UNIQUENESS VIA THE ADJOINT PROBLEM FOR SYSTEMS OF CONSERVATION LAWS

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ABSTRACT. We prove a result of uniqueness of the entropy weak solution to the Cauchy problem for a class of nonlinear hyperbolic systems of conservation laws, that includes in particular the p-system of isentropic gas dynamics. Our result concerns weak solutions satisfying the, as we call it, Wave Entropy Condition or WEC, for short, introduced in this paper. The main feature of this condition is that it concerns both shock waves and rarefaction waves present in a solution. For the proof of uniqueness, we derive an existence result (respectively a uniqueness result) for the backward (resp. forward) adjoint problem associated with the nonlinear system. Our method also applies to obtain results of existence or uniqueness for some linear hyperbolic systems with discontinuous coefficients.

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1. INTRODUCTION

This paper treats the question of uniqueness of entropy weak solutions for a class of systems of conservation laws

$$\partial_t u + \partial_x h(u) = 0, \qquad u(t,x) \in U, \qquad t > 0, \quad x \in \boldsymbol{R}. \tag{1.1}$$

Here U is a convex open subset of $\boldsymbol{R}^p$, h: $U \to \boldsymbol{R}^p$ is a given function of class C^2 and, for each vector u in U, $Dh(u)$ has only real eigenvalues and admits a complete basis of eigenvectors. As is well known, nonlinear hyperbolic systems like (1.1) in general do not admit smooth solutions globally defined in time and one must seek for weak solutions understood in the sense of distributions. Moreover, in order to ensure uniqueness of the weak solutions, one must add to the problem a suitable entropy condition. We refer to Lax [15], [16], [17] and Smoller [30] for the background on weak solutions and Dafermos [3] for a review of the various versions of entropy condition.

The question of whether or not there is a unique entropy weak solution to the Cauchy problem associated with (1.1) was considered by several authors. The case $p = 1$ of a scalar conservation law is very well understood; cf. Volpert [32], Kruskov [14] and Oleinik [25], [26]. Regarding systems of conservation laws, we recall that Oleinik [24] and Liu [22] treated the case of piecewise smooth solutions and, in this class, obtained a uniqueness result for systems of two equations. Oleinik's result holds for the p-system of isentropic gas dynamics and assumes the genuine nonlinearity of the equations. Liu considers the general case of a nonconvex equation of state in the p-system. Next DiPerna [6] developpped a general approach for proving uniqueness in the class of functions of bounded variation. Considering entropy weak solutions to systems of two genuinely nonlinear equations, he proved that if the Cauchy problem admits a piecewise Lipschitz continuous solution, then there cannot exist another solution in the large class of bounded functions of bounded variation. For additional material on the question of uniqueness, we refer to Dafermos-Geng [4], Hurd [13], Lyapidevskii [23], Rozhdestvenskii [27], Smoller [29] and the other references cited in [6]. For results on the Riemann problem, we refer for instance to Li-Yu [20], Liu [21] and Schatzman [28].

The main contribution of this paper is a result of uniqueness of the entropy weak solution to the Cauchy problem for a class of systems of conservation laws. The class of systems under consideration includes the so-called p-system of gas dynamics in Lagrangian coordinates and possibly other physical systems. The solutions are assumed to belong to $L^\infty_{loc}([0,\infty[, BV_{loc}(\boldsymbol{R},\mathrm{U})\,)$ where $BV_{loc}(\boldsymbol{R},\mathrm{U})$ is the space of functions of locally bounded variation. This space is natural in view of the existence result of Glimm [9].

A main idea in our approach lies in the new version of the entropy condition we propose for the system (1.1), cf. inequality (2.16) below. Our condition follows the one given by Oleinik for the scalar equations and the p-system but differs however in an essential way. Indeed, the *Wave Entropy Condition* or WEC for short, as we call it, not only picks up the physical shock waves, but also enforces a specific rate of blow-up for the derivatives of the solution near a center of a centered rarefaction wave. We recall that dealing with the centers of rarefaction, or more generally with the points of interaction of waves, was an important obstacle to a satisfactory result of uniqueness for systems of conservation laws. We believe that our version of entropy condition provides a satisfactory way to handle centered rarefaction waves and interactions of elementary waves. Note however that we can not allow an arbitrary number of such centers in the solution. The technical assumption we demand in the WEC on the set of centers of interaction of waves is not very restrictive. Our result applies to the solutions obtained by piecing together perturbations of solutions to Riemann problems (they are, roughly speaking, the "generic" solutions to system (1.1)).

The proofs in this paper are based on a study of the (linear hyperbolic) adjoint problem associated with system (1.1). As a matter of fact, we shall give two different proofs of our main result. A first method, whose idea goes back to Holmgren and was first used by Oleinik in the context of conservation laws, requires the existence of solutions in the Sobolev space H^1_{loc} for a backward adjoint problem associated with (1.1). The second method, called Haar's method and used by Rozhdestvenskii and Liu, requires the uniqueness of Lipschitz continuous solutions for a forward adjoint problem associated with (1.1). As we show it, the WEC is essential in the derivation of the existence and uniqueness results for the adjoint problem, since in general such a problem is not well-posed. As a byproduct of our proofs, we also get existence and uniqueness results of solutions in the Sobolev space H^1 for some linear hyperbolic systems with discontinuous coefficients.

Regarding the Holmgren approach applied to scalar conservation laws, we refer to the work by Tadmor [31], who shows how to use the scalar Oleinik entropy condition to derive an $W^{1,\infty}$ control for the adjoint problem. In particular, he derives in [31] a local error estimate for approximate solutions to conservation laws. In [18], an existence and uniqueness result is proved for a linear hyperbolic equation with discontinuous coefficients. See also DiPerna-Lions [7] for other related results. The scalar Oleinik condition was also used by Hoff [12], and by Goodman-LeVeque [10] and Brenier-Osher [1] in studying numerical schemes.

Our method applies to the so-called p-system, i.e. the system of isentropic gas dynamics in Lagrangian coordinates [2]:

$$\partial_t v - \partial_x u = 0, \qquad \partial_t u + \partial_x p(v) = 0, \qquad v > 0, \quad u \in \boldsymbol{R} \tag{1.2}$$

with

$$p = p(v), \qquad p > 0,\ p' < 0 \ \text{ and } \ p'' > 0. \tag{1.3}$$

In (1.2), v, u and p are the specific volume, the velocity and the pressure of the gas respectively. For this system, we obtain a uniqueness result provided that the function p satisfies the additional condition

$$\frac{p'}{p''} \ \text{ is a convex function,} \tag{1.4}$$

Condition (1.4) is for instance satisfied in the case of a polytropic perfect gas, i.e. when $p(v) = kv^{-\gamma}$ with $k > 0$ and $\gamma \geq 1$.

An outline of the paper follows. Section 2 contains the statement of the results of this paper concerning system (1.1), including the cases of the scalar conservation laws and the p-system. The proofs based on the backward adjoint problem and the forward one are given in Sections 3 and 4, respectively. Section 5 presents results of existence and uniqueness for linear systems with discontinuous coefficients. Finally in Section 6 we treat several examples.

2. STATEMENT OF THE MAIN RESULTS

This section contains the statements of the results of this paper. For convenience in the presentation, we first state our result when (1.1) is the p-system and then a scalar conservation law, c.f. Theorem 2.1 and 2.2. Theorem 2.3 contains a uniqueness result for two classes of systems. The next two sections, Sections 3 and 4, will yield the proofs of Theorem 2.3. Additional comments on Theorem 2.1 and 2.3 can be found in Section 6.

Throughout this paper, it will be convenient to consider systems of conservation laws in the following symmetric form:

$$\partial_t g(v) + \partial_x f(v) = 0, \qquad v(t,x) \in V,\ t > 0,\ x \in \boldsymbol{R}. \tag{2.1}$$

Here V is a non-empty open subset of $\boldsymbol{R}^p$, g: $V \to \boldsymbol{R}^p$ and f: $V \to \boldsymbol{R}^p$ are mappings of class C^2 and

$$Dg \geq cId, \qquad (Dg)^* = Dg \quad \text{and} \quad (Df)^* = Df. \tag{2.2}$$

In (2.2), c is a positive constant, Id the identity matrix and Dg and Df are the Jacobian matrices of g and f, respectively. If A and B are two $p \times p$ real-valued matrices, A^* denotes the transposition of A and $A \leq B$ means that $\xi^* A \xi \leq \xi^* B \xi$ for all vectors ξ in $\boldsymbol{R}^p$.

In fact, assuming the form (2.1), (2.2) for system (1.1) is not a restriction for the applications we have in mind. As is well known, many systems issued from physics of the form

$$\partial_t u + \partial_x h(u) = 0, \qquad u(t,x) \in U, \quad x \in \boldsymbol{R}, \quad t > 0, \tag{2.3}$$

admit a (uniformly) strictly convex entropy-entropy flux pair (η, q): $U \to \boldsymbol{R} \times \boldsymbol{R}$. By definition, such a pair satisfies

$$(\nabla q)^* = (\nabla \eta)^*\, Dh \quad \text{and} \quad \nabla^2 \eta \geq c' Id \tag{2.4}$$

for some constant $c' > 0$, and each smooth solution u to (2.3) satisfies the additional conservation law

$$\partial_t \eta(u) + \partial_x q(u) = 0.$$

And we recall that any system having an entropy pair can be rewritten in the symmetric form (2.1), (2.2), via the change of variables given by the so-called entropy variables (see, for instance, Friedrichs-Lax [8], Harten-Lax [11] and the reference cited therein)

$$v = \nabla\eta(u), \qquad u \in U.$$

System (2.3) takes the form

$$\partial_t\big\{(\nabla\eta)^{-1}(v)\big\} + \partial_x\big\{h(\,(\nabla\eta)^{-1}(v)\,)\big\} = 0. \tag{2.5}$$

So (2.5) has the form (2.1) with here

$$Dg(v) = \nabla^2\eta(u)^{-1} \quad \text{ and } \quad Df(v) = Dh(u)\,\nabla^2\eta(u)^{-1}.$$

Property (2.2) is satisfied because of (2.4) and the matrices

$$\nabla^2\eta\,Dh = \nabla^2 q - (\nabla\eta)^* D^2 h$$

and then

$$Df = (\nabla^2\eta)^{-1}\big\{\nabla^2\eta\,Dh\big\}(\nabla^2\eta)^{-1}$$

are symmetric matrices.

In this paper, we will use indifferently the formulations (2.1) or (2.3). We are concerned with the weak solutions (in the sense of distributions) to these systems of equations. So we have to add to the problem an entropy condition in order to guarantee uniqueness. It can be checked that the notions of entropy weak solution for system (2.1) and system (2.3) coincide. We first introduce a definition which will help us to formulate the entropy condition that we propose in this paper.

We shall say that a subset E of $\boldsymbol{R}_+$ has *only left accumulation points* if the condition

$$\text{for each } t_* \text{ in } E, \text{ there exists } \epsilon > 0 \text{ such that }]t_*, t_* + \epsilon[\cap E = \emptyset \tag{2.6}$$

is fulfilled. Note that a locally finite set $E \subset \boldsymbol{R}_+$, i.e. such that $E \cap [0,T]$ is finite for all $T > 0$, clearly satisfies (2.6). It is worth also remarking that a closed set satisfying (2.6) is necessarily countable and that $\min(E\cap]t_0,\infty[)$ is well defined (i.e., the infimum is achieved) for all $t_0 \geq 0$.

Using the above definition, we impose a restriction on the location of the centers of rarefaction wave in the solutions of (2.1). Specifically, in the formulation of the entropy condition, we assume that centered rarefaction waves arise at times defining a closed set having only left accumulation points. This will be clear in the formulation given shortly.

