Effective Boundary Conditions for Electro-Magnetic Scattering in Thin Layers

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Abstract

A hierarchy of effective boundary conditions are derived for the Maxwell's and Helmholtz' equations. These generalized impedance boundary conditions approximate the effect of thin layers covering a body. They can be used for accurate numerical calculations of the electro-magnetic field in the exterior of a coated body. The conditions are given in differential, integral and variational form. Well-posedness of the exterior problems with these new boundary conditions are proved and error estimates are given.

1. Introduction. Our problem is to find the electro-magnetic field outside an object Ω in $\mathbf{R^2}$ or $\mathbf{R^3}$. We shall assume that Ω consists of a conducting core Ω_0 covered by a number of homogeneous dielectric layers $\{\Omega_j\}_{j=1}^J$, see Figure 1.1. the outer boundary of Ω is denoted by Γ and the boundary of the core by Γ_δ.

Let the electro-magnetic field we are interested in be given by the Helmholtz equation or the time harmonic Maxwell's equations. Together with the partial differential equations we also have boundary conditions on Γ_δ, far field conditions and interface conditions on Γ and the curves or surfaces between the dielectric layers.

It is the purpose of this paper to approximate the effect of the dielectric layers by boundary conditions on Γ when the layers are relatively thin. The resulting problem will be simpler to approximate numerically. It is also possible to express the new boundary condition at Γ_δ.

The effective boundary conditions on Γ may contain higher order tangential derivatives for the approximation to be accurate for a wide class of electro-magnetic fields. It is essential that the boundary conditions together with the differential equations give well-posed problems.

There are similar generalized impedance conditions of different order for the approximation of thin layers in the literature, see e.g. [A], [B], [C].

1.1. Principles of Derivation.

We will present two principles of derivation for the approximate boundary conditions. The first one start from the differential equation written in some curvilinear coordinates and are obtained directly using the Taylor expansion. This method has the advantage of being simple and give immediately the first approximations. The second method start from the variational formulation. It is more complex, but it leads more easily to the proof of the error estimates and we are able to recover by this way the approximations introduce by the first principle.

1.2. The Example of a Cylinder. Let us use a coated conducting cylinder in order to present examples of the thin layer approximation in a simple setting. We shall consider one dielectric layer and assume that the field is given by the Helmholtz equation. The explicit formulas below follow directly from the general theory developed in Chapter 3.

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Let Ω be a disc with radius R and let the thickness of the dielectric layer be δ,

$$\begin{aligned}
\Omega_0 &= \{x, |x| < R - \delta\}, \\
\Omega_1 &= \{x, R - \delta \le |x| < R\}, \\
\Omega &= \Omega_0 \cup \Omega_1, \quad \Omega_\infty = \mathbf{R^2} \backslash \Omega \\
\Gamma &= \{x, |x| = R\}, \quad x_1 = r\cos\theta, \quad x_2 = r\sin\theta.
\end{aligned}$$

In the Helmholtz equation $k = k_1$ and k_∞ in Ω_1 and Ω_∞ respectively. The boundary and interface conditions for the original problem are,

$$\frac{\partial u}{\partial r} = f(x), \quad r = R - \delta, \quad or \tag{1.1}$$

$$u = g(x), \quad r = R - \delta, \quad together\, with \tag{1.2}$$

$$u(R_+) = u(R_-), \quad R_+(R_-) = \lim_{|x| \to R} x, \quad x \in \Omega_\infty (x \in \Omega_1), \tag{1.3a}$$

$$\frac{\partial u(R_+)}{\partial r} = z \frac{\partial u(R_-)}{\partial r}. \tag{1.3b}$$

The trivial effective boundary condition corresponding to the assumption that u is constant in the radial direction within Ω, is for the Neumann problem (1.1),

$$\frac{\partial u}{\partial r} = z f(x), \quad x \in \Gamma. \tag{1.4}$$

For reference we shall call this type of conditions the zero order approximations. The assumption of linear or quadratic radial dependence gives the first and second order approximation,

$$\frac{\partial u}{\partial r} + z\delta k_1^2 u = z f(x), \quad x \in \Gamma, \tag{1.5}$$

$$\frac{\partial u}{\partial r} + z\delta (\frac{1}{R^2} u_{\theta\theta} + k_1^2 u) = z f(x), \quad x \in \Gamma. \tag{1.6}$$

For a single layer representation of the solution,

$$\begin{aligned}
u(x) &= \int_\Gamma G(x,y)\lambda(y) dy, \quad x \notin \Gamma, \\
G(x,y) &= \frac{i}{4} H_0^{(1)}(k|x-y|),
\end{aligned}$$

we formally get the boundary integral equation,

$$\lambda(x) - 2 \int_\Gamma K(x,y)\lambda(y) dy = -2 z f(x), \quad x \in \Gamma, \tag{1.7}$$

with the kernel defined as follows, corresponding to the different boundary conditions (1.4), (1.5) and (1.6) respectively,

$$K = \frac{\partial G(x,y)}{\partial r}, \tag{1.8}$$

$$K = \frac{\partial G(x,y)}{\partial r}, + z\delta k_1^2 \; G(x,y), \tag{1.9}$$

$$K = \frac{\partial G(x,y)}{\partial r} + \frac{z\delta}{k^2} \; \frac{\partial^2 G(x,y)}{\partial \theta^2} + z\delta R_1^2 \; G(x,y). \tag{1.10}$$

2. Geometry and Notation. We shall consider the $\mathbf{R^3}$ case, the $\mathbf{R^2}$ being similar. Let the bounded domain Ω have a regular simply connected boundary Γ.

Define n as the external unit normal to the domain Ω.

The dielectric layer is assumed to have constant thickness defined as follows,

$$\Gamma_\delta = \{y \in \Omega;\ y = x - \delta n(x),\ x \in \Gamma\}$$

where δ is small enough for Γ_δ to be smooth. We will call Ω_1 the domain between Γ and Γ_δ.

For multiple layers there are several parameters $\{\delta_j\}_{j=1}^J$

$$\delta = \sum_{j=1}^{J} \delta_j$$

$$\Gamma_{\delta_j} = \{y \in \Omega;\ y \in \Omega;\ y = x - \sum_{k=j+1}^{J} \delta_j n(x),\ x \in \Gamma\}.$$

These surfaces define the domains $\Omega_1, \Omega_2, \ldots, \Omega_J$ respectively.

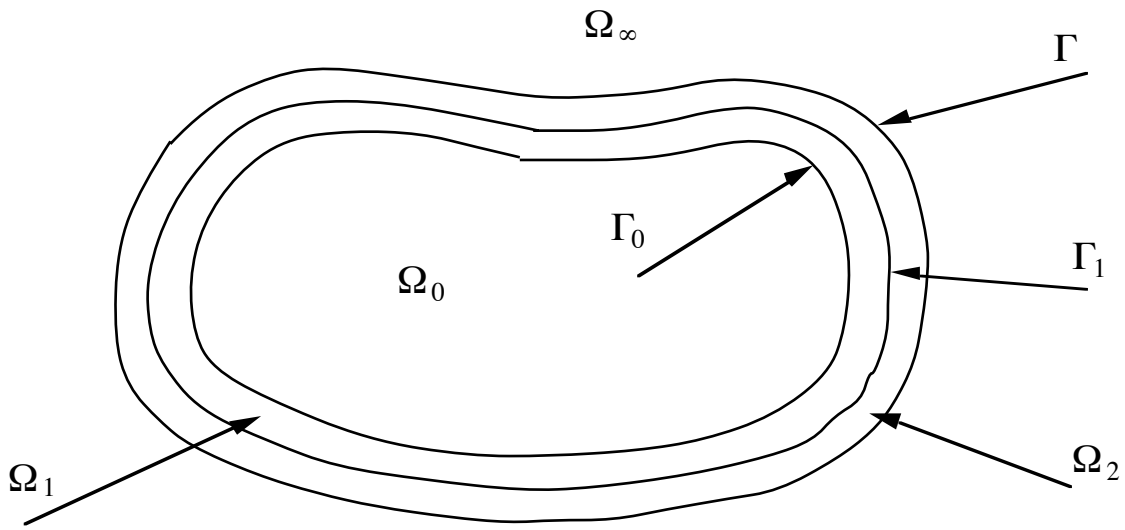

Figure 1

We shall need some differential operators on the surface Γ. We will introduce them. δ being small enough every point y in the domains $\Omega_1, \ldots, \Omega_J$ admits a unique projection $\mathcal{P}(u)$ on the surface Γ. It is the point which realizes the minimum of the distance to the surface Γ denotes by $d(y)$:

