SPREADING OF A PULSE TRAVELLING IN RANDOM MEDIA

J.F. Clouet and J.P. Fouque
Centre de Mathématiques Appliquées
Ecole Polytechnique
91128 Palaiseau Cedex
France

june 1993

Abstract

This paper investigates the deformation of an acoustic pulse travelling in a slab of random medium when its width is large compared to the size of the random inhomogeneities of the medium. A limit theorem is shown that explains how the shape of the transmitted pulse can be obtained as a result of a deterministic gaussian convolution of the initial pulse.

[1] [2]

1 INTRODUCTION

The main problem we are interested in is the following: how has been modified the shape of a pulse when it emerges from a slab of random medium. This analysis takes place in the general framework, based on the separation of scales, introduced by G.Papanicolaou and his co-authors (see for example R.Burridge, G.Papanicolaou, P.Sheng and B.White(1989)[2] for the one-dimensional case and M.Asch, W.Kohler, G.Papanicolaou, M.Postel and B.White(1991)[1] for the three dimensional case); we consider here the problem of acoustic wave propagation in a one-dimensional random medium when the incident pulse wavelength is long compared to the correlation length of the random inhomogeneities but short compared to the size of the slab.

[1]AMS; Primary: 73D70; Secondary: 60B10, 60H10
[2]Keywords: waves in random media, transmitted pulse, diffusion-approximation, stochastic equations

The O'Doherty-Anstey theory predicts that when such a pulse travels in such a medium it retains its shape up to a low spreading; futhermore, the pulse shape is deterministic when observed from the point of view af an observer travelling at the same random speed as the wave whereas it is stochastic when the observer's speed is the mean speed of the wave. This has been proven in [1] (see also J-F.Clouet(1992)[3] for more details) in the case where the random fluctuations are small. We do not make such an assumption but, as we are mainly concerned with the shape of the transmitted pulse, we suppose that the incident energy is small.
Our main result consists in a complete description of the asymptotic law of the emerging pulse: we prove a limit theorem which shows that the pulse spreads in a deterministic way.

2 PROBLEM FORMULATION

We recall the main features of our model. We shall only present the mathematical framework because it has been fully described in [2] . We consider an acoustic wave travelling in a one-dimensional random medium located in the region $0 \leq x \leq L$, satisfying the linear conservation laws:

$$\left\{ \begin{array}{ll} \rho(x)\frac{\partial u}{\partial t}(x,t) + \frac{\partial p}{\partial x}(x,t) & = 0 \\ \frac{1}{K(x)}\frac{\partial p}{\partial t}(x,t) + \frac{\partial u}{\partial x}(x,t) & = 0 \end{array} \right. \tag{1}$$

here $u(x,t)$ and $p(x,t)$ are respectively the speed and pressure of the wave, whereas $\rho(x)$ and $\frac{1}{K(x)}$ are the density and bulk modulus of the medium and admit the following representation:

$$\left\{ \begin{array}{ll} \rho(x) & = \rho_0(x)(1 + \eta(\frac{x}{\epsilon^2})) \\ \frac{1}{K(x)} & = \frac{1}{K_0(x)}(1 + \nu(\frac{x}{\epsilon^2})) \end{array} \right.$$

ρ_0 and K_0 represent the slow varying deterministic parameters of the medium. $\eta(\frac{x}{\epsilon^2})$ and $\nu(\frac{x}{\epsilon^2})$ are the rapidly varying random coefficients describing the inhomogeneities. We shall use the acoustic impedance ζ and the acoustic speed c which are the macroscopic functions defined by:

$$\zeta(x) = \sqrt{\rho_0(x)K_0(x)}$$
$$c(x) = \sqrt{\frac{K_0(x)}{\rho_0(x)}}$$

It is relevant to make a change of variable and to introduce a right-going wave A and a left going wave B in order to work with an hyperbolic system of equations and to precise our boundary conditions. So let:

$$\left\{ \begin{array}{ll} A & = \zeta^{-1/2}p + \zeta^{1/2}u \\ B & = -\zeta^{-1/2}p + \zeta^{1/2}u \end{array} \right. \tag{2}$$

With the following notations:

$$\Lambda = \frac{d \ln \zeta^{1/2}}{dx}$$
$$\left\{ \begin{array}{ll} m(\frac{x}{\epsilon^2}) & = \frac{\eta(\frac{x}{\epsilon^2})+\nu(\frac{x}{\epsilon^2})}{2} \\ n(\frac{x}{\epsilon^2}) & = \frac{\eta(\frac{x}{\epsilon^2})-\nu(\frac{x}{\epsilon^2})}{2} \end{array} \right.$$

from (1),(2), we obtain the system of equations for A and B:

$$\frac{d}{dx}\left[\begin{array}{c} A \\ B \end{array}\right] = \left(\Lambda \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] - \frac{1}{c(x)}\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right] - \frac{1}{c(x)}\left[\begin{array}{cc} m(\frac{x}{\epsilon^2}) & n(\frac{x}{\epsilon^2}) \\ -n(\frac{x}{\epsilon^2}) & -m(\frac{x}{\epsilon^2}) \end{array}\right]\right) \frac{\partial}{\partial t}\left[\begin{array}{c} A \\ B \end{array}\right] \quad (3)$$

The slab of medium we are considering is located in the region $0 \leq x \leq L$ and at $t = 0$ an incident pulse is generated at the interface $x = 0$ between the random medium and the homogenous medium in the outside. According to previous works [2],[1] we choose a pulse which is broad compared to the size of the random inhomogeneities but short compared to the macroscopic variations of the medium. There is no wave entering the medium at $x = L$.

