.

APPROXIMATE CONTROLLABILITY OF THE SEMILINEAR HEAT EQUATION

Caroline FABRE*, Jean-Pierre PUEL** and Enrike ZUAZUA***

Abstract

This article is concerned with the study of approximate controllability for the semilinear heat equation in a bounded domain Ω when the control acts on any open and nonempty subset of Ω or on a part of the boundary. Both, in the case of an internal and a boundary control, the approximate controllability in $L^p(\Omega)$ for $1 \leq p < +\infty$ is proved when the nonlinearity is globally Lipschitz with a control in L^∞. In the case of the interior control we also prove the approximate controllability in $C_0(\Omega)$. The proof combines a variational approach to the controllability problem for linear equations and a fixed point method. We also prove that the control can be taken to be of "quasi bang-bang" form.

1 Introduction

In this paper, we consider the approximate controllability problem for the semilinear heat equation and we develop results which have been announced in [5]. We consider both the case where the controls act in the interior of the domain and the case of boundary control. Let us state the problem in the case where the control acts in the interior of the domain :

Let Ω be an open and bounded set of $\mathbb{R}^N$, $N \geq 1$, with C^2 boundary and ω an open and nonempty subset of Ω. Let f be a real and globally Lipschitz function. Let $c > 0$ and $d > 0$ be such that

$$|f(s)| \leq c|s| + d. \tag{1.1}$$

We consider the following semilinear heat equation :

$$\begin{cases} y' - \Delta y + f(y) = h\chi_\omega & in \quad Q = \Omega \times (0,T) \\ y = 0 \quad on \quad \Sigma = \partial\Omega \times (0,T) \\ y(0) = y^0 \quad in \quad \Omega. \end{cases} \tag{1.2}$$

In (1.2), $h = h(x,t)$ represents the control function and χ_ω is the characteristic function of ω, the set where controls are supported. By $'$ we denote the derivative with respect to the time variable.

We say that system (1.2) is approximately controllable in $L^p(\Omega)$ at time $T > 0$ if the following holds : "For every $y^0 \in L^p(\Omega)$ the set of reachable states at time $T > 0$,

$$E(T) = \{y(x,T), \qquad y \quad \text{is solution of (1.2)} \quad \text{with} \quad h \in L^p(Q)\},$$

is dense in $L^p(\Omega)$."

On the other hand, (1.2) is approximately controllable in $C_0(\Omega)$ (the space of uniformly continuous functions in Ω that vanish on $\partial\Omega$, endowed with the norm of the supremum) at time $T>0$ if for every $y^0 \in C_0(\Omega)$ the set

$$E(T) = \{y(x,T), y \text{ solution of (1.2) with } h \in L^\infty(Q)\}$$

is dense in $C_0(\Omega)$.
We recall that $v \in sgn(s)$ if $v(x,t) = \frac{s(x,t)}{|s(x,t)|}$ if $s(x,t) \neq 0$ and $|v(x,t)| \leq 1$ on the set $\{(x,t), s(x,t) = 0\}$. Furthermore, we will denote by $sgn_0(s)$ the element of $sgn(s)$ which is equal to 0 on $\{(x,t), s(x,t) = 0\}$.

The main results of this paper are as follows :

Theorem 1.1
Under the above assumptions on f, system (1.2) is approximately controllable in $L^p(\Omega)$ for $1 \leq p < \infty$ and in $C_0(\Omega)$ at any time $T > 0$. Moreover we can reach a dense set of final states by using controls of the form

$$h(x,t) \in \left(\int_{\omega\times(0,T)} |\varphi(x,t)|dxdt \right) sgn(\varphi)\chi_{\omega\times(0,T)},$$

where φ is solution of a suitable heat equation.
In the sequel, controls of this form will be referred to as "quasi bang-bang" controls.

Remark 1.1
In order to ensure that our control is of bang-bang form we have to prove that the zero set of φ:

$$\{(x,t) \in Q : \varphi(x,t) = 0\}$$

is of zero $(n+1)$-dimensional Lebesgue measure. By unique continuation ([15], [19]) and backward ([6]) and forward uniqueness it is easy to see that this set has empty interior since φ satisfies a heat equation with a bounded potential. In space dimension $n = 1$, the results of S. Angenent [1] imply that this zero set is of zero measure. Such a result seems to be unknown in several space dimensions. $\square$

As an immediate consequence of Theorem 1.1 and the smoothing effect of the heat equation we have the following result:

Corollary 1.1
Under the hypotheses of Theorem 1.1, for every $y^0 \in L^1(\Omega)$ the reachable set

$$E(T) = \{y(x,T) : y \textit{ is solution of (1.2) with } h \in L^\infty(Q)\}$$

is dense in $C_0(\Omega)$.
When $f = 0$ (or more generally, when f is linear) the approximate controllability problem can

be reduced to the following uniqueness property :

$$(1.3) \qquad \begin{cases} -\varphi' - \Delta\varphi = 0 & in \quad Q \\ \varphi = 0 \quad in \quad \omega \times (0,T) \\ \varphi = 0 \quad on \quad \Sigma \\ \varphi \in L^{p'}(Q) \quad \left(\dfrac{1}{p} + \dfrac{1}{p'} = 1\right) \end{cases} \qquad \Rightarrow \varphi = 0 \quad in \quad \Omega \times (0,T).$$

This uniqueness property is shown to be a consequence of Holmgren's uniqueness Theorem.

However, when f is nonlinear, the approximate controllability problem does not seem to be reductible to a unique continuation property.

Our method of proof of Theorem 1.1 is inspired by recent works by J.L. Lions [13] and [14]. In these works the approximate controllability of linear heat equation is formulated in the following equivalent way in the case $p = 2$: Fix $y^0 \in L^2(\Omega)$, $y^1 \in L^2(\Omega)$ and $\varepsilon > 0$ and find $h \in L^2(Q)$ so that

$$(1.4) \qquad |y(T) - y^1|_{L^2(\Omega)} \leq \varepsilon.$$

The optimal control h satisfying this with minimum $L^2(Q)$-norm is shown to satisfy an optimality system. More precisely, if, for instance, $y^0 = 0$,

$$(1.5) \qquad h = \varphi\chi_\omega$$

where $\varphi = \varphi(x,t)$ is the solution of the linear heat equation

$$(1.6) \qquad \begin{cases} -\varphi' - \Delta\varphi = 0 \quad in \quad Q \\ \varphi = 0 \quad on \quad \Sigma \\ \varphi(T) = \varphi^0 \end{cases}$$

where $\varphi^0 \in L^2(\Omega)$ minimizes

$$(1.7) \qquad J_\varepsilon(\psi^0) = \frac{1}{2}\int_{\omega\times(0,T)} |\psi|^2 dxdt + \varepsilon|\psi^0|_{L^2(\Omega)} - \int_\Omega y^1\psi^0 dx$$

over the solutions of (1.6) with $\psi(T) = \psi^0 \in L^2(\Omega)$.

J.L. Lions in [13] and [14] proved that if the approximate controllability holds, then the optimal control exists and is given by (1.5), (1.6) and (1.7).

In the present paper, we show how one can prove directly approximate controllability results for linear systems by minimizing functionals like (1.7). Once the coercivity of (1.7) is understood one can introduce a number of variants of (1.7) to deal with the approximate controllability in $L^p(\Omega)$ for $1 \leq p < +\infty$ and in $C_0(\Omega)$. Let us point out that the proof of the coercivity requires a unique continuation result. This method only applies to linear systems but combining it with a fixed point technique one can prove Theorem 1.1. This fixed point method was used in [21] to prove the exact controllability of the wave equation under globally Lipschitz perturbations. In order to implement this fixed point argument we must study the approximate controllability of the heat equation perturbed by a bounded potential, obtaining some estimates on the dependence of the control function with respect to the potential. At this level the unique continuation results by S. Mizohata [15] and J.C. Saut and B. Scheurer [19] play a crucial role.

It is important to observe that the assumption on the Lipschitz character of f is optimal in the following sense : when $f(s) = |s|^{r-1}s$ with $r > 1$, system (1.2) is not approximately controllable in any $L^p(\Omega)$ with $p < \infty$ or in $C_0(\Omega)$ since all the reachable states are uniformly bounded on any closed subset of $\bar{\Omega} - \bar{\omega}$ at any time $t > 0$ (see A. Bamberger's result in J. Henry [7]; see also J.I. Diaz [4] where a number of related counterexamples are given). Of course, when the control acts in all Ω, $(\omega = \Omega)$, no restriction on f is needed, except those ensuring the well-posedness of system (1.2)(see [7]). In this case it is easy to check that if $1 < p < \infty$,

$$E(T) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

When f is sublinear at infinity $(i.e\ |f(s)| \leq a|s|^r + b,\ \forall s \in \mathbb{R}, a, b > 0\ and\ 0 \leq r < 1)$, K. Naito and T. Seidman (see [17]) prove that the approximate controllability of the semilinear system can be reduced to the study of the approximate controllability of the linear heat equation perturbed by an affine term. In this sense, our result is close to those of [17] but our methods are different and provide more general results since f is allowed to be any Lipschitz function and it gives "quasi bang-bang " controls.

We are also interested in the approximate boundary controllability of the semilinear heat equation : If Γ_0 is a nonempty and open subset of the boundary Γ, the system considered is

$$\begin{cases} y' - \Delta y + f(y) = 0 & in \quad Q = \Omega \times (0,T) \\ y = g\chi_{\Sigma_0} & on \quad \Sigma = \partial\Omega \times (0,T) \\ y(0) = y^0 & in \quad \Omega. \end{cases} \tag{1.8}$$

In (1.8), $g = g(\sigma, t)$ represents the control function, σ is the boundary variable and χ_{Σ_0} is the characteristic function of $\Sigma_0 = \Gamma_0 \times (0,T)$, the set where controls are supported.

We say that system (1.8) is approximately controllable in $L^p(\Omega)$ at time $T > 0$ if the following holds : "For every $y^0 \in L^p(\Omega)$ the set of reachable states at time $T > 0$

$$E(T) = \{y(x,T), \quad y \quad \text{is solution of (1.8)} \quad \text{with} \quad g \in L^\infty(\Sigma)\}$$

is dense in $L^p(\Omega)$."

We will prove the following

Theorem 1.2

Under the above assumptions on f, system (1.8) is approximately controllable in $L^p(\Omega)$ for any $1 \leq p < \infty$, at any time $T > 0$. Moreover one can take a control of the form

$$g \in \left(\int_{\Sigma_0} (T-t)^\beta |\frac{\partial\varphi}{\partial\nu}(\sigma,t)| d\sigma dt\right) sgn\left(\frac{\partial\varphi}{\partial\nu}\right) \chi_{\Sigma_0},$$

where φ is solution of a suitable heat equation and β is some positive constant that depends on p and the space dimension.square

Remark 1.2

For linear problems with smooth coefficients, existence of boundary bang-bang controls has been proved by E.J.P.G. Schmidt and N. Weck in [20].

One has to remark that it is not obvious that the solution of (1.8) when $g \in L^\infty(\Sigma)$ and $y^0 \in L^p(\Omega)$ satisfies $y(T) \in L^p(\Omega)$. For example, if $g \in L^2(\Sigma)$ and $y^0 \in L^2(\Omega)$, (although claimed in [4]) we don't have $y(T) \in L^2(\Omega)$ (one can see a counter example in [9], p. 217), which explains why we took the space $L^\infty(\Sigma)$ for the controls in the definition of the approximate boundary controllability of (1.8).

The rest of the article is organized as follows. In section 2, we state (in an abstract form) preliminary results concerning the existence and properties of the minima of certain functionals arising in the approximate controllability of linear systems in $L^p(\Omega)$ with $1 < p < \infty$. In section 3, we combine the results of section 2 with a fixed point method mentioned above to prove Theorem 1.1 when $1 < p < \infty$. In Section 4 we prove Theorem 1.1 in $L^1(\Omega)$ and $C_0(\Omega)$. We finish with the section 5 where the boundary control problem is studied and Theorem 1.2 is proved.

2 Study of a functional arising in controllability of linear systems

For sake of simplicity we only consider here the $L^p(\Omega)$-case, with $1 < p < \infty$. We will denote by $|\theta|_p$ the $L^p(\Omega)$-norm of a function θ and by p' the conjugate of p $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$. We denote by (f, g) the integral $\int_Q f(x,t)g(x,t)dxdt$.