The case of a scalar equation with a convex flux is easier. In that case, rarefaction waves can arise only at the time $t = 0$, and the set $E = \{0\}$ trivially satisfies our condition (2.6). For a system ($p \geq 2$), centered rarefaction waves possibly arise either from the initial data at $t = 0$ or, at later times, due to the interactions in the solution between two or more waves. Not so much is known a priori about the amount and the location of the centers of rarefaction. For instance, solutions derived by the random choice scheme with small data possess an at most countable set of such points. We hope that the structural restriction made here will be relaxed in the future.

To be specific, we focus our attention first on the p-system of gas dynamics for which we can prove a uniqueness result in the L^∞ class. Consider the system (c.f. [2], [30] for the background):

$$\partial_t u_1 - \partial_x u_2 = 0, \qquad \partial_t u_2 + \partial_x p(u_1) = 0, \tag{2.7}$$

where $u_1 > 0$ and u_2 represent the specific volume and the velocity of the gas, respectively. The pressure p is given by the equation of state of the gas $p = p(u_1)$ that is assumed to satisfy:

$$p(u_1) > 0, \qquad p'(u_1) < 0 \quad \text{and} \quad p''(u_1) > 0. \tag{2.8}$$

Let us formulate *Wave Entropy Condition* for the p-system as follows:

There exists a closed set $E \subset \boldsymbol{R}_+$ having only left accumulation points such that, for each t_* in E, one has

$$\partial_t u_1 + \frac{2}{(t - t_*)} \frac{p'(u_1)}{p''(u_1)} \le K_* \tag{2.9}$$

in the sense of distributions in the strip $\{x \in \boldsymbol{R},\ t_* < t < t^*\}$, where K_* is a non-negative constant and $t^* = \min(E \cap]t_*, \infty[)$.

Note that, away from the points t_* in the set E, condition (2.9) is equivalent to Oleinik's condition [24]

$$\partial_t u_1 \le K \tag{2.10}$$

for some constant K. Such a condition is also consistent with the Lax entropy inequalities for a shock wave [17]. However, a drawback of (2.10) is that it does not allow rarefaction waves in the solutions.

Condition (2.9) indeed allows both shock waves and rarefaction waves in the solution. It is interesting to note that we have identically

$$\partial_t u_1 + \frac{2}{t} \frac{p'(u_1)}{p''(u_1)} = 0$$

(which implies trivially (2.9)), if (u_1, u_2) is a centered rarefaction wave with its center located on the line $t = 0$. In fact, any C^2-perturbation of a Riemann solution satisfies (2.9). More generally, any solution composed of Lipschitz continuous regions, Lipschitz continuous shock curves and perturbations of Riemann problems do satisfy WEC. We omit the proof. This is an easy consequence of the Lax entropy condition and the explicit formulae known for Riemann solutions and their perturbations (e.g. [17], [20]).

Our main result for the p-system is the following one:

THEOREM 2.1. *Consider the* p *-system* (2.7) *with a function* p *of class* C^4 *satisfying* (2.8) *and the condition*

$$\left(\frac{p'}{p''}\right)'' \ge 0. \tag{2.11}$$

Let (u_1^0, u_2^0) *be a given function in* $L^\infty_{loc}(\boldsymbol{R}, \boldsymbol{R}_+ \times \boldsymbol{R})$*. Then there exists at most one solution* (u_1, u_2) *to system* (2.7) *in the space* $L^\infty_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}, \boldsymbol{R}_+ \times \boldsymbol{R}) \cap C^{0,\delta}_{loc}([0,\infty[, L^1_{loc}(\boldsymbol{R}, \boldsymbol{R}_+ \times \boldsymbol{R}))$ *(with* $\delta > 0$*) which satisfies the Wave Entropy Condition* (2.9) *and assumes the initial condition* (u_1^0, u_2^0) *at* $t = 0$*.*

Note that the convexity condition (2.11) is satisfied by a polytropic perfect gas:

$$p(u_1) = k\, u_1^{-\gamma}, \qquad k > 0 \quad \text{and} \quad \gamma \geq 1,$$

since, in that case, the function

$$\frac{p'(u_1)}{p''(u_1)} = \frac{-u_1}{\gamma + 1}$$

is linear.

Remark 2.1. 1) An equivalent form for the entropy inequality in (2.9) is

$$\partial_t \left(\frac{-1}{(t - t_*)^2\, p'(u_1)} \right) \leq \frac{K_*}{(t - t_*)^2}.$$

2) In view of the Wagner's result of equivalence of the Eulerian and Lagrangian systems [33], the result of uniqueness in Theorem 2.1 could be as well re-written by using the Eulerian coordinates.

We now state a result of uniqueness for scalar conservation laws. Of course, such a result is known from [14], [25] and [32]. Our interest in the scalar case comes from the fact that we can derive a quite elegant method of proof, using the potential method and an estimate in L^p norm, cf. Section 6.

THEOREM 2.2. *Consider a scalar conservation law*

$$\partial_t u + \partial_x f(u) = 0, \qquad u(t,x) \in \boldsymbol{R}^1, \quad t > 0,\ x \in \boldsymbol{R}, \tag{2.12}$$

where the flux-function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ *is of class* C^2 *and satisfies*

$$f'' > 0. \tag{2.13}$$

Let u_0 *be a given function in* $L^\infty_{loc}(\boldsymbol{R})$*. Then there exists at most one weak solution* u *to equation* (2.12) *in the space* $L^\infty_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}) \cap \mathrm{Lip}_{loc}([0,\infty[, L^1_{loc}(\boldsymbol{R}))$ *which assumes the initial data* u_0 *in the* L^1_{loc} *sense and satisfies*

$$\partial_x f'(u) \le \frac{1}{t}. \tag{2.14}$$

Remark 2.2. 1) The approach through the Holmgren method was studied by Oleinik [26] to get an uniqueness result, and by Tadmor [31], who has shown the existence of the adjoint problem in $W^{1,\infty}$ norm.

2) The existence of a solution to equation (2.12) satisfying the requirements in the theorem is well known after the work by Oleinik [25]. Condition (2.14) was first introduced by Hoff [12]. It was studied by Brenier-Osher [1] Goodman-LeVeque [10] and Tadmor [31] in the setting of numerical schemes.

3) Note that (2.14) could be weakened as well to the condition

there exists a closed set $E \subset \boldsymbol{R}_+$ with only left accumulation points such that for each t_* in E, one has

$$\partial_x f'(u) \le \frac{1}{t - t_*} + K_* \tag{2.15}$$

in the sense of distributions in the strip $\{t_* < t < t^*\}$, where K_* is a non-negative constant and $t^* = \min(E \cap]t_*, \infty[)$.

4) Condition (2.14) implies that $f'(u)$ (and thus u itself since the function f' is continuously differentiable) belongs to $L^\infty(\boldsymbol{R}_+, BV_{loc}(\boldsymbol{R}))$.

5) Inequality (2.14) is equivalent to say that:

$$\frac{f'(u(t,x)) - f'(u(t,y))}{x - y} \le \frac{1}{t} \qquad \text{for almost all } x \ne y \text{ and } t,$$

i.e., since f' is increasing,

$$\frac{u(t,x) - u(t,y)}{x - y} \le \frac{u(t,x) - u(t,y)}{f'(u(t,x)) - f'(u(t,y))} \frac{1}{t} \qquad \text{for almost all } x \ne y \text{ and } t,$$

which implies that for some constant M

$$\partial_x u \le \frac{M}{t}. \tag{2.14bis}$$

In the proof of Theorem 2.2, we actually use the condition (2.14bis).

We next consider a class of systems of conservation laws of the form (2.1), (2.2). We will treat solutions having locally bounded variation in space. Specifically, we consider solutions v to system (2.1) (or equivalently (2.3)) which belong to the space $L^\infty_{loc}([0,\infty[, BV_{loc}(\boldsymbol{R}, V))$, and thus to $\mathrm{Lip}_{loc}([0,\infty[, L^1_{loc}(\boldsymbol{R}, V))$. This is the regularity of the solutions given by Glimm's scheme [9]. As before, we impose a restriction on the set of points of rarefaction centers based on the notion of set with only left accumulation points. We introduce the *Wave Entropy Condition* for the system (2.1):

There exists a closed set $E \subset \boldsymbol{R}_+$ with only left accumulation points such that for each t_* in E one has

$$\partial_t(\,(t-t_*)^{-\alpha}\, Dg(v)\,) + \partial_x(\,(t-t_*)^{-\alpha}\, Df(v)\,) \le (t-t_*)^{-\alpha} K_* \tag{2.16}$$

in the sense of distributions in the strip $\{x \in \boldsymbol{R}, t_* < t < t^*\}$, where K_* is a non-negative constant and t^* is defined by $t^* = \min(E \cap]t_*, \infty[)$.

In (2.16), α is an exponent depending on the system under consideration, but not on the specific solution. We shall assume below a range of admissible values for α. The constants K_* as well as the set E can depend on the solution. The inequality in (2.16) bears on matrix-valued functions of the variable (t,x). By definition, it means that

$$\int_{t_*}^{t^*} \int_{\boldsymbol{R}} \Big\{ \xi^*\, Dg(v)\xi\, \partial_t\theta + \xi^* Df(v)\xi\, \partial_x\theta + \xi^*\xi\, K_*\theta \Big\} (t-t_*)^{-\alpha}\, dx\, dt \ge 0$$

for each vector ξ in $\boldsymbol{R}^p$ and each non-negative function θ of class C^1 compactly supported in the strip $\{x \in \boldsymbol{R},\, t_* < t < t^*\}$.

As (2.9) and (2.14), the Wave Entropy Condition allows the presence of shock waves in the solution, as well as rarefaction centers. Moreover the condition enforces an "upper bound" on the rate of blow-up of the derivatives of a solution when approaching a rarefaction center.

This feature of the condition will be very essential in the derivation of the main a priori estimates in Sections 3 and 4. We also emphasize that, although bearing on matrix-valued functions, which seems at first very stringent, condition (2.16) is actually realistic in several situations, e.g. for the scalar equations and the p-system. In these cases and others, (2.16) is consistent with the Lax entropy inequalities [17] and satisfied by (perturbations of) the solution of a Riemann problem.

The class of systems we deal with is motivated by the specific form of WEC and is mainly characterized by some convexity properties of the flux functions of the system. We shall distinguish between two classes of systems. System (2.1), (2.2) is said to belong to the class A if the following three conditions hold:

$$Dg \text{ is a constant matrix,} \tag{2.17i}$$

for all vectors v_0, v_1 in V and w_0, w_1 in $\boldsymbol{R}^p$, all s in $[0,1]$ and any constant $K > 0$, one has:

$$\begin{aligned} &D^2 f(v_i) \cdot w_i \le \alpha\, Dg + K Id \quad \text{for } i = 0 \text{ and } i = 1 \\ &\qquad \Rightarrow D^2 f(v_s) \cdot w_s \le \alpha\, Dg + K Id, \end{aligned} \tag{2.17ii}$$

where $v_s = (1-s)v_0 + s v_1$ and $w_s = (1-s)w_0 + s w_1$

and

for all vectors $v_0^\pm$ and $v_1^\pm$ in V and all s in $[0,1]$, one has :

$$\begin{aligned} &Df(v_i^-) \ge Df(v_i^+) \quad \text{for } i = 0 \text{ and } i = 1 \\ &\qquad \Rightarrow Df(v_s^-) \ge Df(v_s^+), \end{aligned} \tag{2.17iii}$$

where $v_s^\pm = (1-s)v_0^\pm + s v_1^\pm$.

For instance, Burgers' equation (for which $p = 1$, $g(v) = v$ and $f(v) = v^2/2$) belongs to the class A. More generally, (2.17) holds for a conservation law (2.12) when $f'' > 0$ and $\big(-1/f''\big)'' \ge 0$. Another example is given in Section 6.