$$\left\{ \begin{array}{ll} A(0,t) & = f(\frac{t}{\epsilon}) \\ B(L,t) & = 0 \end{array} \right. \quad (4)$$

where f is a function with compact support and C^∞ regularity. Note that the energy entering the medium is $\epsilon \int f(t)^2 dt$ and so is small when ϵ is small. We need to perform another change of variable adapted to our problem so that (3) becomes centered. We are interested in the transmitted wave that is $A(L, \tau(L))$ where $\tau(L)$ is the travel time in the macroscopic medium defined by:

$$\tau(x) = \int_0^x \frac{ds}{c(s)}$$

Moreover, in order to have a complete description of the transmitted pulse we study the process $A(L, \tau(L) + \epsilon\sigma)_{\sigma \in (-\infty,\infty)}$, i.e. we open a window of size ϵ in the neighborhood of the mean travel time. So let:

$$\left\{ \begin{array}{ll} a^\epsilon(x,\sigma) & = A(x, \tau(x) + \epsilon\sigma) \\ b^\epsilon(x,\sigma) & = B(x, -\tau(x) + \epsilon\sigma) \end{array} \right. \quad (5)$$

The solution of (3),(4) takes place in an infinite-dimensional space because of the variable t. So we perform the Fourier transform:

$$\left\{ \begin{array}{ll} \hat{a}^\epsilon(x,\omega) & = \int e^{\imath\omega\sigma} a^\epsilon(x,\sigma) d\sigma \\ \hat{b}^\epsilon(x,\omega) & = \int e^{\imath\omega\sigma} b^\epsilon(x,\sigma) d\sigma \end{array} \right.$$

In the frequency domain, with the change of variable (5), (3)(4) becomes:

$$\frac{d}{dx}\begin{bmatrix} \hat{a}^\epsilon \\ \hat{b}^\epsilon \end{bmatrix} = \frac{1}{\epsilon}P^\epsilon(x,\omega)\begin{bmatrix} \hat{a}^\epsilon \\ \hat{b}^\epsilon \end{bmatrix} + Q^\epsilon(x,\omega)\begin{bmatrix} \hat{a}^\epsilon \\ \hat{b}^\epsilon \end{bmatrix} \tag{6}$$

$$\begin{cases} \hat{a}^\epsilon(0,\omega) & = \hat{f}(\omega) \\ \hat{b}^\epsilon(L,\omega) & = 0 \end{cases} \tag{7}$$

where:

$$\begin{aligned} P^\epsilon(x,\omega) &= \frac{\imath\omega}{c(x)}\begin{bmatrix} m(\frac{x}{\epsilon^2}) & n(\frac{x}{\epsilon^2})e^{-2\imath\omega\frac{\tau(x)}{\epsilon}} \\ -n(\frac{x}{\epsilon^2})e^{2\imath\omega\frac{\tau(x)}{\epsilon}} & -m(\frac{x}{\epsilon^2}) \end{bmatrix} \\ Q^\epsilon(x,\omega) &= \Lambda\omega\begin{bmatrix} 0 & e^{-2\imath\omega\frac{\tau(x)}{\epsilon}} \\ e^{2\imath\omega\frac{\tau(x)}{\epsilon}} & 0 \end{bmatrix} \end{aligned} \tag{8}$$

The transmitted pulse admits the following integral representation:

$$A(L,\tau(L)+\epsilon\sigma) = a^\epsilon(L,\sigma) = \frac{1}{2\pi}\int e^{-\imath\omega\sigma}\hat{a}^\epsilon(L,\omega)d\omega$$

As the problem (6) is linear we can replace (7) by:

$$\begin{cases} \hat{a}^\epsilon(0,\omega) & = 1 \\ \hat{b}^\epsilon(L,\omega) & = 0 \end{cases} \tag{9}$$

and in that case the following representation holds for the transmitted pulse:

$$A(L,\tau(L)+\epsilon\sigma) = a^\epsilon(L,\sigma) = \frac{1}{2\pi}\int e^{-\imath\omega\sigma}\hat{f}(\omega)\hat{a}^\epsilon(L,\omega)d\omega \tag{10}$$

($\hat{a}^\epsilon$ is now the solution of (6)(9)).
We want to prove an asymptotic theorem for this quantity. Instead of working with $(\hat{a}^\epsilon(x,\omega),\hat{b}^\epsilon(x,\omega))$ we will use its propagator i.e. the matrix $Y^\epsilon(x,\omega)$ defined by:

$$\begin{bmatrix} \hat{a}^\epsilon(x,\omega) \\ \hat{b}^\epsilon(x,\omega) \end{bmatrix} = Y^\epsilon(x,\omega)\begin{bmatrix} \hat{a}^\epsilon(0,\omega) \\ \hat{b}^\epsilon(0,\omega) \end{bmatrix}$$

so $Y^\epsilon(x,\omega)$ is solution of the linear differential equation:

$$\begin{aligned} \frac{d}{dx}Y^\epsilon(x,\omega) &= \left(\frac{1}{\epsilon}P^\epsilon(x,\omega) + Q^\epsilon(x,\omega)\right)Y^\epsilon(x,\omega) \\ Y^\epsilon(0,\omega) &= Id_{\mathbb{C}^2} \end{aligned} \tag{11}$$