For $0 < t_1 < t_2$, we denote by $X^p(t_1, t_2)$ the following Banach space :

$$X^p(t_1,t_2) = L^p(t_1,t_2; W_0^{1,p}(\Omega)) \cap W^{1,p}(t_1,t_2; L^p(\Omega)), \tag{2.1}$$

endowed with the natural norm

$$||.||_{X^p(t_1,t_2)} = ||.||_{L^p(t_1,t_2;W_0^{1,p}(\Omega))} + ||.||_{W^{1,p}(t_1,t_2;L^p(\Omega))}.$$

Let $a = a(x,t) \in L^\infty(Q)$. We recall that there exist (see [8], Theorem 9.1, page 341 and [18], pages 226 - 228) constants $C > 0$ (depending on a, Ω and T) and C_{t_1,t_2} (depending on a, Ω, t_1 and t_2) such that, for every $k \in L^p(Q)$ and $w^0 \in L^p(\Omega)$, the solution w of

$$\begin{cases} w' - \Delta w + a(x,t)w = k \\ w = 0 \quad on \quad \Sigma \\ w(0) = w^0 \quad in \quad \Omega \end{cases} \tag{2.2}$$

satisfies

$$\begin{cases} ||w||_{L^\infty(0,T;L^p(\Omega))} \le C(|w^0|_p + ||k||_{L^p(Q)}) \\ ||w||_{X^p(t_1,t_2)} \le C_{t_1,t_2}(|w^0|_p + ||k||_{L^p(Q)}). \end{cases} \tag{2.3}$$

Furthermore, if $w^0 = 0$, we have (see [8], pages 341-342) : and $k \in L^p(Q)$, the solution w of

$$(2.4) \qquad \begin{cases} w' - \Delta w = k & in \quad Q \\ w = 0 & on \quad \Sigma \\ w(0) = 0 & in \quad \Omega, \end{cases}$$

satisfies

$$(2.5) \qquad \begin{cases} w \in X^p(0,T) \\ \text{the mapping} \quad k \in L^p(Q) \to w \in X^p(0,T) \quad \text{is linear continuous.} \end{cases}$$

From (2.5) and by using Gronwall's Lemma, one can easily prove that for $k \in L^p(Q)$, the solution w of

$$(2.6) \qquad \begin{cases} w' - \Delta w + a(x,t)w = k & in \quad Q \\ w = 0 & on \quad \Sigma \\ w(0) = 0 & in \quad \Omega, \end{cases}$$

satisfies

$$(2.7) \qquad \begin{cases} w \in X^p(0,T) \\ ||w||_{X^p(0,T)} \leq C_a(||k||_{L^p(Q)}) \quad \text{with} \quad C_a = O(1 + |a|_\infty exp(|a|_\infty)). \end{cases}$$

We consider a set q which has the following form : $q = \omega \times (0,T)$ where ω is a nonempty and open subset of Ω or $q = \Gamma_0 \times (0,T)$ where Γ_0 is a nonempty and open subset of Γ. The set q is endowed with its natural measure $dxdt$ in the first case and $d\sigma dt$ (where $d\sigma$ is the boundary measure on $\partial\Omega$) in the second one. For simplicity, we will denote it in this section only by $dxdt$ in both cases.

We introduce a mapping L from $L^{p'}(\Omega) \times L^\infty(Q)$ into $L^1(q)$ which satisfies:

(H_1) $\qquad \forall a \in L^\infty(Q), \quad L(\ .\ ,a)$ is linear continuous from $L^{p'}(\Omega)$ in $L^1(q)$

and if φ is solution of

$$(2.8) \qquad \begin{cases} -\varphi' - \Delta\varphi + a(x,t)\varphi = 0 & in \quad Q \\ \varphi = 0 & on \quad \Sigma \\ \varphi(T) = \varphi^0 \in L^{p'}(\Omega) & in \quad \Omega, \end{cases}$$

then

(H_2) $\qquad L(\varphi^0, a) = 0$ almost everywhere in $q \Rightarrow \varphi = 0$ almost everywhere in Q;

$$(H_3) \qquad \begin{cases} \varphi_n^0 \rightharpoonup \varphi^0 \ in \ L^{p'}(\Omega) \text{ weak} \\ a_n \rightharpoonup a \ in \ L^\infty(Q) \text{ weak} - * \end{cases} \Rightarrow L(\varphi_n^0, a_n) \rightharpoonup L(\varphi^0, a) \ in \ L^1(q) \text{ weak;}$$

(H_4) $\qquad \forall \varphi^0 \in L^{p'}(\Omega), \quad L(\varphi^0,\ .\ )$ is compact from $L^\infty(Q)$ into $L^1(q)$.

We will often write $L_a(\varphi^0)$ instead of $L(\varphi^0, a)$.

Remark 2.1

Hypothesis (H_2) is a property of unique continuation for the solutions of (2.8). As we will see in Section 3 and 4, for the mappings L that we consider, they have been proved by Mizohata (see [15]) and Saut-Scheurer (see [19]).

Let $\alpha > 0$ and $y^1 \in L^p(\Omega)$. For $\varphi^0 \in L^{p'}(\Omega)$ and the solution φ of (2.8) with $\varphi(T) = \varphi^0$, we introduce the functional

$$J(\varphi^0; a, y^1) = \frac{1}{2}\left(\int_q |L_a(\varphi^0)(x,t)|dxdt\right)^2 + \alpha|\varphi^0|_{p'} - \int_\Omega y^1\varphi^0 dx. \tag{2.9}$$

and we have the following

Proposition 2.1

For every $\alpha > 0$, $y^1 \in L^p(\Omega)$ and $a \in L^\infty(Q)$, $J(.; a, y^1)$ is a real strictly convex continuous function on $L^{p'}(\Omega)$ and satisfies

$$\liminf_{|\varphi^0|_{p'}\to\infty} \frac{J(\varphi^0; a, y^1)}{|\varphi^0|_{p'}} \geq \alpha. \tag{2.10}$$

The functional $J(.; a, y^1)$ achieves its minimum at a unique point $\hat{\varphi}^0$ in $L^{p'}(\Omega)$. Furthermore,

$$\hat{\varphi}^0 = 0 \Leftrightarrow |y^1|_p \leq \alpha.$$

Proof of Proposition 2.1

Since the $L^{p'}(\Omega)$-norm is strictly convex and L_a is linear, it is clear that $J(a, y^1, .)$ is strictly convex. On the other hand, (2.3) and the continuity of L_a imply the continuity of J_a. In order to prove (2.10), we suppose that there exists a sequence φ_n^0 in $L^{p'}(\Omega)$ such that

$$|\varphi_n^0|_{p'} \to +\infty \tag{2.11}$$

and

$$\liminf_{n\to+\infty} \frac{J(\varphi_n^0; a, y^1)}{|\varphi_n^0|_{p'}} < \alpha. \tag{2.12}$$

For $\tilde{\varphi}_n^0 = \dfrac{\varphi_n^0}{|\varphi_n^0|_{p'}}$ we write $\tilde{\varphi}_n$ the solution of (2.8) with $\tilde{\varphi}_n(T) = \tilde{\varphi}_n^0$. Since $|\tilde{\varphi}_n^0|_{p'} = 1$, we can extract a subsequence (still denoted by $\tilde{\varphi}_n^0$), which weakly converges in $L^{p'}(\Omega)$ to an element $\tilde{\varphi}^0$ in $L^{p'}(\Omega)$. ¿From (2.3), $(\tilde{\varphi}_n)_n$ converges weakly in $L^{p'}(Q)$ to the solution $\tilde{\varphi}$ of (2.8) with $\tilde{\varphi}(T) = \tilde{\varphi}^0$. From (H_1), $L_a(\tilde{\varphi}_n^0)$ weakly converges in $L^1(q)$ to $L_a(\tilde{\varphi}^0)$. But (2.11) and (2.12) imply that there exists a subsequence (still denoted by $(\tilde{\varphi}_n)_n$) such that

$$\int_q |L_a(\tilde{\varphi}_n^0)|dxdt \to 0 \qquad if \qquad n \to +\infty. \tag{2.13}$$

This proves that $L_a(\tilde{\varphi}^0) = 0$ in q hence, by (H_2), $\tilde{\varphi} = 0$ in Q and

$$\tilde{\varphi}^0 = 0. \tag{2.14}$$

We now write

$$J(\varphi_n^0; a, y^1) \geq |\varphi_n^0|_{p'} \left(\alpha - \int_\Omega y^1 \tilde{\varphi}_n^0 dx \right), \tag{2.15}$$

and as $\tilde{\varphi}_n^0$ weakly converges to 0 in $L^{p'}(\Omega)$, we obtain using (2.11)

$$\liminf_{n \to +\infty} \frac{J(\varphi_n^0; a, y^1)}{|\varphi_n^0|_{p'}} \geq \alpha,$$

which contradicts (2.12) and proves (2.10).

Now, if $|y^1|_p \leq \alpha$, we have $J(\varphi^0; a, y^1) \geq 0$ for every $\varphi^0 \in L^{p'}(\Omega)$ and hence $\hat{\varphi}^0 = 0$. If $\hat{\varphi}^0 = 0$, then

$$\forall \varphi^0 \in L^{p'}(\Omega), \quad \forall t > 0, \quad \lim_{t \to 0+} \frac{J(t\varphi^0; a, y^1)}{t} \geq 0.$$

One can easily prove that this implies $|y^1|_p \leq \alpha$. Proposition 2.1 is proved. □

In order to study the nonlinear case, we need to make precise the dependence of the minima with respect to the potential. This is gathered in the following

Proposition 2.2
(i) If we denote by M the mapping

$$\begin{aligned} M : L^\infty(Q) \times L^p(\Omega) &\to L^{p'}(\Omega) \\ (a, y^1) &\to \hat{\varphi}^0 \end{aligned} \tag{2.16}$$

and if K is a compact subset of $L^p(\Omega)$ and B is a bounded subset of $L^\infty(Q)$, then $M(B \times K)$ is a bounded subset of $L^{p'}(\Omega)$.
(ii) Moreover, if $a_n \to a$ in $L^\infty(Q)$ weak-$*$ and if $y_n^1 \to y^1$ strongly in $L^p(\Omega)$ then $\hat{\varphi}_n^0$ strongly converges in $L^{p'}(\Omega)$ to $\hat{\varphi}^0$.

Proof of Proposition 2.2
In order to get (i), we are going to prove first that (2.10) is uniform in $(a, y^1) \in B \times K$. We again argue by contradiction and we follow the same argument as above : suppose that there exist sequences of functions $(a_n)_n$ of $L^\infty(Q)$, $(y_n^1)_n$ in $L^p(\Omega)$ and $(\varphi_n^0)_n$ of $L^{p'}(\Omega)$ such that (we denote by φ_n the solution of (2.8) with respect to the function a_n satisfying $\varphi_n(T) = \varphi_n^0$ and by $L_n(.)$ the function $L_{a_n}(.)$)

$$\exists a \in L^\infty(Q), \quad a_n \rightharpoonup_{n \to +\infty} a \quad in \quad L^\infty(Q) \quad \text{weak} - * \tag{2.17}$$

and

$$\exists y^1 \in L^p(\Omega), \qquad y_n^1 \to_{n \to +\infty} y^1 \qquad strongly \quad in \quad L^p(\Omega), \tag{2.18}$$

such that (2.11) holds and

$$\liminf_{n\to+\infty} \frac{J(\varphi_n^0; a_n, y_n^1)}{|\varphi_n^0|_{p'}} < \alpha. \tag{2.19}$$

As above, we denote by $\tilde{\varphi}_n^0 = \dfrac{\varphi_n^0}{|\varphi_n^0|_{p'}}$ and $\tilde{\varphi}_n$ the solution of (2.8) with respect to a_n and with $\tilde{\varphi}_n(T) = \tilde{\varphi}_n^0$. As $|\tilde{\varphi}_n^0|_{p'} = 1$, we can extract a subsequence, that we still denote by $\tilde{\varphi}_n^0$, which weakly converges in $L^{p'}(\Omega)$ to an element $\tilde{\varphi}^0$ of $L^{p'}(\Omega)$ and (from (H_3)) $L_n(\tilde{\varphi}_n^0)$ weakly converges in $L^1(q)$ to $L_a(\tilde{\varphi}^0)$. We denote by $\tilde{\varphi}$ the weak limit in $L^{p'}(Q)$ of $\tilde{\varphi}_n$. Using (2.3) and passing to the limit in the equation satisfied by φ_n, one can easily prove that $\tilde{\varphi}$ is the solution of (2.8) with respect to a and with $\tilde{\varphi}(T) = \tilde{\varphi}^0$.

To finish the proof, we just have now to follow what was done before : briefly, from (2.11) and (2.19), we have (after extraction of a subsequence)

$$\int_q |L_n(\tilde{\varphi}_n^0)| dxdt \to_{n\to+\infty} 0. \tag{2.20}$$

By (H_3) and (H_2), we get

$$\tilde{\varphi}^0 = 0. \tag{2.21}$$

Let us now prove that if $J_n = \frac{J(\varphi_n^0; a_n, y_n^1)}{|\varphi_n^0|_{p'}}$, we have $\liminf_{n\to+\infty} J_n \geq \alpha$. For this, we write

$$J_n \geq \left(\alpha - \int_\Omega y_n^1 \tilde{\varphi}_n^0 dx\right), \tag{2.22}$$

and as $\tilde{\varphi}_n^0$ weakly converges to 0 in $L^{p'}(\Omega)$ and y_n^1 strongly converges in $L^p(\Omega)$, we have

$$\liminf_{n\to+\infty} J_n \geq \alpha,$$

which contradicts (2.19).
Now, $J(0; a, y^1) = 0$ and therefore

$$J(\hat{\varphi}^0; a, y^1) \leq 0,$$

so that one can easily prove that the range of M is a bounded set of $L^{p'}(\Omega)$. This ends the proof of (i).