System (2.1), (2.2) is said to belong to the class B if the following three conditions hold:

$$Df \text{ is a constant and invertible matrix,} \tag{2.18i}$$

for all vectors u_0, u_1 in U, w_0, w_1 in $\boldsymbol{R}^p$ and all s in $[0,1]$ and any constant $K>0$, one has:

$$D^2g(v_i)\cdot Dg(v_i)^{-1}w_i \leq \alpha\, Dg(v_i) + KId \quad \text{for } i=0 \text{ and } i=1 \tag{2.18ii}$$
$$\Rightarrow D^2g(v_s)\cdot Dg(v_s)^{-1}w_s \leq \alpha\, Dg(v_s) + KId,$$

where $g(v_s) = (1-s)g(v_0) + sg(v_1)$ and $w_s = (1-s)w_0 + sw_1$

and

for all vectors $u_0^{\pm}$ and $u_1^{\pm}$ in U and all s in $[0,1]$, one has:

$$Dg(v_i^-) \geq Dg(v_i^+) \quad \text{for } i=0 \text{ and } i=1 \tag{2.18iii}$$
$$\Rightarrow Dg(v_s^-) \geq Dg(v_s^+),$$

where $g(v_s^{\pm}) = (1-s)g(v_0^{\pm}) + sg(v_1^{\pm})$.

The p-system (2.7) belongs to the class B when (2.11) is satisfied. C.f. Section 6.

We will need the regularity properties for functions of bounded variation established by Volpert. According to [32], one has, for each function v in $L^\infty_{loc}([0,\infty[, BV_{loc}(\boldsymbol{R}, V))$, the decomposition

$$]0,\infty[\times\boldsymbol{R} = E^1(v)\cup E^2(v)\cup E^3(v),$$

where $E^1(v)$ is the set of points of approximate continuity for v, $E^2(v)$ is the set of points of approximate jump for v and $E^3(v)$ is a set of one-dimensional Hausdorff measure zero, i.e. $H_1(E^3(v)) = 0$. Cf. [32] for the definitions. We also recall that a normal $\nu(t,x)$ and left and right limiting values $v^{\pm}(t,x)$ and $v^{\pm}(t,x)$ exist at any point of $E^2(v)$.

We finally state our result concerning the system (2.1).

THEOREM 2.3. *Let v_0 be a given function in $BV_{loc}(\boldsymbol{R}, V)$. Consider a system of the form* (2.1), (2.2) *belonging to either the class* A *or the class* B *characterized by conditions (2.17) and (2.18) respectively. Suppose that the exponent α (introduced in* (2.16)*) belongs to the interval* $[1,3[$. *Then there exists at most one function v in $L^\infty_{loc}([0,\infty[, BV_{loc}(\boldsymbol{R}, V))$ which satisfies the system of conservation laws* (2.1), *the initial condition v_0 in the L^1_{loc} sense and the Wave Entropy Condition* (2.16) *and, on the set $E^2(v)$, is absolutely continuous with respect to the Lebesgue measure.*

Sections 3 and 4 contain two proofs of Theorem 2.3. Section 3 however requires that $\alpha \in [1, 5/2[$. Theorem 2.1 and 2.2 are essentially corollaries of Theorem 2.3. See Section 6 for the details.

Remark 2.3. 1) The assumption that v is absolutely continuous when restricted to the set $E^2(v)$ is technical, and is used in deriving the entropy inequality for the average between two solutions (Lemma 3.3 below). For instance, this assumption is satified if v is locally piecewise Lipschitz continuous on $E^2(v)$.

2) One can check from the proofs in Sections 3 and 4 that a result of continuous dependence of the solution with respect to its initial data holds in the class of entropy solutions that have no centered rarefaction waves. This kind of result was already proved by DiPerna in [6] when comparing a Lipschitz continuous solution with an arbitrary entropy solution having bounded variation.

3) All the results in this paper remain valid if we replace the Wave Entropy Condition by the following weakened version of it: for each point (t_*, x_*) there exists an ϵ small enough and a function $K_* = K_*(t)$ in $L^1(t_*, t_* + \epsilon)$ such that:

$$\partial_t(\,(t - t_*)^{-\alpha}\, Dg(v)\,) + \partial_x(\,(t - t_*)^{-\alpha}\, Df(v)\,) \le (t - t_*)^{-\alpha} K_*(t) Id$$

holds in the sense of distributions in the set $\big\{t_* < t < t_* + \epsilon, x_* - \epsilon < x < x_* + \epsilon\big\}$.

3. UNIQUENESS VIA EXISTENCE FOR THE BACKWARD ADJOINT PROBLEM

This section contains a (first) proof of Theorem 2.3 based on Holmgren's method. The problem of proving uniqueness of weak solutions in BV for (2.1) is reduced to proving existence of strong solutions in H^1_{loc} for the backward adjoint problem associated with (2.1). This latter problem consists of a linear hyperbolic system whose coefficients are only BV functions. The existence of a solution is not guaranteed in general. However, when the Wave Entropy Condition (2.16) holds, this problem admits a solution which is at least continuous and in general non-unique. The uniqueness result for the nonlinear system follows from that. Note that this proof requires that $\alpha < 5/2$ instead of the assumption $\alpha < 3$ made in Theorem 2.3.

Let v_1 and v_2 be two solutions to (2.1) in $L^\infty_{loc}([0,\infty[, BV_{loc}(\boldsymbol{R}, V))$ satisfying the Wave Entropy Condition (2.16). Set $u_1 = g(v_1)$ and $u_2 = g(v_2)$. To begin with, we consider the region $]0,T[\times\boldsymbol{R}$ where T is defined by

$$T = \min\{t \in E_1 \cup E_2 \,|\, t > 0\}$$

and E_1 and E_2 are the sets in the entropy condition (2.16) associated with the solutions v_1 and v_2, respectively. We establish first the uniqueness result in $]0,T[\times\boldsymbol{R}$, by assuming (later in this section) that $v_1(0,\cdot) = v_2(0,\cdot)$. At the end of this section, we will explain how to deal with the times $t > T$. Let us first assume the existence of a solution to the adjoint problem associated with the system (2.1) and deduce from that the uniqueness result for the nonlinear problem. We shall next prove that the adjoint problem admits a solution having the required properties.

Let C be the matrix-valued function defined by

$$C = \int_0^1 Dh\big((1-s)\,u_1 + s\,u_2\big)\,ds. \tag{3.1}$$

We assume that the following backward adjoint problem admits (at least) one solution ϕ in the space $H^1_{loc}(]0,T] \times \boldsymbol{R}, \boldsymbol{R}^p)$:

$$\begin{aligned} &\partial_t \phi + C^* \, \partial_x \phi = k, && t \le T, x \in \boldsymbol{R}, \\ &\phi(T,x) = 0, && x \in \boldsymbol{R}, \\ &\phi(t,x) = 0, && |x| \ge b + M(T-t), \end{aligned} \tag{3.2}$$

satisfying the additional property:

$$t^\alpha \int_{\boldsymbol{R}} |\partial_x \phi(x)|^2 \, dx \in L^\infty(0,T). \tag{3.3}$$

Here α is the exponent introduced in the entropy condition (2.16). In (3.2), the constants $b > 0$ and $M > 0$ are assumed to be sufficiently large (cf. below), and k is a given function in $W^{1,\infty}(]0,T[\times \boldsymbol{R},\, \boldsymbol{R}^p)$ compactly supported in the open set $\{|x| < b + M(T-t)\,,\, 0 < t < T\}$.

Lemma 3.1. *For all times t in $[0,T]$, one has*

$$\int_t^T \int_{\boldsymbol{R}} \big(u_2 - u_1\big)^* \, k \, dx \, dt = - \int_{\boldsymbol{R}} \big(u_2(t) - u_1(t)\big)^* \, \phi(t) \, dx. \tag{3.4}$$

Proof: We observe that the functions u_1 and u_2 are locally Lipschitz continuous in time with values in L^1_{loc} in space and satisfy the equations (2.1) in the sense of distributions. On the other hand, ϕ belongs to $H^1([t,T] \times \boldsymbol{R})$ and has there compact support, thus can be used as a test-function. These facts imply that

$$\begin{aligned} &\int_t^T \int_{\boldsymbol{R}} \big\{ \partial_t \phi^* (u_2 - u_1) + \partial_x \phi^* (h(u_2) - h(u_1)) \big\} \, dx \, dt \\ &\quad + \int_{\boldsymbol{R}} \phi(t)^* \big(u_2(t) - u_1(t)\big)\big) \, dt = 0 \end{aligned} \tag{3.5}$$

where we have used the fact that $\phi(T) = 0$. Using the equations in (3.2) and the definition of C (cf. (3.1)), one obtains

$$\big(u_2 - u_1\big)^* \, \partial_t \phi + \big(h(u_2) - h(u_1)\big)^* \, \partial_x \phi = \big(u_2 - u_1\big)^* \, k. \tag{3.6}$$

Thus (3.4) follows from (3.5) and (3.6). The proof is complete. $\square$

We next estimate the right-hand side of (3.4) by using the property (3.3).

Lemma 3.2. *Suppose that* $\alpha \geq 1$. *For all times* t *in* $]0, T]$, *one has*

$$
\begin{aligned}
&\Big| \int_t^T \int_{\boldsymbol{R}} \big(u_2 - u_1\big)^* \, k \, dx \, dt' \Big| \\
&\leq O(1) \, t^{(-2\alpha+1)/4} \big\{ t + \| u_2(0) - u_1(0) \|_{L^1(-b-MT, b+MT)} \big\}
\end{aligned}
\tag{3.7}
$$

where the constant $O(1)$ *is independent of* t *but can depend on* T.

Proof: Let us first check that (3.3) implies the following bound for ϕ:

$$
\| \phi(t) \|_{L^\infty(\boldsymbol{R})} \leq O(1) \, t^{(-2\alpha+1)/4} \qquad textfor \;\; 0 < t < T.
\tag{3.8}
$$

On one hand, by Sobolev's inequality, one has

$$
\| \phi(t) \|_{L^\infty(\boldsymbol{R})} \leq \; \| \phi(t) \|^{1/2}_{L^2(\boldsymbol{R})} \| \partial_x \phi(t) \|^{1/2}_{L^2(\boldsymbol{R})},
$$

and, on the other hand, by Poincaré's inequality and using (3.2)

$$
\begin{aligned}
\| \phi(t) \|^2_{L^2(\boldsymbol{R})} &= \| \phi(t) \|^2_{L^2(\boldsymbol{R})} - \| \phi(T) \|^2_{L^2(\boldsymbol{R})} \\
&\leq \int_t^T 2 \| \phi(s) \|_{L^2(\boldsymbol{R})} \| \partial_t \phi(s) \|_{L^2(\boldsymbol{R})} \, ds \\
&\leq O(1) \int_t^T \| \partial_t \phi(s) \|^2_{L^2(\boldsymbol{R})} \, ds \\
&\leq O(1) \int_t^T \big\{ \| \partial_x \phi(s) \|^2_{L^2(\boldsymbol{R})} + \| k(s) \|^2_{L^2(\boldsymbol{R})} \big\} ds,
\end{aligned}
$$

where the constant $O(1)$ depend on T. Hence we obtain

$$
\| \phi(t) \|_{L^\infty(\boldsymbol{R})} \leq O(1) \| \partial_x \phi(t) \|^{1/2}_{L^2(\boldsymbol{R})} \big\{ \int_t^T \big\{ \| \partial_x \phi(s) \|^2_{L^2(\boldsymbol{R})} + \| k(s) \|^2_{L^2(\boldsymbol{R})} \big\} ds \big\}^{1/4}.
\tag{3.9}
$$

Then, we use (3.3) in inequality (3.9) and get

$$
\begin{aligned}
\| \phi(t) \|_{L^\infty(\boldsymbol{R})} &\leq O(1) t^{-\alpha/4} \big\{ \int_t^T s^{-\alpha} \, ds + \int_t^T \| k(s) \|^2_{L^2(\boldsymbol{R})} \big\}^{1/4} \\
&\leq O(1) \, t^{(-2\alpha+1)/4}
\end{aligned}
$$

since $\alpha \geq 1$. The proof of (3.8) is complete.