If (α,β) is a solution of (6)(9) then $(\overline{\beta},\overline{\alpha})$ is another solution linearly independent from the previous one so we can write $Y^\epsilon(x,\omega)$ as:

$$Y^\epsilon(x,\omega) = \begin{bmatrix} \alpha(x,\omega) & \overline{\beta}(x,\omega) \\ \beta(x,\omega) & \overline{\alpha}(x,\omega) \end{bmatrix} \tag{12}$$

Now, since:

$$\det Y^\epsilon(x,\omega) = \exp(\mathrm{tr} \int_0^x \left(\frac{1}{\epsilon}P^\epsilon(y,\omega) + Q^\epsilon(y,\omega)\right) dy) = 1$$

we have

$$|\alpha(x,\omega)|^2 - |\beta(x,\omega)|^2 = 1\,,\forall x$$

From:

$$\begin{bmatrix} \hat{a}^\epsilon(L,\omega) \\ 0 \end{bmatrix} = Y(L,\omega) \begin{bmatrix} 1 \\ \hat{b}^\epsilon(0,\omega) \end{bmatrix}$$

we deduce that:

$$\left\{ \begin{array}{ll} \hat{a}^\epsilon(L,\omega) & = \frac{1}{\overline{\alpha}(L,\omega)} \\ \hat{b}^\epsilon(0,\omega) & = -\frac{\beta(L,\omega)}{\overline{\alpha}(L,\omega)} \end{array} \right. \tag{13}$$

so we get the conservation of energy relation:

$$|\hat{a}^\epsilon(L,\omega)|^2 + |\hat{b}^\epsilon(0,\omega)|^2 = 1 \tag{14}$$

which shows that $\hat{a}^\epsilon(L,\omega)$ is uniformly bounded.

3 SHAPE OF THE TRANSMITTED PULSE

Let us define the correlation coefficients of the noise by:

$$\left\{ \begin{array}{l} \alpha_m = \int_0^\infty m(0)m(x)dx \\ \alpha_n = \int_0^\infty n(0)n(x)dx \end{array} \right. \tag{15}$$

Let $\theta_L = \int_0^L \frac{dy}{c^2(y)}$ and Z_L be a gaussian variable such that:

$$\mathbb{E}Z_L = 0 \quad \mathbb{E}Z_L^2 = 2\alpha_m\theta_L \tag{16}$$

Let $G_L(t)$ be the gaussian kernel given by: $G_L(t) = \frac{1}{\sqrt{2\alpha_n\theta_L}}\exp(-\frac{t^2}{4\alpha_n\theta_L})$. Our main result is the following:

Theorem 3.1
The process $(A(L,\tau(L)+\epsilon\sigma))_{-\infty<\sigma<\infty}$ converges in distribution as ϵ goes to zero to the process $(f * G_L(\sigma+Z_L))_{-\infty<\sigma<\infty}$.

This tells us that the pulse retains its shape during the travel in the randon medium but that there is a deterministic spreading due to the gaussian kernel and a random centering at Z_L which do not affect the shape. As usual, we fisrt show the tightness which is very easy because of the uniform boundness of $\hat{a}^\epsilon$ and the spectral representation (10) and then we identify the limiting distribution through all its moments by the use of a diffusion approximation theorem.

Lemma 3.2
The family of processes $((a^\epsilon(L,\sigma))_{-\infty<\sigma<\infty})_{\epsilon>0}$ *is tight in* $C[(-\infty,\infty);\mathbb{R}]$ *with the sup norm.*

Proof :
$|a^\epsilon(L,\sigma)|$ is bounded by $\frac{1}{2\pi}\int|\hat{f}(\omega)|d\omega$ and the modulus of continuity:

$$M^\epsilon(\delta) = \max_{\substack{|\sigma_1-\sigma_2|\leq\delta \\ -\infty<\sigma_1,\sigma_2<\infty}} |a^\epsilon(L,\sigma_1)-a^\epsilon(L,\sigma_2)|$$

is bounded by

$$M^\epsilon(\delta) \leq \frac{1}{2\pi}\int \sup_{|\sigma_1-\sigma_2|\leq\delta} |1-e^{\imath\omega(\sigma_1-\sigma_2)}|.|\hat{f}(\omega)|d\omega$$

By Lebesgue's theorem:

$$\lim_{\delta\to 0}\sup_{\epsilon} \mathbb{P}\{M(\delta)>\alpha\} = 0,\ \forall\alpha>0$$

which is enough for the tightness. □

We shall now characterize the limits of the finite-dimensional distributions of the process $(a^\epsilon(L,\sigma))_{-\infty<\sigma<\infty}$ by computing the moments $\mathbb{E}[a^\epsilon(L,\sigma_1)^{p_1}\ldots a^\epsilon(L,\sigma_k)^{p_k}]$ for every real numbers $\sigma_1<\ldots<\sigma_k$ and every integers $p_1,\ldots,p_k$. We have:

$$\mathbb{E}[a^\epsilon(L,\sigma_1)^{p_1}\ldots a^\epsilon(L,\sigma_k)^{p_k}] = \frac{1}{(2\pi)^{\sum_{j=1}^k p_j}}\int \exp\left(-\imath\sum_{\substack{1\leq j\leq k\\ 1\leq l\leq p_j}}\sigma_j\omega_j^l\right)\left(\prod_{\substack{1\leq j\leq k\\ 1\leq l\leq p_j}}\hat{f}(\omega_j^l)\right)\mathbb{E}\left[\prod_{\substack{1\leq j\leq k\\ 1\leq l\leq p_j}}\hat{a}^\epsilon(L,\omega_j^l)\right]d\omega_1^1\ldots d\omega_k^{p_k} \tag{17}$$