We now prove (ii). From (i), the functions $\hat{\varphi}_n^0$ are bounded in $L^{p'}(\Omega)$ hence they weakly converge to an element $\tilde{\varphi}^0 \in L^{p'}(\Omega)$ and $\hat{\varphi}_n$ weakly converge in $L^{p'}(Q)$ to the solution $\tilde{\varphi}$ of (2.8) with $\tilde{\varphi}(T) = \tilde{\varphi}^0$. Furthermore, (H_3) implies that $L_n(\hat{\varphi}_n^0)$ weakly converges in $L^1(q)$ to $L_a(\hat{\varphi}^0)$.
Let us prove that for every $\varphi^0 \in L^{p'}(\Omega)$,

$$\lim_{n\to+\infty} J(\varphi^0; a_n, y_n^1) = J(\varphi^0; a, y^1). \tag{2.23}$$

We denote by φ_n the solution of (2.8) with the potential a_n and $\varphi_n(T) = \varphi^0$. From (2.7) and since these functions have the same final data, one can prove that they weakly converge in $X^{p'}(0,T)$

(hence strongly in $L^{p'}(Q)$) to the solution φ of (2.8) with potential a and $\varphi(T) = \varphi^0$. Using (H_4), one can pass to the limit in each term of the functional, and prove (2.23).

Now, we write

$$\forall \varphi^0 \in L^{p'}(\Omega), \quad J(\hat{\varphi}_n^0; a_n, y_n^1) \le J(\varphi^0; a_n, y_n^1). \tag{2.24}$$

Since $y_n^1 \to y^1$ strongly in $L^p(\Omega)$, we have

$$J(\tilde{\varphi}^0, a, y^1) \le \liminf_{n\to+\infty} J(\hat{\varphi}_n^0; a_n, y_n^1). \tag{2.25}$$

From (2.24) and (2.25), we get

$$\begin{aligned} \forall \varphi^0 \in L^{p'}(\Omega), \quad J(\tilde{\varphi}^0, a, y^1) \le \liminf_{n\to+\infty} J(\hat{\varphi}_n^0; a_n, y_n^1) &\le \limsup_{n\to+\infty} J(\hat{\varphi}_n^0; a_n, y_n^1) \\ &\le \lim_{n\to+\infty} J(\varphi^0; a_n, y_n^1) \end{aligned} \tag{2.26}$$

hence we obtain

$$\forall \varphi^0 \in L^{p'}(\Omega), \quad J(\tilde{\varphi}^0, a, y^1) \le J(\varphi^0, a, y^1), \tag{2.27}$$

which implies, since $J(.; a, y^1)$ is strictly convex, that $\tilde{\varphi}^0 = \hat{\varphi}^0$. But, having this equality, (2.26) gives

$$J(\hat{\varphi}^0, a, y^1) = \lim_{n\to+\infty} J(\hat{\varphi}_n^0; a_n, y_n^1). \tag{2.28}$$

On the other hand, we have

$$\lim_{n\to+\infty} \int_\Omega y_n^1(x)\hat{\varphi}_n^0(x)dx = \int_\Omega y^1(x)\hat{\varphi}^0(x)dx,$$

$$\int_q |L_a(\hat{\varphi}^0)(x,t)|dxdt \le \liminf_{n\to+\infty} \int_q |L_n(\hat{\varphi}_n^0)(x,t)|dxdt$$

and

$$|\hat{\varphi}^0|_{p'} \le \liminf_{n\to+\infty} |\hat{\varphi}_n^0|_{p'}.$$

Equation (2.28) then implies that

$$\lim_{n\to+\infty} |\hat{\varphi}_n^0|_{p'} = |\hat{\varphi}^0|_{p'}.$$

As $L^{p'}(\Omega)$ is uniformly convex and $\hat{\varphi}_n^0$ is weakly convergent to $\hat{\varphi}^0$ in $L^{p'}(\Omega)$, we deduce that the convergence is strong in $L^{p'}(\Omega)$ which ends the proof of Proposition 2.2. □

We now are going to interpret the results of Proposition 2.1.

Since the functional $J(.; a, y^1)$ is convex continuous with real values, it possesses a subdifferential at every point of $L^{p'}(\Omega)$. At its minimum, we have $0 \in \partial J(\hat{\varphi}^0; a, y^1)$. Let us now prove the

Proposition 2.3

For every $\varphi^0 \in L^{p'}(\Omega), \varphi^0 \neq 0$, *denoting by* φ *the solution of (2.8) with* $\varphi(T) = \varphi^0$, *we have*

$$\partial J(\varphi^0; a, y^1) = \Big\{ \xi \in L^p(\Omega), \exists v \in sgn(L_a(\varphi^0))\chi_q \quad \textit{satisfying}$$

$$\int_\Omega \xi(x)\theta^0(x)dx = \int_q |L_a(\varphi^0)(x,t)|dxdt \int_q v(x,t)L_a(\theta^0)(x,t)dxdt \tag{2.29}$$

$$\alpha \int_\Omega \frac{|\varphi^0(x)|^{p'-2}\varphi^0(x)}{|\varphi^0|_{p'}^{p'-1}}\theta^0(x)dx - \int_\Omega y^1(x)\theta^0(x)dx,$$

$$\textit{for every } \theta^0 \in L^{p'}(\Omega), \textit{ where } \theta \textit{ is solution of } \quad (1.6) \textit{ with } \theta(T) = \theta^0. \Big\}$$

Proof of Proposition 2.3

Since the functions a and y^1 are fixed, we write $J(\varphi^0) = J(\varphi^0; a, y^1)$ and $L = L_a$. We have $J(\varphi^0) = j_1(\varphi^0) + j_2(\varphi^0)$ where

$$j_1(\varphi^0) = \frac{1}{2}\left(\int_q |L(\varphi^0)(x,t)|dxdt\right)^2$$

and

$$j_2(\varphi^0) = \alpha|\varphi^0|_{p'} - \int_q y^1(x)\varphi^0(x)dx.$$

Since j_2 is Gateaux-differentiable at every $\varphi^0 \neq 0$ in $L^{p'}(\Omega)$ and j_1 subdifferentiable at every $\varphi^0 \neq 0$ in $L^{p'}(\Omega)$, we have for every $\varphi^0 \neq 0$ in $L^{p'}(\Omega)$:

$$\partial J(\varphi^0) = \partial j_1(\varphi^0) + \partial j_2(\varphi^0).$$

It is easy to check that

$$\forall \theta^0 \in L^{p'}(\Omega), \quad (\partial j_2(\varphi^0), \theta^0) = \alpha \int_\Omega \frac{|\varphi^0(x)|^{p'-2}\varphi^0(x)}{|\varphi^0(x)|_{p'}^{p'-1}}\theta^0(x)dx - \int_\Omega y^1(x)\theta^0(x)dx. \tag{2.30}$$

We now determine the set $\partial j_1(\varphi^0)$. Let $\xi \in \partial j_1(\varphi^0)$. By definition, we have

$$\forall \theta^0 \in L^{p'}(\Omega), \quad (\xi, \theta^0) \leq \lim_{\varepsilon \to 0+} \frac{j_1(\varphi^0 + \varepsilon\theta^0) - j_1(\varphi^0)}{\varepsilon}. \tag{2.31}$$

Now, since L is linear, one can prove that

$$\lim_{\varepsilon \to 0+} \frac{j_1(\varphi^0 + \varepsilon\theta^0) - j_1(\varphi^0)}{\varepsilon} = |L(\varphi^0)|_{L^1(q)} \Bigg(\int_{[L(\varphi^0)\neq 0]\cap q} sgn(L(\varphi^0))L(\theta^0)dxdt + \int_{[L(\varphi^0)=0]\cap q} |L(\theta^0)|dxdt \Bigg). \tag{2.32}$$

Using (2.31) and (2.32), we get

$$
(2.33)\qquad
\begin{aligned}
&\xi \in \partial j_1(\varphi^0) \Leftrightarrow \forall \theta^0 \in L^{p'}(\Omega),\\
&(\xi,\theta^0) \leq |L(\varphi^0)|_{L^1(q)} \left(\int_{[L(\varphi^0)\neq 0]\cap q} sgn(L(\varphi^0))L(\theta^0)dxdt + \int_{[L(\varphi^0)=0]\cap q} |L(\theta^0)|dxdt \right).
\end{aligned}
$$

The mapping $\theta^0 \rightarrow (\xi,\theta^0)$ is then a linear form on $R(L) = L(L^{p'}(\Omega))$ and applying Hahn-Banach Theorem, there exists a linear form V on $L^1(q)$ such that

$$
(2.34)\qquad \forall \theta^0 \in L^{p'}(\Omega), \quad (\xi,\theta^0) = V(\theta)
$$

and

$$
(2.35)\qquad
\begin{aligned}
&\forall \Theta \in L^1(q),\\
&V(\Theta) \leq |L(\varphi^0)|_{L^1(q)} \left(\int_{[L(\varphi^0)\neq 0]\cap q} sgn(L(\varphi^0))\Theta dxdt + \int_{[L(\varphi^0)=0]\cap q} |\Theta|dxdt \right).
\end{aligned}
$$

From (2.35), we deduce that V is continuous on $L^1(q)$ hence $V \in L^\infty(q)$ and that

$$
(2.36)\qquad
\begin{aligned}
&\forall \Theta \in L^1(q),\\
&\left| \int_q V(x,t)\Theta(x,t)dxdt - |L(\varphi^0)|_{L^1(q)} \int_{[L(\varphi^0)\neq 0]\cap q} sgn_0(L(\varphi^0))\Theta dxdt \right| \leq\\
&\qquad\qquad |L(\varphi^0)|_{L^1(q)} \left(\int_{[L(\varphi^0)=0]\cap q} |\Theta|dxdt \right).
\end{aligned}
$$

Take first $\Theta \in L^1(q)$ whose support is contained in $[L(\varphi^0) \neq 0]\cap q$ to obtain that $V = |L(\varphi^0)|_{L^1(q)} \frac{L(\varphi^0)}{|L(\varphi^0)|}$ almost everywhere on $[L(\varphi^0) \neq 0]\cap q$ and then $\Theta \in L^1([L(\varphi^0) = 0]\cap q)$ to obtain that $|V(x,t)| \leq |L(\varphi^0)|_{L^1(q)}$ almost everywhere on $[L(\varphi^0) = 0]\cap q$. This proves that $V = |L(\varphi^0)|_{L^1(q)} v$ with $v \in sgn(L(\varphi^0))\chi_q$.

For the reciprocal, if a function $V \in |L(\varphi^0)|_{L^1(q)} sgn(L(\varphi^0))\chi_q$ then (θ is the solution of (2.8) with $\theta(T) = \theta^0$)

$$
\theta^0 \rightarrow \int_q V(x,t)\theta(x,t)dxdt
$$

is a linear continuous form on $L^{p'}(\Omega)$. Thus there exists an unique $\xi \in L^p(\Omega)$ such that

$$
(\xi,\theta^0) = \int_q V(x,t)\theta(x,t)dxdt.
$$

One can easily prove that ξ satisfies the right hand side of (2.33) and hence $\xi \in \partial J(\varphi^0)$. This ends the proof of (2.29). □

3 Internal controllability of the semilinear heat equation: case $1 < p < \infty$.

3. 1 The linear heat equation with a potential

In this section, we prove the approximate controllability of the linear heat equation with a potential with "quasi bang-bang" controls. This is a direct consequence of the results of section 2.

We take $q = \omega \times]0, T[$ where ω is an open and nonempty subset of Ω and

$$L(\varphi^0, a) = \varphi \chi_q,$$

where φ is the solution of (2.8) with $\varphi(T) = \varphi^0$.
Let us prove that L satisfies the hypotheses (H_1), (H_2), (H_3) and (H_4) :
(H_1) : It is a simple corollary of (2.3).

(H_2) : It can be written here as follows :

$$(3.1) \qquad \begin{cases} -\varphi' - \Delta\varphi + a\varphi = 0 \quad in \quad Q \\ \varphi = 0 \quad in \quad \omega \times (0,T) \\ \varphi = 0 \quad on \quad \Sigma \\ \varphi \in L^{p'}(Q) \end{cases} \qquad \Rightarrow \varphi = 0 \quad in \quad \Omega \times (0,T).$$

This uniqueness property has been proved by Mizohata (see [15]) and Saut and Scheurer (see [19]).

(H_3) : We write $\varphi = \phi + \psi$ where ϕ is solution of (1.6) with $\phi(T) = \varphi^0$ and ψ is solution of (2.5) with $k = -a\phi$. It is then a consequence of (2.3) and (2.7).

(H_4) : We do the same decomposition as above. The function ϕ is fixed and using (2.7), the functions ψ_n are uniformly bounded in $X^{p'}(0,T)$ with respect to the potential a_n.

Applying the results and the notations of Section 2, we obtain

Proposition 3.1
If $|y^1|_p > \alpha$, denoting by $\hat{\varphi}$ the solution of (2.8) with $\hat{\varphi}(T) = \hat{\varphi}^0$, there exists $v \in sgn(\hat{\varphi})\chi_q$ such that the solution y of

$$(3.2) \qquad \begin{cases} y' - \Delta y + a(x,t)y = |\hat{\varphi}|_{L^1(q)} v \chi_\omega \\ y = 0 \quad on \quad \Sigma \\ y(0) = 0 \quad in \quad \Omega, \end{cases}$$

satisfies $y(T) = y^1 - \alpha \frac{|\hat{\varphi}^0|^{p'-2}\hat{\varphi}^0}{|\hat{\varphi}^0|_{p'}^{p'-1}}$ and hence $|y(T) - y^1|_p = \alpha$.