$$d(y) = |y - \mathcal{P}(y)| = \inf_{x \in \Gamma} |y - x|. \tag{2.1}$$

Then the normal vector on the surface Γ is the opposite of gradient of the function d and we use this property to define

$$n(y) = -\nabla d(y) = n(\mathcal{P}(y)); \quad y \in \Omega_1, \ldots \Omega_J. \tag{2.2}$$

Any point in the domains $\Omega_1, \ldots \Omega_J$, can be parametrized as

$$y = \mathcal{P}(y) + s\, n(\mathcal{P}(y)) \qquad -\delta \leq s \leq 0; \tag{2.3}$$

To any function u defined on the surface Γ, we associate the function $\tilde{u}$ defined on the interior domains by

$$\tilde{u}(y) = u(\mathcal{P}(y)). \tag{2.4}$$

We also introduce the family of parallel surfaces Γ_s

$$\Gamma_s = \{y \in \Omega_1, \ldots, \Omega_J; \quad y = x + sn(x); \quad x \in \Gamma\}. \tag{2.5}$$

The field of the vectors n is the field of the normal vectors to the surface Γ_s. We will denote as T_s the tangent plane to this surface.

The tangential gradient of a function u is

$$\nabla_\Gamma u = \nabla \tilde{u}|_\Gamma. \tag{2.6}$$

This vector belongs to the tangent plan to the surface Γ.

The tangential rotational (or curl) is defined as the tangential vector:

$$curl_\Gamma u = curl(\tilde{u}n)|_\Gamma. \tag{2.7}$$

As we have curl $n = 0$, we can rewrite (2.7) as

$$curl_\Gamma u = \nabla_\Gamma u \wedge n. \tag{2.8}$$

The curvature operator $\mathcal{R}_s$ is defined as

$$\mathcal{R}_s = \nabla n. \tag{2.9}$$

It is a symmetric operator from the tangent plane into itself. So it admits two real eigenvalues denoted by $\frac{1}{\mathcal{R}_1(s)}$ and $\frac{1}{\mathcal{R}_2(1)}$. Then the *mean curvature* is defined as

$$H_s = \frac{1}{2} div\ n = \tfrac{1}{2}\left(\tfrac{1}{R_1(s)} + \tfrac{1}{R_2(s)}\right). \tag{2.10}$$

the *Gauss curvature* is defined as

$$G_s = \frac{1}{R_1(s)\, R_2(s)} = det\ (\mathcal{R}_s|_{T_s}). \tag{2.11}$$

The area element $d\gamma_s$ on the surface Γ_s at the point $y = x + sn(x)$, is related to the area element of the surface Γ at the point x by

$$d\gamma_s(y) = (1 + 2sH(x) + s^2 G(x)) d\gamma(x). \tag{2.12}$$

To any field of vectors v in the tangent plane T_Γ, we associate a field of vectors in the tangent plane T_{Γ_s} to the surface Γ_s by the formula ($y = x + sn(x)$)

$$\tilde{v}(y) = v(x) - s\mathcal{R}(y)v(x). \tag{2.13}$$

We define now the *surface divergence* of a tangent vector as

$$div_\Gamma v = div\ \tilde{v}_{|\Gamma}. \tag{2.14}$$

The *surface rotational* of a tangent vector is defined as

$$rot_\Gamma v = (curl\ \tilde{v} \cdot n)_{|\Gamma}. \tag{2.15}$$

The operator of Laplace-Beltrami is defined as

$$\Delta_\Gamma u = div_\Gamma \nabla_\Gamma u - rot_\Gamma\ curl_\Gamma u. \tag{2.16}$$

Lemma 2.1. *Let u be a function in $C^1(\Gamma)$ and v a tangent vector in $(C^1(\Gamma))^2$. We have*

$$\int_\Gamma (\nabla_\Gamma u \cdot v) d\gamma + \int_\Gamma u\ div_\Gamma v d\gamma = 0 \tag{2.17}$$

$$\int_\Gamma (curl_\Gamma u \cdot v) d\gamma - \int_\Gamma u\ rot_\Gamma v d\gamma = 0 \tag{2.18}$$

Moreover, if w is a function in $C^2(\Gamma)$, we have

$$\int_\Gamma \Delta_\Gamma w u d\gamma = -\int_\Gamma (\nabla_\Gamma w \cdot \nabla_\Gamma u) d\gamma = -\int_\Gamma (curl_\Gamma w \cdot curl_\Gamma u) d\gamma. \tag{2.19}$$

Let u be a smooth functional defined in $\Omega_1 \ldots \Omega_J$, and v a smooth vector defined on the same domain, we have

$$\nabla u = \nabla_{\Gamma_s} u + \frac{\partial u}{\partial s} n \tag{2.20}$$

$$v = (v \cdot n) n + v_{\Gamma_s} \tag{2.21}$$

$$div v = div_{\Gamma_s} v_{\Gamma_s} + 2H_s (v \cdot n) + \tfrac{\partial}{\partial s}(v \cdot n) \tag{2.22}$$

$$\begin{cases} curl v = rot_{\Gamma_s} v_{\Gamma_s} n + curl_{\Gamma_s}(v \cdot n) + \mathcal{R}_s (v \wedge n) \\ \qquad\qquad -2H_s\, v \wedge n - \tfrac{\partial}{\partial s}(v \wedge n) \end{cases} \tag{2.23}$$

$$\Delta u = \Delta_{\Gamma_s} u + 2H_s \frac{\partial u}{\partial s} + \frac{\partial^2}{\partial s^2} u \tag{2.24}$$

3. The Helmholtz Equation.

We will consider the case of one layer. The problem is posed in the exterior of the domain Ω_0

$$\Omega_0^{ext} = \Omega_1 \cup \Omega_\infty .$$

The differential equations are

$$\begin{cases} \alpha_\infty \Delta u + \beta_\infty u = g; & x \in \Omega_\infty; \\ \alpha_1 \Delta u_1 + \beta_1 u_1 = 0; & x \in \Omega_1; \end{cases} \tag{3.1}$$

The interfaces conditions are

$$\begin{cases} u_1|_\Gamma = u|_\Gamma; \\ \alpha_1 \frac{\partial u_1}{\partial n}|_\Gamma = \alpha_\infty \frac{\partial u}{\partial n}|_\Gamma. \end{cases} \tag{3.2}$$

The solution u is a complex value function and must satisfy in addition the Sommerfeld' condition

$$\frac{\partial u}{\partial r} - iku = 0\left(\frac{1}{r}\right);\ k = \sqrt{\frac{\beta_\infty}{\alpha_\infty}},\ \ on\ the\ sphere\ of\ radius\ r,\ \ r \to \infty. \tag{3.3}$$

We will consider the problems corresponding to the following two boundaries' conditions

$$u_1|_{\Gamma_0} = 0 \quad Dirichlet'problem; \tag{3.4}$$

$$\frac{\partial u_1}{\partial n}|_{\Gamma_0} = 0 \quad Neumann'problem. \tag{3.5}$$

We want to replace in each case (Dirichlet or Neumann) the differential equation in the domain Ω_1 by an approximate boundary condition. The parameter of approximation is the thickness δ. We will present two methods which lead to such approximations.

