(\frac{j^m}{n_D}\right)^2 \psi\left(\frac{n_D^{3/2}}{j^m}(x - r_m)\right). \tag{4.11}$$

Therefore ψ satisfies

$$\begin{cases} \dfrac{d^2\psi}{dx^2} \;=\; \dfrac{1}{\sqrt{\psi}} - 1, \quad on\ J^m \;=\; (0, \dfrac{n_D^{3/2}}{j^m}(r_{m+1} - r_m)) \\ \psi(0) \;=\; \psi\left(\dfrac{n_D^{3/2}}{j^m}(r_{m+1} - r_m)\right) \;=\; \dfrac{d\psi}{dx}(0) \;=\; \dfrac{d\psi}{dx}\left(\dfrac{n_D^{3/2}}{j^m}(r_{m+1} - r_m)\right) \;=\; 0, \\ \psi > 0 \quad on\ J^m. \end{cases} \tag{4.12}$$

From section 4, we deduce that $\frac{n_D^{3/2}}{j^m}(r_{m+1} - r_m) \;=\; 2x_0$ where x_0 is given by formula (3.20), with $\beta \;=\; 0$. Then we have

$$\frac{n_D^{3/2}}{j^m}(r_{m+1} - r_m) \;=\; 2\sqrt{2}\,\pi.$$

This ends the proof.

$\blacksquare$

Proposition 4.8 *The system (4.8) has a unique solution (j, φ); It is given by:*

$$j \;=\; \frac{1}{(1 - r_p)^2}\, j_{CL}((1 - r_p)^2\, n_D), \tag{4.13}$$

where j_{CL} is defined in Theorem 3.6, and

$$\varphi(x) \;=\; \left(\frac{j}{n_D}\right)^2 \psi_0\left(\frac{n_D^{3/2}}{j}(x - r_p)\right) \tag{4.14}$$

Proof: The proof is similar to that of the previous proposition, We set

$$\varphi(x) \;=\; \psi\left(\frac{x - r_p}{1 - r_p}\right). \tag{4.15}$$

Therefore, ψ satisfies

$$\begin{cases} \dfrac{d^2\psi}{dx^2} \;=\; \dfrac{j(1 - r_p)^2}{\sqrt{\psi}} - n_D(1 - r_p)^2, \quad on\ (0, 1] \\ \psi(0) \;=\; \dfrac{d\psi}{dx}(0) \;=\; 0, \quad \psi(1) \;=\; 1 \\ \psi > 0 \quad on\ (0, 1]. \end{cases} \tag{4.16}$$

This gives:

$$j(1-r_p)^2 \;=\; j_{CL}\left(n_D(1-r_p)^2\right),$$

and by (3.12)

$$\psi(x) \;=\; \left(\frac{j(1-r_p)^2}{n_D(1-r_p)^2}\right)^2 \psi_0\left(\frac{(n_D(1-r_p)^2)^{\frac{3}{2}}}{j(1-r_p)^2}\,x\right).$$

In view of (4.15) we obtain the result.

∎

4.3 Passing to the limit

Lemma 2.13 applies and gives

Lemma 4.9 $\frac{d\varphi^\varepsilon}{dx} \to \frac{d\varphi}{dx}$ *in the* $C^0[0,1]$ *topology.*

Consequently, we have

Theorem 4.10 (i) *We have the following convergence results:*

$$n^\varepsilon \to n \qquad L^1[0,1] \text{ strong and } L^\infty(I_\alpha) \text{ strong } (\forall\alpha>0) \tag{4.17}$$
$$\varphi^\varepsilon \to \varphi \qquad W^{2,1}[0,1] \text{ strong and } W^{2,\infty}(I_\alpha) \text{ strong } (\forall\alpha>0) \tag{4.18}$$

where

$$I_\alpha \;=\; [\alpha, r_2-\alpha] \;\cup\; [r_2+\alpha, r_3-\alpha] \;\cup\; \cdots \cup\; [r_p+\alpha, 1].$$

(ii) *The distribution function is given by:*

$$f(x,v) \;=\; \frac{j_+^m}{\sqrt{\varphi(x)}}\,\delta(v-\sqrt{\varphi(x)}) + \frac{j_-^m}{\sqrt{\varphi(x)}}\,\delta(v+\sqrt{\varphi(x)}), \qquad (x,v)\in I^m \times \mathbb{R}$$

where j_+^m *and* j_-^m *are defined by*

$$j_+^m + j_-^m \;=\; j^m, \quad j_+^m - j_-^m \;=\; j.$$

Proof: (i) The proof is identical to that of Theorem 3.10.
(ii) From the support properties of f, we deduce that

$$f(x,v) \;=\; n_1(x)\,\delta(v-\sqrt{\varphi(x)}) \;+\; n_2(x)\,\delta(v+\sqrt{\varphi(x)}).$$

Since on I^m

$$n_1+n_2 \;=\; n \;=\; \int_{\mathbb{R}} f\,dv \;=\; \frac{j^m}{\sqrt{\varphi}}$$

and

$$j = \int_{\mathbb{R}} vf\,dv = (n_1(x) - n_2(x))\sqrt{\varphi(x)}$$

then

$$n_1 = \frac{j_+^m}{\sqrt{\varphi}}$$

and

$$n_2 = \frac{j_-^m}{\sqrt{\varphi}}.$$

∎

5 Conclusion

In this paper we investigated the influence of a background density (doping density) on the transport characteristics in the Child-Langmuir regime. The main conclusion that comes out is that an electron beam which flows through a background of positive ions behaves in the collisionless case like a vacuum diode when the ion density is small

$$N_D \le C\,\frac{\varepsilon_0 \Phi_L}{eL^2}.$$

When the doping density is large the limit potential could vanish and therefore we cannot characterise uniquely the limit solution.

Another factor, namely the collisions, can have an important influence on the behavior of $N^+ - N^- - N^+$ diode. In a forthcoming paper, [Ba3], the Child-Langmuir asymptotics for the Boltzmann equation of semiconductors is investigated. Finally, in [Ba1] the modelling of the potential barrier lying in an $N^+ - N_-$ junction is derived via a Child-Langmuir type asymptotics.

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