Remark 3.1
If $|y^1|_p \leq \alpha$, then we can take $y = 0$ to obtain $|y(T) - y^1|_p \leq \alpha$ with the nul control. □

Proof of Proposition 3.1

As $|y^1|_p > \alpha$, $\hat{\varphi}^0$ realizes the minimum of $J(\varphi^0; a, y^1)$ and $J(.; a, y^1)$ is subdifferentiable at $\hat{\varphi}^0 \neq 0$, then $0 \in \partial J(\hat{\varphi}^0; a, y^1)$. From (2.29), there exists $v \in |\hat{\varphi}|_{L^1(q)} sgn(\hat{\varphi})\chi_q$ such that

$$\forall \theta^0 \in L^{p'}(\Omega), \tag{3.3}$$
$$|\hat{\varphi}|_{L^1(q)} \int_q v(x,t)\theta(x,t)dxdt + \left(\alpha \frac{|\hat{\varphi}^0|^{p'-2}\hat{\varphi}^0}{|\hat{\varphi}^0|_{p'}^{p'-1}} - y^1, \theta^0\right) = 0,$$

where θ is the solution of (2.8) with $\theta(T) = \theta^0$. Now, if we multiply equation (3.2) by θ, we obtain

$$(y(T), \theta^0) = |\hat{\varphi}|_{L^1(q)} \int_q v(x,t)\theta(x,t)dxdt. \tag{3.4}$$

Equations (3.3) and (3.4) imply

$$\forall \theta^0 \in L^{p'}(\Omega), \quad (y(T), \theta^0) = \left(y^1 - \alpha \frac{|\hat{\varphi}^0|^{p'-2}\hat{\varphi}^0}{|\hat{\varphi}^0|_{p'}^{p'-1}}, \theta^0\right) \tag{3.5}$$

hence $y(T) = y^1 - \alpha \frac{|\hat{\varphi}^0|^{p'-2}\hat{\varphi}^0}{|\hat{\varphi}^0|_{p'}^{p'-1}}$ which ends the proof of Proposition 3.1. □

We now consider the nonlinear case.

3. 2 The nonlinear case

In order to prove the approximate controllability of system (1.2), we first suppose that f is in $C^1(\mathbb{R})$ and we use a fixed point argument applying Kakutani 's Theorem (see [2], p. 126). The general case of a globally Lipschitz function f will be obtained later on by a density argument. Our hypotheses on f imply that the function g defined by

$$g(s) = \frac{f(s) - f(0)}{s}, \text{ if } s \neq 0, \text{ and } g(0) = f'(0) \tag{3.6}$$

is continuous and, in view of (1.1), belongs to $L^\infty(\mathbb{R})$.

For $z \in L^p(Q)$, we write y as $y = u + Y$ where u is solution of

$$\begin{cases} u' - \Delta u + g(z)u = -f(0) & in \quad Q, \\ u = 0 \quad on \quad \Sigma \\ u(0) = y^0 \quad in \quad \Omega. \end{cases} \tag{3.7}$$

Since g is in $L^\infty(\mathbb{R})$ and due to the smoothing effects of the heat equation, the set

$$\{y^1 - u(T), \quad \text{when} \quad z \in L^p(Q)\}$$

is a compact subset of $L^p(\Omega)$.

From Propositions 2.1, 2.2 and 2.3, the functional

$$\begin{aligned} L^{p'}(\Omega) &\rightarrow \mathbb{R} \\ \varphi^0 &\rightarrow J(\varphi^0; g(z), y^1 - u(T)) \end{aligned} \tag{3.8}$$

possesses a unique minimum $\varphi^0(z, y^0, y^1)$ and there exists $v(z, y^0, y^1) \in sgn(\varphi(z, y^0, y^1))\chi_q$ such that the solution Y of

$$\begin{cases} Y' - \Delta Y + g(z)Y = |\varphi(z, y^0, y^1)|_{L^1(q)} v(z, y^0, y^1)\chi_\omega & in \quad Q \\ Y = 0 \quad on \quad \Sigma \\ Y(0) = 0 \quad in \quad \Omega, \end{cases} \tag{3.9}$$

satisfies $|Y(T) - y^1 + u(T)|_p \leq \alpha$. We then deduce that $y = u + Y$ is solution of

$$\begin{cases} y' - \Delta y + g(z)y = -f(0) + |\varphi(z, y^0, y^1)|_{L^1(q)} v(z, y^0, y^1)\chi_\omega & in \quad Q \\ y = 0 \quad on \quad \Sigma \\ y(0) = y^0 \quad in \quad \Omega, \\ |y(T) - y^1|_p \leq \alpha. \end{cases} \tag{3.10}$$

For $v \in sgn(\varphi(z, y^0, y^1))\chi_q$, we denote by $y(v)$ the solution of

$$\begin{cases} y' - \Delta y + g(z)y = -f(0) + |\varphi(z, y^0, y^1)|_{L^1(q)} v\chi_\omega & in \quad Q \\ y = 0 \quad on \quad \Sigma \\ y(0) = y^0 \quad in \quad \Omega. \end{cases} \tag{3.11}$$

and we consider the following mapping with set values : $\Lambda : L^p(Q) \rightarrow \mathcal{P}(L^p(Q))$ with

$$\Lambda(z) = \left\{ y(v), \quad v \in sgn(\varphi(z, y^0, y^1))\chi_q \quad \text{and} \quad |y(T) - y^1|_p \leq \alpha. \right\}.$$

From (3.10), $\Lambda(z)$ is always a nonempty subset of $L^p(Q)$ and we have the

Proposition 3.2

If f is C^1 and satisfies (1.1), then
(i)There exists a compact subset X of $L^p(Q)$, such that for every $z \in L^p(Q)$, $\Lambda(z) \subset X$,
(ii) For all $z \in L^p(Q)$, $\Lambda(z)$ is a nonempty convex compact subset of $L^p(Q)$,
(iii) Λ is upper hemicontinuous on $L^p(Q)$.

Proof of Proposition 3.2

(i) : Since $g \in L^\infty(Q)$ and from Propositions 2.1 and 2.2 (i), the solutions of (2.8) with initial data $\varphi^0(z, y^0, y^1)$ are bounded in $L^{p'}(Q)$. From (3.3), one can easily deduce that

$$\{|\varphi(z, y^0, y^1)|_{L^1(q)} v, v \in sgn(\varphi(z, y^0, y^1)), z \in L^p(Q)\}$$

is bounded in $L^\infty(q)$ hence there exists a bounded set X in $L^p(Q)$ such that for every $z \in L^p(Q)$, $\Lambda(z) \subset X$. Let us now prove that we can choose X being compact. For this, it is sufficient to prove that the set $\mathcal{Y} = \{y(v), v \in sgn(\varphi(z, y^0, y^1)), z \in L^p(Q)\}$ is relatively compact in $L^p(Q)$.

If $y = y(v) \in \mathcal{Y}$, there exist $z \in L^p(Q)$ and $v \in sgn(\varphi(z, y^0, y^1))\chi_q$ such that we can write $y = u_1 + u_2 + Y(v)$ where u_1, u_2 and $Y = Y(v)$ are defined by the following equations :

$$
(3.12) \qquad \begin{cases} u_1' - \Delta u_1 = -f(0) \quad in \quad Q \\ u_1 = 0 \qquad on \quad \Sigma \\ u_1(0) = y^0 \quad in \quad \Omega \end{cases}
$$

$$
(3.13) \qquad \begin{cases} u_2' - \Delta u_2 + g(z)(u_1 + u_2) = 0 \quad in \quad Q \\ u_2 = 0 \qquad on \quad \Sigma \\ u_2(0) = 0 \quad in \quad \Omega. \end{cases}
$$

$$
(3.14) \qquad \begin{cases} Y' - \Delta Y + g(z)Y = |\varphi(z, y^0, y^1)|_{L^1(q)} v \chi_\omega \quad in \quad Q \\ Y = 0 \qquad on \quad \Sigma \\ Y = 0 \quad in \quad \Omega. \end{cases}
$$

The function u_1 is fixed in $L^p(Q)$. When z takes its values in $L^p(Q)$, $g(z)u_1$ describes a bounded set of $L^p(Q)$. It follows from (2.7) that u_2 is in a bounded set of $X^p(0,T)$. As $X^p(0,T)$ is compactly imbedded in $L^p(Q)$, the function u_2 describes a compact set K_1 of $L^p(Q)$.

As the functions $\varphi(z, y^0, y^1)$ are bounded in $L^{p'}(Q)$, the functions $|\varphi(z, y^0, y^1)|_{L^1(q)} v$ are bounded in $L^\infty(Q)$ thus (using again (2.7)) Y(v) describes a bounded set of $X^p(0,T)$ hence a compact set K_2 of $L^p(Q)$. This proves that $\mathcal{Y} \subset u_1 + K_1 + K_2$ is relatively compact in $L^p(Q)$. It suffices to choose for X the closure of $\mathcal{Y}$ in $L^p(Q)$ to obtain (i).

(ii) : We have already seen that for all $z \in L^p(Q)$, $\Lambda(z)$ is a nonempty subset of $L^p(Q)$. Since the sets $sgn(\varphi(z, y^0, y^1))$ and $B(y^1, \alpha)$ are convex, it is easy to prove that $\Lambda(z)$ is convex so we are just going to prove that it is compact. As we already have $\Lambda(z) \subset X$ with X compact, we just have to prove that it is closed. Let $(y_n)_n$ be a sequence of elements of $\Lambda(z)$ which converges in $L^p(Q)$ to an element $y \in X$. Let us prove that $y \in \Lambda(z)$. There exist functions $v_n \in sgn(\varphi(z, y^0, y^1))$ such that

$$
(3.15) \qquad \begin{cases} y_n' - \Delta y_n + g(z)y_n = |\varphi(z, y^0, y^1)|_{L^1(q)} v_n \chi_\omega \quad in \quad Q \\ y_n = 0 \quad on \quad \Sigma \\ y_n(0) = y^0 \quad in \quad \Omega \\ |y_n(T) - y^1|_p \leq \alpha. \end{cases}
$$

Since $|v_n|_\infty \leq 1$, the functions v_n converge for the weak-$*$ topology (and after extraction of a subsequence) to an element $v \in L^\infty(q)$. Furthermore, as v_n are in the same set $sgn(\varphi(z, y^0, y^1))$, we have $|v|_\infty \leq 1$ and $v = sgn_0(\varphi(z, y^0, y^1))$ on $[\varphi(z, y^0, y^1) \neq 0] \cap q$. This proves that $v \in sgn(\varphi(z, y^0, y^1))$. Now, using (2.3) and passing to the limit in (3.15), we obtain

$$\begin{cases} y' - \Delta y + g(z)y = |\varphi(z, y^0, y^1)|_{L^1(q)} v\chi_\omega & in \quad Q \\ y = 0 \quad on \quad \Sigma \\ y(0) = y^0 \quad in \quad \Omega. \end{cases} \tag{3.16}$$

Due to the smoothing effects, $y_n(T)$ converges in $L^p(\Omega)$ to $y(T)$ and therefore $|y(T) - y^1|_p \leq \alpha$. This proves that $y \in \Lambda(z)$ and ends the proof of (ii).

(iii) : We first recall that Λ is hemicontinuous at $z_0 \in L^p(Q)$ if

$$\forall k \in L^{p'}(Q), \quad z \to \sigma(\Lambda(z), k) = \sup_{y \in \Lambda(z)} \int_Q k(x,t)y(x,t)dxdt \tag{3.17}$$

is upper semicontinuous at z_0.

We then have to show that

$$\forall z_0 \in L^p(Q), \qquad \limsup_{z_n \to z_0} \sigma(\Lambda(z_n), k) \leq \sigma(\Lambda(z_0), k). \tag{3.18}$$

We denote by $\langle u, v\rangle$ the integral $\int_Q u(x,t)v(x,t)dxdt$. From (ii), $\Lambda(z)$ is compact thus for every $n \in \mathbb{N}$, there exist $y_n \in \Lambda(z_n)$ such that

$$\sigma(\Lambda(z_n), k) = \langle y_n, k\rangle.$$

¿From (i), $(y_n)_n \subset X$ which is compact in $L^p(Q)$ and hence there exists $y \in L^p(Q)$ such that (after extraction of a subsequence) $y_n \to y$ in $L^p(Q)$. Let us prove that $y \in \Lambda(z_0)$.