3.1. Derivation using the Differential Equations.

We obtain these boundary conditions starting from the hypothesis that the function u_1 is approximated *by a polynomial of order k in the normal variable s*. We will say that this approximate boundary condition is then of order k.

The process of derivation of the boundary condition differ from one case to another and so we will present then separately.

3.1.1. Dirichlet problem.

The conditions of order 1.

We approximate our solution setting u_1 as a polynomial of degree 1 in the normal variable s. Then starting from the boundary condition on Γ_0, we obtain

$$\frac{\partial u_1}{\partial n} - \frac{1}{\delta} u_1 = 0. \tag{3.6}$$

Then using the transmission conditions (3.2), we obtain the approximate boundary condition

$$u - \delta \frac{\alpha_\infty}{\alpha_1} \frac{\partial u}{\partial n} = 0; \qquad x \in \Gamma. \tag{3.7}$$

The condition of order 2.

Here we suppose that u_1 is a polynomial of order 2 in s; so at the point $y = x + sn(x)$, we have

$$u_1(y) = u_1(x) + s\frac{\partial u_1}{\partial n}(x) + \frac{s^2}{2}\frac{\partial^2 u_1}{\partial n^2}(x) \tag{3.8}$$

$$\frac{\partial u_1}{\partial n}(y) = \frac{\partial u_1}{\partial n}(x) + s\frac{\partial^2 u_1}{\partial n^2}(x). \tag{3.9}$$

Using the boundary condition on Γ_0, we find

$$u_1(x) = \delta \frac{\partial u_1}{\partial n}(x) - \frac{\delta^2}{2}\frac{\partial^2 u_1}{\partial n^2}(x). \tag{3.10}$$

Then, we eliminate the quantity $\frac{\partial^2 u_1}{\partial n^2}(x)$ using the equation (3.1) in the domain Ω_1, that can be rewritten as

$$\alpha_1(\Delta_\Gamma u_1(x) + 2H(x)\frac{\partial u_1}{\partial n}(x) + \frac{\partial^2 u_1}{\partial n^2}(x)) + \beta_1 u_1(x) = 0. \tag{3.11}$$

If we only keep the terms up to the order δ^2, we can forget u_1 and $\Delta_\Gamma u_1$ and we find the approximate condition

$$u(x) = \delta(1 + \delta H(x))\frac{\alpha_\infty}{\alpha_1}\frac{\partial u}{\partial n}(x); \qquad x \in \Gamma. \tag{3.12}$$

To summarize, in the case of a sphere of radius R and for several layers, we have obtained the following conditions

Order 1.

$$u(x) = \left(\sum_{j=1}^{J} \frac{\delta_j}{\alpha_j} \right) \alpha_\infty \frac{\partial u}{\partial n}(x). \tag{3.13}$$

Order 2.

$$u(x) = \left(1 + \frac{\delta_1}{R}\right) \left[\frac{\delta_1}{\alpha_1} + \left(1 + \frac{\delta_2}{R}\right) \left[\frac{\delta_2}{\alpha_2} + \cdots \right] \right] \alpha_\infty \frac{\partial u}{\partial n}(x). \tag{3.14}$$

3.1.2. The Neumann problem.

To obtain the approximate boundary condition of order k, we integrate the equation (3.1) along the normal and then we neglect the terms of order more that k.

The condition of order 1.

We integrate the equation (3.11) along the normal and use the boundary condition on Γ_0. We obtain

$$\alpha_1 \left[\int_0^\delta \left(\Delta_{\Gamma_s} u + 2H_s \frac{\partial u}{\partial n} \right) ds + \frac{\partial u_1}{\partial n}(x) \right] + \beta_1 \int_0^\delta u ds = 0 \tag{3.15}$$

and also

$$u_1(x) = u_1(-\delta) - \delta \frac{\partial u}{\partial n}(x). \tag{3.16}$$

The terms containing $\partial u/\partial n$ are of order δ. So up to order 1 the equation (3.15) is

$$\alpha_1 \left[\delta \Delta_\Gamma u_1(x) + \frac{\partial u_1}{\partial n}(x) \right] + \beta_1 \delta u_1(x) = 0. \tag{3.17}$$

Using now the transmission condition, we obtain the approximate boundary condition of order 1:

$$\frac{\partial u}{\partial n}(x) = -\frac{\delta}{\alpha_\infty} \left[\alpha_1 \Delta_\Gamma u(x) + \beta_1 u(x) \right]. \tag{3.18}$$

The condition of order 2.

The process is exactly the same, except when neglecting the terms of order 2.

The equation (3.17) is replaced by the following (up to terms of order 3)

$$\alpha_1 \delta \left(\Delta_\Gamma u_1(x) + H(x) \frac{\partial u_1}{\partial n}(x) \right) + \alpha_1 \frac{\partial u_1}{\partial n}(x) + \beta_1 \delta u_1(x) = 0. \tag{3.19}$$

This equation and the transmission condition yield the second order approximate boundary condition

$$(1 + \delta H(x)) \frac{\partial u}{\partial n}(x) = -\frac{\delta}{\alpha_\infty} (\alpha_1 \Delta_\Gamma u(x) + \beta_1 u(x)). \tag{3.20}$$

Comments.

It is possible in each case (Dirichlet or Neumann) to obtain by the same way the conditions of order k. Growing in order, it will progressively appear tangentials derivatives of higher order, and also quantities of higher order related to the geometry.

3.2. Derivation Using the Variational Formulation.

We shall start from the variational formulation for the exact problem and then introduce a variational approximation of the solution. We shall present the case of one layer in three dimensions.

We introduce a ball B_R of radius R containing the body Ω , $\Omega_R = \Omega_\infty \cap B_R \cdot S_R$ will denote the boundary of the sphere B_R. If the right hand side g has a support in B_R the system of equation (3.1)(3.2) yields:

$$\left\{\begin{array}{l} \alpha_1 \int_{\Omega_1}(\nabla u \cdot \nabla v)dx - \beta_1 \int_{\Omega_1} uvdx \\ +\alpha_\infty \int_{\Omega_R}(\nabla u \cdot \nabla v)dx - \beta_\infty \int_{\Omega_R} uvdx - \alpha_\infty \int_{S_R} vd\gamma \\ \qquad -\alpha_1 \int_{\Gamma_0} \frac{\partial u}{\partial n} vd\gamma = \int_{\Omega_R} gvdx; \quad \forall v \in H^1(\Omega_R \cup \Omega_1). \end{array}\right. \tag{3.21}$$

In order to obtain a variational formulation "equivalent" to the original system, we introduce a coupling. The solution of the Helmholtz equation in the exterior of the sphere B_R can be expressed in terms of an expansion in terms of Spherical Hankel functions and Spherical Harmonic functions. Let (r, θ, φ) be the spherical coordinates. Let $Y_\ell^m(\theta, \varphi)$ be an orthonormal basis of the spherical harmonics functions ($\ell \geq 0$; $m = -\ell$ to ℓ).

