Kinetic formulation for chromatography and some other hyperbolic systems

F. James[(1)] Y.-J. Peng[(2)] B. Perthame[(3)]
Département de Mathématiques, Faculté des Sciences,
Université d'Orléans, BP 6759,
45067 Orléans Cedex 2, FRANCE

Abstract

We present several new examples of kinetic formulations for some hyperbolic systems, which is to write an equation for a whole family of entropies. It appears that, in fact, several families of entropies can be represented in such a way. For instance, in the $N \times N$ chromatography case, we obtain $N+1$ kinetic representations from which we recover the invariant regions and the stability for weakly convergent initial data by a compensated compactness argument, including the non-strictly hyperbolic case.

Résumé

Nous présentons quelques nouveaux exemples de formulations cinétiques de systèmes hyperboliques, ce qui consiste à écrire une équation pour toute une famille d'entropies. Il apparaît qu'en fait plusieurs familles différentes peuvent se représenter ainsi. Par exemple, dans le cas du système $N \times N$ de la chromatographie, nous obtenons $N+1$ représentations cinétiques. Nous retrouverons ainsi les zones invariantes et la stabilité, pour des données initiales convergeant faiblement, grâce à un argument de compacité par compensation, incluant le cas non-strictement hyperbolique.

Key-words : hyperbolic systems - kinetic equations - entropy - compensated compactness - chromatography.

Mots clés : systèmes hyperboliques - équations cinétiques - entropies - compacité par compensation - chromatographie.

AMS Class numbers : 35L65, 35L67, 35Q20

1 Introduction

We complement, on some new examples such as the system of chromatography, the theory of kinetic formulations of hyperbolic systems. This theory, introduced by P.L. Lions, B. Perthame and E. Tadmor [11], consists in representing a whole family of entropy inequalities by a single equation, using a supplementary variable, for the fundamental solution of the wave equation for entropies. It can be used to derive rather easily some properties of the system based on the manipulation of entropies : invariant regions, regularizing effects in L^∞ for averages in time, compensated compactness applied to prove the stability for weakly convergent initial data. Our progresses are going into two directions.

First, we are able to derive a kinetic formulation for the $N \times N$ chromatography system, very close to that of [12]. The new point here, is that we obtain $N+1$ kinetic equilibria and equations, which are necessary to get the N maximum *and* minimum principles on the Riemann Invariants, a very specific property of this kind of systems. Also, $N+1$ kinetic equations are necessary to perform the compensated compactness argument as in [12] for $\gamma \geq 3$, or [13] for the p-system. Let us recall that A. Heibig [9] has proved the weak stability of solutions to systems in the Temple class, including chromatography, using compensated compactness. The electrophoresis was also previously studied by D. Serre [19]. For the chromatography system, we extend Heibig's result to treat the non-strictly hyperbolic case which corresponds to a more general set of possible initial data.

Secondly, it turns out that in general the possible representation formulæ for entropies is larger than expected. For 2×2 systems the choice can be made at least among six families and not only two as in [12]. In order to be more specific, let us consider the case of our examples (2×2 chromatography, Leroux system, p-system, gas dynamics with a γ-law). Then, the usual families of entropies can be written in terms of the Riemann Invariants w_1, w_2 (say $w_1 \leq w_2$) as

$$\chi_{\rm W}(x,\xi,t) = \alpha(w_1,w_2)\ [(w_2-\xi)(\xi-w_1)]_+^\lambda, \tag{1.1}$$

$$\chi_{\rm S}(x,\xi,t) = \beta(w_1,w_2)\ [(w_2-\xi)(\xi-w_1)]_+^\mu, \tag{1.2}$$

where $x_+ = \max(0,x)$ and α, β and λ, μ depend on the examples. $\chi_{\rm W}$ and $\chi_{\rm S}$ here denote the "weak" and "strong" entropy waves and they depend upon a parameter ξ ; they are the fundamental solutions of the wave equation for entropies (see A.V. Bitsadze [1]). Considering for instance the weak entropy wave, two other families of entropies are just given by

$$\chi_{\rm W}^-(x,\xi,t) = \alpha(w_1,w_2)\ [(w_2-\xi)(w_1-\xi)_+]^\lambda, \tag{1.3}$$

$$\chi_{\rm W}^+(x,\xi,t) = \alpha(w_1,w_2)\ [(\xi-w_2)_+(\xi-w_1)]^\lambda. \tag{1.4}$$

Similar formulæ on the strong family yield the six families on entropies of the 2×2 system that we consider. From these "fundamental solutions", we recover entropies by

integration, whenever this is meaningful (since λ can be negative)

$$S(w_1, w_2) = \int \chi(w_1, w_2, \xi) g(\xi) d\xi. \tag{1.5}$$

The main difficulty is to characterize the functions $g(\xi)$ which give, in (1.5), convex entropies. We have no automatic rule, and this generally selects, among the six possible χ, the good one or the good ones. For the chromatography system, five among them will work, for isentropic gas dynamics only one works. This is strongly related to the invariant regions which are much stronger for the chromatography system than for the isentropic gas dynamics. Of course, this analysis cannot be performed for $N \times N$ systems, even for the "rich systems" introduced by D. Serre [17, 18], because there is no wave equation underlying the representation of entropies, but a Goursat problem which is somewhat different. As noticed by A. Heibig [8], the kinetic formulation deduced from these representations is rather hard to use.

We shall give the details of the kinetic formulation for the chromatography system in Section 2. In Section 3, we show how to use it in order to recover the invariant regions and weak stability by the compensated compactness method. Finally, we will illustrate in Section 4 the way to use the strong entropy waves on several examples: p-system, chromatography... Notice that the Section 4 is widely independent of the beginning. Also, particular results for chromatography are proved together in an appendix. Several extensions to electrophoresis or other state laws for chromatography, are possible. We leave these systems for a future work. In these cases the dispersion lemmas of [12] seem to apply because large velocities can occur in the system. Also let us recall that a possible investigation is to apply averaging lemmas, as in [11], to recover compactness.

2 Kinetic formulation of the chromatography system.

We first consider a $N \times N$ system which appears in chemical engineering and called chromatography for Langmuir isotherms (see H.K. Rhee, R. Aris and N.R. Admundson [15] for the applications, variants and origins of this model). The unknowns are N functions $u_i(x,t)$ solutions of

$$\partial_t u_i + \partial_x \frac{k_i u_i}{D} = 0,\ t \geq 0,\ x \in \mathbb{R},\ 1 \leq i \leq N, \tag{2.1}$$

$$u_i(x,0) = u_i^\circ(x) \geq 0, \tag{2.2}$$

where the k_i are given numbers,

$$0 < k_1 < k_2 < \cdots < k_N, \tag{2.3}$$

$$D = 1 + u_1 + u_2 + \cdots + u_N. \tag{2.4}$$

We set in the following

$$u(x,t) = (u_1(x,t), ..., u_N(x,t)). \tag{2.5}$$

For this system we shall exhibit $N+1$ one-parameter families of entropies, which forms are very close to those used in [12]. We will use these entropies to write a kinetic formulation of the system (2.1)-(2.4). This means that we can easily identify the convex entropies in this family and it turns to be simpler than the isentropic gas dynamics case. We give bounds on the measure appearing in this formulation and applications are left for the next section. This requires to know some algebraic properties of the system which are recalled in a first subsection, the special entropies we will use are presented in subsection 2.2. Then, we give the kinetic formulation (subsection 2.3). The results are proved in subsection 2.4.