We denote by $\varphi_n = \varphi(z_n, y^0, y^1)$. There exist $v_n \in sgn(\varphi_n)$ such that y_n satisfies

$$\begin{cases} y_n' - \Delta y_n + g(z_n)y_n = |\varphi_n|_{L^1(q)} v_n\chi_\omega & in \quad Q \\ y_n = 0 \quad on \quad \Sigma \\ y_n(0) = y^0 \quad in \quad \Omega \\ |y_n(T) - y^1|_p \leq \alpha. \end{cases} \tag{3.19}$$

We will need the following

Lemma 3.1

We have

$$\varphi^0(z_n, y^0, y^1) \to \varphi^0(z_0, y^0, y^1) \textit{ strongly in } L^{p'}(\Omega) \textit{ when } z_n \to z_0 \textit{ strongly in } L^p(Q) \tag{3.20}$$

Proof of Lemma 3.1

We denote by u_n (resp. u) the solution of (3.7) with the potential $g(z_n)$ (resp. $g(z_0)$.) The functions $g(z_n)$ are bounded in $L^\infty(Q)$ hence they converge for the weak-$*$ topology (and after extraction of a subsequence) to an element $Z \in L^\infty(Q)$. But, since g is continuous and $z_n \to z_0$ almost everywhere, we have $Z = g(z_0)$. We deduce that (after extraction of a subsequence) u_n weakly converges to u in $L^p(Q)$. Due to the smoothing effects of the heat equation $u_n(T) \to u(T)$ strongly in $L^p(\Omega)$, hence (3.20) is proved if we apply Proposition 3.2 (ii). ▫

End of the proof of Proposition 3.2

From Lemma 3.1 and (2.7), φ_n converges strongly in $L^{p'}(Q)$ to $\varphi(z_0, a, y^1)$ hence

$$|\varphi_n|_{L^1(q)} \to |\varphi(z_0; y^0, y^1)|_{L^1(q)}$$

and $\varphi_n \to \varphi(z_0; y^0, y^1)$ almost everywhere in Q. One can easily prove that $v_n \in sgn(\varphi_n)$ implies that the functions v_n converge (after extraction of a subsequence) for the weak-$*$ topology of $L^\infty(q)$ and almost everywhere to a function $v \in sgn(\varphi(z_0; y^0, y^1))$. We then deduce that y is the solution of

$$\begin{cases} y' - \Delta y + g(z_0)y = |\varphi(z_0; y^0, y^1)|_{L^1(q)} v\chi_\omega & in \quad Q \\ y = 0 \quad on \quad \Sigma \\ y(0) = y^0 \quad in \quad \Omega \end{cases} \tag{3.21}$$

Moreover, due to the smoothing effects of the heat equation, we keep the condition $y(T) \in B(y^1, \alpha)$ thus $y \in \Lambda(z_0)$.

Now, we have $\langle k, y_n \rangle \to \langle k, y \rangle$ with $y \in \Lambda(z_0)$, which proves that Λ is upper semicontinuous in z_0 and ends the proof of Proposition 3.2. ▫

We now prove the

Proposition 3.3

If f is C^1 and satisfies (1.1), there exist $y \in L^p(Q)$ such that $y \in \Lambda(y)$. Furthermore, y is solution of (1.2).

Proof of Proposition 3.3

The restriction of the mapping Λ to the convex hull conv(X) of X (which is compact in $L^p(Q)$) satisfies the hypotheses of Kakutani 's Theorem. We deduce that Λ has a fixed point y. We then have the existence of $\varphi^0 \in L^{p'}(\Omega)$ and $v \in sgn(\varphi)\chi_q$ such that

$$\begin{cases} -\varphi' - \Delta\varphi + g(y)\varphi = 0 \quad in \quad Q \\ \varphi = 0 \quad on \quad \Sigma \\ \varphi(T) = \varphi^0 \quad in \quad \Omega, \\ y' - \Delta y + f(y) = |\varphi|_{L^1(q)} v\chi_\omega \quad in \quad Q \\ y = 0 \quad on \quad \Sigma \\ y(0) = y^0 \quad in \quad \Omega, \\ |y(T) - y^1|_p \leq \alpha, \end{cases} \tag{3.22}$$

which proves that y is solution to the controllability problem stated in Theorem 1.1. ▫

Remark 3.2

If we suppose that f is globally Lipschitz and that $f'(s_0)$ exists for some $s_0 \in \mathbb{R}$, all what we have done before is valid for the following new function g_{s_0} instead of g : $g_{s_0}(s) = \frac{f(s)-f(s_0)}{s-s_0}$ if $s \neq s_0$ and $g_{s_0}(s_0) = f'(s_0)$. ▫

In order to complete the proof of Theorem 1.1, we have to get the general case of a globally Lipschitz function f. This is gathered in the following

Proposition 3.4

Let f be a globally Lipschitz function. There exist $A > 0$ and a sequence $(f_n)_n$ in $C^1(\mathbb{R})$ such that

$$\begin{aligned} &\forall n \in \mathbb{N}, \quad \forall s \in \mathbb{R}, \quad \left|\frac{f_n(s) - f_n(0)}{s}\right| \leq A, \\ &\lim_{n\to+\infty} f_n = f \quad \text{in} \quad C_c(\mathbb{R}). \end{aligned} \tag{3.23}$$

For each $n \in \mathbb{N}$, if we denote by φ_n, $v_n \in sgn(\varphi_n)$ and y_n the solutions of (3.22) associated to f_n, there exists $G \in L^\infty(Q)$ such that : φ_n^0 converges strongly in $L^{p'}(\Omega)$ to the minimum φ^0 of $J(.;G, y^1 - u(T))$ and (φ_n, y_n) converges strongly in $L^{p'}(Q) \times L^p(Q)$ to the solution of

$$\begin{cases} -\varphi' - \Delta\varphi + G\varphi = 0 \quad in \quad Q \\ \varphi = 0 \quad on \quad \Sigma \\ \varphi(T) = \varphi^0 \quad in \quad \Omega, \\ y' - \Delta y + f(y) = |\varphi|_{L^1(q)} v\chi_\omega \quad in \quad Q \\ y = 0 \quad on \quad \Sigma \\ y(0) = y^0 \quad in \quad \Omega, \\ |y(T) - y^1|_p \leq \alpha, \end{cases} \tag{3.24}$$

where $v \in sgn(\varphi)\chi_q$. Furthermore, $G(x,t) = g(y(x,t))$ on the set $[y(x,t) \neq 0]$.

Proof of Proposition 3.4

For a regularizing sequence ρ_n ($\rho_n \in C^\infty(\mathbb{R})$, $\rho_n \geq 0$, $\text{supp}(\rho_n) \subset [-1/n, 1/n]$ and $\int_{-1/n}^{1/n} \rho_n(s)ds = 1$), we take $f_n(s) = \rho_n * f(s) = \int_{-1/n}^{1/n} f(s-\tau)\rho_n(\tau)d\tau$. It is well-known that $f_n \in C^1(\mathbb{R})$ and that $f_n \to f$ uniformly in compact intervals. On the other hand, we have (denoting $g_n(s) = \frac{f_n(s)-f_n(0)}{s}$)

$$g_n(s) = \int_{-1/n}^{1/n} \frac{f(s-\tau) - f(-\tau)}{s}\rho_n(\tau)d\tau.$$

There exist $\beta > 0, k > 0$ such that

$$\forall (s_1, s_2) \in [-\beta, \beta], \quad |f(s_1) - f(s_2)| \leq k|s_1 - s_2|,$$

hence

$$|s| \leq \frac{\beta}{2} \quad \Rightarrow \quad |g_n(s)| \leq k.$$

Now, for $|s| \geq \frac{\beta}{2}$, using (1.1), we get

$$|g_n(s)| \leq \frac{2}{\beta}(a\beta + 2a + 2b)$$

which proves (3.23).
In order to prove (3.24), we write again $y_n = u_n + Y_n$ where

$$\begin{cases} u_n' - \Delta u_n + g_n(y_n)u_n = -f_n(0) & in \quad Q \\ u_n = 0 \quad on \quad \Sigma \\ u_n(0) = y^0 \quad in \quad \Omega, \end{cases} \tag{3.25}$$

and

$$\begin{cases} Y_n' - \Delta Y_n + g_n(Y_n + u_n)Y_n = |\varphi_n|_{L^1(q)} v_n \chi_\omega & in \quad Q \\ Y_n = 0 \quad on \quad \Sigma \\ Y_n(0) = 0 \quad in \quad \Omega. \end{cases} \tag{3.26}$$

Since the norms $|g_n|_\infty$ are uniformly bounded, there exists $G \in L^\infty(Q)$ such that $g_n(y_n) \rightarrow G$ in $L^\infty(Q)$ weak-$*$. As $f_n(0) \rightarrow f(0)$, if u_0 denotes the solution of

$$\begin{cases} u_0' - \Delta u_0 = 0 \quad in \quad Q \\ u_0 = 0 \quad on \quad \Sigma \\ u_0(0) = y^0 \quad in \quad \Omega, \end{cases} \tag{3.27}$$

one can easily see (using (2.7)) that (after extraction of a subsequence) $u_n - u_0$ converges in $X^p(0,T)$ to the solution U of

$$\begin{cases} U' - \Delta U + GU = -Gu_0 - f(0) \quad in \quad Q \\ U = 0 \quad on \quad \Sigma \\ U(0) = 0 \quad in \quad \Omega, \end{cases} \tag{3.28}$$

hence u_n strongly converges in $L^p(Q)$ to the solution u of

$$\begin{cases} u' - \Delta u + Gu = -f(0) \quad in \quad Q \\ u = 0 \quad on \quad \Sigma \\ u(0) = y^0 \quad in \quad \Omega. \end{cases} \tag{3.29}$$

Due to the smoothing effects of the heat equation, $u_n(T)$ converges strongly in $L^p(\Omega)$ to $u(T)$. From this, (3.23) and Proposition 2.2, we deduce that the functions φ_n^0 strongly converge in $L^{p'}(\Omega)$ to the minimum $\varphi^0 \in L^{p'}(\Omega)$ of the functional $J(.; G, y^1 - u(T))$ and φ_n strongly converges in $L^{p'}(Q)$ to the solution φ of

$$(3.30) \qquad \begin{cases} -\varphi' - \Delta\varphi + G\varphi = 0 & in \quad Q \\ \varphi = 0 & on \quad \Sigma \\ \varphi(T) = \varphi^0 & in \quad \Omega. \end{cases}$$

This proves that there exists $v \in sgn(\varphi)\chi_q$ such that the functions $|\varphi_n|_{L^1(q)} v_n \chi_\omega$ converges in $L^p(q)$ to $|\varphi|_{L^1(q)} v\chi_\omega$. From (2.7), Y_n weakly converges in $X^p(0,T)$ to the solution Y of

$$(3.31) \qquad \begin{cases} Y' - \Delta Y + GY = |\varphi|_{L^1(q)} v\chi_\omega & in \quad Q \\ Y = 0 & on \quad \Sigma \\ Y(0) = 0 & in \quad \Omega. \end{cases}$$

We saw that u_n and Y_n converge strongly in $L^p(Q)$ hence $y_n = u_n + Y_n$ also converges strongly in $L^p(Q)$. This proves (using Tonelli's Theorem, (3.23) and Lebesgue's Theorem) that $f_n(y_n) \to f(y)$ almost everywhere and in $L^p(Q)$. Having this strong convergence, one can easily prove that $g_n(y_n) \to \frac{f(y)-f(0)}{y}$ almost everywhere on the set $[y(x,t) \neq 0]$ hence $G = g(y)$ on this set.
We can pass to the limit directly in the equation satisfied by y_n and we obtain that there exist $G \in L^\infty(Q), G = g(y)$ on $[y(x,t) \neq 0]$ such that (y, φ) is solution of

$$(3.32) \qquad \begin{cases} -\varphi' - \Delta\varphi + G\varphi = 0 & in \quad Q \\ \varphi = 0 & on \quad \Sigma \\ \varphi(T) = \varphi^0 & in \quad \Omega, \\ y' - \Delta y + f(y) = |\varphi|_{L^1(q)} v\chi_\omega & in \quad Q \\ y = 0 & on \quad \Sigma \\ y(0) = y^0 & in \quad \Omega, \\ |y(T) - y^1|_p \leq \alpha, \end{cases}$$

and Proposition 3.3 is proved. □

Remark 3.3

One can prove that if $f'(s_0)$ exists for some $s_0 \in \mathbb{R}$, then if we take the function g_{s_0} defined in Remark 3.2, we have $G = g_{s_0}(y)$. □

Remark 3.4

In order to prove Theorem 1.1, one can use a method which avoids the use of the subdifferential and multi-valued mappings. Indeed, if we consider for $r > 1$

$$J_r(\varphi^0; a, y^1) = \frac{1}{2}\left(\int_q |\varphi(x,t)|^r dxdt\right)^2 + \alpha|\varphi^0|_{p'} - \int_\Omega y^1\varphi^0 dx,$$

the functional J_r is strictly convex continuous and coercive hence it possesses a unique minimum $\hat{\varphi}_r^0$. Furthermore, J_r is differentiable at $\hat{\varphi}_r^0 \neq 0$. We then obtain a control in the linear problems of the form

$$h(x,t) = \left(\int_q |\hat{\varphi}_r(x,t)|^r dxdt\right) r|\hat{\varphi}_r(x,t)|^{r-2}\hat{\varphi}_r(x,t) \in L^{\frac{p'}{r-1}}(Q).$$

We can then pass to the limit when $r \to 1$ to obtain (3.2) or else we can apply a fixed point argument (which will be here the Schauder's Theorem) with fixed r and then pass to the limit in the nonlinear problems when $r \to 1$ to obtain (3.22). □

Remark 3.5

The method used here is flexible and can be adapted to other situations : it suffices essentially to have a unique continuation property as (1.3) to obtain the coercivity of the functional J. For example, Theorem 1.1 is valid for parabolic equations with variable coefficients if the continuation property is fulfilled. □

4 Internal controllability for the semilinear heat equation: Cases $L^1(\Omega)$ and $C_0(\Omega)$

4. 1 The $L^1(\Omega)$-case

The results obtained in sections 2 and 3.1 concerning approximated controllability in $L^p(\Omega)$, $1 < p < \infty$, of the *linear* heat equation with a potential can be adapted without major changes to the $L^1(\Omega)$-case.