We next estimate the right hand side of (3.4):

$$\begin{aligned}&\Big|\int_{\boldsymbol{R}} \phi(t)^* \big(u_2(t) - u_1(t)\big)\, dx\Big| \\ &\qquad \le O(1)\, \|\phi(t)\|_{L^\infty(\boldsymbol{R})} \|u_2(t) - u_1(t)\|_{L^1(-b-M(T-t),b+M(T-t))}.\end{aligned}$$

¿From (3.8) and the Lipschitz continuity in time in L^1 norm of the functions u_2 and u_1, we deduce form the above inequality that

$$\begin{aligned}&\Big|\int_{\boldsymbol{R}} \phi(t)^* \big(u_2(t) - u_1(t)\big)\, dx\Big| \\ &\qquad \le O(1)\, t^{(-2\alpha+1)/4} \big\{t + \|u_2(0) - u_1(0)\|_{L^1(-b-MT,b+MT)}\big\}.\end{aligned}$$

Using the above inequality in (3.4) yields (3.7). The proof of Lemma 3.2 is complete. □

In view of Lemma 3.2, we can conclude our proof of uniqueness for the nonlinear problem. Namely, when $u_2(0) = u_1(0)$ i.e. when the solutions u_2 and u_1 assume the same initial data, estimate (3.7) as t goes to zero implies

$$\int_0^T \int_{\boldsymbol{R}} \big(u_2 - u_1\big)^*\, k\, dx\, dt = 0$$

if the exponent α in the entropy condition belongs to the interval $[1, \frac{5}{2}[$. Henceforth, since the function k is arbitrary, we obtain in that case $u_2 = u_1$ almost everywhere in $]0,T[\times \boldsymbol{R}$.

It now remains (and this is the central issue of this section) to prove the existence of a solution to the backward adjoint problem (3.2) that also satisfies the condition (3.3). To this end, we are going to regularize the (discontinuous) coefficients of system (3.2). The existence of a smooth solution ϕ^ϵ to the regularized problem is trivial, and we end up with the need of deriving suitable uniform estimates on ϕ^ϵ to prove that ϕ^ϵ converges to a solution of (3.2), (3.3).

We decompose the matrix C defined by (3.1), in the form $C^* = AB^{-1}$ as follows. If system (2.1) belongs to the class A, we set

$$A = \int_0^1 Df\big(\,(1-s)v_1 + s v_2\big)\, ds,\ \ B = Dg.$$

If the system belongs to the class B, we set

$$A = Df \ , \qquad B^{-1} = \int_0^1 D\big(g^{-1}\big)((1-s)\,u_1 + s\,u_2\,)\;ds.$$

We consider first the case of a system in the class A and give the definition of the regularizations B^ϵ and A^ϵ of B and A. Let ω^ϵ be a standard smoothing kernel in the two variables (t,x), say $\omega^\epsilon(t,x) = \dfrac{1}{\epsilon^2}\omega(\dfrac{t}{\epsilon}, \dfrac{x}{\epsilon})$ where ω is a non-negative and smooth function compactly supported in $[-1,1]\times[-1,1]$, and satisfying $\int_{-1}^1\int_{-1}^1 \omega(t,x)\,dt\,dx = 1$. Let $\tilde{A}$ be the function

$$\tilde{A}(t,x) = \begin{cases} A(t,x) & \text{for} \quad t>\epsilon,\ x\in \boldsymbol{R}, \\ \\ 0 & \text{for} \quad t<\epsilon,\ x\in \boldsymbol{R}, \end{cases} \tag{3.10}$$

and define A^ϵ by

$$A^\epsilon = t^\alpha(\,(t^{-\alpha}\tilde{A}) * \omega^\epsilon). \tag{3.11a}$$

Since in that case B is constant, we simply set

$$B^\epsilon = B. \tag{3.11b}$$

Of course B^ϵ and A^ϵ are smooth functions of the variables (t,x). Moreover, one has

$$A^\epsilon = t^\alpha\big((\,t^{-\alpha}A) * \omega^\epsilon\big) \qquad \text{for} \qquad t > 2\epsilon.$$

Consider now the case of a system in the class B. The matrix A is a constant, so we set

$$A^\epsilon = A \tag{3.12a}$$

Let ω^ϵ be a smoothing kernel in the *single* variable x say $\omega^\epsilon(x) = \frac{1}{\epsilon}\omega(\frac{x}{\epsilon})$, where ω is a non-negative and smooth function compactly supported in $[-1,1]$ and satisfying $\int_{-1}^1 \omega(x)\,dx = 1$. Define first u_1^ϵ, u_2^ϵ, v_1^ϵ and v_2^ϵ by

$$u_i^\epsilon = u_i * \omega^\epsilon \qquad \text{and} \qquad v_i^\epsilon = g(u_i^\epsilon) \tag{3.13}$$

and then B^ϵ by

$$(B^\epsilon)^{-1} = \int_0^1 D(g^{-1})\big((1-s)\,u_1^\epsilon + s\,u_2^\epsilon\big)\,ds. \tag{3.12b}$$

We note that, although ω^ϵ depends only on the variable x, the functions u_1^ϵ, u_2^ϵ and B^ϵ are Lipschitz continuous. This is because (2.3) implies

$$\partial_t u_i^\epsilon = -h(u_i) * \partial_x \omega^\epsilon \in L^\infty_{loc}(\,(0,T \times \boldsymbol{R}\,,\,\boldsymbol{R}^p).$$

Hence, in both cases, we have constructed A^ϵ and B^ϵ which are (at least) Lipschitz continuous functions in (t,x). Let ϕ^ϵ be the unique solution to the following regularized version of problem (3.2)

$$\partial_t \phi^\epsilon + (B^\epsilon)^{-1} A^\epsilon\, \partial_x \phi^\epsilon = k, \quad t \le T,\; x \in \boldsymbol{R},$$

$$\phi^\epsilon(T,x) = 0, \quad x \in \boldsymbol{R}, \tag{3.14}$$

$$\phi^\epsilon(t,x) = 0, \quad |x| \ge b + M(T-t).$$

Here we require that M is sufficiently large (larger than the maximum characteristic speed of the system).

Due to the specific regularization we have made, the Wave Entropy Condition (2.16) is satisfied by the coefficients B^ϵ and A^ϵ of system (3.14).

Lemma 3.3. *There exists a constant K independent of ϵ such that*

$$\partial_t(t^{-\alpha} B^\epsilon) + \partial_x(t^{-\alpha} A^\epsilon) \le t^{-\alpha} K Id \tag{3.15}$$

in the whole strip $]0,T[\times\boldsymbol{R}$.

Proof: Since the functions v_1 and v_2 have locally bounded total variation in space and are solutions to system (2.1), they have locally bounded variation in both variables (t,x) in the set $]0,T[\times\boldsymbol{R}$. We can decompose $]0,T[\times\boldsymbol{R}$ in the form

$$]0,T[\times\boldsymbol{R} = E^1 \cup E^2 \cup E^3,$$

where E^1 is the set of points of approximate continuity for both v_1 and v_2, E^2 is the set of points of approximate jump for either v_1 or v_2 and E^3 is a set of one-dimensional Hausdorff measure zero, i.e. $H_1(E^3) = 0$. We also have a normal $\nu(t,x)$ and left and right limiting values $v_1^{\pm}(t,x)$ and $v_2^{\pm}(t,x)$ exist at any point of E^2.

Since v_1 and v_2 satisfy the entropy condition (2.16), we have

$$\partial_t(t^{-\alpha} Dg(v_i)\,) + \partial_x(t^{-\alpha} Df(v_i)\,) \le t^{-\alpha} K_i Id \tag{3.16}$$

for some constants K_1 and K_2. Set $K = \max(K_1, K_2)$. We are going to deduce from (3.16) and the convexity properties (2.17) and (2.18) that

$$\partial_t(t^{-\alpha} Dg(r_s)) + \partial_x(t^{-\alpha} Df(r_s)) \le t^{-\alpha} K Id \tag{3.17}$$

where $r_s = g^{-1}(z_s)$ and $z_s = (1-s)\,g(v_1) + s\,g(v_2) = (1-s)u_1 + su_2$ and $s \in [0,1]$.

On the set E^1, both functions v_1 and v_2 are approximately continuous, so we can write:

$$\partial_t(t^{-\alpha} Dg(v_i)\,) = -\alpha\, t^{-\alpha-1}\, Dg(v_i) + t^{-\alpha}\, D^2 g(v_i) \cdot Dg(v_i)^{-1}\, \partial_t u_i$$

and

$$\partial_x(t^{-\alpha} Df(v_i)\,) = t^{-\alpha}\, D^2 f(v_i) \cdot \partial_x v_i.$$

Inequality (3.16) thus gives the inequality

$$D^2 g(v_i) \cdot Dg(v_i)^{-1}\, t\partial_t u_i + D^2 f(v_i) \cdot t\partial_x v_i \le\ \alpha\, Dg(v_i) + K Id \quad \text{in the set } E^1. \tag{3.18}$$

By our assumption, the measures $t\partial_t u_i$ and $t\partial_x v_i$ restricted to the set E^1 have densities with respect to the Lebesgue measure, which are locally integrable functions. Inequality (3.18) holds almost everywhere for the density function and we can use the properties (2.17ii) and (2.18ii) and obtain:

$$D^2 g(r_s) \cdot Dg(r_s)^{-1}\, t\partial_t z_s + D^2 f(r_s) Dg(r_s)^{-1}\, t\partial_x z_s \le\ \alpha\, Dg(r_s) + K Id \quad \text{in } E^1.$$

This last inequality can be easily rewritten in the form (3.17). This proves (3.17) in the set E^1.

We now check (3.17) in the set E^2. Let (t,x) be in E^2. By (3.16), we have

$$-\sigma(Dg(v_i^+) - Dg(v_i^-)) + Df(v_i^+) - Df(v_i^-) \le 0,$$

where σ is the speed of the discontinuity at the point (t,x). In view of the properties (2.17iii) and (2.18iii), we obtain

$$-\sigma\big(Dg(r_s^+) - Dg(r_s^-)\big) + Df(r_s^+) - Df(r_s^-) \le 0,$$

where $g(r_s^+) = (1-s)\,g(r_1^+) + s\,g(r_s^+)$ and $g(r_s^-) = (1-s)\,g(r_1^-) + s\,g(r_2^-)$. This proves that (3.17) holds in E^2.

Since $H_1(E^3) = 0$, it follows finally that (3.17) is satisfied in the sense of distributions in the whole strip $]0,T[\times\boldsymbol{R}$.

Suppose first that system (2.1) belongs to the class A. Then (3.17) can be rewritten in the form

$$-\alpha\, t^{-\alpha-1}\, Dg + t^{-\alpha}\, \partial_x(Df(r_s)) \le t^{-\alpha} K Id,$$

that is

$$-\frac{\alpha}{t}\, Dg + \partial_x\big(Df(r_s))\big) \le K Id$$

with, in this case, $r_s = (1-s)\, v_1 + s\, v_2$. By integration with respect to s and using the fact that

$$-\frac{\alpha}{t} Dg \le 0 \le K \text{ for all times } t > 0,$$

we see that $\tilde{A}$ (cf. (3.10)) satisfies

$$-\frac{\alpha}{t} Dg + \partial_x \tilde{A} \le K Id.$$

This latter inequality is equivalent to

$$\partial_t(t^{-\alpha} Dg) + \partial_x(t^{-\alpha}\tilde{A}) \le t^{-\alpha} K$$

which, after regularization by the kernel ω^ϵ, cf (3.11), gives immediately

$$\partial_t\big(t^{-\alpha} Dg\big) + \partial_x\big(t^{-\alpha} A^\epsilon\big) \le t^{-\alpha} K Id.$$

This proves (3.15) for the systems of the class A.