Hence the quantity of interest is $\mathbb{E}[\hat{a}^\epsilon(L,\omega_1)\ldots\hat{a}^\epsilon(L,\omega_n)]$ where $\omega_1,\ldots,\omega_n$ are n distinct frequencies. So we define a n-dimensional propagator

$$Y^\epsilon_x = Y^\epsilon(x,\omega_1,\omega_2,\ldots,\omega_n) = \begin{bmatrix} Y^\epsilon(x,\omega_1) & & \\ & \ddots & \\ & & Y^\epsilon(x,\omega_n) \end{bmatrix}$$

which satisfies:

$$\begin{cases} \dfrac{d}{dx}Y^\epsilon_x & = \frac{1}{\epsilon}P(x,\frac{x}{\epsilon},\frac{x}{\epsilon^2})Y^\epsilon_x + Q(x,\frac{x}{\epsilon},\frac{x}{\epsilon^2})Y^\epsilon_x \\ Y^\epsilon_0 & = Id_{\mathbb{C}^{2n}} \end{cases}$$

with:

$$\begin{aligned} P(x,\frac{x}{\epsilon},\frac{x}{\epsilon^2}) &= \begin{bmatrix} P^\epsilon(x,\omega_1) & & \\ & \ddots & \\ & & P^\epsilon(x,\omega_n) \end{bmatrix} \\ Q(x,\frac{x}{\epsilon},\frac{x}{\epsilon^2}) &= \begin{bmatrix} Q^\epsilon(x,\omega_1) & & \\ & \ddots & \\ & & Q^\epsilon(x,\omega_n) \end{bmatrix} \end{aligned}$$

where $P(x,\omega)$ and $Q(x,\omega)$ have been defined in (8). We shall use a diffusion-approximation theorem for $(Y^\epsilon_x)_{x\geq 0}$. We recall the general result from G.Papanicolaou,D.Stroock and S.Varadhan(1977)[5](Th 2.8); in our case the variable x will play the role of a time in the process $(Y^\epsilon_x)_{x\geq 0}$ but, as usual, we shall denote it by $(X_t)_{t\geq 0}$.

Theorem 3.3
Assume that:

- *i/ $(q_t)_{t\geq 0}$ is an ergodic markov process with state space S (S being a compact metric space for example).*
- *ii/ $P(t,h,q,x)$ and $Q(t,h,q,x)$ are two smooth bounded functions from $\mathbb{R}\mathrm{x}\mathbb{R}\mathrm{x}S\mathrm{x}\mathbb{R}^d$ to $\mathbb{R}^d$, periodic in h with period T_0 independent of t,q and x.*
- *iii/ $\mathbb{E}P(t,h,q_0,x) = 0$ where the expectation is taken with respect to the unique invariant probability measure of $(q_t)_{t\geq 0}$.*

Then the solution of the ordinary differential equation with stochastic coefficients:

$$\begin{aligned} \tfrac{dX^\epsilon_t}{dt} &= \tfrac{1}{\epsilon}P(t,\tfrac{t}{\epsilon},q_{\frac{t}{\epsilon^2}},X^\epsilon_t) + Q(t,\tfrac{t}{\epsilon},q_{\frac{t}{\epsilon^2}},X^\epsilon_t) \\ X^\epsilon_0 &= x_0 \end{aligned} \tag{18}$$

converges in distribution as ϵ goes to 0 to a diffusion process whose infinitesimal generator *is given by:*

$$\begin{array}{ll} \mathcal{L}_t F(x) & = \int_0^\infty \frac{1}{T_0}\int_0^{T_0} \mathbb{E}P(t,h,q_0,x).\nabla_x[P(t,h,q_u,x).\nabla_x F(x)]dhdu \\ & +\frac{1}{T_0}\int_0^{T_0} \mathbb{E}Q(t,h,q_0,x)\nabla_x F(x)dh \end{array} \tag{19}$$

(assuming that all the integrals are well-defined and finite)

The theorem is also true under general mixing conditions on the coefficients but we shall not enter into the details (see for example [3] for the computation of the infinitesimal generator with the averaging introduced by the periodic variable in that case); we only assume that the processes $m(x)$ and $n(x)$ are either ergodic markov processes or bounded processes with enough decorrelation so that the previous theorem applies and $(Y_x^\epsilon)_{x\in[0,L]}$ converges in distribution as ϵ goes to 0 to a diffusion process $(Y_x)_{x\in[0,L]}$.
We would like to characterize $(Y_x)_{x\in[0,L]}$ as the solution of a stochastic differential equation. This is not so easy because, Y_x^ϵ being a complex valued matrix, we have to consider separately the real and imaginary parts of each coefficient. But taking advantage of the linearity of the problem it is possible to simplify the computation.
The first step is to find a stochastic differential equation in the real case. So let us adopt the following notation for matrices $M(t,h,q)$ and $N(t,h,q)$ such that the product MN makes sense:

$$\begin{array}{ll} < M|N >_t & = \int_0^\infty \frac{1}{T_0}\int_0^{T_0} \mathbb{E}M(t,h,q_0)N(t,h,q_u)dhdu \\ < M >_t & = \frac{1}{T_0}\int_0^{T_0} \mathbb{E}M(t,h,q_0)dh \end{array}$$

We denote also by ^{T} the transposition for matrices. Then, we have:

Lemma 3.4
Let $(X_t^\epsilon)_{t\in[0,T]}$ be the $\mathbb{R}^d$-valued random process solution of:

$$\begin{array}{ll} \frac{dX_t^\epsilon}{dt} & = \frac{1}{\epsilon}P(t,\frac{t}{\epsilon},q_{\frac{t}{\epsilon^2}})X_t^\epsilon + Q(t,\frac{t}{\epsilon},q_{\frac{t}{\epsilon^2}})X_t^\epsilon \\ X_0^\epsilon & = x_0 \end{array} \tag{20}$$

where P and Q are two dxd real matrices satisfying the hypothesis i/,ii/ and iii/ of the previous theorem.
Then there exists matrices $P_t^k, 1\le k\le n, n\le d^2$ such that, for all vector x of $\mathbb{R}^d$:

$$< Px|(Px)^T >_t = \frac{1}{2}\sum_{k=1}^n P_t^k x(P_t^k x)^T \tag{21}$$

and (X_t^ϵ) converges in distribution to the diffusion process solution of the stochastic differential equation:

$$\left\{ \begin{array}{ll} dX_t & = \sum_k P_t^k X_t dB_t^k + (< P|P >_t + < Q >_t)X_t dt \\ X_0 & = x_0 \end{array} \right. \tag{22}$$

where $B^1,\ldots,B^n$ are n standard independent brownian motions.

Proof :
For $f(x) \in C^2(\mathbb{R}^d)$ and $F(x) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ we note:

$$\begin{array}{l} \nabla^2 f = (\frac{\partial^2 f}{\partial x_i \partial x_j})_{1\leq i,j\leq d} \\ \nabla f = (\frac{\partial f}{\partial x_i})_{1\leq i\leq d} \\ \nabla\{F\} = (\frac{\partial F_j}{\partial x_i})_{1\leq i,j\leq d} \end{array}$$

$\nabla\{Px\}_{i,j} = \dfrac{\partial \sum_k P_{j,k}x_k}{\partial x_i} = P_{j,i}$ so $\nabla\{Px\} = P^T$ and:

$$\begin{aligned} Px.\nabla[Px.\nabla f] &= PX.(\nabla\{Px\}\nabla f + [\nabla^2 f]Px) \\ &= Px.PT\nabla f + Px.[\nabla^2 f]Px \\ &= (P^2x)^T\nabla f + \text{tr } (Px(Px)^T\nabla^2 f) \end{aligned}$$

So in the linear case the generator (19) can be written:

$$\mathcal{L}_t F(x) = \text{tr } (< Px|(Px)^T >_t \nabla^2 F) + [(< P|P >_t x)^T + < Q >^T]\nabla F \tag{23}$$

there always exists matrices P_t^k such that (21) holds for all vector x in $\mathbb{R}^d$ (see J.P.Fouque and E.Merzbach(1992)[4]) and the solution of (22) which exists and is unique has (23) for infinitesimal generator. □
We turn now to the complex case:

Lemma 3.5
Suppose that $(Z_t^\epsilon)_{t\in[0,T]}$ is a $\mathbb{C}^d$-valued random process such that:

$$\left\{ \begin{array}{ll} \frac{dZ_t^\epsilon}{dt} &= \frac{1}{\epsilon}P(t, \frac{t}{\epsilon}, q_{\frac{t}{\epsilon^2}})Z_t^\epsilon + Q(t, \frac{t}{\epsilon}, q_{\frac{t}{\epsilon^2}})Z_t^\epsilon \\ Z_0^\epsilon &= z_0 \end{array} \right. \tag{24}$$

Where P and Q are two dxd complex matrices satisfying the hypothesis i/,ii/ and iii/ of the previous theorem.
We suppose that there exists complex matrices $P_t^k, 1 \leq k \leq n, n \leq d^2$ such that for all $\mathbb{C}^d$ vectors z one has:

$$\left\{ \begin{array}{ll} < Pz|(Pz)^T >_t &= \frac{1}{2}\sum_{k=1}^n P_t^k z(P_t^k z)^T \\ < Pz|(\overline{Pz})^T >_t &= \frac{1}{2}\sum_{k=1}^n P_t^k z(\overline{P_t^k z})^T \end{array} \right. \tag{25}$$

then the law of (Z_t^ϵ) converges to the law of the diffusion process solution of the following stochastic differential equation:

$$\left\{ \begin{array}{ll} dZ_t &= \sum_{k=1}^n P_t^k Z_t dB_t^k + (< P|P >_t + < Q >_t)Z_t dt \\ Z_0 &= z_0 \end{array} \right. \tag{26}$$

where $B^1, \ldots, B^n$ are n independent standard real brownian motions.

Proof :
Note first that matrices P_t^k do not always exist (consider for example $d = 1, P(t, \frac{t}{\epsilon}, q_{\frac{t}{\epsilon^2}}) = q_{\frac{t}{\epsilon^2}} e^{\imath \frac{t}{\epsilon}}, Q = 0$). Let $C \in \mathcal{M}_{d,2d}(\mathbb{C})$ and $D \in \mathcal{M}_{2d,d}(\mathbb{C})$ defined by:

$$C = \begin{bmatrix} 1 & \imath & 0 & \ldots \\ 0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ \ldots & 0 & 1 & \imath \end{bmatrix} \quad D = \frac{1}{2} \begin{bmatrix} 1 & 0 & \ldots & \ldots \\ -\imath & \ddots & \ldots\ldots & \\ \ldots & \ldots & \ddots & 1 \\ \ldots & \ldots & 0 & -\imath \end{bmatrix}$$

If $z = (z_1, \ldots, z_d)^T$ and $x = x(z) = (\Re z_1, \Im z_1, \ldots, \Re z_d, \Im z_d)^T$ then:

$$\begin{aligned} x &= Dz + \overline{Dz} \\ z &= Cx \\ CD &= Id_{\mathbb{C}^d} \\ C\overline{D} &= 0_{\mathbb{C}^d} \end{aligned}$$