2.1 Some properties of the chromatography system

The system (2.1)-(2.4) has been widely studied (E. Canon and F. James [2],[3], H.K. Rhee and al [15], D. Serre [16]) and we recall some fundamental points in the next proposition. First, it is a non-strictly hyperbolic system i.e. the Jacobian matrix $A(u) = \left(\frac{\partial}{\partial u_j}\left(\frac{k_i u_i}{D}\right)\right)_{1\le i,j\le N}$ admits N real eigenvalues. We have $A(u) = J(u)/D(u)$, where

$$J(u) = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_N \end{pmatrix} - \frac{1}{D}\begin{pmatrix} k_1u_1 & \dots & k_1u_1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ k_Nu_N & \dots & k_Nu_N \end{pmatrix}, \tag{2.6}$$

and it admits N strong Riemann Invariants.

Proposition 2.1 *We consider the system (2.1) for $u_i \ge 0$, $1 \le i \le N$.*
(i) it admits N eigenvalues

$$0 < \lambda_1(u) \le \lambda_2(u) \le \cdots \le \lambda_N(u);$$

(ii) $w_i = \lambda_i D$ is a i-Riemann Invariant, i.e. for smooth solutions of (2.1), we have

$$\partial_t w_i + \lambda_i \partial_x w_i = 0; \tag{2.7}$$

(iii)
$$0 \le w_1 \le k_1 \le w_2 \le k_2 \le \cdots \le k_{N-1} \le w_N \le k_N;$$
and $u_i = 0$ is equivalent to $w_i = k_i$ or $w_{i+1} = k_i$;

(iv)
$$D = \prod_{i=1}^{N} \frac{k_i}{w_i},$$

(v)
$$u_i \prod_{j\ne i}\left(1 - \frac{k_i}{k_j}\right) = -\prod_{j=1}^{N}\left(1 - \frac{k_i}{w_j}\right);$$

(vi) $$\sigma_j(u) = \sum_{1\le i_1,\dots,i_j\le N} \frac{1}{w_{i_1}\dots w_{i_j}}, \quad \text{for } 1\le j\le N, \quad \sigma_0 = \prod_{i=1}^{N} k_i,$$
are $N+1$ independent affine functions of $(u_1,\dots,u_N)$.

Of course these properties are very strong. D. Serre [17], [18], called (i), (ii) richness. In fact this system also belongs to the Temple class (B. Temple [21], D. Serre [18], A. Heibig [9]) for which some existence and uniqueness results are known when they are strictly hyperbolic. Notice that the results in Proposition 2.1, as well as in Lemma 2.3 below, were already mentioned by D. Serre in [18, 19].

In order to go directly to the essential point, the particular entropies which give our kinetic formulation, we postpone the proof of Proposition 2.1 to an appendix.

Remark 2.2 Let us point out an important point (see [3] for further details). The properties (i) and (iii) allow a degeneracy of the system (two equal eigenvalues). This can only happen for $w_i = w_{i+1} = k_i$, then $\lambda_i = \lambda_{i+1}$, and $u_i = 0$. It requires that initially $w_i^0(x) = w_{i+1}^0(x) = k_i$ for some $x \in \mathbb{R}$.

2.2 Some special entropies

In this subsection, we build some particular pairs entropy-entropy fluxes (E,F), this means additional conservation laws, valid for smooth solutions of (2.1),

$$\partial_t E + \partial_x F = 0. \tag{2.8}$$

It is easier to look for functions depending upon the Riemann Invariants

$$E = E(w_1,...,w_N),\ F = F(w_1,...,w_N), \tag{2.9}$$

rather than the conservative variables. Using the equation (2.7) for w_i, (2.8) then reduces to the following $N(N-1)/2$ Euler-Darboux equations on E, for $i \neq j$,

$$(\lambda_i - \lambda_j)\frac{\partial^2 E}{\partial w_i\,\partial w_j} + \frac{\partial \lambda_i}{\partial w_j}\frac{\partial E}{\partial w_i} - \frac{\partial \lambda_j}{\partial w_i}\frac{\partial E}{\partial w_j} = 0. \tag{2.10}$$

When E is known, we recover F by

$$\frac{\partial F}{\partial w_i} = \lambda_i \frac{\partial E}{\partial w_i}, \quad 1\le i\le N. \tag{2.11}$$

A first step toward the kinetic formulation is to notice that

Lemma 2.3 *For each $\xi \in \mathbb{R}^+$ the function*

$$E_0(\xi; w_1,...,w_N) = \prod_{i=1}^{N}\left(1 - \frac{\xi}{w_i}\right) \tag{2.12}$$

is an entropy, affine with respect to u. The entropy flux is given by

$$F_0(\xi; w_1, ..., w_N) = \frac{\xi E_0}{D}. \tag{2.13}$$

Proof of Lemma 2.3 Using the relation $w_i = \lambda_i D$ (Proposition 2.1), we obtain

$$\frac{\partial \lambda_i}{\partial w_j} = \frac{\delta_{ij}}{D} - \frac{w_i}{D^2}\frac{\partial D}{\partial w_j} = \left(\delta_{ij} + \frac{w_i}{w_j}\right)\frac{1}{D}, \tag{2.14}$$

since $\dfrac{\partial D}{\partial w_i} = -\dfrac{D}{w_i}$ (Proposition 2.1, (iv)). Therefore the relation (2.10) becomes, in the chromatography case,

$$(w_i - w_j)\frac{\partial^2 E}{\partial w_i\, \partial w_j} + \frac{w_i}{w_j}\frac{\partial E}{\partial w_i} - \frac{w_j}{w_i}\frac{\partial E}{\partial w_j} = 0 \tag{2.15}$$

It is clear that E_0 defined by the formula (2.12) solves (2.15), and thus, it is an entropy.