Suppose next that (2.1) belongs to the class B, so that (3.17) is equivalent to

$$\partial_t(t^{-\alpha} Dg(g^{-1}(z_s))) \leq t^{-\alpha} K Id.$$

Indeed this latter inequality holds with z_s replaced by z_s^ϵ, i.e.

$$\partial_t(t^{-\alpha} Dg(g^{-1}(z_s^\epsilon))) \leq t^{-\alpha} K Id, \tag{3.19}$$

where $z_s^\epsilon = (1-s)\, g(v_1^\epsilon) + s\, g(v_2^\epsilon) = z_s * \omega^\epsilon$. This because (3.16) holds with v_i replaced by v_i^ϵ, as it can be checked by using the arguments of the proof of (3.17). Integrating (3.19) with respect to s gives (3.15). The proof of Lemma 3.3 is complete. □

The following yields a uniform estimate for ϕ^ϵ in the H^1 norm with the weight $t^{\alpha/2}$.

Lemma 3.4. *For all times t in the interval $]0,T]$, one has*

$$\|t^{\alpha/2}\phi^\epsilon(t)\|_{H^1(\boldsymbol{R})} + \|t^{\alpha/2}\partial_t\phi^\epsilon(t)\|_{L^2(\boldsymbol{R})} \leq O(1)\, \|k\|_{H^1((0,T)\times\boldsymbol{R})}. \tag{3.20}$$

Proof: We treat first the case of a system of the class A. In this case, $B = B^\epsilon = Dg$ is a constant matrix. We set

$$\psi^\epsilon = \partial_x \phi^\epsilon.$$

We differentiate with respect to x the equations in (3.14) and multiply the resulting equation by $t^\alpha \psi_\epsilon^*$. After integration over $\boldsymbol{R}$, we obtain

$$\int_{\boldsymbol{R}} t^\alpha \psi_\epsilon^* \partial_t(B\psi^\epsilon)\, dx + \int_{\boldsymbol{R}} t^\alpha \psi_\epsilon^* \partial_x(A^\epsilon \psi^\epsilon)\, dx = \int_{\boldsymbol{R}} t^\alpha \psi_\epsilon^* \partial_x k\, dx.$$

Since B and A^ϵ are symmetric, this is equivalent to:

$$\frac{d}{dt}\int_{\boldsymbol{R}} \frac{1}{2} t^\alpha \psi_\epsilon^* B \psi^\epsilon\, dx = \int_{\boldsymbol{R}} \frac{1}{2}\psi_\epsilon^* \partial_t(t^\alpha B)\psi^\epsilon\, dx - \int_{\boldsymbol{R}} \frac{1}{2}\psi_\epsilon^* \partial_x(t^\alpha A^\epsilon)\psi^\epsilon\, dx + \int_{\boldsymbol{R}} t^\alpha \psi_\epsilon^* \partial_x k\, dx,$$

i.e.

$$\frac{d}{dt}\int_{\boldsymbol{R}} \frac{1}{2} t^\alpha \psi_\epsilon^* B \psi^\epsilon\, dx = \frac{1}{2}\int_{\boldsymbol{R}} \psi_\epsilon^* \big\{\partial_t(t^\alpha B) - \partial_x(t^\alpha A^\epsilon)\big\}\psi_\epsilon\, dx + \int_{\boldsymbol{R}} t^\alpha \psi_\epsilon^* \partial_x k\, dx. \tag{3.21}$$

In view of the entropy inequality (3.15) in Lemma 3.3, we have

$$\begin{aligned}\partial_t(t^\alpha B) - \partial_x(t^\alpha A^\epsilon) &= \alpha\, t^{\alpha-1} B - t^\alpha \partial_x A^\epsilon \\ &= -t^{2\alpha}\big\{\partial_t(t^{-\alpha}B) + \partial_x(t^{-\alpha}A^\epsilon)\big\} \geq -t^\alpha K Id.\end{aligned} \tag{3.22}$$

Using (3.22) in (3.21) yields

$$\begin{aligned}\frac{d}{dt}\int \frac{1}{2} t^\alpha \psi_\epsilon^* B \psi^\epsilon\, dx &\geq -\frac{K}{2}\int_{\boldsymbol{R}} t^\alpha \psi_\epsilon^* \psi^\epsilon\, dx + \int_{\boldsymbol{R}} t^\alpha \psi_\epsilon^* \partial_x k\, dx \\ &\geq -\big(\frac{K}{2} + O(1)\big)\int_{\boldsymbol{R}} t^\alpha \psi_\epsilon^* \psi_\epsilon\, dx - O(1)\int_{\boldsymbol{R}} t^\alpha \partial_x k^* \partial_x k\, dx.\end{aligned}$$

By Gronwall's inequality and since B is a positive definite matrix (c.f. (2.2)), we get

$$\int_{\boldsymbol{R}} t^\alpha \psi^\epsilon(t)^* \psi^\epsilon(t)\, dx \leq O(1)\int_{\boldsymbol{R}} T^\alpha \psi^\epsilon(T)^*\, \psi^\epsilon(T)\, dx + O(1)\int_t^T \int_{\boldsymbol{R}} {t'}^\alpha k_x^* k_x\, dx\, dt'.$$

Because $\psi^\epsilon(T) \equiv 0$ (C.f. (3.14)), we have in fact proved that

$$\|t^{\alpha/2}\partial_x \phi^\epsilon(t)\|_{L^2(\boldsymbol{R})} \leq O(1)\|t^{\alpha/2}\partial_x k\|_{L^2((0,T)\times\boldsymbol{R})}. \tag{3.23}$$

Using (3.23) with the identity $\partial_t\phi^\epsilon = -B^{-1}A^\epsilon\partial_x\phi^\epsilon + B^{-1}k$ yields:

$$\|t^{\alpha/2}\partial_t \phi^\epsilon(t)\|_{L^2(\boldsymbol{R})} \leq O(1)\big\{\|t^{\alpha/2}\partial_x h\|_{L^2(\,(0,T)\times\boldsymbol{R})} + \|t^{\alpha/2} k\|_{L^2(\,(0,T)\times\boldsymbol{R})}\big\}. \tag{3.24}$$

The estimate for $\phi^\epsilon(t)$ follows from (3.23) and Poincaré's inequality, namely:

$$\|t^{\alpha/2}\phi^\epsilon(t)\|_{L^2(\boldsymbol{R})} \leq O(1)\|t^{\alpha/2}\partial_x\phi^\epsilon(t)\|_{L^2(\boldsymbol{R})} \leq O(1)\|k\|_{H^1(\,(0,T)\times\boldsymbol{R})}. \tag{3.25}$$

Estimates (3.23)–(3.25) give the desired result (3.20) for the systems of the class A.

We now deal with the systems in the class B. In this case, $A^\epsilon = A = Df$ is a constant and invertible matrix. Set $\psi^\epsilon = \partial_t\phi^\epsilon$. We differentiate the equations in (3.14) with respect to the variable t. After integration over $\boldsymbol{R}$ and integration by parts, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\boldsymbol{R}} \frac{1}{2} t^\alpha \psi_\epsilon^* B^\epsilon \psi^\epsilon\, dx \\ &= \int_{\boldsymbol{R}} \frac{1}{2}\psi_\epsilon^* \partial_t(t^\alpha B^\epsilon)\psi^\epsilon\, dx - \int_{\boldsymbol{R}} \psi_\epsilon^* t^\alpha \partial_t(B^\epsilon)\psi^\epsilon\, dx \\ &+ \int_{\boldsymbol{R}} \frac{1}{2}\psi_\epsilon^* \partial_x(t^\alpha A)\psi^\epsilon\, dx + \int_{\boldsymbol{R}} t^\alpha \psi_\epsilon^* \partial_t h\, dx.\end{aligned} \tag{3.26}$$

We have $\partial_x(t^\alpha A)=0$ and

$$\begin{aligned}\partial_t(t^\alpha B^\epsilon)-2t^\alpha\partial_t B^\epsilon &= \alpha t^{\alpha-1}B^\epsilon - t^\alpha\partial_t B^\epsilon = -t^{2\alpha}\partial_t(t^{-\alpha}B^\epsilon)\\ &= -t^{2\alpha}\big\{\partial_t(t^{-\alpha}B^\epsilon)+\partial_x(t^{-\alpha}A)\big\} \geq -t^\alpha K.\end{aligned} \tag{3.27}$$

¿From (3.26) and (3.27), we deduce that

$$\frac{d}{dt}\int_{\boldsymbol{R}}\frac{1}{2}t^\alpha\psi_\epsilon^* B^\epsilon\psi^\epsilon\,dx \geq -\big(\frac{K}{2}+O(1)\big)\int_{\boldsymbol{R}}t^\alpha\psi_\epsilon^*\psi^\epsilon\,dx - O(1)\int_{\boldsymbol{R}}t^\alpha k_t^* k_t\,dx.$$

Since B^ϵ is a uniformly positive definite matrix (c.f. (2.2) and the definition of B^ϵ and by Gronwall's inequality, we finally obtain:

$$\|t^\alpha\psi^\epsilon(t)\|^2_{L^2(\boldsymbol{R})} \leq O(1)\|T^\alpha\psi^\epsilon(T)\|^2_{L^2(\boldsymbol{R})} + O(1)\int_t^T\int_{\boldsymbol{R}}t^\alpha k_t^* k_t\,dx\,ds.$$

But $\psi^\epsilon(T)=(B^\epsilon)^{-1}\big\{-A\partial_x\phi^\epsilon(T)-k(T)\big\}\equiv 0$, thus

$$\|t^\alpha\partial_t\phi^\epsilon(t)\|_{L^2(\boldsymbol{R})} \leq O(1)\|t^\alpha\partial_t k\|_{L^2(\,(0,T)\times\boldsymbol{R})}. \tag{3.28}$$

¿From (3.28), Poincaré inequality and the relation

$$\partial_x\phi^\epsilon = A^{-1}\big\{-B^\epsilon\partial_t\phi^\epsilon + k\big\},$$

similar bounds for ϕ^ϵ and $\partial_x\phi^\epsilon$ are easily deduced. The proof of Lemma 3.4 is complete. □

We now complete the proof of existence for the adjoint problem. In view of the a priori estimates (3.20), a subsequence of ϕ^ϵ must converge to an element ϕ in the space $H^1_{loc}(]0,T]\times\boldsymbol{R},\boldsymbol{R}^p)$ in the following sense: for all t in $]0,T[$

$$\phi^\epsilon\to\phi \text{ strongly in } L^2(\,(t,T)\times\boldsymbol{R},\boldsymbol{R}^p) \tag{3.29}$$

and

$$(\partial_t\phi^\epsilon,\partial_x\phi^\epsilon)\rightharpoonup(\partial_t\phi,\partial_x\phi) \text{ weakly in } L^2(\,(t,T)\times\boldsymbol{R},\boldsymbol{R}^p). \tag{3.30}$$

The function ϕ also satisfies

$$\|t^{\alpha/2}\phi\|_{H^1(\boldsymbol{R})}+\|t^{\alpha/2}\partial_t\phi\|_{L^2(\boldsymbol{R})}\in L^\infty(0,T). \tag{3.31}$$

Properties (3.29) and (3.30) are sufficient for the passage to the limit in the equations (3.14) satisfied by ϕ^ϵ. Thus the function ϕ satisfies (3.2). Moreover (3.31) implies (3.3). The proof of existence of a solution to (3.2), (3.3) is complete.

It remains to complete the proof of Theorem 2.3. The above arguments have proved that, if $u_2(0) = u_1(0)$, then $u_2(t) = u_1(t)$ for all times t in the interval $[0, T]$. Note that u_1 and u_2 are Lipschitz continuous in time. Next, in view of the entropy inequality (2.16), there exists a small interval $[T_1, T_2]$ such that no rarefaction waves arise at the times t in the interval $]T_1, T_2[$. We can apply the same arguments as previously and conclude that $u_1(t) = u_2(t)$ for all $t \in [T_1, T_2]$. We can continue this process inductively since the set occurring in condition (2.16) is countable and has only left accumulation points. It follows that $u_1(t) = u_2(t)$ for all times $t \geq 0$. The (first) proof of Theorem 2.3 is thus completed provided that $\alpha < 5/2$.