We can apply the diffusion-approximation theorem to the real process $X_t^\epsilon = X(Z_t^\epsilon)$.

$$\frac{dX_t^\epsilon}{dt} = \frac{1}{\epsilon}(DPC + \overline{DPC})X_t^\epsilon + (DQC + \overline{DQC})X_t^\epsilon$$

By (23) and (25) the diffusion matrix is:

$$\begin{aligned} \mathcal{D} &= < (DPC + \overline{DPC})x|((DPC + \overline{DPC})x)^T >_t \\ &= D < Pz|(Pz)^T >_t D^T + D < Pz|(\overline{Pz})^T >_t \overline{D}^T \\ &+ \overline{D} < \overline{Pz}|(Pz)^T >_t D^T + \overline{D < Pz|(Pz)^T >_t D^T} \\ &= \frac{1}{2} \sum_{k=1}^{n} [(DP_t^k C + \overline{DP_t^k C})x][(DP_t^k C + \overline{DP_t^k C})x]^T \end{aligned}$$

And the drift vector is:

$$\begin{aligned} \mathcal{V} &= < DPC + \overline{DPC}|DPC + \overline{DPC} >_t + < DQC + \overline{DQC} >_t \\ &= D < P|P >_t C + \overline{D < P|P >_t C} + D < Q >_t C + \overline{D < Q >_t C} \end{aligned}$$

Thus (X_t^ϵ) converges in law to (X_t) solution of:

$$\begin{aligned} dX_t &= \sum_k (DP_t^k C + \overline{DP_t^k C})X_t dB_t^k \\ &+ (D < P|P >_t C + \overline{D < P|P >_t C} + D < Q >_t C + \overline{D < Q >_t C})X_t dt \end{aligned}$$

Multiplying on the left by C gives (26). $\square$

Thanks to this lemma, it is now easy to see that the limit in distribution of the process $(Y_x^\epsilon)_{x\in[0,L]}$ is solution of the following stochastic differential equation:

$$\begin{cases} dY_x = P_x Y_x^\epsilon dB_x + \sum_{k=1}^n (Q_x^k Y_x dB_x^k + \tilde{Q}_x^k Y_x d\tilde{B}_x^k) + R_x Y_x dx \\ Y_0 = Id \end{cases} \tag{27}$$

where $W, B^1, \ldots, B^n, \tilde{B}^1, \ldots, \tilde{B}^n$ are $2n+1$ standard real brownian motions independent one from another and the matrices $P, Q^1, \ldots, Q^n, \tilde{Q}^1, \ldots, \tilde{Q}^n, R$ are defined as follow:

- α_m and α_n are the correlation coefficient (15). They measure the randomness of the medium:
- the drift matrix is zero when only one between the density and the bulk modulus is random:

$$R = \frac{\alpha_n - \alpha_m}{c(x)^2} \begin{bmatrix} \omega_1^2 & & & & \\ & \omega_1^2 & & & \\ & & \ddots & & \\ & & & \omega_n^2 & \\ & & & & \omega_n^2 \end{bmatrix}$$

- Each matrix Q^k or $\tilde{Q}^k$ acts only on $Y(x, \omega_k)$

$$Q_x^k = \frac{\sqrt{\alpha_n}}{c(x)} \begin{bmatrix} 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & \ldots & 0 & \omega_k & 0 & 0 \\ 0 & 0 & \omega_k & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \end{bmatrix} \quad \tilde{Q}^k = \imath \frac{\sqrt{\alpha_n}}{c(x)} \begin{bmatrix} 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & \ldots & 0 & \omega_k & 0 & 0 \\ 0 & 0 & -\omega_k & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \end{bmatrix}$$

- The matrix P is the only one which creates a coupling between distinct frequencies.

$$P_x = \frac{\imath\sqrt{2\alpha_m}}{c(x)} \begin{bmatrix} \omega_1 & & & & \\ & -\omega_1 & & & \\ & & \ddots & & \\ & & & \omega_n & \\ & & & & -\omega_n \end{bmatrix}$$

From (13), we need to compute:

$$\hat{a}(L,\omega_k) = \frac{1}{\overline{\alpha}_k} = \frac{\Re\alpha_k + \imath\Im\alpha_k}{(\Re\alpha_k)^2 + (\Im\alpha_k)^2}$$

From the stochastic differential equation (27) we get:

$$\left\{\begin{array}{l} \left\{\begin{array}{ll} d(\Re\alpha_k)_x & = -\frac{\sqrt{2\alpha_m}}{c(x)}\omega_k\Im\alpha_k dW_x + \frac{\sqrt{\alpha_n}}{c(x)}\omega_k\Re\beta_k(dB_x^k - \Im\beta_k d\tilde{B}_x^k 0 + \frac{\alpha_n-\alpha_m}{c(x)^2}\omega_k^2 dx \\ (\Re\alpha_k)_0 & = 1 \end{array}\right. \\ \left\{\begin{array}{ll} d(\Im\alpha_k)_x & = \frac{\sqrt{2\alpha_m}}{c(x)}\omega_k\Re\alpha_k dW_x + \frac{\sqrt{\alpha_n}}{c(x)}\omega_k(\Im\beta_k dB_x^k - \Re\beta_k d\tilde{B}_x^k) + \frac{\alpha_n-\alpha_m}{c(x)^2}\omega_k^2 dx \\ (\Im\alpha_k)_0 & = 0 \end{array}\right. \end{array}\right.$$