We consider the solution φ of (2.8) with final data $\varphi^0 \in L^\infty(\Omega)$. Then $\varphi \in L^\infty(Q)$ and $\varphi \in C([0,T]; L^p(\Omega))$ for every $1 \leq p < \infty$. Given $y^1 \in L^1(\Omega)$ and $\alpha > 0$ we define

$$J(\varphi^0; a, y^1) = \frac{1}{2}\left(\int_q |\varphi| dxdt\right)^2 + \alpha|\varphi^0|_\infty - \int_\Omega y^1\varphi^0 dx. \tag{4.1}$$

As a a consequence of the unique-continuation property (2.8) we deduce that J is strictly convex. The analogous of Propositions 2.1 and 2.2 for $p = 1$ and $p' = \infty$ may be proved making use of course of the weak-$*$ topology in $L^\infty(\Omega)$. However, in part (ii) of Proposition 2.2 we can only prove the convergence of $\hat\varphi_n^0$ to $\hat\varphi^0$ in the weak-$*$ topology of $L^\infty(\Omega)$. We obtain then:

Proposition 4.1

If $|y^1|_1 > \infty$, denoting by $\hat\varphi$ the solution of (2.8) with $\hat\varphi(T) = \hat\varphi^0$ where $\hat\varphi^0$ is the minimum in $L^\infty(\Omega)$ of $J(\cdot; a, y^1)$, there exists $v \in sgn(\hat\varphi)\chi_q$ such that the solution y of

$$\begin{cases} y' - \Delta y + ay = |\hat\varphi|_{L^1(q)} v\chi_q & in \quad Q \\ y = 0 \quad on \quad \Sigma \\ y(0) = 0 \quad in \quad \Omega \end{cases} \tag{4.2}$$

satisfies $|y(T) - y^1|_1 \leq \alpha$.

Proof of Proposition 4.1

Notice that the $L^\infty(\Omega)$-norm is not differentiable but, as $\hat{\varphi}^0$ is the minimum of $J(\cdot;a,y^1)$, there exists $v \in |\hat{\varphi}|_{L^1(q)} sgn(\hat{\varphi})\chi_q$ such that, for every $\theta^0 \in L^\infty(\Omega)$

$$|\hat{\varphi}|_{L^1(q)} \int_q v\theta dxdt + \alpha[|\hat{\varphi}^0 + \theta^0|_\infty - |\hat{\varphi}^0|_\infty] - \int_\Omega y^1\theta^0 dx \geq 0 \tag{4.3}$$

where θ is the solution of (2.8) with $\theta(T) = \theta^0$.

Multiplying (4.2) by θ we obtain

$$\int_\Omega y(T)\theta^0 dx = |\hat{\varphi}|_{L^1(g)} \int_q v\theta dxdt$$

so that

$$\forall \theta^0 \in L^\infty(\Omega), \quad \int_\Omega (y^1 - y(T))\theta^0 dx \leq \alpha|\theta^0|_\infty. \tag{4.4}$$

As we know that $y(T) - y^1 \in L^1(\Omega)$, this proves that

$$|y^1 - y(T)|_1 \leq \alpha. \ \square$$

Concerning the nonlinear case, we follow the same procedure as in section 3.2. The main difference lies in the compactness property, due to the fact that the initial data for y belongs to $L^1(\Omega)$ (analogous of (2.3) and (2.7) are no longer valid). Nevertheless we can prove the following result:

Lemma 4.1

Let $a = a(x,t)$ be an element of a bounded set in $L^\infty(Q)$ and h be bounded in $L^\infty(0,T;L^1(\Omega))$. Then, if $y^0 \in L^1(\Omega)$ the solution y of

$$\begin{cases} y' - \Delta y + ay = h & in \quad Q \\ y = 0 \quad on \quad \Sigma \\ y(0) = y^0 \quad in \quad \Omega \end{cases} \tag{4.5}$$

stays in a compact set of $C([0,T];L^1(\Omega))$, uniformly in a and h.

Proof of Lemma 4.2

It is standard to show that y is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$. Thus, it is sufficient to consider the case where $a \equiv 0$ and $y^0 \equiv 0$. By using the smoothing effect of the heat equation it is easy to see that y is uniformly bounded in $L^\infty(\delta,T;L^p(\Omega))$ for some $p > 1$ and for every $\delta > 0$. The results of section 2 show that y remains in a compact set of $C([\delta,T];L^1(\Omega))$ for every $\delta > 0$. Since y is continuous at $t = 0$ with values in $L^1(\Omega)$, uniformly on h, we deduce that y remains in a compact set of $C([0,T];L^1(\Omega))$. $\square$

Lemma 4.2 enables us to proceed as in section 3.2 and to prove Theorem 1.1 for the case $p = 1$.

Remark 4.1

In this case we can also use the method described in Remark 3.4 that avoids the use of the subdifferential and multivalued mappings. $\square$

4. 2 The $C_0(\Omega)$-case

We begin with the linear case for which, again, the method developed in sections 2 and 3.1 can be easily adapted.

Let us introduce the dual space of $C_0(\Omega)$, $M(\Omega)$, which is the space of bounded measures on Ω. The norm in $M(\Omega)$ will be denoted by $|\cdot|$ and it is defined as follows

$$|\mu| = \sup_{\varphi \in C_0(\Omega)} |\langle \mu, \varphi \rangle| / |\varphi|_\infty$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $M(\Omega)$ and $C_0(\Omega)$.

For every $\varphi^0 \in M(\Omega)$, (2.8) has a unique solution $\varphi \in L^\infty(0,T;L^1(\Omega)) \cap C([0,T];L^1(\Omega))$ that takes the final data φ^0 in the following weak sense:

$$\forall \theta \in C_0(\Omega), \quad \int_\Omega \varphi(x,t)\theta(x)dx \to \langle \varphi^0, \theta \rangle, \quad as \quad t \to T.$$

For $y^1 \in C_0(\Omega)$ and $\alpha > 0$ fixed we define

$$J(\varphi^0; a, y^1) = \frac{1}{2}\left(\int_q |\varphi| dx dt\right)^2 + \alpha|\varphi^0| - \quad \langle \varphi^0, y^1 \rangle. \tag{4.6}$$

In view of the unique-continuation property, J is strictly convex. We can prove its coercivity without changes using the fact that the unit ball of $M(\Omega)$ is relatively compact with respect to the weak-$*$ topology of $M(\Omega)$. Thus J achieves its minimum on $M(\Omega)$ at a unique $\hat{\varphi}^0 \in M(\Omega)$ and we obtain the analogous of Proposition 4.1.

Proposition 4.2

If $|y^1|_\infty > \alpha$ and $\hat{\varphi}$ is the solution of (2.8) with $\hat{\varphi}(T) = \hat{\varphi}^0$, there exists $v \in sgn(\hat{\varphi})\chi_\omega$ such that the solution y of

$$\begin{cases} y' - \Delta y + ay = |\hat{\varphi}|_{L^1(q)} v \chi_\omega & in \quad \Theta \\ y = 0 \quad on \quad \Sigma \\ y(0) = 0 \quad in \quad \Omega \end{cases} \tag{4.7}$$

satisfies $|y(T) - y^1|_\infty \le \alpha$.

For the nonlinear case, the proof of the uniform coercivity of J with respect to a in a bounded set of $L^\infty(Q)$ and y^1 in a compact set of $C_0(\Omega)$ being excepted, the application of the fixed point argument is straightforward.

Thus, let us conclude the proof of Theorem 1.1 giving a sketch of the proof of the uniform coercivity of J. Arguing, as usual, by contradiction, the key point consists in proving that if

$$\begin{aligned} a_n \to a \quad & in \quad L^\infty(Q) \quad \text{weak}* \\ \varphi_n^0 \to \varphi^0 \quad & in \quad M(\Omega) \quad \text{weak}* \end{aligned}$$

then, φ_n (solution of (2.8) corresponding to the potential a_n and the final data φ_n^0) converges to φ (solution of (2.8) with the potential a and final data φ^0) in $L^\infty(0,T;M(\Omega))$ weak$*$. The smoothing effect of the heat equation allows us to prove easily that φ_n converges to φ strongly in $C([0, T-\delta]; L^1(\Omega))$ for every $\delta > 0$. This allows us to pass to the limit in the equation. On the other hand, this implies that $\varphi \in L^\infty(0,T; L^1(\Omega)) \cap C([0,T); L^1(\Omega))$ even if, a priori, φ_n only converges to φ in $L^\infty(0,T; M(\Omega))$ weak$*$.

Let us finally show that φ takes the data φ^0 at time $t = T$. We have

$$\forall \theta \in C^2(\overline{\Omega}) \cap C_0(\Omega) : \int_\Omega \varphi_n(t)\theta dx - \langle \varphi_n^0, \theta\rangle - \int_t^T \int_\Omega \varphi_n \Delta\theta dx dt + \int_t^T \int_\Omega a_n \varphi_n \theta dx dt = 0$$

so that, when $n \to \infty$, we obtain

$$|\int_\Omega \varphi(t)\theta dx - \langle \varphi^0, \theta\rangle| \leq c(T-t), \quad \forall \theta \in C^2(\overline{\Omega}) \cap C_0(\Omega)$$

where c is a constant that only depends on the uniform $L^\infty(0,T; L^1(\Omega))$-bound of φ_n and the $L^\infty(Q)$-bound of a_n. As $\varphi \in L^\infty(0,T; L^1(\Omega))$, this implies that

$$\int_\Omega \varphi(t)\theta dx \to \int \langle \varphi^0, \theta\rangle \quad as \quad t \to T^-, \quad \forall \theta \in C_0(\Omega).$$

5 Boundary approximate controllability

Before proving Theorem 1.2, we begin by a subsection where we prove regularity results concerning the non homogeneous linear heat equation. For example, it is not obvious that the solution of (1.8) exists but moreover satisfies $y(T) \in L^p(\Omega)$. Furthermore, if we take a boundary term in $L^2(\Sigma)$ in (1.8) (with $f = 0$!) and $y^0 \in L^2(\Omega)$ then we don't have $y(T) \in L^2(\Omega)$ and we refer to [9], p. 217 for a counter example.

In order to chose $L(\varphi^0, a) = \frac{\partial\varphi}{\partial\nu}(\sigma, t)\chi_{\Sigma_0}$, we will then have to prove regularity results for the linear homogeneous heat equation with a potential. This is done in a second subsection where we also prove that, with this choice of L, hypotheses $(H_1), (H_2), (H_3), (H_4)$ are fulfilled.

We then end with the approximate boundary controllability problem of (1.8) : we first study the case of the linear equation with a potential and then we will apply again Kakutani 's Theorem to treat the nonlinear case.

In all this section $1 < p < \infty$.

5. 1 Regularity for the nonhomogeneous heat equation

We denote by $H^{\frac{1}{2},\frac{1}{4}}(Q)$ the set $L^2(0,T; H^{\frac{1}{2}}(\Omega)) \cap H^{\frac{1}{4}}(0,T; L^2(\Omega))$. We first recall that J. L. Lions and E. Magenes proved in [10], vol. 2, p. 86 that for $g \in L^2(\Sigma)$, there exists a unique solution $Y \in H^{\frac{1}{2},\frac{1}{4}}(Q)$ of

(5.1)
$$\begin{cases} Y' - \Delta Y = 0 & in \quad Q \\ Y = g & on \quad \Sigma \\ Y(0) = 0 & in \quad \Omega. \end{cases}$$

Such a solution does not satisfy $y(T) \in L^2(\Omega)$ hence one can not hope to obtain a result of approximate controllability in $L^2(\Omega)$ with controls in $L^2(\Sigma)$.