4. UNIQUENESS VIA UNIQUENESS FOR THE FORWARD ADJOINT PROBLEM

This section contains a second proof of Theorem 2.3. Our approach is based on Haar's method which reduces the question of proving uniqueness for the nonlinear system to proving uniqueness for a forward adjoint problem. Using again the Wave Entropy Condition (2.16), we obtain an uniqueness result in L^2 for this latter problem.

Let v_1 and v_2 in $L^\infty_{loc}([0,\infty[, BV_{loc}(\boldsymbol{R}, V))$ be two weak solutions of system (2.1) satisfying the Wave Entropy Condition (2.16). It is sufficient to consider the interval $]0,T[$ with T defined in Section 3. Larger times can be treated by induction. As before, we also use the notation $u_i = g(v_i)$.

Let us define the *potential functions* ϕ_1 and ϕ_2 associated with the functions v_1 and v_2, respectively :

$$\phi_i(t,x) = \int_0^x g(v_i(0,y))\,dy - \int_0^t f(v_i(s,x))\,ds \qquad \text{for almost every } t>0,\ x\in\boldsymbol{R}. \tag{4.1}$$

Clearly ϕ_1 and ϕ_2 belong to $L^\infty_{loc}(\boldsymbol{R}_+\times\boldsymbol{R},\boldsymbol{R}^p)$.

Lemma 4.1. *In the sense of distributions, one has*

$$\partial_t\phi_i = -f(v_i) \quad \text{and} \quad \partial_x\phi_i = g(v_i). \tag{4.2}$$

As a consequence of Lemma 4.1 we get

$$\phi_i \in W^{1,\infty}_{loc}([0,\infty[\times\boldsymbol{R},\boldsymbol{R}^p), \tag{4.3}$$

and thus $\phi_i(t,x)$ is defined for all (t,x).

Proof: To show that $\partial_t\phi_i = -f(v_i)$, we have to prove

$$\int\!\!\int \phi_i\partial_t\theta\,dx\,dt = \int\!\!\int f(v_i)\,\theta\,dx\,dt \tag{4.4}$$

for each (say, scalar valued) test-function $\theta = \theta(t,x)$. In view of (4.1) and using Fubini's theorem and the fact that $\partial_t\int_0^t f(v_i(s,x))ds = f(v_i(t,x))$ almost everywhere (since $f(v_i)\in$

L^∞), one has

$$\begin{aligned}
&\int\int \partial_t\theta(t,x)\,\phi_i(t,x)\,dx\,dt\\
&= \int\int \partial_t\theta(t,x)\int_0^x g(v_i(0,y)\,)\,dy\,dx\,dt - \int\int \partial_t\theta(t,x)\int_0^t f(v_i(s,x)\,)\,ds\,dx\,dt\\
&= \int\int \theta(t,x)\,f(v_i(t,x)\,)\,dx\,dt,
\end{aligned}$$

which proves (4.4).

We now check that $\partial_x\phi_i = g(v_i)$, or more precisely

$$\int \phi_i(t,x)\,\theta'(x)\,dx = -\int g(v_i(t,x)\,)\,\theta(x)\,dx \qquad \text{for almost every } t>0, \tag{4.5}$$

where $\theta = \theta(x)$ is a test-function of the single variable x. Indeed, since v_i are weak solutions to (2.1) and Lipschitz continuous in time, one has

$$\frac{d}{dt}\int \theta(x)\,g(v_i(t,x)\,)\,dx - \int \theta'(x)\,f(v_i(t,x)\,)\,dx = 0, \quad \text{for almost every } t>0. \tag{4.6}$$

Indeed we observe that

$$\int \theta(x)\,g(v_i(t,x)\,)\,dx$$

is a Lipschitz continuous function of the time variable since $v_i \in \mathrm{Lip}_{loc}([0,\infty[, L^1_{loc}(\boldsymbol{R}, V))$ and thus

$$\frac{d}{dt}\int \theta(x)\,g(v_i(t,x)\,)\,dx \in L^\infty_{loc}([0,\infty[, \boldsymbol{R}^p).$$

We integrate (4.6) over $]0,t[$ and get

$$\int \theta(x)\,g(v_i(t,x)\,)\,dx - \int \theta(x)\,g(v_i(0,x)\,)\,dx = \int_0^t\int \theta'(x)\,f(v_i(s,x)\,)\,dx\,ds. \tag{4.7}$$

Then, combining (4.1) and (4.7), we find

$$\begin{aligned}
&\int \theta'(x)\,\phi_i(t,x)\,dx = \int \theta'(x)\int_0^x g(v_i(0,y)\,)\,dy\,dx - \int \theta'(x)\int_0^t f(v_i(s,x)\,)\,ds\,dx\\
&= -\int \theta(x)\,g(v_i(0,x)\,)\,dx - \int \theta(x)\,g(v_i(t,x)\,)\,dx + \int \theta(x)\,g(v_i(0,x)\,)\,dx\\
&= -\int \theta(x)\,g(v_i(t,x)\,)\,dx,
\end{aligned}$$

which proves (4.5). The proof of Lemma 4.1 is complete. □

Let ϕ, B and A be defined by: $\phi = \phi_2 - \phi_1$ and

$$B^{-1} = \int_0^1 D(g^{-1})(\,(1-s)u_1 + s\,u_2)\,ds \quad \text{and} \quad A = \int_0^1 Df(\,(1-s)v_1 + sv_2)\,ds.$$

Note that B and A are measurable and bounded functions, while ϕ is a locally Lipschitz continuous function. Thus the product of $AB^{-1}\partial_x\phi$ is well-defined in a classical sense.

Lemma 4.2. *The function ϕ is a solution to the following linear hyperbolic system:*

$$\partial_t\phi + AB^{-1}\partial_x\phi = 0 \qquad \textit{almost everywhere in }]0,\infty[\times\boldsymbol{R}. \tag{4.8}$$

Proof: From (4.1) and the definition of B and A, we deduce:

$$\partial_t\phi = \partial_t\phi_2 - \partial_t\phi_1 = -f(v_2) + f(v_1)$$

and

$$\begin{aligned} AB^{-1}\partial_x\phi &= AB^{-1}(\partial_x\phi_2 - \partial_x\phi_1) = AB^{-1}(g(v_2) - g(v_1)\,) \\ &= A(v_2 - v_1) = f(v_2) - f(v_1) \end{aligned}$$

and (4.8) follows immediately. □

Our goal now is to prove that the only solution to problem

$$\begin{aligned} &\partial_t\phi + AB^{-1}\partial_x\phi = 0, \quad t > 0,\ x \in \boldsymbol{R}, \\ &\phi(0,x) = 0, \quad x \in \boldsymbol{R}, \end{aligned} \tag{4.9}$$

is $\phi \equiv 0$. This result would imply that $v_1(t,x) = v_2(t,x)$, for almost every $x \in \boldsymbol{R}$ and for all times $0 \le t \le T$ (cf. property (4.2) satisfied by ϕ_1 and ϕ_2). As explained previously, we only need consider the strip $]0,T[\times\boldsymbol{R}$.

For convenience, we regularize the matrix-valued functions A and B in the same way as it was done in Section 3; cf. formula (3.11) and (3.12) for the definition of A^ϵ and B^ϵ. Of course, we have

$$\partial_t\phi + A^\epsilon(B^\epsilon)^{-1}\partial_x\phi = k^\epsilon, \quad t > 0, \quad x \in \boldsymbol{R}, \tag{4.10}$$

where $k^\epsilon = (A^\epsilon (B^\epsilon)^{-1} - AB^{-1})\partial_x \phi$ satisfies:

$$k^\epsilon \to 0 \quad \text{in} \quad L^2_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}, \boldsymbol{R}^p). \tag{4.11}$$

We also recall from Section 3, Lemma 3.3, that the Wave Entropy Condition

$$\partial_t (t^{-\alpha} B^\epsilon) + \partial_x (t^{-\alpha} A^\epsilon) \le t^{-\alpha} K Id \tag{4.12}$$

holds in the whole strip $]0, T[\times \boldsymbol{R}$.

In what follows, we denote by $M > 0$ a sufficiently large constant and by $N > 0$ an arbitrary positive constant. We set : $\Omega_t = \big\{ |x| \le N + Mt \big\}$.

Lemma 4.3. *For all times* s *and* t *satisfying* $0 < s \le t \le T$, *one has*

$$t^{-\alpha} \int_{\Omega_t} \phi(t)^* \, \phi(t)\, dx \le O(1)\, s^{-\alpha} \int_{\Omega_s} \phi(s)^* \, \phi(s)\, dx, \tag{4.13}$$

where $O(1)$ *is a constant independent of* s and t but possibly on T.

Proof: We consider first a system in the class A. We set: $\psi = B^{-1}\phi$. In view of (4.10), one has

$$B\partial_t \psi + A^\epsilon \partial_x \psi = k^\epsilon.$$

We multiply this equation by $t^{-\alpha}\psi^*$ and integrate the resulting equation over Ω_t. We get:

$$\int_{\Omega_t} t^{-\alpha} \psi^* B \partial_t \psi + \int_{\Omega_t} t^{-\alpha} \psi^* A^\epsilon \partial_x \psi = \int_{\Omega_t} t^{-\alpha} \psi^* k^\epsilon$$

and, by integration by parts, using the fact that A^ϵ and B^ϵ are symmetric matrices

$$\frac{d}{dt} \int_{\Omega_t} t^{-\alpha} \psi^* B \psi \le \int_{\Omega_t} \psi^* \partial_t (t^{-\alpha} B) \psi + \int_{\Omega_t} \psi^* \partial_x (t^{-\alpha} A^\epsilon)_x \psi + 2 \int_{\Omega_t} t^{-\alpha} \psi^* k^\epsilon.$$

Here we take M large enough so that the boundary term arising from the integration by parts in the above inequality is non-positive. Next we use (4.12) and Gronwall's lemma:

$$t^{-\alpha} \int_{\Omega_t} \psi^\epsilon(t)^* B \psi^\epsilon(t)\, dx \le O(1)\, s^{-\alpha} \int_{\Omega_s} \psi^\epsilon(s)^* B \psi^\epsilon(s)\, dx + O(1) \int_s^t \int_{\Omega_\tau} \tau^{-\alpha} k_\epsilon^* k^\epsilon \, dx\, d\tau$$

for $0 < s \leq t \leq T$. Using (4.11), we get

$$t^{-\alpha} \int_{\Omega_t} \psi(t)^* B \psi(t)\, dx \leq O(1)\, s^{-\alpha} \int_{\Omega_s} \psi(s)^*\, B\, \psi(s)\, dx$$

which is equivalent to (4.13).

We next treat the case of a system of the class B. Setting $\psi = A^{-1}\phi$, the equations (4.10) give:

$$B^\epsilon \partial_t \psi + A \partial_x \psi = B^\epsilon A^{-1} k_\epsilon.$$

We multiply this equation by $t^{-\alpha}\psi_\epsilon^*$ and integrate it over Ω_t:

$$\int_{\Omega_t} \big\{ t^{-\alpha} \psi^* B^\epsilon \partial_t \psi + t^{-\alpha} \psi^* A \partial_x \psi \big\}(t)\, dx = \int_{\Omega_t} t^{-\alpha} \big(\psi^* B^\epsilon A^{-1} k^\epsilon \big)(t)\, dx.$$

Because of the symmetry property of the matrices A and B^ϵ, we can easily integrate by parts and obtain:

$$\frac{d}{dt} \frac{1}{2} \int_{\Omega_t} t^{-\alpha} \psi^* B^\epsilon \psi\, dx \leq \int_{\Omega_t} \frac{1}{2} \psi^* \partial_t (t^{-\alpha} B^\epsilon) \psi^\epsilon\, dx + \int_{\Omega_t} t^{-\alpha} \psi^* B^\epsilon A^{-1} k_\epsilon\, dx.$$

Note here that A and B^ϵ are Lipschitz continuous functions of the variables (t, x), so the above integration by parts is possible. We use the entropy inequality (4.12) to deduce from the above inequality that:

$$\frac{d}{dt} \frac{1}{2} \int_{\Omega_t} t^{-\alpha} \psi^* B^\epsilon \psi\, dx \leq \big(K + O(1)\big) \int_{\Omega_t} t^{-\alpha} \psi^* \psi\, dx + O(1) \int_{\Omega_t} t^{-\alpha} k_\epsilon^* k^\epsilon\, dx, \tag{4.15}$$

where we have used the fact that $c_1 Id \geq B^\epsilon \geq c_2 I$. By Gronwall's lemma, it follows that

$$t^{-\alpha} \int_{\Omega_t} (\psi^* \psi)(t)\, dx \leq O(1)\, s^{-\alpha} \int_{\Omega_s} (\psi^* \psi)(s)\, dx + O(1) \int_s^t \int_{\Omega_\tau} \tau^{-\alpha} k_\epsilon^* k^\epsilon\, dx\, d\tau$$

for all $0 < s \leq t \leq T$. Letting $\epsilon \to 0$ and using (4.11) and the invertibily of A give the estimate (4.13). The proof of Lemma 4.3 is complete. □

It remains to determine the limit of the right hand side of (4.13) when $s \to 0$. We have the following estimate for ϕ.