Now a long but straightforward use of Itô's formula leads to the following equation for $\hat{a}$:

$$\left\{\begin{array}{ll} d\hat{a}(x,\omega_k) & = \frac{\imath\sqrt{2\alpha_m}\omega_k}{c(x)}\hat{a}(x,\omega_k)dB_x - \frac{\omega_k^2(\alpha_n+\alpha_m)}{c(x)^2}\hat{a}(x,\omega_k)dx + F(Y(x,\omega_k))(dB_x^k - \imath d\tilde{B}_x^k) \\ \hat{a}(0,\omega_k) & = 1 \end{array}\right. \tag{28}$$

where $F(Y(x,\omega_k))$ is only a function of α_k and β_k. Now we compute $\mathbb{E}[\hat{a}(L,\omega_1)\dots\hat{a}(L,\omega_n)]$ for distinct frequencies. Itô's formula gives:

$$\begin{aligned} d\prod_i \hat{a}(x,\omega_i) &= \sum_i \prod_{j\neq i}\hat{a}(x,\omega_j)d\hat{a}(x,\omega_i) + \sum_{i\neq j}\prod_{k\neq i,j}\hat{a}(x,\omega_k)d < \hat{a}(x,\omega_i)\hat{a}(x,\omega_j) > \\ &= \left(\prod_i \hat{a}(x,\omega_i)\right)\sum_i\left(\frac{\imath\sqrt{2\alpha_m}\omega_i}{c(x)}\hat{a}(x,\omega_k)dB_x - \frac{\omega_k^2(\alpha_n+\alpha_m)}{c(x)^2}\hat{a}(x,\omega_k)dx\right) \\ &+ \left(\prod_i \hat{a}(x,\omega_i)\right)\sum_i\left(F(Y(x,\omega_i))(dB_x^i - \imath d\tilde{B}_x^i\right) \\ &+ \left(\prod_i \hat{a}(x,\omega_i)\right)\sum_{i\neq j}\frac{\imath\sqrt{2\alpha_m}\omega_i}{c(x)}\frac{\imath\sqrt{2\alpha_m}\omega_j}{c(x)}dx \end{aligned}$$

Therefore:

$$d\mathbb{E}\left[\prod_i \hat{a}(x,\omega_i)\right] = -\frac{(\alpha_n+\alpha_m)\sum_i\omega_i^2 + 2\alpha_m\sum_{i\neq j}\omega_i\omega_j}{c(x)^2}\mathbb{E}\left[\prod_i \hat{a}(x,\omega_i)\right]dx$$

This equation has a unique solution but instead of solving it we can easily see it is also satisfied by $\mathbb{E}\left[\prod_i \tilde{a}(x,\omega_i)\right]$ where $\tilde{a}(x,\omega)$ is solution of:

$$\left\{\begin{array}{ll} d\tilde{a}(x,\omega_k) & = \frac{\imath\sqrt{2\alpha_m}\omega_k}{c(x)}\tilde{a}(x,\omega_k)dB_x - \frac{\omega_k^2(\alpha_n+\alpha_m)}{c(x)^2}\tilde{a}(x,\omega_k)dx \\ \tilde{a}(0,\omega_k) & = 1 \end{array}\right. \tag{29}$$

and so:

$$\mathbb{E}[\hat{a}(L,\omega_1)\dots\hat{a}(L,\omega_n)] = \mathbb{E}[\tilde{a}(L,\omega_1)\dots\tilde{a}(L,\omega_n)]$$

Futhermore we can solve explicitely (29):

$$\tilde{a}(x,\omega_k) = \exp(\imath\omega_k\sqrt{2\alpha_m}\int_0^x \frac{dB_y}{c(y)} - \omega_k^2\alpha_n\int_0^x \frac{dy}{c(y)^2})$$

We can now conclude: $Z_L = \sqrt{2\alpha_m}\int_0^L \frac{dB_y}{c(y)}$ is a centered gaussian variable with the right variance (16) and:

$$\begin{aligned}\frac{1}{2\pi}\int e^{-\imath\omega\sigma}\hat{f}(\omega)\tilde{a}(L,\omega)d\omega &= \int f(t)\frac{1}{\sqrt{2\pi}}\int e^{\imath\omega(t-\sigma+Z_L)}\exp(-\omega^2\alpha_n\theta_L)d\omega dt \\ &= \frac{1}{\sqrt{2\alpha_n T_L}}\int f(t)\exp(-\frac{(t-\sigma+Z_L)^2}{4\alpha_n\theta_L})dt \\ &= f * G_L(\sigma+Z_L)\end{aligned}$$

hence:

$$\lim_{\epsilon\to 0}\mathbb{E}[A(L,\tau(L)+\epsilon\sigma_1)^{p_1}\dots A(L,\tau(L)+\epsilon\sigma_k)^{p_k}] = \mathbb{E}[f*G_L(\sigma_1+Z_L)^{p_1}\dots f*G_L(\sigma_k+Z_L)^{p^k}]$$

which is enough to conclude because we know already that the process is tight.
The description of the shape of the transmitted pulse is now complete: there is a deterministic spreading due to the convolution by the gaussian function G_L and a stochastic translation by a gaussian variable which does not affect the shape.

Remark 3.6
We have in fact proved a stronger result because it is easy to see that the tightness of the process $(Y_x^\epsilon)_{x\in[0,L]}$ implies that of the process $A(x,\tau(x)+\epsilon\sigma)_{x\in[0,L]}$ and so the field $A(x,\tau(x)+\epsilon\sigma)_{\sigma\in[-T,T],x\in[0,L]}$ converges in distribution to $f * G_x(\sigma_k + Z_x)$.