Next, we develop the polynomial in ξ defined by (2.12).

$$E_0 = \sum_{j=0}^{N} (-1)^j\ \sigma_j \xi^j, \tag{2.16}$$

where the σ_j are defined in Proposition 2.1 (v), from which we deduce that E_0 is affine in u. Finally, the function F_0 defined in (2.13) satisfies indeed

$$\begin{aligned}
\frac{\partial F_0}{\partial w_i} &= \frac{\xi}{D}\left(\frac{\partial E_0}{\partial w_i} - \frac{E_0}{D}\frac{\partial D}{\partial w_i}\right) \\
&= \frac{\xi}{D}\left(\frac{\partial E_0}{\partial w_i} + \frac{E_0}{w_i}\right) \\
&= \frac{\xi}{D}\left(1 + \frac{w_i - \xi}{\xi}\right)\frac{\partial E_0}{\partial w_i},
\end{aligned}$$

because $\frac{\partial E_0}{\partial w_i} = \frac{\xi}{w_i}\,\frac{\partial E_0}{w_i - \xi}$. Since $w_i = \lambda_i D$ we obtain the relation (2.11) and thus F_0 is the entropy flux defined up to a constant $C(\xi)$.

A particular phenomenon occurs for the pair (E_0, F_0) given by (2.12), (2.13). The functions both vanish for $\xi = w_i$, so that (2.8) also yields

$$\partial_t \left(E_0 \mathbb{1}_{]w_i, w_{i+1}[}\right) + \partial_x \left(F_0 \mathbb{1}_{]w_i, w_{i+1}[}\right) = 0. \tag{2.17}$$

This means that we have now built $N+1$ families of nonlinear entropies. We take the convention $w_0 = 0,\ w_{N+1} = +\infty$ and we deduce from (2.17) the

Corollary 2.4 *For $0 \leq i \leq N$, the function*

$$\chi_i(\xi; w_1, \ldots, w_N) = \prod_{j=1}^{N} |1 - \frac{\xi}{w_j}| . \mathbb{1}_{\{\xi \in]w_i, w_{i+1}[\}} \tag{2.18}$$

is an entropy of the system (2.1), convex in u, with the entropy flux

$$\phi_i(\xi; w_1, \ldots, w_N) = \frac{\xi}{D} \chi_i. \tag{2.19}$$

Proof of corollary 2.4. After (2.17), it remains to notice that, since E_0 is affine in u and vanishes of sign at w_j, $|E_0| \mathbb{1}_{]w_i, w_{i+1}[}$ is indeed convex in u.

Remark 2.5 This way to build new entropies using the positive part of another one is rather usual in the literature. That is the way we can obtain the entropies used by R. Di Perna [6] [7], G.Q. Chen [4], or [12]. Also, for general systems in the Temple class, D. Serre [18] builds another N parameters family of entropies using the positive part of an affine entropy. The relation with ours is still unclear to the authors.

2.3 Kinetic formulation

We are now interested in solutions with shocks of the system (2.1). Therefore, the entropy equality (2.8) cannot hold true, see P.D. Lax [10] for instance, and it has to be imposed (for uniqueness) as an inequality:

$$\partial_t E + \partial_x F \leq 0 \quad \text{in } \mathcal{D}'([0, +\infty[\times \mathbb{R}). \tag{2.20}$$

We shall call "entropy solution of (2.1)", a solution satisfying (2.20) for any convex entropy in the particular class defined by

$$E(u) = \int_{\mathbb{R}^+} g(\xi) \chi_i(\xi; u) d\xi, \tag{2.21}$$

for some $0 \leq i \leq N$, $g \in L^1(\mathbb{R}^+)$. Following [12], from (2.20)-(2.21) we derive the kinetic formulation of the system (2.1) :

$$\begin{cases} \text{there are } N+1 \text{ non-positive measures } m_i(x, \xi, t) \text{ such that} \\ \partial_t \chi_i(\xi, u(x,t)) + \partial_x \dfrac{\xi}{D} \chi_i(\xi, u(x,t)) = m_i \text{ in } \mathcal{D}'(\mathbb{R}_x \times \mathbb{R}^+_\xi \times \mathbb{R}^+_t) \end{cases} \tag{2.22}$$

This is very close to a transport equation because of the additional kinetic variable ξ, and explains the terminology "kinetic formulation" : $\chi_i(\xi, u)$ looks like a Gibbs equilibrium in physical kinetics.

The relation between (2.1) and (2.22) is more precisely stated in the

Theorem 2.6 *Let $u \in L^\infty(\mathbb{R}^+ ; L^1(\mathbb{R}))^N$, $u_i \geq 0$ a.e. for $1 \leq i \leq N$. Then u is an entropy solution of (2.1),(2.2) if and only if $\chi_i(\xi, u)$, $0 \leq i \leq N$, satisfy (2.22), (2.2) and (iii) below. Moreover, the measures m_i in (2.22) satisfy*
(i) For some constant $C_1(k_1, ..., k_N)$

$$0 \leq -\int_{t=0}^{+\infty} \int_{x\in\mathbb{R}} \sum_{i=0}^{N} m_i(x,\xi,t)dxdt \leq C_1(1+\xi^N)\|u^\circ\|_{L^1(\mathbb{R})^N}$$

(ii)

$$\left|\frac{\partial}{\partial\xi}\int_{t=0}^{\infty} \int_{x\in\mathbb{R}} \sum_{i=0}^{N} m_i(x,\xi,t)dxdt\right| \leq C_1(1+\xi^{N-1})\|u^\circ\|_{L^1(\mathbb{R})^N}$$

(iii)

$$m_i(x,k_i,t) = 0 \quad , \quad 0 \leq i \leq N.$$

The following result is just an easy variant of Theorem 2.6, because of the finite speed of propagation. We give it here because it will be necessary later on to state a precise result on the invariant regions.

Theorem 2.7 *Let $u \in L^\infty(0,T; L^1_{loc}(\mathbb{R}))$, $u_i \geq 0$ a.e. for $1 \leq i \leq N$. Then u is an entropy solution of (2.1) if and only if $\chi_i(\xi, u)$, $0 \leq i \leq N$, satisfy (2.22) and (iii) of Theorem 2.6. Moreover we have*

$$0 \leq -\int_{[0,T]\times[-R,R]} \sum_{i=0}^{N} m_i(x,\xi,t)dxdt \leq C(R,T)(1+\xi^N)\|u\|_{L^\infty(0,T;L^1(-R,R))};$$

the same estimate, replacing $\int \sum m_i$ by $\left|\frac{\partial}{\partial\xi}\int \sum m_i\right|$, and ξ^N by ξ^{N-1}, also holds.

Remark 2.8

- For a given i, χ_i, and thus m_i, has its support in ξ included in the interval $[k_i, k_{i+1}]$. Thus, the estimates (i) and (ii) give a uniform bound on m_i except for $i = N$ (then $k_{N+1} = +\infty$). A consequence of (i) and (ii) is that the total mass in x, t of m_i is continuous in ξ and thus (iii) is meaningful.
- In particular, we recover the original system (2.1) from (2.22), up to a multiplicative constant, taking $\xi = k_i$ in the equation for χ_i. Indeed, from Proposition 2.1 (v) we deduce that

$$\chi_i(k_i, u) = u_i \prod_{j\neq i} \left|1 - \frac{k_i}{k_j}\right|, \quad 1 \leq i \leq N, \tag{2.23}$$

and (iii) concludes.