Let $h_\ell^{(1)}(r)$ be the Spherical Hankel function of order ℓ determined by the fact that

$$h_\ell^{(1)}(r) \sim \frac{1}{r} e^{ir}; \quad r \to \infty. \tag{3.22}$$

We can expand the exterior solution in the form

$$u(r, \theta, \varphi) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} u_\ell^m Y_\ell^m(\theta, \varphi) h_\ell^{(1)}(kr)/h_\ell^{(1)}(kR). \tag{3.23}$$

The complex coefficients u_ℓ^m are known as soon as u is given on S_R. We can now express the exterior normal derivative

$$\frac{\partial u}{\partial n} = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} u_\ell^m k \frac{\frac{d}{dr} h_\ell^{(1)}(kR)}{h_\ell^{(1)}(kR)} Y_\ell^m(\theta, \varphi) \tag{3.24}$$

and this gives us the Dirichlet-Neumann operator

$$Tu = \frac{\partial u}{\partial n}|_{S_R}. \tag{3.25}$$

This operator is such that

$$0 \leq Im(Tu, \bar{u})_{L^2(S_R)} \leq \frac{k}{R^2} \|u\|^2_{L^2(S_R)} \tag{3.26}$$

$$\frac{1}{R^3} \|u\|^2_{L^2(S_R)} \leq -Re(Tu, \bar{u})_{L^2(S_R)} \leq \frac{1}{R^3} \|u\|^2_{H^{\frac{1}{2}}(S_R)}. \tag{3.27}$$

We have the theorem

Theorem 3.1. *the Helmholtz equation (Dirichlet or Neumann problem) admit a unique solution in $H^1_{loc}(\Omega_0^{ext})$ satisfying the Sommerfield' condition. It is also a solution of the following variational equation*

$$\left\{\begin{array}{c} \alpha_1 \int_{\Omega_1}(\nabla u \cdot \nabla v)dx - \beta_1 \int_{\Omega_1} uvdx + \alpha_\infty \int_{\Omega_R}(\nabla u \cdot \nabla v)dx - \beta_\infty \int_{\Gamma_R} uvdx \\ -\alpha_\infty \int_{S_R} Tuvd\gamma = \int_{\Omega_R} gvdx \end{array}\right. \tag{3.28}$$

for every v in $H^1(\Omega_R \cup \Omega_1)$ (Neumann problem) or every v in $H^1(\Omega_R \cup \Omega_1)$ such that $v|_{\Gamma_0} = 0$ (Dirichlet problem).

Proof. It can be found in [E] . It is based on the inequalities (3.26) and (3.27), and on the Rellich' theorem. ∎

The approximate boundary conditions are now obtained by setting an approximate space of functions for u and the test function v in the domain Ω_1. Namely, the natural choice consist of taking for *u and v functions which are polynomials of degree k in the variable s*. We will develop in detail the *degree 1* which will lead to approximate conditions of order 1 and 2.

In that case, the functions u and v are uniquely determined by their values on Γ_0 and Γ respectively. At the point $y = x + sn(x),\ x \in \Gamma$, we have

$$\begin{cases} u(y) = (1 + \frac{s}{\delta})u(x) - \frac{s}{\delta}u_0(x); \\ v(y) = (1 + \frac{s}{\delta})v(x) - \frac{s}{\delta}v_0(x). \end{cases} \tag{3.29}$$

Using the decomposition (2.20) and the linear variation, we have

$$\nabla u = \nabla_{\Gamma_s} u + \frac{u(x) - u_0(x)}{\delta} n(x). \tag{3.30}$$

If we use these expressions in the variational formulation, we obtain the variational formulation of the approximate problems.

Approximate problems

Dirichlet conditions. In that case u_0 and v_0 are zero. So the problem is:

Find $u \in H^1(\Omega_R)$ such that for every v in $H^1(\Omega_R)$

$$\begin{cases} \alpha_\infty \int_{\Omega_R} (\nabla u \cdot \nabla u) dx - \beta_\infty \int_{\Omega_R} uv dx - \alpha_\infty \int_{S_R} Tuv d\gamma \\ \quad + \alpha_1 \int_{\Omega_1} (\nabla_{\Gamma_s} u \cdot \nabla_{\Gamma_s} v) dy + \alpha_1 \int_{-\delta}^{0} \int_\Gamma \frac{u(x)v(x)}{\delta^2} dy \\ \quad - \beta_1 \int_{\Omega_1} u(y)v(y) dy = \int_{\Omega_R} gv dx. \end{cases} \tag{3.31}$$

All the integrals on the domain Ω_1 depends only on the values of u, v and $\nabla_\Gamma u, \nabla_\Gamma v$ on the surface Γ.

Lemma 3.1. *We have*

$$\nabla_{\Gamma_s} u(x + sn(x)) = (1 + \frac{s}{\delta})(I + sR(x))^{-1} \nabla_\Gamma u(x). \tag{3.32}$$

Proof. By construction ∇_{Γ_s} is the gradient in y of the function $u(z)$ which consist of keeping the value in z equal to the value of the projection of z on the surface Γ_s. We have by construction

$$\begin{cases} y = x + sn(x) \ \ and \ \ x = y - sn(y) \\ (I + sR(x))^{-1} = I - sR_s(y) \end{cases} \tag{3.33}$$

$$\nabla_{\Gamma_s} u(y) = (1 + \frac{s}{\delta}) \nabla_y u(x) = (1 + \frac{s}{\delta})(I + sR(x))^{-1} \nabla_x u(x). \tag{3.34}$$

∎

We can now express the integrals associated to the domain Ω_1.

$$\begin{cases} \int_{\Omega_1} (\nabla_{\Gamma_s} u \cdot \nabla_{\Gamma_s} v) dy = \int_\Gamma (B(x, \delta) \nabla_\Gamma u \cdot \nabla_\Gamma v) d\gamma \\ B(x, \delta) = \int_{-\delta}^{0} (1 + \frac{s}{\delta})^2 (I + sR(x))^{-2} (1 + 2sH(x) + s^2 G(x)) ds \end{cases} \tag{3.35}$$

$$\begin{cases} \int_{\Omega_1} u(y)v(y)dy = \int_\Gamma b(x,\delta)u(x)v(x)d\gamma \\ b(x,\delta) = \int_{-\delta}^0 (1+\frac{s}{\delta})^2(1+2sH(x)+s^2G(x))dx. \end{cases} \tag{3.36}$$

The function b is

$$b(x,\delta) = \frac{\delta}{3} - \frac{1}{6}\delta^2 H(x) + \frac{1}{30}\delta^3 G(x). \tag{3.37}$$

$B(x,\delta)$ can be computed exactly by recursion. We will only give the two first terms in the development in δ which are

$$B(x,\delta) = \frac{\delta}{3} I - \frac{1}{6}\delta^2 (H(x)I - R(x)) + \cdots \tag{3.38}$$

If we order in increasing powers of δ, the terms coming from the approximation, the variational formulation take the form:

$$\begin{cases} \alpha_1 \frac{1}{\delta}\int_\Gamma u(x)v(x)d\gamma - \alpha_1 \int_\Gamma H(x)u(x)v(x)d\gamma \\ +\delta \int_\Gamma \left[\frac{\alpha_1}{3}G(x) - \frac{\beta_1}{3}\left(1 - \frac{1}{2}\delta H(x) + \frac{\delta^2}{10}G(x)\right)\right] u(x)v(x)d\gamma \\ +\alpha_1 \int_\Gamma \left(B(x,\delta)\nabla_\Gamma u(x)\cdot\nabla_\Gamma v(x)\right) d\gamma(x) \\ +\alpha_\infty \int_{\Omega_R} (\nabla u\cdot\nabla v)dx - \beta_\infty \int_{\Omega_R} uvdx - \alpha_\infty \int_{S_R} Tuvd\gamma \\ \qquad = \int_{\Omega_R} gvdx. \end{cases} \tag{3.39}$$