Remark 3.7
With the chosen normalization of the incident pulse, the only observable signal is the coherent transmission whose amplitude is of order one. Indeed if we observe the transmitted signal far from the mean arrival time of the pulse, i.e. if we look at:

$$A(L,\tau(L)+t) = \frac{1}{2\pi}\int e^{-\imath\frac{\omega t}{\epsilon}}\hat{f}(\omega)\hat{a}^\epsilon(L,\omega)d\omega$$

for t strictly non-negative, then:

$$\begin{aligned}\mathbb{E}A(L,\tau(L)+t)^n &= \frac{1}{(2\pi)^n}\int e^{-\imath\frac{\sum_j\omega_j t}{\epsilon}}\prod_j\hat{f}(\omega_j)\mathbb{E}\prod_j\hat{a}(L,\omega_j)d\omega_1\dots d\omega_n \\ &+\frac{1}{(2\pi)^n}\int e^{-\imath\frac{\sum_j\omega_j t}{\epsilon}}\prod_j\hat{f}(\omega_j)\mathbb{E}\prod_j(\hat{a}^\epsilon(L,\omega_j)-\hat{a}(L,\omega_i))d\omega_1\dots d\omega_n\end{aligned}$$

The first term goes to zero with ϵ owing to the boundness of $\hat{a}(L,\omega)$ and the regularity of $\hat{f}$ and the second term goes to zero because of the convergence result previously proved.

The study of the reflected signal, defined for t positive by:

$$B(0,t) = \frac{1}{2\pi}\int e^{-\imath\frac{\omega t}{\epsilon}}\hat{f}(\omega)\hat{b}^{\epsilon}(0,\omega)d\omega$$

can be deduced from the previous analysis. If we want to work with forward stochastic differential equations we make the change of variable:

$$\begin{array}{l} \tilde{a}^{\epsilon}(x,\omega) = \hat{a}^{\epsilon}(L-x,\omega) \\ \tilde{b}^{\epsilon}(x,\omega) = \hat{b}^{\epsilon}(L-x,\omega) \end{array}$$

then we find an equation satisfied by the limit in distribution of the propagator matrix $\tilde{Y}^{\epsilon}(x,\omega_1,\ldots,\omega_n)$ for $(\tilde{a}^{\epsilon},\tilde{b}^{\epsilon})$ with distinct frequencies and we find the following stochastic differential equation for $\tilde{b}(x,\omega_k)$:

$$\left\{\begin{array}{ll} d\tilde{b}(x,\omega_k) & = \frac{\imath 2\sqrt{2\alpha_m}\omega_k}{c(x)}\tilde{b}(x,\omega_k)dB_x - \frac{2\omega_k^2(2\alpha_m+\alpha_n)}{c(x)^2}\tilde{b}(x,\omega_k)dx + \tilde{F}(\tilde{Y}(x,\omega_k))(dB_x^k - \imath d\tilde{B}_x^k) \\ \tilde{b}(0,\omega_k) & = 0 \end{array}\right.$$

The equation for the moments is then:

$$\begin{array}{ll} d\mathbb{E}\left[\prod_j \tilde{b}(x,\omega_j)\right]_x & = \frac{1}{c(x)^2}\left((2\alpha_m+\alpha_n)\sum_j \omega_j^2 - 8\alpha_m\sum_{j\neq l}\omega_j\omega_l\right)\mathbb{E}\left[\prod_j \tilde{b}(x,\omega_j)\right]dx \\ \mathbb{E}\left[\prod_j \tilde{b}(0,\omega_j)\right] & = 0 \end{array}$$

so:

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\prod_j \hat{b}^{\epsilon}(L,\omega_j)\right] = 0$$

hence $B(0,t)$ converges in probability to zero.

Remark 3.8
Although the amplitude of the reflected pulse is of order less than one and cannot be observed in the limit its contribution to the outcoming energy is non-zero because the energy of the coherent transmission is less than the total incoming energy (remember the conservation of energy relation (14))

$$\frac{\int A(L,\tau(L)+t)^2dt}{\int f(\frac{t}{\epsilon})^2dt} = \frac{\int A(L,\tau(L)+\epsilon t)^2dt}{\int f(t)^2dt} \to_{\epsilon\to 0} \frac{\int \hat{f}(\omega)^2 e^{\imath\omega Z_L - \omega^2\alpha_m\theta_L}d\omega}{\int \hat{f}(\omega)^2 d\omega} < 1$$

References

[1] M.Asch, W.Kohler, G.Papanicolaou, M.Postel and B.White *Frequency content of randomly scattered signals* SIAM review,Vol.33,No.4,pp.519-625(12/1991)

[2] R.Burridge, G.Papanicolaou, P.Sheng and B.White *Probing a Random Medium with a Pulse* SIAM J. Appl Math, Vol 49, N02. pp582-607, (4/1989)

[3] J.F.Clouet *Théorie d'O'Doherty-Anstey et approximation-diffusion* Rapport interne 273 CMAP Ecole Polytechnique (1992)

[4] J.P.Fouque and E.Merzbach *A Limit Theorem for Linear Boundary Value Problems in Random Media* to appear

[5] G.Papanicolaou,D.Stroock and S.Varadhan *Martingale Approach to some Limit Theorems* Duke Turbulence Conference Duke.Univ.Math. Series III (1977)

[6] D.Stroock and S.Varadhan *Multidimensionnal Diffusion Processes* Springer-Verlag,233,(1979)