- Another simple entropy is obtained by (see Proposition 2.1 (iv))

$$\lim_{\xi\to+\infty}\frac{\chi_N(\xi,u)}{\xi^N}=\frac{D}{\prod_{j=1}^N k_j},$$

but we do not know how to recover the physical entropy $\sum u_i\log u_i$.

- The nonnegativity of solutions $u_i(x,t)$ for $u_i^0\geq 0$, is in fact a consequence of the invariant regions that we present in Section 3.1.

Finally, let us point out that, compared to the kinetic formulation of the isentropic gas dynamics or p-system given in [11], [12], (2.22) is rather simpler because the right hand side is just a measure and not a derivative. Notice also that the $N+1$ equations in (2.22) seem necessary to use this formulation in order to derive all the set of invariant regions and to perform the compensated compactness method in the "easy way", see Section 2.

2.4 Proof of Theorem (2.6)

If (2.20) holds for convex entropies given by (2.21), then (2.22) holds also. Indeed, it is enough to take $g(\xi)=\delta(\xi-\xi_0)$ (or a sequence of $g_n\geq 0$, which converges weakly to this g), and we know from corollary 2.4 that $\chi(\xi_0,u)$ is convex in u. The other sense is also clear : from (2.22) we recover (2.20), for any entropy of the form (2.21) with $g(\xi)\geq 0$ (except for $\xi=k_i$), which is a necessary and sufficient condition for E to be convex. Finally, we also recover the equalities in the system (2.1) thanks to (iii) as indicated in Remark 2.8.

It remains to obtain the bounds (i) (ii) on the m_i. Integrating (2.22), we obtain

$$-\int_{[0,T]\times[-R,R]} m_i(x,\xi,t)dxdt=\int_0^T\frac{\xi}{D}\chi_i(\xi,u(x,t))|_R^{-R}\,dt+\int_{-R}^R\chi_i(\xi,u(x,t))|_T^0\,dx. \tag{2.24}$$

Adding these equalities, we obtain since $|a-b|\leq|a-c|+|b-c|$ and, by (2.16)

$$\sum_i\chi_i=|E_0|=\left|\sum_{j=0}^N(-1)^j\sigma_j(u)\xi^j\right|,$$

$$\begin{aligned}-\int_{[0,T]\times[-R,R]}\sum_{i=0}^N m_i(x,\xi,t)dxdt&=\int_0^T\frac{\xi}{D}|E_0(\xi,u)|dt|_R^{-R}+\int_{-R}^R|E_0(\xi,u)|dx|_T^0\leq\\&\leq\sum_{j=0}^N\xi^j\left\{\int_0^T\xi(|\sigma_j(u(R,t))-\sigma_j(0)|+|\sigma_j(u(-R,t))-\sigma_j(0)|)dt\right.\\&\qquad\left.+\int_{-R}^{+R}(|\sigma_j(u(x,T))-\sigma_j(0)|+|\sigma_j(u^\circ(x))-\sigma_j(0)|)dx\right\}\end{aligned} \tag{2.25}$$

But we have, for some constant M,

$$|\sigma_j(u) - \sigma_j(0)| \ \leq \ M(u_1 + u_2 + \cdots + u_n)$$

because $u \mapsto \sigma_j(u) - \sigma_j(0)$ is a linear function. Hence, we can choose a sequence $R_n \to \infty$ of Lebesgue points of $\int_0^T (u_1 + \cdots + u_N)(R,t)dt$ and the first integral in the right hand side of (2.25) vanishes. Thus, we obtain

$$\begin{aligned} -\int_{[0,T]\times\mathbb{R}} \sum_{i=1}^{N} m_i(x,\xi,t)dxdt \ \leq \\ \leq MN(1+\xi^N)\left[\int_{\mathbb{R}} (u_1 + \cdots + u_N)(x,T)dx + \int_{\mathbb{R}} (u_1 + \cdots + u_N)(x,0)dx\right] \\ \leq 2MN(1+\xi^N)\|u^0\|_{L^1(\mathbb{R})} \end{aligned}$$

and (i) is proved by letting $T \to +\infty$. The proof of (ii) goes in the same way after differentiating (2.24) in ξ. The proof of the Theorem 2.7 is similar and we skip it.

3 Applications of the kinetic formulation to the chromatography system.

In this section, we shall give the applications of the kinetic formulation (2.22) of the chromatography system (2.1), along the lines of [12]. We have two applications in mind. First, we recover the maximum and minimum principle on the Riemann Invariants, which are clearly related to the existence of $N+1$ kinetic equations for convex entropies. Second, we show how to handle the compensated compactness method on the χ_i.

3.1 Invariant regions.

Here we prove the

Theorem 3.1 *Let u be an entropy solution of (2.1) satisfying the assumptions of Theorem 2.7. Then a.e.*

$$k_{i-1} \leq \inf_{y\in\mathbb{R}} w_i(y,0) \leq w_i(x,t) \leq \sup_{y\in\mathbb{R}} w_i(y,0) \leq k_i \tag{3.1}$$

Remark 3.2 This result is already known for the Riemann problem or for BV solutions (see [2], [16]).

Proof of Theorem 3.1. To prove the maximum principle on w_i, we use the equation on χ_{i-1} (therefore χ_0 is used here). We take

$$g(\xi) = \mathbb{1}_{\{\xi \geq \sup w_i(y,0)\}}, \tag{3.2}$$

so that

$$\int_{\mathbb{R}^+\times\mathbb{R}} g(\xi)\chi_{i-1}(\xi,u^0(x))d\xi dx = 0.$$

From $m_{i-1} \leq 0$, we deduce

$$0 \leq \int_{\mathbb{R}^+\times\mathbb{R}} g(\xi)\chi_{i-1}(\xi,u(x,t))d\xi dx \leq 0,$$

and thus $w_i(x,t) \leq \sup_{y\in\mathbb{R}} w_i(y,0)$. The minimum principle goes in the same way, using now χ_i, $1 \leq i \leq N$.

In particular, we obtain the

Corollary 3.3 *For some constant $C > 0$*

$$C \inf_{y\in\mathbb{R}} u_j^0(y) \leq u_i(x,t) \leq \frac{1}{C} \sup_{y\in\mathbb{R}} u_j^0(y), \qquad 1 \leq j \leq N. \tag{3.3}$$

See Proposition 2.1.