We will denote as u_1^δ the solution of (3.39). The usual formulation of this problem is

$$\begin{cases} \alpha_\infty \Delta u_1^\delta + \beta u_1^\delta = g;\ in\ \Omega_\infty; \\ \frac{\partial u^\delta}{\partial n} = \frac{1}{\delta\alpha_\infty}\left[\alpha_1 - \alpha_1\delta H(x) + \delta^2(\frac{\alpha_1}{3}G(x) - \frac{\beta_1}{3}(1+\delta H(x) + \frac{\delta^2}{10}G(x)))\right] u_1^\delta \\ \qquad -\frac{\alpha_1}{\alpha_\infty}\ div_\Gamma(B(x',\delta)\nabla_\Gamma u_1^\delta); \\ \qquad + Sommerfeld\, condition. \end{cases} \tag{3.40}$$

If we neglect the terms of order more than 2 in δ, the approximate boundary condition become

$$\frac{\partial u_2^\delta}{\partial n} = \frac{\alpha_1}{\delta\alpha_\infty}(1-\delta H(x))u_2^\delta. \tag{3.41}$$

We will denote as u_2^δ the corresponding solution. If we neglect also the terms of order δ, the solution, denote by u_3^δ, satisfied the approximate boundary condition:

$$\frac{\partial u_3^\delta}{\partial n} = \frac{\alpha_1}{\delta\alpha_\infty} u_3^\delta. \tag{3.42}$$

The problem (3.42) is nothing more than the problem (3.7) (condition of order 1). The problem (3.41) is very close (but different) from the problem (3.12) (condition of order 2).

Neumann condition.

In that case u_0 and v_0 given in (3.29) are free variables and so the approximate problem have two unknowns: u (in $H^1(\Omega_R)$) and u_0 (in $H^{1/2}(\Gamma)$). It will be seen that, in this case (but it is very specific !), we can simplify the approximation by choosing $u_0 \equiv u|_\Gamma$; i.e. the normal derivative is zero in the layer Ω_1. The approximate problem admit the variational formulation

$$\begin{cases} \alpha_\infty \int_{\Omega_R} (\nabla u_1^\delta\cdot\nabla v)dx - \beta_\infty \int_{\Omega_R} u_1^\delta v dx - \alpha_\infty \int_{S_R} Tu_1^\delta v d\gamma \\ \qquad +\alpha_1 \int_\Gamma (C(x,\delta)\nabla_\Gamma u_1^\delta\cdot\nabla_\Gamma v)d\gamma \\ \qquad -\beta_1 \int_\Gamma c(x,\delta)u_1^\delta v d\gamma = \int_{\Omega_R} gvdx. \end{cases} \tag{3.43}$$

$$\begin{cases} C(x,\delta) = \int_{-\delta}^{0} (I + sR(x))^{-2}(1 + 2sH(x) + s^2 G(x))ds \\ C(x,\delta) = \delta I - \delta^2 (H(x)I - R(x)) + \cdots \end{cases} \tag{3.44}$$

$$c(x,\delta) = \delta - \delta^2 H(x) + \frac{\delta^3}{3} G(x). \tag{3.45}$$

The approximate boundary condition is now

$$\alpha_\infty \frac{\partial u_1^\delta}{\partial n}(x) = -\beta_1 c(x,\delta) u_1^\delta(x) - \alpha_1 \ div_\Gamma(C(x,\delta)\nabla_\Gamma u_1^\delta(x)). \tag{3.46}$$

If we neglect the terms of order more than 2 in δ, we obtain the following approximate boundary condition

$$\alpha_\infty \frac{\partial u_2^\delta}{\partial n} = -\beta_1 \delta (1 - \delta H(x)) u_2^\delta - \alpha_1 \delta \ div_\Gamma [(I - \delta(H(x)I - R(x)))\nabla_\Gamma u_2^\delta. \tag{3.47}$$

If we neglect also the terms of order δ^2 we obtain

$$\alpha_\infty \frac{\partial u_3^\delta}{\partial n} = -\delta(\beta_1 u_3^\delta + \alpha_1 \Delta_\Gamma u_3^\delta). \tag{3.48}$$

The condition (3.48) is exactly the condition (3.18) (order 1). The condition (3.47) is different from the condition of order 2: (3.19).

3.3. Error Estimates.

The error estimate are based on the use of the following theorem that can be found for example in [J].

Theorem 3.2. *Let V and H two Hilbert spaces such $V \subset H$,the injection being compact. Let a a bilinear form on V that can be decomposed in*

$$a(u,v) = a_0(u,v) + k(u,v). \tag{3.49}$$

We suppose that a_0 is coercive i.e.

$$Re \ a_0(u,\bar{u}) \geq \alpha \|u\|_V^2; \quad \alpha > 0. \tag{3.50}$$

We suppose that k is bilinear continue on H. Let $V_\delta \subset V$ an "approximation" of the space V such that we can associate to u an interpolate $\pi_\delta u$ in V_δ. We suppose that

$$|u - \pi_\delta u|_H \leq \varepsilon(\delta)\|u\|_V; \qquad \varepsilon(\delta) \to 0 \ \ with \ \delta; \tag{3.51}$$

Then the exact problem

$$a(u,v) = \langle g, v \rangle, \quad \forall v \in V; \tag{3.52}$$

and the approximate problem

$$a(u_\delta, v_\delta) = \langle g, v_\delta \rangle, \quad \forall v_\delta \in V_\delta; \tag{3.53}$$

are such that

$$\|\pi_\delta u - u_\delta\|_v \leq \sup_{v_\delta \in V_\delta} \frac{a(u - \pi_\delta u, v_\delta)}{\|v_\delta\|_V}. \tag{3.54}$$

If the approximate problem is associated to an approximate bilinear form a_δ

$$a_\delta(u_\delta, v_\delta) = \langle g, v_\delta \rangle; \quad \forall v_\delta \in V_\delta \tag{3.55}$$

and if a_δ admit also a split form analogous to (3.49)*, which satisfies also* (3.50)*, we have*

$$\|\pi_\delta u - u_\delta\|_V \leq c \left[\sup_{v_\delta \in V_\delta} \frac{a(u - \pi_\delta u, v_\delta) + a(\pi_\delta u, v_\delta) - a_\delta(\pi_\delta u, v_\delta)}{\|v_\delta\|_V} \right]. \tag{3.56}$$

■

Concerning the application of this theorem to the error estimates, we will use, as a_0 and k, the following bilinear forms

$$a_0(u,v) = \alpha_1 \int_{\Omega_1} (\nabla u \cdot \nabla v) dx + \alpha_\infty \int_{\Omega_R} (\nabla u \cdot \nabla v) dx - \alpha_\infty \int_{S_R} T u v d\gamma \tag{3.57}$$

$$k(u,v) = -\beta_1 \int_{\Omega_1} uv dx - \beta_\infty \int_{\Omega_R} uv dx. \tag{3.58}$$

In the case of the approximation u_2^δ or u_3^δ, we are in the situation where the initial bilinear form is modified. The difference $a_\delta - a$ are just the terms that are neglected. As soon as we insure the positivity of the terms that are kept, we can apply the theorem with a modified form a_0.

The coercivity (hypothesis 3.44) of the bilinear form a_0 result of the inequalities (3.27) and (3.26). The Hilbert space V is just $H^1(\Omega_1 \cup \Omega_R)$. The Hilbert space H is $L^2(\Omega_1 \cup \Omega_R)$. The injection is compact as it is well-known.

The space V_δ is the space of functions in $H^1(\Omega_1 \cup \Omega_R)$ which have a linear variation in the variable s in the domain Ω_1, and are zero on Γ_δ in the case of Dirichlet problems. In case of the Neumann problem, the space V_δ is the space of functions in $H^1(\Omega_1 \cup \Omega_R)$ which are constants in the variable s in the domain Ω_1.

Dirichlet problem.