This explains why we need the following

Proposition 5.1

If $g \in L^\infty(\Sigma)$ and $y^0 \in L^p(\Omega)$, there exists a unique solution $y \in L^p(Q)$ of

(5.2)
$$\begin{cases} y' - \Delta y + ay = 0 & in \quad Q \\ y = g & on \quad \Sigma \\ y(0) = y^0 & in \quad \Omega. \end{cases}$$

Furthermore,
(i) If $p = 2$, $y \in H^{\frac{1}{2},\frac{1}{4}}(Q)$ and there exists a constant $c > 0$ such that

(5.3)
$$||y||_{H^{\frac{1}{2},\frac{1}{4}}(Q)} \leq c(||g||_{L^\infty(\Sigma)} + |y^0|_2).$$

(ii) $\forall t > 0$, $y(t) \in L^p(\Omega)$ and there exists a constant $d > 0$ such that

(5.4)
$$\forall t > 0, \quad |y(t)|_p \leq d(||g||_{L^\infty(\Sigma)} + |y^0|_p).$$

Proof of Proposition 5.1

We write $y = Y + Z$ where Y is solution of (5.1) and

(5.5)
$$\begin{cases} Z' - \Delta Z + aZ = -aY & in \quad Q \\ Z = 0 & on \quad \Sigma \\ Z(0) = y^0 & in \quad \Omega. \end{cases}$$

From J. L. Lions and E. Magenes ' result just mentioned above, we have $Y \in H^{\frac{1}{2},\frac{1}{4}}(Q)$. Hence $aY \in L^2(Q)$ which proves that $Z \in L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$. One, now, can easily deduce (i).

Let us now prove (ii) for $p = 2$. We already have $Z(t) \in L^2(\Omega)$ with

(5.6)
$$|Z(t)|_2 \leq c_1(|y^0|_2 + |Y|_{L^2(Q)}).$$

Let us prove that $Y(t) \in L^\infty(\Omega)$. By the maximum principle, we have $Y \in L^\infty(Q)$ and $||Y||_{L^\infty(Q)} \leq ||g||_{L^\infty(\Sigma)}$. On the other hand, it is well-known that $Y \in C([0,T]; H^{-1}(\Omega))$. Now, we take regular functions $g_n \in L^\infty(\Sigma)$ converging to g in $L^\infty(\Sigma)$ weak-$*$. If Y_n is the solution of (5.1) associated to g_n, one can easily prove that for every $t > 0$, $Y_n(t)$ is bounded in $H^{-1}(\Omega) \cap L^\infty(\Omega)$. This proves that for every $t > 0$, $Y(t) \in H^{-1}(\Omega) \cap L^\infty(\Omega)$. As $y = Y + Z$, (ii) is proved. Let us now prove (ii) for $p \neq 2$. The solution y of (5.2) can be written as $y = Y + Z$ where Y satisfies (5.2) with $g = 0$ and Z satisfies (5.2) with $Z(0) = 0$. We know that $Y \in C([0,T]; L^p(\Omega))$ and

$$||Y||_{L^\infty(0,T;L^p(\Omega))} \leq c|y^0|_p.$$

On the other hand, from (i) and (ii) above we have

$$||Z||_{L^\infty(Q)} \leq ||g||_{L^\infty(\Sigma)}$$

and $Z \in C([0,T]; H^{-1}(\Omega))$. This implies $Z(t) \in L^p(\Omega)$ for every t in $[0,T]$ and concludes the proof of Proposition 4.1. □

We have given a sense to the non homogeneous linear heat equation.

5. 2 Regularity results for the homogeneous equation and choice of L

In order to have controls in $L^\infty(\Sigma)$ we will need regularity results concerning the solutions of (2.8). We begin with some results on the L^2-setting.

Lemma 5.1
If $\phi^0 \in L^2(\Omega)$, then the solution ϕ of (1.6) satisfies $(T-t)^{\frac{1}{4}}\dfrac{\partial \phi}{\partial \nu} \in L^2(\Sigma)$ and there exists $C > 0$ such that

$$||(T-t)^{\frac{1}{4}}\frac{\partial \phi}{\partial \nu}||_{L^2(\Sigma)} \leq C|\phi^0|_2. \tag{5.7}$$

Proof of lemma 5.1
We use a multiplier method. Let ρ be in $(W^{1,\infty}(\Omega))^N$ with $\rho = \nu$ on $\partial\Omega$. If we multiply the equation satisfied by ϕ by $(T-t)^{\frac{1}{2}}\rho.\nabla\phi$ and we integrate over Q, we obtain

$$\begin{aligned}
-\frac{1}{2}\int_\Sigma (T-t)^{\frac{1}{2}}|\frac{\partial\phi}{\partial\nu}(\sigma,t)|^2 d\sigma dt &= \frac{1}{2}\sum_{i,k}\int_Q (T-t)^{\frac{1}{2}}\frac{\partial\phi}{\partial x_i}(x,t)\frac{\partial\rho_k}{\partial x_i}(x,t)\frac{\partial\phi}{\partial x_k}(x,t)dxdt \\
&- \frac{1}{2}\int_Q (T-t)^{\frac{1}{2}}|\nabla\phi(x,t)|^2 div(\rho(x))dxdt - \int_Q (T-t)^{\frac{1}{2}}\phi'(x,t)\rho(x).\nabla\phi(x,t)dxdt.
\end{aligned} \tag{5.8}$$

On the other hand, if we multiply the equation by $(T-t)\phi'$, we get

$$-\int_Q (T-t)\phi'^2(x,t)dxdt - \frac{T}{2}\int_\Omega |\nabla\phi(x,0)|^2 dx + \frac{1}{2}\int_Q |\nabla\phi(x,t)|^2 dxdt = 0,$$

hence there exists $C > 0$ such that

$$\int_Q (T-t)\phi'^2(x,t)dxdt \leq \frac{1}{2}\int_Q |\nabla\phi(x,t)|^2 dxdt \leq C|\phi^0|_2.$$

Using this upper bound in (5.8), we obtain Lemma 5.1. □

Remark 5.1

Lemma 4.1 proves that $\frac{\partial \phi}{\partial \nu} \in L^1(\Sigma)$. □

We deduce from Lemma 5.1 the following

Proposition 5.2

Let $0 < s < \frac{1}{3}$. If $\phi^0 \in H^{-s}(\Omega)$, the solution ϕ of (1.6) with $\phi(T) = \phi^0$ satisfies $\frac{\partial \phi}{\partial \nu} \in L^1(\Sigma)$. Furthermore the mapping

$$\phi^0 \in H^{-s}(\Omega) \to \frac{\partial \phi}{\partial \nu} \in L^1(\Sigma)$$

is linear continuous.

Proof of Proposition 5.2

Let us first consider $\phi^0 \in H^{-1}(\Omega)$ and denote by ϕ the solution of (1.6) with $\phi(T) = \phi^0$. Then, using that $\phi \in L^2(Q)$, one can easily prove that $(T-t)\phi \in L^2(0,T;H^2(\Omega))$ and that there exists a constant $D > 0$ such that

$$\forall \phi^0 \in H^{-1}(\Omega), \quad ||(T-t)\frac{\partial \phi}{\partial \nu}||_{L^2(\Sigma)} \leq D||\phi^0||_{H^{-1}(\Omega)}. \tag{5.9}$$

Consider the mapping L defined by $L(\phi^0) = \frac{\partial \phi}{\partial \nu}$. We have proved in Lemma 5.1 and just above that $L \in \mathcal{L}(L^2(\Omega;dx), L^2(\Sigma;\sqrt{T-t}dtd\sigma))$ and $L \in \mathcal{L}(H^{-1}(\Omega), L^2(\Sigma;(T-t)^2dtd\sigma))$. By interpolation, we deduce that

$$\forall s \in [0,1], \quad L \in \mathcal{L}([L^2(\Omega), H^{-1}(\Omega)]_s, [L^2(\Sigma;\sqrt{T-t}dtd\sigma), L^2(\Sigma;(T-t)^2dtd\sigma)]_s).$$

For $0 < s < \frac{1}{2}$, we have (see [10], vol. 1) $H^{-s}(\Omega) = [L^2(\Omega), H^{-1}(\Omega)]_s$ and one can prove that $[L^2(\Sigma;\sqrt{T-t}dtd\sigma), L^2(\Sigma;(T-t)^2dtd\sigma)]_s = L^2(\Sigma;(T-t)^{\frac{1+3s}{2}}dtd\sigma)$. Now, for $0 < s < \frac{1}{3}$, $L^2(\Sigma;(T-t)^{\frac{1+3s}{2}}dtd\sigma) \subset L^1(\Sigma)$. This proves Proposition 5.2. □

Now, for the homogeneous heat equation with a potential, we have

Proposition 5.3

The linear operator

$$B : \varphi^0 \in L^2(\Omega) \to \frac{\partial \varphi}{\partial \nu} \in L^1(\Sigma), \tag{5.10}$$

where φ is the solution of (2.8) with $\varphi(T) = \varphi^0$, is continuous and compact. Furthermore

$$||B||_{\mathcal{L}(L^2(\Omega),L^1(\Sigma))} = O(1 + |a|^2_\infty exp(C|a|_\infty)).$$

Proof of Proposition 5.3

We write $\varphi = \psi + \phi$ where ϕ is solution of (1.6) with $\phi(T) = \varphi^0$ and ψ is solution of

$$\begin{cases} -\psi' - \Delta\psi + a\psi = a\phi & in \quad Q \\ \psi = 0 \quad on \quad \Sigma \\ \psi(T) = 0 \quad in \quad \Omega. \end{cases} \tag{5.11}$$

It is well-known (see [8]) that $\psi \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ hence the mapping

$$\varphi^0 \in L^2(\Omega) \to \frac{\partial \psi}{\partial \nu} \in L^1(\Sigma)$$

is compact. From Proposition 5.2, the operator

$$\varphi^0 \in L^2(\Omega) \to \frac{\partial \phi}{\partial \nu} \in L^1(\Sigma),$$

is continuous and compact. As the estimate on the norm of B can be easily deduced from (2.7) and [8], this ends the proof of Proposition 5.2. □

Having Proposition 5.3, we choose $q = \Sigma_0$ and

$$L(\varphi^0, a) = \frac{\partial \varphi}{\partial \nu}(\sigma, t)\chi_{\Sigma_0},$$

where $\varphi^0 \in L^2(\Omega), a \in L^\infty(Q)$ and φ is the solution of (2.8) with $\varphi(T) = \varphi^0$.

Let us prove that L satisfies $(H_1), (H_2), (H_3)$ and (H_4).
(H_1) : it is a consequence of Proposition 5.3.
(H_2) : the following uniqueness property is a consequence of the results proved by Mizohata [15] and Saut and Scheurer [19] :

$$\begin{cases} -\varphi' - \Delta\varphi + a\varphi = 0 \quad in \quad Q \\ \dfrac{\partial \varphi}{\partial \nu} = 0 \quad on \quad \Sigma_0 \\ \varphi = 0 \quad on \quad \Sigma \\ \varphi \in L^2(Q) \end{cases} \qquad \Rightarrow \varphi = 0 \quad in \quad \Omega \times (0,T)$$

which is exactly (H_2).
Assertions (H_3) and (H_4) are simple corollaries of Proposition 4.3.

Using Proposition 2.1, 2.2 and 2.3 and the notations of Section 2, we obtain the approximate controllability of the linear heat equation perturbed with a potential :

Proposition 5.4

If $|y^1|_2 > \alpha$, denoting by $\hat\varphi$ the solution of (2.8) with $\hat\varphi(T) = \hat\varphi^0$, there exists $v \in sgn\left(\frac{\partial \hat\varphi}{\partial \nu}\right)\chi_{\Sigma_0}$ such that the solution y of

$$\begin{cases} y' - \Delta y + a(x,t)y = 0 \\ y = |\dfrac{\partial \hat\varphi}{\partial \nu}|_{L^1(\Sigma_0)} v\chi_{\Sigma_0} \quad on \quad \Sigma \\ y(0) = 0 \quad in \quad \Omega, \end{cases}$$

satisfies $y(T) = y^1 - \alpha\frac{\hat\varphi^0}{|\hat\varphi^0|_2}$ hence $|y(T) - y^1|_2 = \alpha$.

We will not give the proof which follows the ideas of Proposition 3.1.

Let us now give the L^p-version of these results.

Proposition 5.5

Let be $1 < p < \infty$ *and* $a = a(x,t) \in L^\infty(Q)$. *Then,*
(i) There exists $\beta = \beta(p,n) > 0$ *such that for every* $\varphi^0 \in L^p(\Omega)$ *the solution* $\varphi = \varphi(x,t)$ *of (2.8) satisfies* $(T-t)^\beta \frac{\partial\varphi}{\partial\nu} \in L^1(\Sigma)$.
(ii) The linear operator

$$B : \varphi^0 \in L^p(\Omega) \rightarrow (T-t)^\beta \frac{\partial\varphi}{\partial\nu} \in L^1(\Sigma)$$

is continuous and compact.

Proof of Proposition 5.5

(i) We decompose φ as in the proof of Prop. 5.3. We have $L^p(\Omega) \subset D((-\Delta)^{-k})$ for some $k \in N$ (depending on p and the dimension n) with compact embedding. Since $\varphi^0 \in D((-\Delta)^{-k})$ there exists some $\beta > 0$ such that $(T-t)^\beta \phi \in L^2(0,T;H^2(\Omega))$ and therefore $(T-t)^\beta \frac{\partial\phi}{\partial\nu} \in L^1(\Sigma)$.

Let us now consider the solution ψ of (5.11). Clearly

$$-\psi' - \Delta\psi = -a\varphi \in L^p(Q)$$

and therefore

$$\psi \in L^p(0,T;W^{2,p}(\Omega)) \cap W^{1,p}(0,T;L^p(\Omega)). \tag{5.12}$$

This implies that $\frac{\partial\psi}{\partial\nu} \in L^1(\Sigma)$.