3.2 Compensated compactness

In this subsection we consider a uniformly bounded family $(u_{i,n}(x,t))_{n\geq 0}$ of entropy solutions to the chromatography system (2.1). From (3.3) this is to say that the initial data $u_{i,n}(x)$ are uniformly bounded. In such a situation we may assume that as n tends to $+\infty$

$$u_{i,n}(x,t) \rightharpoonup u_i(x,t), \quad 1 \leq i \leq N \tag{3.4}$$

in $L^\infty((0,T)\times\mathbb{R})$ weak-$*$, for all $T \in (0,\infty)$.
The question, answered first by R.J. Di Perna [6], [7], for isentropic gas dynamics and strictly hyperbolic, genuinely non-linear systems, is to prove that this convergence is strong, which allows to show that u_i is an entropy solution of (2.1).

R.J. Di Perna's method was extended by G.Q. Chen [4] for isentropic gas dynamics, D. Serre [19] for electrophoresis and A. Heibig [9] for strictly hyperbolic systems in the Temple class. The advantage of the method in [9, 19] is its simplicity because it only uses explicit entropies rather than high frequency approximations like in [10], [7], [4].

The kinetic formulation allows the same kind of method because we indeed know enough entropies. It was achieved first in [12] for isentropic gas dynamics with $\gamma \geq 3$, a particularly simple case which we will mimic here. The case of isentropic gas dynamics, for the full range $1 < \gamma < 3$ is first achieved in P.L. Lions, B. Perthame, P.E. Souganidis [13] combining some ideas of [7], [4], and the kinetic formulation. In [13], the p-system is also treated.

The isentropic gas dynamics, for the full range $\gamma \geq 3$, the p-system and the $N \times N$ chromatography system can be treated in the same way, with adaptations that we present below. The case $1 < \gamma < 3$ is harder because of both the degeneracy of the system close to the vacuum and the lack of entropies. Here we shall allow the "vacuum", which corresponds to $\lambda_i = \lambda_{i+1} = k_i$, this implies $u_i = 0$. Of course, it is impossible

for bounded from below solutions. But the lack of hyperbolicity is compensated by the numerous family of entropies.

More precisely we are going to prove the

Theorem 3.4 *Let $u_{i,n}$ be as above and satisfy (3.4), then $(u_{i,n})_n$ converges pontwise to u_i, an entropy solution of (2.1).*

Proof of Theorem 3.4 Let us denote by $d\nu_{x,t}(w),\ w=(w_1,...,w_N)$, or $d\nu$, the Young measures associated to the weak limit on the Riemann Invariants. We refer to L. Tartar [20], F. Murat [14] for the definition of Young measures and the original ideas about compensated compactness. We also set for any continuous function

$$\overline{\phi} = \int \phi(w) d\nu(w).$$

First step : general proof

Now, let us notice that adding the kinetic formulations (2.22) for $0 \leq i \leq N$, we obtain

$$\partial_t G_0(\xi, w) + \partial_x \frac{\xi}{D} G_0(\xi, w) = m := \sum_{i=0}^{N} m_i, \tag{3.5}$$

with $G_0 = |E_0|$ where E_0 was defined in Lemma 2.3. Next, we apply the compensated compactness lemma to (3.5), as it was proved to hold in [12]. We obtain

$$\overline{G_0(\xi)}\ \overline{\xi' G_0(\xi')/D} - \overline{G_0(\xi')}\ \overline{\xi G_0(\xi)/D}\ =\ (\xi'-\xi)\overline{G_0(\xi)G_0(\xi')/D}. \tag{3.6}$$

Dividing by $\overline{G_0(\xi)}\ \overline{G_0(\xi')}\ (\xi'-\xi)$ and letting ξ' go to ξ, we get

$$\frac{\partial}{\partial \xi} \frac{(\xi \overline{G_0(\xi)/D})}{\overline{G_0(\xi)}}\ =\ \frac{\overline{G_0(\xi)^2/D}}{\overline{G_0(\xi)}^2}. \tag{3.7}$$

Of course (3.7) has to be justified at points where $\overline{G_0(\xi)} = 0$. This is a difficulty in particular in the non-strictly hyperbolic case, and we postpone the justification until step 2. Let us continue the proof assuming that (3.7) holds. (3.7) is not completely satisfactory because its righthand side does not vanish when $d\nu$ is a Dirac mass. Therefore, we again apply compensated compactness to (3.5) and

$$\partial_t D + \partial_x \frac{\alpha}{D} = 0, \tag{3.8}$$

$$\alpha = k_1 u_1 + \cdots + k_N u_N. \tag{3.9}$$

This yields

$$\overline{G_0(\xi)}\ \overline{\alpha/D} - \xi \overline{G_0(\xi)/D}\ \overline{D} = \overline{G_0(\xi)\alpha/D} - \xi \overline{G_0(\xi)}.$$

After dividing it by $\overline{D}\ \overline{G_0(\xi)}$ we can combine to the lefthand side of (3.7) and get

$$-\partial_\xi \frac{\overline{G_0(\xi)\alpha/D}}{\overline{G_0(\xi)}} = \frac{\overline{G_0(\xi)^2/D}}{\overline{G_0(\xi)}^2} - \frac{1}{\overline{D}}\ \geq 0, \tag{3.10}$$

this last inequality being just Cauchy-Schwarz'. The inequality (3.10) is the keystone to prove that $d\nu$ is in fact a Dirac mass.

If in (3.10) the inequality is strict at some point we get, comparing the values of the non-increasing function

$$\frac{\overline{G_0(\xi)\alpha/D}}{\overline{G_0(\xi)}},$$

at the points $\xi = 0$ and $\xi = +\infty$, then $\xi = k_i$ and $\xi = \infty$,

$$\overline{\alpha/D} > \frac{\overline{\alpha}}{\overline{D}} \quad (\text{since } G_0(0) = 1, \frac{G_0(\xi)}{\xi^N} \to_{\xi\to+\infty} \frac{D}{\prod k_i}), \tag{3.11}$$

$$\frac{\overline{u_i\alpha/D}}{\overline{u}_i} \geq \frac{\overline{\alpha}}{\overline{D}}, \tag{3.12}$$

Since $D = 1 + u_1 + \cdots + u_n$, (3.11) added to (3.12) multiplied by $\bar{u}_i$ yields

$$\overline{\alpha} = \overline{D\alpha/D} > \frac{\overline{\alpha}}{\overline{D}}\overline{D} = \overline{\alpha}.$$

Of course this means that the inequality in (3.10) is just an equality for all $\xi \geq 0$. Then, the equality case in Cauchy-Schwarz' inequality asserts the existence of a function $\lambda(\xi)$ such that, for all $\xi \geq 0$, $G_0(\xi, w) = \lambda(\xi)\, D(w)$ a.e. in the support of $d\nu(w)$.
From this we deduce that, for two possible elements w^1, w^2 of the support of $d\nu(w)$, we have

$$G_0(\xi, w^1)D(w^2) \;=\; G_0(\xi, w^2)D(w^1).$$

For $\xi = 0$, $G_0(\xi, w) = 1$, so that $D(w^1) = D(w^2)$. Next we obtain (since $D \neq 0$)

$$G_0(\xi, w^1) = G_0(\xi, w^2).$$

Thus $\sigma_j(w^1) = \sigma_j(w^2)$ and by Proposition 2.1 (iv), this proves that $w^1 = w^2$ and the support of ν is a single point.