We have the following theorem

Theorem 3.3. *If g is sufficiently regular, the error estimates between u solution of (3.28) with Dirichlet condition and $u_1^\delta, u_2^\delta, u_3^\delta$ respectively solution of (3.40) (3.41) and (3.42) are*

$$\|u - u_1^\delta\|_{H^1(\Omega_R)} \leq c\delta^{5/2}; \tag{3.59}$$

$$\|u - u_2^\delta\|_{H^1(\Omega_R)} \leq c\delta^{5/2}; \tag{3.60}$$

$$\|u - u_3^\delta\|_{H^1(\Omega_R)} \leq c\delta^{3/2}. \tag{3.61}$$

Proof. It is a consequence of the Theorem 3.2. We need to estimate in each case the quantities appearing in (3.56). We choose as v^δ the function which value is u on the domain Ω_R and is the interpolate of u on the domain Ω_1. We will denote as $\pi_\delta u$ this function. The error of interpolation is:

$$\sup_{v_\delta \in V_\delta} \frac{a(u - \pi_\delta u, v_\delta)}{\|v_\delta\|_V} \tag{3.62}$$

$$\pi_\delta u(y) = (1 + \frac{s}{\delta}) \int_{-\delta}^{0} \frac{\partial u}{\partial s}(y) ds; \quad (y = x + sn(x)). \tag{3.63}$$

So we have, using the Taylor formula with integral remainder

$$u(x,s) - \pi_\delta u(x,s) = -\int_{-\delta}^{0} [[\tau - s]^- - (1 + \frac{s}{\delta})\tau] \frac{\partial^2 u}{\partial s^2}(x,\tau) d\tau. \tag{3.64}$$

If we derive this equation with respect to $x = (x_1, x_2)$ we obtain:

$$\frac{\partial}{\partial x_i} u(x,s) - \frac{\partial}{\partial x_i} \pi_\delta u(x,s) = -\int_\delta^0 [[\tau - s]^- - (1 + \frac{s}{\delta}\tau] \frac{\partial^3 u}{\partial x_i \partial s^2}(x,\tau) d\tau. \tag{3.65}$$

The tangential derivative $\nabla_{\Gamma_s} u$ can be expressed in a local system of coordinates in terms of these partial derivatives $\partial u / \partial x_i$. Using now the regularity of the data g and known results on elliptic equations, we know that the derivatives $\frac{\partial^2 u}{\partial s^2}$, $\frac{\partial^3 u}{\partial x_i \partial s^2}$, ... are bounded uniformly with respect to δ. Altogether we have already proved that

$$|u(x,s) - \pi_\delta u(x,\Delta)| \leq c\delta^2; \tag{3.66}$$

$$|\nabla_{\Gamma_s} u(x,s) - \nabla_\Gamma \pi_\delta u(x,s)| \leq c\delta^2 \tag{3.67}$$

$$|\Delta_{\Gamma_s} u(x,s) - \Delta_\Gamma \pi_\delta u(x,v)| \leq c\delta^2. \tag{3.68}$$

we also have using the Taylor formula:

$$\frac{\partial}{\partial s} u(x,s) - \frac{\partial}{\partial s} \pi_\delta u(x,s) = -\int_{-\delta}^0 (1_{[-\delta,s]} - (1 + \frac{s}{\delta})) \frac{\partial^2 u}{\partial s^2}(x,\tau) d\tau. \tag{3.69}$$

The bilinear form $a(u - \pi_\delta u, v_\delta)$ contains some terms in

$$\begin{cases} |\int_{\Omega_1} (u - \pi_\delta u) v_\delta dx| \leq c \left(\int_{\Omega_1} |u - \pi_\delta u|^2 dx \right)^{1/2} \left(\int_{\Omega_1} |v_\delta|^2 dx \right)^{1/2} \\ \qquad\qquad \leq c\delta^{5/2} |\int_{\Omega_1} |v_\delta|^2 dx|^{1/2}. \end{cases} \tag{3.70}$$

The integral $\int_{\Omega_1} |v_\delta|^2 dx$ can be expressed in terms of integrals on Γ and we obtain using these expressions

$$\int_{\Omega_1} |v_\delta|^2 dx \leq c\delta \int_\Gamma |v_\delta|^2 d\gamma(x) \leq c\delta^2 \int_{\Omega_1} |\frac{\partial v_\delta}{\partial s}|^2 dx. \tag{3.71}$$

So that these terms are bounded by

$$|\int_{\Omega_1} (u - \pi_\delta u) v_\delta dx| \leq c\delta^{7/2} \|v_\delta\|_{H^1(\Omega_1)}. \tag{3.72}$$

We have

$$\int_{\Omega_1} (\nabla_{\Gamma_s} (u - \pi_\delta u) \cdot \nabla_{\Gamma_s} v_\delta) dy = -\int_{\Omega_1} \Delta_{\Gamma_s} (u - \pi_\delta u) v_\delta dy \tag{3.73}$$

$$|\int_{\Omega_1} (\nabla_{\Gamma_s} (u - \pi_\delta u) \cdot \nabla_{\Gamma_s} v_\delta) dy| \leq c\delta^{7/2} \|v_\delta\|_{H^1(\Omega_1)}. \tag{3.74}$$

The last term in the bilinear form $a(u - \pi_\delta u, v_\delta)$ is

$$\int_{\Omega_1} \left(\frac{\partial u}{\partial s} - \frac{\partial}{\partial s} \pi_\delta u \right) \frac{\partial v_\delta}{\partial s} dx = \int_\Gamma \int_{-\delta}^0 \left(\frac{\partial u}{\partial s} - \frac{\partial}{\partial s} \pi_\delta u \right) \frac{\partial v_\delta}{\partial s} (1 + 2sH(x) + s^2 G(x)) ds d\gamma(x). \tag{3.78}$$

But $\partial v_\delta / \partial s = v_\delta(x)/\delta(x)$ is constant for x fixed, and s_0 using (3.68) we obtain

$$\int_{-\delta}^0 \left(\frac{\partial u}{\partial s} - \frac{\partial}{\partial s} \pi_\delta u \right) \frac{\partial v_\delta}{\partial s} ds = -\int_{-\delta}^0 (u - \pi_\delta u) \frac{\partial^2 v_\delta}{\partial s^2} dx + [(u - \pi_\delta u) \frac{\partial v_\delta}{\partial s}]_{-\delta}^0 = 0. \tag{3.77}$$

We can give bounds for this last term using (3.69) (3.75) and (3.76)

$$|\int_{\Omega_1}\left(\frac{\partial u}{\partial s}-\frac{\partial}{\partial s}\pi_\delta u\right)\frac{\partial v_\delta}{\partial s}dx|\leq c\delta^{5/2}\int_{\Omega_1}|\frac{\partial v_\delta}{\partial s}|^2dx. \tag{3.78}$$

This inequality with (3.72) and (3.74) gives the result (3.59), using the fact that $\pi_\delta u$ is u on the domain Ω_R.

Concerning the approximate problem u_2^δ at u_3^δ, they are obtained by neglecting some integrals in the bilinear form $a(u_\delta, v_\delta)$. We take that in account in the inequality (3.56) and so the extra error terms are exactly the neglected part of the bilinear form. All the terms containing uv are bounded by

$$|\int_{\Omega_1}\alpha(x)\pi_\delta u(x)v_\delta(x)dx|\leq c\delta^{5/2}\int_{\Omega_1}|\frac{\partial v_\delta}{\partial s}|^2dx \tag{3.75}$$

and can be neglected.