Lemma 4.4. *For all* $t \in [0,T]$ *we have*

$$\|\phi(t)\|_{L^2(\Omega_t)} \le \|w_2 - w_1\|_{L^2(\Omega_0)} + O(1)\, t \|v_2(0) - v_1(0)\|_{L^2(\Omega_0)} + O(1)\, t^{3/2}, \tag{4.16}$$

where

$$w_i(x) = \int_0^x g(v_i(y,0)\,)\, dy \quad \text{for } x \in \boldsymbol{R}. \tag{4.17}$$

Proof: In view of (4.1), one has:

$$\begin{aligned}\phi(x,t) &= \phi_2(x,t) - \phi_1(x,t) \\ &= \int_0^x \big(g(v_2(y,0)\,) - g(v_1(y,0)\,)\,\big)\, dy - \int_0^t \big(f(v_2(x,s)\,) - f(v_1(x,s)\,)\,\big)\, ds,\end{aligned}$$

which yields

$$\begin{aligned}\|\phi(t)\|_{L^2(\Omega_t)} &\le \|w_2 - w_1\|_{L^2(\Omega_t)} + \Big\{ \int_{\Omega_t} \Big(\int_0^t |f(v_2(x,s)\,) - f(v_1(x,s)\,)|\, ds \Big)^2 dx \Big\}^{1/2} \\ &\le \|w_2 - w_1\|_{L^2(\Omega_t)} + O(1) \Big\{ \int_{\Omega_t} t \int_0^t |v_2(x,s) - v_1(x,s)|^2\, ds\, dx \Big\}^{1/2} \\ &\le \|w_2 - w_1\|_{L^2(\Omega_t)} + O(1)\, t^{1/2} \Big\{ \int_0^t \|v_2(s) - v_1(s)\|_{L^1(\Omega_s)}\, ds \Big\}^{1/2}.\end{aligned}$$

But $\|v_i(s) - v_i(0)\|_{L^1(\Omega_s)} \le O(1)\, s$, thus we get

$$\begin{aligned}\|\phi(t)\|_{L^2(\Omega_t)} \le\ & \|w_2 - w_1\|_{L^2(\Omega_0)} + O(1)\, t^{1/2} \Big\{ \int_0^t \|v_2(s) - v_2(0)\|_{L^1(\Omega_s)}\, ds \\ & + \int_0^t \|v_2(0) - v_1(0)\|_{L^1(\Omega_s)}\, ds + \int_0^t \|v_1(s) - v_1(0)\|_{L^1(\Omega_s)}\, ds \Big\}^{1/2} \\ \le\ & \|w_2 - w_1\|_{L^2(\Omega_t)} + O(1)\, t^{3/2} + O(1) t \|v_2(0) - v_1(0)\|_{L^1(\Omega_s)}.\end{aligned}$$

The proof of Lemma 4.4 is complete. □

We finally combine the results in Lemmas 4.3 and 4.4.

Lemma 4.5. *For all* $0 < s < t < T$, *we have*

$$\|\phi(t)\|_{L^2(\Omega_t)} \le \frac{t^{\alpha/2}}{s^{\alpha/2}} \big\{ \|w_2 - w_1\|_{L^2(\Omega_0)} + O(1) s \|v_2(0) - v_1(0)\|_{L^1(\Omega_0)} \big\} + O(1)\, t^{\alpha/2} s^{(3-\alpha)/2}.$$

When v_1 and v_2 assume the same initial condition, i.e. when $v_1(0) = v_2(0)$, then by (4.17), $w_2 \equiv w_1$ and Lemma 4.5 gives

$$\|\phi(t)\|_{L^2(\Omega_t)} \le O(1)\, t^{\alpha/2}\, s^{(3-\alpha)/2} \quad \text{for all } \ 0 < s \le t \le T,$$

which implies $\phi \equiv 0$ if $\alpha \in [0, 3[$. The (second) proof of Theorem 2.3 is complete.

5. LINEAR HYPERBOLIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

In this section, we prove results of existence and uniqueness for some linear hyperbolic systems with discontinuous coefficients. Such a kind of result was proved and used throughout Sections 3 and 4. The systems considered there were adjoint problems associated with the nonlinear equations (2.1). The present section deals with a class of linear hyperbolic systems that are not necessarily an adjoint problem to a nonlinear system.

Generally speaking such systems are ill-posed: neither existence nor uniqueness of solutions is a priori ensured. However, we can prove existence or uniqueness of solutions in the space $H^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}, \boldsymbol{R}^p)$ in case that the coefficients of the system satisfy the Wave Entropy Condition. We emphasize that this condition was introduced in the context of nonlinear problems but, as we show it here, is also suitable for linear problems. Results of this type were previously proved for scalar equations and under the restriction that the coefficients have "no singularity" of the kind $1/t$, as can be the case here. Our results indeed hold for some systems of equations. For scalar equations, we refer to DiPerna-Lions [7], Le Floch [18] and Tadmor [31].

Note also that an existence and uniqueness result is proved in [18] for some linear hyperbolic equations with discontinuous coefficients. The solutions found are not continuous in general, but simply bounded functions of bounded variation. Thus the concept of non-conservative product in the sense of Volpert [32] (c.f. also Dal Maso-Le Floch-Murat [5] and Le Floch-Liu [19]) is needed there.

We consider the Cauchy problem for a linear hyperbolic system

$$B\partial_t\phi + A\partial_x\phi = k, \qquad t > 0, \quad x \in \boldsymbol{R}, \tag{5.1}$$

$$\phi(0, x) = \phi_0(x), \tag{5.2}$$

where $\phi_0 \in H^1(\boldsymbol{R})$, $k \in H^1(\boldsymbol{R}_+ \times \boldsymbol{R}, \boldsymbol{R}^p)$ and A and B are bounded measurable $m \times m$ matrix-valued functions such that A and B are symmetric matrices and either

$$B \text{ is a positive definite constant matrix} \tag{5.3a}$$

or

$$(5.3b) \qquad B \text{ is positive definite and } A \text{ is an invertible constant matrix.}$$

We prove first an existence theorem for the Cauchy problem (5.1), (5.2).

Theorem 5.1. *In addition to assumption (5.3), we assume that there exist two non-negative constants T and α and a locally bounded non-negative function $K = K(t,x)$ satifying*

$$\int_0^T \max_{x\in \boldsymbol{R}} K(t,x)\,dt < \infty, \tag{5.4}$$

such that

$$\partial_t(\,(T-t)^{-\alpha}B) + \partial_x(\,(T-t)^{-\alpha}A) \geq -K\,(T-t)^{-\alpha} \tag{5.5}$$

in the sense of distributions in the strip $\boldsymbol{R}\times(0,T)$. Then the Cauchy problem (5.1), (5.2) has at least one solution ϕ in $H^1_{loc}([0,T)\times\boldsymbol{R},\boldsymbol{R}^p)$. Furthermore, we have the following estimate

$$(T-t)^{\alpha/2}\|\phi(t)\|_{H^1(\boldsymbol{R})} \leq O(1)\|\phi_0\|_{H^1(\boldsymbol{R})} + O(1)\|k\|_{H^1(\boldsymbol{R}\times(0,T))} \tag{5.6}$$

for all $t\in[0,T)$.

Proof: The proof is very similar to the one given in Section 3. So we only sketch it. Using a smoothing kernel as was done in Section 3, we construct the regularizations A^ϵ and B^ϵ associated with A and B. Then (5.5) implies

$$\partial_t(\,(T-t)^{-\alpha}B^\epsilon) + \partial_x(\,(T-t)^{-\alpha}A^\epsilon) \geq -K(T-t)^{-\alpha}.$$

A solution ϕ to (5.1), (5.2) is obtained as the limit of the solutions ϕ^ϵ to the regularized problem

$$B^\epsilon\partial_t\phi^\epsilon + A^\epsilon\partial_x\phi^\epsilon = k \qquad 0\leq t<T,\ x\in\boldsymbol{R},$$

$$\phi^\epsilon(0,x) = \phi_0(x),$$

for which a priori estimates are derived in the same way as was done in Lemma 3.4. □

We next have the following uniqueness.

Theorem 5.2. *Suppose that* (5.3) *holds and the matrices* B *and* A *satisfy the inequality*

$$\partial_t(t^{-\alpha}B) + \partial_x(t^{-\alpha}A) \le K\,t^{-\alpha} \tag{5.7}$$

in the sense of distributions, where T *and* α *are non-negative constants and* $K = K(t,x)$ *is a measurable non-negative function satisfying (5.4). Then the Cauchy problem* (5.1), (5.2) *has at most one solution* ϕ *in* $L^2_{loc}((0,T)\times \boldsymbol{R})$ *such that*

$$\lim_{t\to 0^+} \frac{\|\phi(t)-\phi_0\|_{L^2(\boldsymbol{R})}}{t^{\alpha/2}} = 0. \tag{5.8}$$

The proof of this theorem follows the lines of the proof given in section 4, so we omit it. Note that (5.8) is automatically satisfied if $0 \le \alpha \le 1$.

6. EXAMPLES AND REMARKS

In this section, we apply the results of Sections 3 and 4 to the cases of the system of isentropic gas dynamics in Lagrangian coordinates (Theorem 2.1), the scalar conservation laws with convex flux (Theorem 2.2) and a symmetric system of two equations that provides us with another application of Theorem 2.3. For scalar conservation laws, the existence of an infinite number of entropy functions (trivial in the scalar case) allows us to relax the assumptions of convexity (2.17).

We deal first with Theorem 2.1. We have to check that the hypotheses of Theorem 2.3 hold for system (2.7). Let η be the following entropy function associated with system (2.7):

$$\eta(u) = \eta(u_1, u_2) = \frac{1}{2}(u_2)^2 + W(u_1),$$

where $W' = -p$. In view (2.8), η is a strictly convex function, so the following formula defines a change of variables:

$$v = (v_1, v_2) = \nabla \eta(u) = \big(- p(u_1), u_2 \big).$$

System (2.7) takes the form (2.1) with

$$g(v) = u = \big(p^{-1}(v_1), v_2\big) \qquad \text{and} \qquad f(v) = \big(- u_2, p(u_1)\big) = \big(- v_2, v_1 \big).$$

The matrix Df is constant and invertible and

$$Dg(v) = \begin{pmatrix} -1/p'(p^{-1}(-v_1)) & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.1}$$

We shall define the sound speed c by $c = \big(- p'(u_1)\big)^{1/2}$. Condition (2.18i) holds in the case of system (2.7). The entropy condition (2.16) here reads:

$$\partial_t \left(\frac{1}{(t-t_*)^{-\alpha}\, c(u_1)^2} \right) \leq \frac{K_*}{(t-t_*)^{-\alpha}},$$

which is exactly (2.9) with $\alpha = 2$ (C.f. our argument in Remark 2.2).