Second step : Justification of (3.10), strictly hyperbolic case.

We now justify (3.10) at points ξ_0 such that $\overline{G_0(\xi_0)} = 0$. In particular we will see here how the degeneracy of the hyperbolicity appears. We have

$$G_0(\xi_0, w) = 0, \ \forall\, w \;\in\; \text{supp}\ \nu$$

that is $w_j = \xi_0$ for some j. We can define, for ξ in a neighborhood of ξ_0,

$$\varphi(\xi) = \frac{\overline{G_0(\xi)\alpha/D}}{\overline{G_0(\xi)}},$$

because $\overline{G_0(\xi)} \neq 0$. Since (3.10) holds when $\overline{G_0(\xi)} \neq 0$, we only have to prove the continuity of φ at ξ_0. By l'Hospital's rule, when $\xi \to \xi_0$ with $\xi > \xi_0$ we have

$$\varphi(\xi) \to \frac{\overline{G_0'(\xi_0^+)\alpha/D}}{\overline{G_0'(\xi_0^+)}} \;:= \varphi(\xi_0^+)$$

provided that $\overline{G_0'(\xi_0^+)} \neq 0$. In the same way, for $\xi < \xi_0$, $\varphi(\xi)$ tends to a value $\varphi(\xi_0^-)$. If ξ_0 is not a k_i, which is possible by imposing strict inequalities on the initial Riemann Invariants for the initial data (see Remark 2.2), then E_0 has only simple roots. Thus all the $E_0'(\xi_0, w)$ do not vanish and have the same sign depending only on the parity of the index i such that ξ belongs to (k_i, k_{i+1}). Thus $\varphi_0(\xi_0^+) = \varphi_0(\xi_0^-) = \overline{E_0'(\xi_0)\alpha/D} \;/\; \overline{E_0'(\xi_0)}$ and we are done.

Third step : Non-strictly hyperbolic case.

It remains now the case where $\xi_0 = k_i$ for some i and $\overline{G'(\xi_0)} = 0$. This means that the support of ν is the union of three sets E_1, E_2, E_3 corresponding respectively to $w_i = k_i < w_{i+1}$, $w_i < w_{i+1} = k_i$ and $w_i = w_{i+1} = k_i$. On E_3, ξ_0 is a double root of E_0, and we have $G_0'(\xi_0^+) = E_0'(\xi_0) = G'(\xi_0^-) = 0$. On E_1 and E_2, ξ_0 is a simple root, and $0 \neq G_0'(\xi_0^+) = |E_0'(\xi_0)| = -G_0'(\xi_0^-)$ but $E_0'(\xi_0)$ have opposite signs on E_1 and E_2. Hence, if $E_1 \cup E_2$ is not empty we find $\varphi(\xi_0^+) = \varphi(\xi_0^-)$ and we are done.

If E_1 and E_2 are empty, when $\overline{G_0'(\xi_0^+)} = \overline{G_0'(\xi_0^-)} = 0$ and thus using again l'Hospital's rule, $\varphi(\xi_0^+)$ has now to be defined by

$$\frac{\overline{G"_0(\xi_0^+)\alpha/D}}{\overline{G"_0(\xi_0^+)}} := \varphi(\xi_0^+).$$

Now ξ_0 is a double root of E_0, but cannot be triple (see Proposition 2.1). Therefore $G_0''(\xi_0^+) = G_0''(\xi_0^-) = |E_0"(\xi_0)|$ and again $\varphi(\xi_0^+) = \varphi(\xi_0^-)$ and thus φ is continuous at ξ_0 which concludes the proof of Theorem 3.4.

Remark 3.5 Up to our knowledge, this is the first time that the compensated compactness method is applied to a $N \times N$ system, $N > 2$, which is not strictly hyperbolic. The situation is however better than the gas dynamics case where the Young measure in (ρ, u) is not a Dirac mass when $\rho = 0$, that is at vacuum points.

4 Strong entropy waves for 2×2 systems

We now give several examples of systems with a kinetic formulation based on the so-called strong entropy wave. Roughly speaking, the 2×2 systems allow to reduce the entropy equation to a wave equation in the appropriate set of variables. The weak entropy waves correspond to the "initial data" $E(\rho = 0) = 0$, $E_\rho(\rho = 0)$ given, for this wave equation on the entropy. The strong entropy waves correspond to the other possible initial data, $E(\rho = 0)$ given, $E_\rho(\rho = 0) = 0$. For the classical systems (p-system, isentropic gas dynamics), the strong entropy waves are difficult to use for integrability reasons. Here, we show how to use them. The main idea, deduced from the chromatography case in Section 2, is to use the ξ out of the interval $[w_1, w_2]$ defined by the Riemann Invariants.

We first consider again the chromatography system and we show that the strong entropy waves indeed give easy to characterize convex entropies. Next, we will show the representation formulæ of strong entropies for gas dynamics looking-like of systems.

This is a way to complete, in this particular case, the set of invariant regions. Indeed all of them cannot be obtained from the weak entropies. Finally, we give two more examples, unphysical variants of the isentropic gas dynamics, to show the generality of the kinetic approach.

4.1 Chromatography again

In the 2×2 case, we can precise the results of Proposition 2.1. We have, with the notations of Section 1 :

$$\begin{gathered} \lambda_1, \lambda_2 = \frac{1}{2D^2}\left(k_1 + k_2 + k_1 u_2 + k_2 u_1 \pm 2\sqrt{\eta^2 - D k_1 k_2}\right) \\ \eta = \frac{1}{2}(k_1 + k_2 + k_1 u_2 + k_2 u_1), \\ w_i = \lambda_i D \end{gathered} \tag{4.1}$$

To obtain a wave equation for the entropies, we introduce two new variables (η, ρ), η defined by (4.1) and

$$\rho(u_1, u_2) = (\eta^2 - D k_1 k_2)^{3/2} = (k_1 k_2)^3 \ \left(\frac{1}{w_1} - \frac{1}{w_2}\right)^3 .$$

In terms of these new variables, the equation for entropies is now

$$E_{\rho\rho} - \frac{1}{9}\rho^{-4/3} E_{\eta\eta} = 0. \tag{4.2}$$

This equation corresponds to that of the gas dynamics with $\gamma = 5/3$. The solutions are well known, see [1]. The weak entropies ($E(\rho = 0, \eta) = 0,\ E_\rho(\rho = 0, \eta) = \delta(\eta - \underline{\xi})$) are given by the χ_2 in Section1. We can write it also setting $\underline{\xi} = \dfrac{k_1 k_2}{\xi}$ and up to a multiplicative constant $C(\underline{\xi})$,

$$\chi_2 = \left[\rho^{2/3} - (\underline{\xi} - \eta)^2\right]_+ \ . \tag{4.3}$$