The interpolate $\pi_\delta u$ is u on Γ and the following term is

$$\begin{cases}\int_\Gamma(B(x,\delta)\nabla_\Gamma\pi_\delta u(x)\cdot\nabla_\Gamma v_\delta(x))d\gamma(x)\\ \qquad=-\int_\Gamma\ div_\Gamma(B(x,\delta)\nabla_\Gamma u(x))v_\delta(x)d\gamma(x).\end{cases} \tag{3.76}$$

It can be bounded by using Poincaré inequality

$$\begin{cases}|\int_\Gamma(B(x,\delta)\nabla_\Gamma u(x)\cdot\nabla_\Gamma v_\delta(x))d\gamma(x)|\\ \qquad\leq c\delta^{1/2}\left(\int_\Gamma|div_\Gamma(B(x,\delta)\nabla_\Gamma u(x))|^2d\gamma(x)\right)^{1/2}\left(\int_{\Omega_1}|\frac{\partial v_\delta}{\partial s}|^2dx\right)^{1/2}.\end{cases} \tag{3.77}$$

All the tangential derivatives of u are bounded by δ; and B and its tangential derivatives are also in δ. So that this term is bounded by

$$|\int_\Gamma(B(x,\delta)\nabla_\Gamma u(x)\cdot\nabla_\Gamma v_\delta(x))d\gamma_{(x)}|\leq c\delta^{5/2}\|v_\delta\|_{H^1(\Omega_1)}. \tag{3.78}$$

The bounds (3.75) and (3.78) proves (3.60).

If we also neglect the term

$$\int_\Gamma H(x)u(x)v_\delta(x)d\gamma(x) \tag{3.79}$$

it can be bounded by

$$|\int_\Gamma H(x)u(x)v_\delta(x)d\gamma(x)|\leq c\delta^{3/2}\left(\int_{\Omega_1}|\frac{\partial v_\delta}{\partial s}|^2dx\right)^{1/2} \tag{3.80}$$

which gives then the result (3.61). ∎

Comments.

The error estimates of Theorem 3.3 implies similar error estimates in the norm $H^{1/2}(\Gamma)$ that are

$$\|u-u_1^\delta\|_{H^{1/2}(\Gamma)}\leq c\delta^{5/2} \tag{3.81}$$

$$\|u-u_2^\delta\|_{H^{1/2}(\Gamma)}\leq c\delta^{5/2} \tag{3.82}$$

$$\|u-u_3^\delta\|_{H^{1/2}(\Gamma)}\leq c\delta^{3/2}. \tag{3.82}$$

These estimates correspond to an error in δ^3 for u_1^δ and u_2^δ and an error in δ^2 for u_3^δ, says in the $L^2(\Gamma)$ norm.

Neumann problem.

We have the following theorem.

Theorem 3.4. *If g is sufficiently regular, the error estimates between u solution of (3.28) with Neumann condition, and respectively $u_1^\delta, u_2^\delta, u_3^\delta$ solution of the Helmholtz equation with boundary conditions (3.46) (3.47) and (3.48) are*

$$\|u - u_1^\delta\|_{H^1(\Omega_R)} \leq c\delta^3; \tag{3.84}$$

$$\|u - u_2^\delta\|_{H^1(\Omega_R)} \leq c\delta^3; \tag{3.85}$$

$$\|u - u_3^\delta\|_{H^1(\Omega_R)} \leq c\delta^2. \tag{3.86}$$

Proof. The proof is very similar to that of the Theorem 3.4. The interpolate $\pi_\delta u$ is the function constant with value $u(x)$

$$\pi_\delta u(y) = u(x); \qquad y = x + sn(x). \tag{3.87}$$

So that we have

$$u(y) - \pi_\delta u(y) = \int_0^s \frac{\partial u}{\partial s}(x,\tau)d\tau \tag{3.88}$$

$$\frac{\partial u}{\partial x_i}u(y) - \frac{\partial}{\partial x_i}\pi_\delta u(y) = \int_0^s \frac{\partial^2 u}{\partial x_i \partial s}(x,\tau)d\tau. \tag{3.89}$$

The normal derivative $\partial u/\partial s$ and its tangential derivatives are bounded by $c\delta$. So we have

$$|u(y) - \pi_\delta u(y)| \leq c\delta^2; \tag{3.90}$$

$$|\nabla_{\Gamma_s} u(y) - \nabla_{\Gamma_s}\pi_\delta u(y)| \leq c\delta^2; \tag{3.91}$$

We have

$$\begin{cases} a(u - \pi_\delta u, v_\delta) = -\alpha_1 \int_{\Omega_1} \Delta_{\Gamma_s}(u - \pi_\delta u)v_\delta dy \\ \qquad\qquad\qquad -\beta_1 \int_{\Omega_1}(u - \pi_\delta u)v_\delta dy. \end{cases} \tag{3.92}$$

The operator Δ_{Γ_s} can be expressed in terms of the first and second derivates with respect to x_i. So we have

$$|\Delta_{\Gamma_s} u(y) - \Delta_{\Gamma_s} u(x)| \leq c\delta^2. \tag{3.93}$$

Using (3.90) (3.93) we obtain

$$|a(u - \pi_\delta u, v_\delta)| \leq c\delta^3 \|v\|_{L^2(\Gamma)}. \tag{3.94}$$

This ends the proof for the error estimate (3.84).

To prove (3.85) and (3.86) we need to estimate the neglected terms. they are of the form

$$\left| \int_\Gamma \alpha(x,\delta)u(x)v_\delta(x)d\gamma \right| \leq \sup|\alpha(x,\delta)u(x)| \left(\int_\Gamma |v_\delta(x)|^2 d\gamma \right)^{1/2} \tag{3.95}$$

and

$$\begin{cases} \int_\Gamma (A(x,\delta)\nabla_\Gamma u(x)\cdot\nabla_\Gamma v_\delta(x))d\gamma(x) \\ \qquad = -\int_\Gamma \ div_\Gamma(A(x,\delta)\nabla_\Gamma u(x))v_\delta(x)d\gamma(x) \end{cases} \tag{3.96}$$

$$|\int_\Gamma (A(x,\delta)\nabla_\Gamma u(x)\cdot\nabla_\Gamma v_\delta(x))d\gamma(x)| \le \sup|div(_\Gamma A(x,\delta)\nabla_\Gamma u(x))|\left(\int_\Gamma |v_\delta(x)|^2 d\gamma\right)^{1/2}. \tag{3.97}$$

The tangential derivatives of u appearing in (3.97) are bounded and so this term is the same order as $A(x,\delta)$. In case u_2^δ the neglected terms are in δ^3 and this gives (3.85). In case u_3^δ the neglected terms are in δ^2 and this gives (3.86). ∎

3.4. A Model Problem with Explicit Error Expressions.

Let us consider the simple example of the two dimensional Helmholtz equation in a half space. In this case we can get explicit error formulas for the different boundary approximations

$$\begin{aligned} \Delta u + ku = 0, \qquad & x > 0, \quad -\infty < y < \infty, \quad (\Omega_\infty), \\ \Delta u + k_1 u = 0, \qquad & -\delta < x < 0, \quad -\infty < y < \infty, \quad (\Omega_1). \end{aligned}$$

The Neumann boundary and interface conditions are,

$$\begin{aligned} x = -\delta: \qquad & \frac{\partial u}{\partial x} = 0, \\ x = 0: \qquad & u \ \ continuous, \\ x = 0: \qquad & \frac{\partial u(x_+,y)}{\partial x} = z\frac{\partial u(x_-,y)}{\partial x}. \end{aligned}$$