We have proved, in particular, that $(T-t)^\beta \frac{\partial\varphi}{\partial\nu} \in L^1(\Sigma)$. The continuity of the map B is obvious from the proof. Its compactness is a consequence of the compactness of the embedding $L^p(\Omega) \subset D((-\Delta)^{-k})$ and (5.12). □

Proposition 5.5 allows us to apply the results of section 2 to deduce the approximate controllability in $L^p(\Omega)$ with "quasi bang-bang" controls. In this case we choose

$$L(\varphi^0, a) = (T-t)^\beta \frac{\partial\varphi}{\partial\nu}(x,t)\chi_{\Sigma_0}$$

where β is the exponent that corresponds to p' in Proposition 5.5. Minimizing the corresponding functional J in $L^{p'}(\Omega)$ and denoting $\hat{\varphi}^0 \in L^{p'}(\Omega)$ the minimizer we obtain the following result:

Proposition 5.6

Let us take $1 < p < \infty$, $\alpha > 0$ *and* $y^1 \in L^p(\Omega)$ *with* $|y^1|_p > \alpha$. *Let* $\hat{\varphi}$ *be the solution of (2.8) with* $\hat{\varphi}(T) = \hat{\varphi}^0$. *Then, there exists* $v \in sgn\left(\frac{\partial\hat{\varphi}}{\partial\nu}\right)\chi_{\Sigma_0}$ *such that the solution* y *of*

$$\begin{cases} y' - \Delta y + a(x,t)y = 0 & in\ Q \\ y = |(T-t)^\beta \frac{\partial\hat{\varphi}}{\partial\nu}|_{L^1(\Sigma_0)}\left(sgn\frac{\partial\hat{\varphi}}{\partial\nu}\right)\chi_{\Sigma_0} & on\ \Sigma \\ y(0) = 0 \end{cases} \tag{5.13}$$

satisfies $y(T) = y^1 - \alpha\frac{|\hat{\varphi}^0|^{p'-2}\hat{\varphi}^0}{|\hat{\varphi}^0|_{p'}^{p'-1}}$ *and hence* $|y(T) - y^1| = \alpha$.

If $y^1 \in L^1(\Omega)$ *and* $\alpha > 0$ *with* $|y^1|_1 > \alpha$, *there exists a solution* $\hat{\varphi}$ *of (2.8) with final data* $\hat{\varphi}^0 \in L^\infty(\Omega)$ *such that the solution* y *of (5.13) with* $\beta = 0$ *satisfies*

$$|y(T) - y^1|_1 \leq \alpha.$$

We have given the proof of Proposition 5.6 when $1 < p < \infty$. The $L^1(\Omega)$-case can be proved in a similar way taking into account the developments in section 4.1. This time we have to minimize

$$J(\varphi^0) = \frac{1}{2}\left(\int_{\Sigma_0} |\frac{\partial \varphi}{\partial \nu}| d\Sigma\right)^2 + \alpha ||\varphi^0||_\infty - \int_\Omega y^1 \varphi^0 dx$$

over $L^\infty(\Omega)$. Note that we do not get $|y^1 - y(T)|_1 = \alpha$.

5. 3 The nonlinear case : end of the proof of Theorem 1.2

We give a brief sketch of it in the case $p = 2$. When $1 \leq p < \infty$ the proof is very similar.

For $z \in L^2(Q)$, let u be the solution of (3.7). Since g is in $L^\infty(\mathbb{R})$ and due to the smoothing effects of the heat equation, the set $\{y^1 - u(T), \quad \text{when} \quad z \in L^2(Q)\}$ is a compact subset of $L^2(\Omega)$. From Proposition 5.4, the functional

$$\begin{aligned} &L^2(\Omega) \to \mathbb{R} \\ &\quad \varphi^0 \to J(\varphi^0; g(z), y^1 - u(T)) \end{aligned} \tag{5.14}$$

possesses a unique minimum $\varphi^0(z, y^0, y^1)$ and there exists

$$v(z, y^0, y^1) \in sgn\left(\frac{\partial \varphi(z, y^0, y^1)}{\partial \nu}\right) \chi_{\Sigma_0}$$

such that the solution Y of

$$\begin{cases} Y' - \Delta Y + g(z) Y = 0 \quad in \quad Q \\ Y = |\dfrac{\partial \varphi(z, y^0, y^1)}{\partial \nu}|_{L^1(\Sigma_0)} v(z, y^0, y^1) \chi_{\Sigma_0} \quad on \quad \Sigma \\ Y(0) = 0 \quad in \quad \Omega, \end{cases} \tag{5.15}$$

satisfies $|Y(T) - y^1 + u(T)|_2 \leq \alpha$. We then deduce that $y = u + Y$ is solution of

$$\begin{cases} y' - \Delta y + g(z) y = -f(0) \quad in \quad Q \\ y = |\dfrac{\partial \varphi(z, y^0, y^1)}{\partial \nu}|_{L^1(\Sigma_0)} v(z, y^0, y^1) \chi_{\Sigma_0} \quad on \quad \Sigma \\ y(0) = y^0 \quad in \quad \Omega, \\ |y(T) - y^1|_2 \leq \alpha. \end{cases} \tag{5.16}$$

For $v \in sgn\left(\frac{\partial \varphi(z, y^0, y^1)}{\partial \nu}\right) \chi_{\Sigma_0}$, we denote by $y(v)$ the solution of

$$\begin{cases} y' - \Delta y + g(z) y = -f(0) \quad in \quad Q \\ y = |\dfrac{\partial \varphi(z, y^0, y^1)}{\partial \nu}|_{L^1(\Sigma_0)} v \chi_{\Sigma_0} \quad on \quad \Sigma \\ y(0) = y^0 \quad in \quad \Omega. \end{cases} \tag{5.17}$$

and we consider the following set-valued mapping : $\Lambda : L^2(Q) \to \mathcal{P}(L^2(Q))$ with

$$\Lambda(z) = \left\{ y(v), \quad v \in sgn\left(\frac{\partial \varphi(z, y^0, y^1)}{\partial \nu}\right) \chi_{\Sigma_0} \quad \text{and} \quad |y(T) - y^1|_2 \leq \alpha. \right\}.$$

From (5.16), $\Lambda(z)$ is always a nonempty subset of $L^2(Q)$ and one can prove (following the proof of Proposition 3.2) the

Proposition 5.7

If f is C^1 and satisfies (1.1), then
(i)There exists a compact subset X of $L^2(Q)$, such that for every $z \in L^2(Q)$, $\Lambda(z) \subset X$,
(ii) For all $z \in L^2(Q)$, $\Lambda(z)$ is a nonempty convex compact subset of $L^2(Q)$,
(iii) Λ is upper hemicontinuous on $L^2(Q)$.

The restriction of the mapping Λ to the convex hull $\mathrm{conv}(X)$ of X (which is compact in $L^2(Q)$) satisfies the hypotheses of Kakutani 's Theorem. We deduce that Λ has a fixed-point y. We then have the existence of $\varphi^0 \in L^2(\Omega)$ and $v \in sgn\left(\frac{\partial\varphi}{\partial\nu}\right)\chi_{\Sigma_0}$ such that

$$(5.18) \qquad \begin{cases} -\varphi' - \Delta\varphi + g(y)\varphi = 0 \quad in \quad Q \\ \varphi = 0 \quad on \quad \Sigma \\ \varphi(T) = \varphi^0 \quad in \quad \Omega, \\ y' - \Delta y + f(y) = 0 \quad in \quad Q \\ y = |\dfrac{\partial\varphi}{\partial\nu}|_{L^1(\Sigma_0)} v\chi_{\Sigma_0} \quad on \quad \Sigma \\ y(0) = y^0 \quad in \quad \Omega, \\ |y(T) - y^1|_2 \leq \alpha, \end{cases}$$

which proves that y is solution the controllability problem stated in Theorem 1.2 in the case of a C^1 function f. If f is just locally Lipschitz and satisfies (1.1) then we obtain (as in Proposition 3.4) the existence of functions $G \in L^\infty(Q)$, $\varphi^0 \in L^2(\Omega)$ and $v \in sgn\left(\frac{\partial\varphi}{\partial\nu}\right)\chi_{\Sigma_0}$ such that

$$(5.19) \qquad \begin{cases} -\varphi' - \Delta\varphi + G\varphi = 0 \quad in \quad Q \\ \varphi = 0 \quad on \quad \Sigma \\ \varphi(T) = \varphi^0 \quad in \quad \Omega, \\ y' - \Delta y + f(y) = 0 \quad in \quad Q \\ y = |\dfrac{\partial\varphi}{\partial\nu}|_{L^1(\Sigma_0)} v\chi_{\Sigma_0} \quad on \quad \Sigma \\ y(0) = y^0 \quad in \quad \Omega, \\ |y(T) - y^1|_2 \leq \alpha, \end{cases}$$

with $G(x,t) = g(y(x,t))$ on the set $[y(x,t) \neq 0]$. This completes the proof of Theorem 1.2. □

Remark 5.2

The arguments of Remark 1.1 allow us to prove that the zero set of $\frac{\partial\varphi}{\partial\nu}$ over Σ_0 has empty interior on Σ_0. Whether it has zero boundary-measure seems to be an open problem.
□

References

[1] S. Angenent. *The zero set of a solution of a parabolic equation.* J. reine angew. Math. 390 (1988), p. 79-96.

[2] J.P. Aubin. *L'analyse non linéaire et ses motivations économiques.* Masson, 1984.

[3] T. Cazenave and A. Haraux. *Introduction aux problèmes d'évolution semi-linéaires.* Collection S.M.A.I Mathématiques et applications. Ellipses, 1990.

[4] J.I. Diaz. Sur la contrôlabilité approchée des inéquations variationnelles et d'autres problèmes paraboliques non linéaires. C. R. Acad. Sci. Paris, t 312, Série 1, p. 519-522, 1991.

[5] C. Fabre, J.P. Puel and E. Zuazua. Contrôlabilité approchée de l'équation de la chaleur semilinéaire. C. R. Acad. Sci. Paris, t. 315, Série 1, p. 807-812, 1992.

[6] A. Friedman. *Partial differential equations of parabolic type*, Prentice-Hall Inc., 1964.

[7] J. Henry. Etude de la contrôlabilité de certaines équations paraboliques. Thèse d'Etat de l'Université Paris VI, 1978.

[8] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Ural'ceva. *Linear and quasilinear equations of parabolic type.* A.M.S, Rhode Island, 1968.

[9] J.L. Lions. *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles.* Dunod, collection E.M, 1968.

[10] J.L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications.* Vol 1 and 2, Dunod, 1968.

[11] J.L. Lions. Exact controllability, stabilization and perturbations for distributed systems. SIAM, Boston 1986, Siam Review, 30, p. 1-68, 1988.

[12] J.L. Lions. *Contrôlabilité exacte, perturbations et stabilization des systèmes distribués.* Tome 1, *Contrôlabilité exacte.* Collection R.M.A 8, Masson, 1988.

[13] J.L. Lions. Remarks on approximate controllability. To appear in the volume dedicated to F. Browder.

[14] J.L. Lions. Remarques sur la contrôlabilité approchée. Proceedings of "Jornadas Hispano-Francesas sobre Control de Sistemas Distribuidos", University of Málaga, Spain, October 1990.

[15] S. Mizohata. Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques. Mem. Coll. Sci. Univ. Kyoto, Ser. A31 (3) (1958), p. 219-239.

[16] K. Naito. On controllability for a nonlinear Volterra equation. Nonlinear Analysis Theory, Methods and Appl., vol 18, n.1, p. 99-108, 1992.

[17] K. Naito and T.I. Seidman. Invariance of the approximately reachable set under nonlinear perturbations. SIAM J. Cont. Optim., vol 29 (3), p. 731-750, 1991.

[18] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Springer-Verlag, vol 44, 1983.

[19] J.C. Saut and B. Scheurer. Unique continuation for some evolution equations. J. Diff. Equations, 66 (1) (1987), p. 118-139.

[20] E.J.P.G. Schmidt and N. Weck. On the boundary behavior of solutions to elliptic and parabolic equations - Applications to boundary control for parabolic equations, Siam Jour. on Cont. and Opt., vol 16 (4), p. 593 - 598, 1978.

[21] E. Zuazua. Exact boundary controllability for the semilinear wave equation. *Nonlinear differential equations and their applications*, H. Brezis and J.L. Lions Eds., Séminaire du Collège de France 1987-1988, vol X, Research Notes in Mathematics, Pitman, (1991), p. 357-391.

* Université Paris XII-Val de Marne, U.F.R. Sciences, Laboratoire de Mathématiques, Av. du Général de Gaulle, 94010 Creteil Cedex and Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 PALAISEAU CEDEX, France.

** Université de Versailles Saint-Quentin, Département de Mathématiques and Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 PALAISEAU CEDEX, France.

*** Departamento de Matemática Aplicada, Universidad Complutense, 28040 Madrid, Spain.

Supported by project PB90-0245 of the "Dirección General de investigatión Científico y Técnica (MEC-*España*)" and by Eurhomogenization Project SC1*-CT91-0732 of the EEC.