Classically, the strong entropy wave corresponds to

$$\zeta = \rho \chi_2^{-2} \ . \tag{4.4}$$

ζ is not integrable, but the "exteriors" of the interval $[w_1, w_2]$ can be used. Let us define

$$\zeta_1 = \rho\left[\left(\xi - \frac{1}{w_1}\right)^{-2}\right]_+ \left(\xi - \frac{1}{w_2}\right)^{-2}, \quad \zeta_2 = \rho\left[\left(\frac{1}{w_2} - \xi\right)^{-2}\right]_+ \left(\frac{1}{w_1} - \xi\right)^{-2} \ . \tag{4.5}$$

Then, we have

Lemma 4.1 *For any $g(\xi)$, the function*

$$E_i(u_1, u_2) = \int_{\mathbb{R}} g(\xi)\, \zeta_i(\xi, u) d\xi, \quad i = 1, 2 \tag{4.6}$$

is an entropy. It is convex in u if and only if g is convex.

Of course, to define the integration in (4.6), we impose that g used for E_i vanishes on a neighborhood of the interval

$$\left[\min_{y\in\mathbb{R}} w_i^{-1}(y,0),\ \max_{y\in\mathbb{R}} w_i^{-1}(y,0)\right] \ :=\ I_i, \quad i = 1, 2.$$

From Lemma 4.1, we deduce a kinetic formulation of the chromatography system

$$\partial_t \zeta_i(\xi, u) + \partial_x \psi_i(\xi, u) = \partial^2_{\xi\xi} m_i, \quad i = 1, 2 \tag{4.7}$$

for some nonpositive measures $m_i(x, \xi, t)$ on $\mathbb{R} \times (\mathbb{R}^+ \setminus I_i) \times \mathbb{R}^+$. Here ψ_i denotes the kinetic flux associated to the kinetic entropy ζ_i; it has no explicit formulation. We skip the proof of the Lemma 4.1 which can be performed following [12] or the proof of the similar result in subsection 4.2 below.

We do not know what are the equivalent strong entropies for the $N \times N$ case, neither the relation with those of Serre [18]. Notice however that the two families of entropies (4.7) are certainly contained in the three entropy families built on ξ.

4.2 Other examples.

Let us just indicate for the sake of completeness some other examples where convex entropies can be built in the same way. These two systems are issued from unphysical modifications of the gas dynamics system.

The "Leroux" system ([16]) is

$$\begin{cases} \partial_t u + \partial_x\, uv \ =\ 0 \\ \partial_t v + \partial_x (v^2 + u) \ =\ 0. \end{cases} \tag{4.8}$$

It also belongs to the Temple class. The two Riemann Invariants we are going to use are

$$w_1,\ w_2 \ =\ v \pm \sqrt{D}\ , \quad D = v^2 + 4u,$$

and the eigenvalues of the system are

$$\lambda_1, \lambda_2 \ =\ \frac{1}{2}\,(\pm\sqrt{D} + 3v).$$

Again, the wave equation for the entropies, in the variables $\rho = D^{3/2}$, v, corresponds to the gas dynamics with $\gamma = 5/3$.

The weak entropies are generated by

$$\chi_1 = (w_1 - \xi)_+ \, (w_2 - \xi), \tag{4.9}$$

$$\chi_2 = [(\xi - w_1) \, (w_2 - \xi)]_+, \tag{4.10}$$

$$\chi_3 = (\xi - w_1) \, (\xi - w_2)_+, \tag{4.11}$$

and they are convex in (u, v), because they are the positive parts of an affine function as in the chromatography case. The strong entropies can also be used, the formulæ are again (4.5) with $\rho = D^{3/2}$ and Lemma 4.1 still holds for the Leroux system. We recover, as for the chromatography system, five kinetic equations, corresponding to the five families of convex entropies.

Our last example is

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x \, \rho u \; = \; 0 \\ \partial_t u + \partial_x (u^2 + p(\rho)) \; = \; 0, \\ p(\rho) = \kappa \rho^{\gamma - 1}, \; \gamma > 1, \; \kappa = \frac{\gamma - 1}{4}. \end{array} \right. \tag{4.12}$$

For smooth solutions, this system is equivalent to the isentropic gas dynamics, but of course the convexity of entropies is modified.

Weak entropies are no longer easy to use, but an easy calculation shows that the "exteriors" of the strong entropy waves give again convex entropies for g convex, so that Lemma 4.1 still holds, using

$$w_1, w_2 = u \pm \rho^{(\gamma - 1)/2},$$

$$\zeta_2 = \rho \, [(\xi - w_2)^\mu]_+ \, (\xi - w_1)^\mu, \quad \zeta_1 = \rho \, [(w_1 - \xi)^\mu]_+ \, (w_2 - \xi)^\mu,$$

with $\mu = -(\gamma + 1)/2(\gamma - 1)$, for $\gamma \neq 3$.

Again, for $\gamma \leq 3$, g has to be supported away from an interval defined by one of the Riemann Invariants, because ζ_i is not integrable against $d\xi$. We also deduce a kinetic formulation like (4.7). For $\gamma > 3$, we obtain six meaningful families of entropies, which, again, are certainly not independant.

APPENDIX : Proof of Proposition 2.1.

The proof of this proposition is a combination of [2], [15], [18] and we give it for the sake of completeness. Defining $w_i = \lambda_i D$, the w_i are now the eigenvalues of the matrix $J(u)$ in (2.6). They are the solutions of the algebraic equation

$$\varphi(w) := \sum_{i=1}^{N} \frac{k_i u_i}{k_i - w} - D = w \; \sum_{i=1}^{N} \frac{u_i}{k_i - w} - 1 = 0. \tag{4.13}$$

It clearly has exactly N roots with the order indicated in (iii) so that (i) and (iii) are proved. To prove (ii), i.e. (2.7), we just fix a root w_j of (4.13) and

$$0 = \partial_t w_j S + w_j \sum_{i=1}^N \frac{\partial_t u_i}{k_i - w_j},$$

$$S = \sum_{i=1}^N \left(\frac{u_i}{k_i - w_j} + \frac{u_i}{(k_i - w_j)^2} \right).$$

Therefore, writing the same equation for $\partial_x w_j$

$$\begin{aligned} -\left(\partial_t w_j + \lambda_j \partial_x w_j\right) \sum_{i=1}^N (\cdots) \frac{1}{w_j} &= \sum_{i=1}^N \frac{\partial_t u_i + \lambda_j \partial_x u_i}{k_i - w_j} \\ &= \frac{1}{D^2} \sum_{i=1}^N \frac{-D k_i \partial_x u_i + k_i u_i \partial_x D + w_j D \partial_x i_i}{k_i - w_j} \\ &= \frac{1}{D} \sum_{i=1}^N \frac{(-k_i + w_j)\partial_x u_i}{k_i - w_j} + \frac{\partial_x D}{D}, \end{aligned}$$

using (4.13) again. And this is clearly zero so that we have obtained (ii) since $S > 0$ thanks to (4.13).