We shall study reflection of an incoming plane wave,

$$\begin{aligned} u &= e^{i\alpha x} + Re^{-i\alpha x}, \quad x \in \Omega_\infty, \\ \alpha &= \sqrt{k^2-\eta^2}, \quad k^2 > \eta^2, \end{aligned}$$

where R is the reflection coefficient and η the wave number in the y-direction. The solution in Ω_1,

$$\begin{aligned} u &= Ae^{i\beta x} + Be^{-i\beta x}, \quad x \in \Omega_1 \\ \beta &= \sqrt{k_1^2-\eta^2} \end{aligned}$$

is coupled to the solution in Ω_∞ through the interface conditions. We get the following equations for A, B and R,

$$\begin{aligned} i\beta\gamma^{-1}A - i\beta\gamma B = 0 \qquad & (from \ \ (2.2.9)), \\ A + B - R = 1 \qquad & (from \ \ (2.2.6)), \\ i\beta zA - i\beta zB + i\alpha R = i\alpha \qquad & (from \ \ (2.2.c)), \\ \gamma = e^{i\beta\delta}. \qquad & \end{aligned}$$

The solution of this linear system implies,

$$R = \frac{\alpha(1+\gamma^{-2}) - \beta z(1-\gamma^{-2})}{\alpha(1+\gamma^{-2}) + \beta z(1-\gamma^{-2})}.$$

For real z we have,

$$|R| = 1.$$

Note also that,

$$\begin{aligned} R = 1 \quad for \quad & \beta\delta = 0, \ mod\ \pi \ (\gamma = 1), \\ R = -1 \quad for \quad & \rho\delta = \tfrac{\pi}{2}, \ mod\ \pi \ (\gamma = -1). \end{aligned}$$

From Section 3.1 we get the half order boundary approximation,

$$x = 0 : \frac{\partial u}{\partial x} = -z\delta k_1^2 u,$$

the first order approximation

$$x = 0 : \frac{\partial u}{\partial x} = -z\delta(k_1^2 u + \frac{\partial^2 u}{\partial y^2}), \quad or$$
$$x = 0 : \frac{\partial u}{\partial x} = -z\delta(k_1^2 - \eta^2)u,$$

and the second order approximation

$$x = 0 : \frac{\partial u}{\partial x} = -z\delta(k_1^2 + (1 + \frac{z\delta^2 k_1^2}{3})\frac{\partial^2}{\partial y^2} + \frac{\delta^2}{3}\frac{\partial^4}{\partial y^4})u$$

If these boundary conditions are applied to a solution of the form (2.3.a) we get the reflection coefficients,

$$R_{\frac{1}{2}} = \frac{i\alpha + z\delta k_1^2}{i\alpha - z\delta k_1^2},$$
$$R_1 = \frac{i\alpha + z\delta\beta^2}{i\alpha - z\delta\beta^2},$$
$$R_2 = \frac{3i\alpha + z\delta\beta^2(3 + \beta^2\delta^2)}{3i\alpha - z\delta\beta^2(3 + \beta^2\delta^2)}$$

The *Dirichlet boundary condition* leads to

$$x = -\delta \qquad u = 0$$
$$R = \frac{\alpha(1 - \gamma^{-2}) - \beta z(1 + \gamma^{-2})}{\alpha(1 - \gamma^{-2}) + \beta z(1 + \gamma^{-2})}.$$

First order approximation.

$$\frac{\partial u}{\partial x} = \frac{z}{\delta}u$$
$$R_1 = -\frac{z - i\alpha\delta}{z + i\alpha\delta}$$

Third order approximation.

$$\frac{\partial u}{\partial x} = \frac{z}{\delta}(u - \frac{\delta z}{3}\Delta_\Gamma u)$$
$$R_3 = -\frac{z(1 + \frac{\delta\alpha^2}{3}) - i\alpha\delta}{z(1 + \frac{\delta\alpha^2}{3}) + i\alpha\delta}$$

4. Maxwell's Equations.

The geometry of the problem is the same as in the case of Helmholtz equation. We will consider only the case of one layer to simplify the exposure.

The unknowns are the electric field E and the magnetic field H. If ω is the pulsation, μ_∞ and ε_∞ the permeability and the permitivity in the domain Ω_∞, μ_1 and ε_1 the permeability and the permitivity in the domain Ω_1,

The equations are

$$
\begin{aligned}
&curl\ E - i\omega\mu_0\ H = K; \quad in\ \ \Omega_\infty;\\
&curl\ H + i\omega\varepsilon_0\ E = L; \quad in\ \ \Omega_\infty\\
&curl\ E_1 - i\omega\mu_1\ H_1 = 0; \quad in\ \ \Omega_1;\\
&curl\ H_1 + i\omega\varepsilon_1\ E_1 = 0; \quad in\ \ \Omega_1
\end{aligned}
\tag{4.1}
$$

The transmission condition on the surface Γ are

$$
\begin{cases} H_1 \times n = H \times n \\ E_1 \times n = E \times n. \end{cases}
\tag{4.2}
$$

We must add a radiation condition at infinity which in case of a 3D problem is

$$
\frac{\partial E}{\partial n} - ikE = o\left(\frac{1}{|x|}\right)\ ; \quad k = \omega_0\sqrt{\mu_\infty\varepsilon_\infty}
\tag{4.3}
$$

or equivalently

$$
\frac{\partial H}{\partial n} - ikH = o\left(\frac{1}{|x|}\right)
\tag{4.4}
$$

We also need a boundary condition on the boundary Γ_δ. We will here consider the perfectly conducting condition which is

$$
E \times n = 0 \quad on\ \Gamma_\delta.
\tag{4.5}
$$

To obtain the approximate boundary condition, we start from the formula (2.23). We have up to order δ

$$
curl E \times n = \tfrac{\partial}{\partial n} E_T - \nabla_{\Gamma_s}(E \cdot n)
\tag{4.6a}
$$

$$
curl H \cdot n = curl_{\Gamma_s} H_T = div_{\Gamma_s}(H \times n)
\tag{4.6b}
$$

We integrate these equations along the normal. Using the boundary condition and the differential equations we obtain,

$$
E_T(x) = \int_0^\delta \nabla_{\Gamma_s}(E_1 \cdot n)ds + i\omega_0\mu_1 \int_0^\delta H_1 \times n ds
\tag{4.7}
$$

We also have from the second equation,

$$
i\omega_0\varepsilon_1 \int_0^\delta E_1 \cdot n ds + \int_0^\delta \ div_{\Gamma_s}(H_s \times n)ds = 0.
\tag{4.8}
$$

The effective conditions of order zero is just $E \times n = 0$.

For the first order conditions suppose that E and H has linear variation along the normal n. Then (4.7) is approximated by

$$
E_T(x) = \delta\ \nabla_\Gamma(E_1 \cdot n) + i\omega_0\mu_1\delta\ H_1 \times n
$$

and (4.8) is approximated by,

$$
i\omega_0\varepsilon_1\delta(E_1 \cdot n) + \delta_1\ \ div_\Gamma(H_1 \times n) = 0.
$$

These lead to the approximate boundary condition

$$E_T(x) = i\delta_1(\frac{1}{\omega_0\varepsilon_1}\nabla_\Gamma div_\Gamma(H(x)\times n) + \omega_0\mu_1\ H(x)\times n) \tag{4.9}$$

5. Conclusions and Extensions. Let us finally give a few different applications for which the techniques discussed in this paper may be useful.

Our assumption that the core Ω_0 is perfectly conducting can be replaced by a problem for which the electro-magnetic equations are also valid in Ω_0. The concentration of the layers outside Ω_0 will then produce an approximative interface condition on Γ.

Another variant of our original problem is layers which are not homogeneous in the directions tangential to Γ. If these inhomogeneities are regular, a tangential homogenization can first be applied and then the concentration to approximative boundary conditions. If the thickness vary smoothly as a function of the tangential variables the lower order approximative boundary conditions can be applied directly with a locally defined δ.

The technique discussed above is not only applicable to electro-magnetic problems. The equations for elastic wave propagations are similar and the thin layer approximations are also possible in that case.

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