Next, and following [18], $\varphi(w)$ is a rational fraction with poles k_i and roots w_i, therefore

$$\varphi(w) = -D \prod_{i=1}^N \frac{w - w_i}{w - k_i}, \tag{4.14}$$

since $\varphi(w) \to -D$ as $w \to \infty$. For $w = 0$, we find

$$\varphi(0) = -1 = -D \prod_{i=1}^N \frac{w_i}{k_i}, \tag{4.15}$$

and (iv) is proved. To prove (v) write the equality of φ in (4.13) and (4.14), that we multiply by $\prod(w - k_i)$. We obtain

$$D \prod_{i=1}^N (w - w_j) = \sum_{i=1}^N k_i u_i \prod_{j \neq i} (w - k_j) - D \prod_{i=1}^N (w - k_i). \tag{4.16}$$

Choosing $w = k_i$, for some fixed i, in this formula, and replacing D by the formula (iv) gives exactly (v). Finally, we write (4.13), (4.14) as

$$\begin{aligned} P(w) &= D \prod_j (k_j - w) - \sum_{i=1}^N k_i u_i \prod_{j \neq i} (k_j - w) \\ &= \sum_{i=0}^N a_i w^i \\ &= D \prod_{j=1}^N (w_j - w) \\ &= 0 \end{aligned} \tag{4.17}$$

for some coefficients a_i affine in u, with $a_N = (-1)^N D$. From (4.17) and (4.15), we deduce

$$(-1)^k a_{n-k} = (-1)^N D \sum w_{i_1} \dots w_{i_k} = (-1)^N \sigma_k(u) \prod k_i$$

and thus $\sigma_k(u)$ is indeed affine in u. Finally, $a_0 = D \prod w_j = \prod k_j$.

Finally, if $\sigma_j(u^1) = \sigma_j(u^2)$ for all $j, 0 \leq j \leq N$, in particular $D(u^1) = D(u^2)$ and $P^1(w) = P^2(w)$, which also means that $\varphi^1(w) = \varphi^2(w)$ [superscripts correspond to u^1 and u^2 respectively]. Then, from (4.13) we clearly obtain $u^1 = u^2$ and Proposition 2.1 is proved.

References

[1] A.V. Bitsadze *Equations of Mixed type*, Macmillan, New York, 1964.

[2] E. Canon and F. James *Resolution of the Cauchy problem for several hyperbolic system arising in chemical engineering.*, Ann. Inst. H. Poincaré, Analyse non linéaire, 9, 2, pp 219-238, (1992).

[3] E. Canon and F. James *Resolution of the Cauchy problem for non strictly hyperbolic systems arising in chemical engineering*, Proceedings 3rd International conference on hyperbolic problems, Uppsala 1990, Engquist B, Gustafsson B. eds, Studentlitteratur, Chartwell Bratt, pp 212-225, (1991).

[4] G.Q. Chen *The theory of compensated compactness and the system of isentropic gas dynamics*, Preprint MCS-P154-0590, University of Chicago.

[5] C.M. Dafermos *Estimates for conservation laws with little viscosity*, SIAM J. Math. Anal., 18, 2, pp 409-421, (1987).

[6] R.J. Di Perna *Convergence of approximate solutions to conservation laws* Arch. Rat. Mec. and Anal., 82, 1, pp 27-70, (1983).

[7] R.J. Di Perna *Convergence of the viscosity method for isentropic gas dynamics* Comm. Math. Phys., 91, 1, pp 1-30, (1983).

[8] A. Heibig *Existence and uniqueness of solutions for some hyperbolic systems of conservation laws* To appear in Arch. Rat. Mech. Anal.

[9] A. Heibig *Existence et unicité des solutions pour certains systèmes de lois de conservation*, C.R. Acad. Sc. Paris, t. 311, série I, pp. 861-866.

[10] P.D. Lax *Hyperbolic systems of conservation laws and the mathematical theory of shock waves* CBMS-NSF regional conferences series in applied mathematics, 11, (1973).

[11] P.L. Lions, B. Perthame, and E. Tadmor *Kinetic formulation of scalar conservation laws*, To appear in Journal of American Society.

[12] P.L. Lions, B. Perthame, and E. Tadmor *Kinetic formulation of the isentropic gas dynamics and p-system*, preprint.

[13] P.L. Lions, B. Perthame, and P.E. Souganidis, In preparation.

[14] F. Murat *Compacité par compensation*, Ann. Sc. Normale. Sup. Pisa, 5, pp 489-507, (1978).

[15] H.K. Rhee, R. Aris and N.R. Admundson *On the theory of multicomponent chromatography*, Philos. Trans. Roy. Soc. London, A267, pp 419-455, 1970.

[16] D. Serre *Solutions à variations bornées pour certains systèmes hyperboliques de lois de conservation*, J. Diff. Eq. 68, pp 137-168, (1987).

[17] D. Serre *Systèmes hyperboliques riches de lois de conservation*, Collège de France, Seminar X, Pitman Res. Notes Math. Series, Longmann (1993).

[18] D. Serre *Richness and the classification of quasilinear hyperbolic systems*, I.M.A. Vol. in Mathematics and its applications n. 29, J. Glimm and A. Majda eds, Springer-Verlag, Heidelberg.

[19] D. Serre *Oscillations non linéaires des systèmes hyperboliques : méthodes et résultats qualitatifs*, Ann. Inst. H. Poincaré, Analyse non linéaire, 8, 3-4 (1991), pp 351-417.

[20] L. Tartar *Compensated compactness and applications to partial differential equations*, In research notes in mathematics, nonlinear analysis and mechanics: Heriot-Watt, symposium., vol. 4, ed R.J. Knops, Pitman press (1979).

[21] B. Temple *Systems of conservation laws with invariant submanifolds*, Trans. A.M.S., vol. 280, pp 781-795, (1983).

(1) F. J. : Centre de Mathématiques Appliquées, URA CNRS 756, Ecole Polytechnique, 91128 Palaiseau Cedex, FRANCE

(2) Y.-J. P. (current address): CeReMaB, URA CNRS 226, Université Bordeaux I, 351 cours de la Libération, 33405 Talence Cedex, FRANCE

(3) B. P. (current address): Laboratoire d'Analyse Numérique, URA CNRS 189, Université Paris 6, 4 place Jussieu, 75252 Paris Cedex 05, FRANCE