Concerning the conditions (2.18ii) in the case of system (2.7), we note that (with the notation introduced in (2.18ii)):

$$D^2 g(v_i) \cdot Dg(v_i)^{-1} w_i = \begin{pmatrix} p''(u_1)w_1/p'(u_1)^2 & 0 \\ 0 & 0 \end{pmatrix},$$

thus (2.18ii) is equivalent to:

$$\frac{p''(u_1^i)}{p'(u_1^i)^2} w_1^i + \frac{2}{p'(u_1^i)} \le K \qquad \Rightarrow \qquad \frac{p''(u_1^s)}{p'(u_1^s)^2} + \frac{2}{p'(u_1^s)} \le K. \tag{6.2}$$

But this implication is true in view of the convexity assumption (2.11). This proves (2.18ii). It is not difficult to see also that (2.8) implies (2.18iii).

Henceforth, Theorem 2.3 applies to system (2.7) and provides us with the statement in Theorem 2.1 but under the assumption that the solutions have bounded variation in x. To complete the proof of Theorem 2.1, we have now to relax this assumption. We observe that, in the proofs of Sections 3 and 4, the BV regularity is used only in the proof of Lemma 3.3. As a matter of fact, due to the form of the entropy inequality (2.9), we can somewhat simplify the proof of Lemma 3.3 in the case of system (2.7). We explain that now. For the convenience of the reader, we shall sketch the main lines of the proof of Section 4 in this case.

Let (u_1^i, u_2^i) for $i = 1, 2$ be two weak solutions of (2.7) satisfying the same initial condition and all the properties stated in Theorem 2.1. To show that $(u_1^1, u_2^1) = (u_1^2, u_2^2)$, we consider the potential functions:

$$\begin{aligned} \phi_1^i(t,x) &= \int_{(0,0)}^{(t,x)} u_1^i \, dx + u_2^i \, dt, \\ \psi_2^i(t,x) &= \int_{(0,0)}^{(t,x)} u_2^i \, dx + p(u_1^i) \, dt, \end{aligned} \tag{6.3}$$

which are well defined since the equation (2.7) has a conservative form. The potential functions are Lipschitz continuous and satisfy

$$\begin{aligned} \partial_x \phi_1^i = u_1^i, \quad \partial_t \phi_1^i = u_2^i, \\ \partial_x \psi_2^i = u_2^i, \quad \partial_t \psi_2^i = -p(u_1^i). \end{aligned}$$

The function $\phi = \phi_2 - \phi_1$ satisfies the following linear hyperbolic problem

$$\begin{aligned} \partial_t \phi_1 - \partial_x \phi_2 &= 0, \\ \partial_t \phi_2 - c^2 \partial_x \phi_1 &= 0, \\ (\phi_1, \phi_2)(0, x) = (0, 0) \quad &\text{for all } x \in \boldsymbol{R}, \end{aligned} \tag{6.4}$$

where

$$c^2 = -\int_0^1 p'(\,(1-\theta)u_1^1 + \theta u_1^2)\, d\theta.$$

In view of (2.9), there exists a time $T > 0$ so that

$$\partial_t u_1^i + \frac{2}{t}\,\frac{p'(u_1^i)}{p''(u_1^i)} \le K_i \quad \text{in} \quad (0, T) \times \boldsymbol{R}. \tag{6.5}$$

We set $K = \max(K_1, K_2)$. We rewrite (6.4) in the form

$$\begin{aligned} &\partial_t \phi_1 - \partial_x \phi_2 = 0, \\ &\partial_t \phi_2 - c_\epsilon^2 \partial_x \phi_1 = (c^2 - c_\epsilon^2)\partial_x \phi_1 \equiv k^\epsilon, \\ &(\phi, \psi)(0, x) = (0, 0), \quad \text{for all } x \in \boldsymbol{R}, \end{aligned} \tag{6.6}$$

where

$$(c^\epsilon)^2 = -\int_0^1 p'(\,(1-\theta)u_1^{1,\epsilon} + \theta u_1^{2,\epsilon})\, d\theta \qquad \text{and} \qquad u_1^{i,\epsilon} = u_1^i * \omega^\epsilon(x).$$

It follows from the definition of k^ϵ that

$$\|k^\epsilon\|_{L^2(\boldsymbol{R}\times(0,T))} \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{6.7}$$

Using the convexity condition (2.11) and Jensen's inequality, we deduce from (6.5) that

$$\partial_t u_1^{i,\epsilon} + \frac{2}{t}\,\frac{p'(u_1^{i,\epsilon})}{p''(u_1^{i,\epsilon})} \le K.$$

Consequently, we have

$$t\,\partial_t(-c^\epsilon)^2 - 2(c^\epsilon)^2 \le t\,K. \tag{6.8}$$

This proves Lemma 3.3 in the case of system (2.7).

For the convenience of the reader, we reproduce the a priori estimates of Section 4 in this case. We estimate the solutions of (6.6) by the method of Section 4. We fix a time $t > 0$ and s such that $0 < s < t$. We take the inner product of (6.6) with $(\tau^{-2}\phi_1,\, \tau^{-2}c^{\epsilon^{-2}}\phi_2)$ and we integrate the resulting expression over $(s,t) \times \boldsymbol{R}$. After integration by parts we get

$$\frac{1}{2}\Big[\int_{\boldsymbol{R}} \frac{1}{2}\tau^{-2}\phi^2\,dx\Big]_s^t + \frac{1}{2}\Big[\int_{\boldsymbol{R}} (\tau c^\epsilon)^{-2}\psi^2\,dx\Big]_s^t + \int_s^t\int_{\boldsymbol{R}} \tau^{-3}\phi^3\,dx\,d\tau$$
$$\int_s^t\int_{\boldsymbol{R}} \partial_\tau\Big(\frac{-1}{\tau^2(c^\epsilon)^2}\Big)\frac{\psi^2}{2}\,dx\,d\tau$$
$$\le \frac{1}{2}\int_s^t\int_{\boldsymbol{R}} (\tau^2(c^\epsilon)^2)^{-1}\psi^2\,dx\,d\tau + \frac{1}{2}\int_s^t\int_{\boldsymbol{R}} (\tau c^\epsilon)^{-2}(k^\epsilon)^2\,dx\,d\tau.$$

The inequality (6.9) yields

$$\int_s^\tau\int_{\boldsymbol{R}} \frac{\partial}{\partial\tau}\Big(\frac{1}{\tau^2(c^\epsilon)^2}\Big)\frac{\psi^2}{2}\,dxd\tau = \int_s^\tau\int_{\boldsymbol{R}} \frac{\tau(\tau\partial\tau(-(c^\epsilon)^2) + 2(-(c^\epsilon)^2))}{2(\tau^2(c^\epsilon)^2)^2}\,\psi^2\,dx\,d\tau$$
$$\le \int_s^t\int_{\boldsymbol{R}} (K(c^\epsilon)^{-2})\frac{1}{2}(\tau c^\epsilon)^{-2}\psi^2\,dx\,d\tau,$$

and so

$$\frac{1}{2}\Big[\int_{\boldsymbol{R}} \tau^{-2}\phi^2\,dx\Big]_s^t + \frac{1}{2}\Big[\int_{\boldsymbol{R}} (\tau c^\epsilon)^{-2}\psi^2\,dx\Big]_s^t + \int_s^t\int_{\boldsymbol{R}} \tau^{-3}\phi^3\,dx\,d\tau$$
$$\le \int_s^\tau\int_{\boldsymbol{R}} (K(c^\epsilon)^{-2}+1)\frac{1}{2}(\tau c^\epsilon)^{-2}\psi^2\,dx\,d\tau + \frac{1}{2}\int_s^t\int_{\boldsymbol{R}} (\tau c^\epsilon)^{-2}(k^\epsilon)^2\,dx\,d\tau.$$

Letting $\epsilon \to 0^+$ in the above inequality and using (6.7), we get

$$\Big[\int_{\boldsymbol{R}} \frac{1}{2}\tau^{-2}(c^{-2}\psi^2 + \phi^2)\,dx\Big]_s^t \le \int_s^t\int_{\boldsymbol{R}} \big(Kc^{-2}+1\big)\frac{1}{2}(\tau^2c^2)^{-1}\,\psi^2(\tau,x)\,dx\,d\tau,$$

which leads to

$$t^{-2}\|(\phi,\psi)(t)\|^2_{L^2(\boldsymbol{R})} \le O(1)\,s^{-2}\,\|(\phi,\psi)(s)\|^2_{L^2(\boldsymbol{R})}. \tag{6.9}$$

To get that the right-hand side of (6.9) tends to zero as $s \to 0^+$, we use the property that the potential functions are path independent, so that:

$$\phi_1(s,x) = \int_{(0,x)}^{(s,x)} (u_2^2 - u_2^1)\,d\tau$$
$$\phi_2(s,x) = \int_{(o,x)}^{(s,x)} \big(p(u_1^2) - p(u_1^1)\big)\,d\tau.$$

Therefore we have

$$\|\phi_1(s)\|^2_{L^2(\boldsymbol{R})} \le O(1)s^2\big(\sup_{0<\tau\le s}\|u_2^1(\tau)-u_2^1(0)\|_{L^1(\boldsymbol{R})}+\sup_{0<\tau\le s}\|u_2^2(\tau)-u_2^2(0)\|_{L^1(\boldsymbol{R})}\big),$$

and

$$\|\phi_2(s)\|^2_{L^2(\boldsymbol{R})} \le O(1)s^2\big(\sup_{0<\tau\le s}\|u_1^1(\tau)-u_1^1(0)\|_{L^1(\boldsymbol{R})}+\sup_{0<\tau\le s}\|u_2^2(\tau)-u_2^2(0)\|_{L^1(\boldsymbol{R})}\big).$$

It follows that

$$\lim_{s\to 0^+} s^{-2}\|(\phi,\psi)(s)\|^2_{L^2(\boldsymbol{R})}=0,$$

which, together with (6.9), shows that

$$(\phi,\psi)(t,x)\equiv 0 \quad \text{for almost all} \quad (t,x)\in(0,T)\times\boldsymbol{R}.$$

This completes the discussion of Theorem 2.1.

We now give the proof of Theorem 2.2. For convenience, we focus on the potential method that will give a quite elegant argument. Let bounded measurable functions u_1 and u_2 be solutions to (2.12) with the same initial data u_0 and satisfying Oleinik's entropy condition (2.14). We set

$$\phi_i(t,x)=\int_{(0,0)}^{(t,x)} u_i\,dx - f(u_i)\,dt, \qquad i=1,2. \tag{6.10}$$

The function $\phi_i(t,x)$ is Lipschitz continuous and satisfies

$$\partial_t\phi_i=-f(u_i), \qquad \partial_x\phi_i=u_i. \tag{6.11}$$

The function $\phi=\phi_2-\phi_1$ then solves the following problem

$$\begin{aligned}&\partial_t\phi+a(x,t)\partial_x\phi=0 \quad t>0,\\ &\phi(x,0)\equiv 0,\end{aligned} \tag{6.12}$$

where

$$a(x,t)=\int_0^1 f'(1-\theta)u_1+\theta u_2)\,d\theta.$$

To show that (6.12) has only the null solution in the class of Lipschitz continuous functions, we set

$$a^\epsilon(x,t) = \int_0^1 f'(\,(1-\theta)u_1^\epsilon + \theta\, u_2^\epsilon)\, d\theta,$$

where u_i^ϵ is the regularization of u_i defined by a kernel $\omega_\epsilon(x)$

$$u_i^\epsilon = u_i * \omega_\epsilon(x).$$

The entropy condition (2.14) implies that

$$\partial_x a^\epsilon \leq \frac{E}{t}, \tag{6.13}$$

where E is a positive constant independent of ϵ. Let p be a large positive even integer to be chosen later and let $t > 0$ be fixed and s such that $0 < s < t$. Multiplying (6.12) by $p\phi^{p-1}$ and integrating over $(s,t) \times \boldsymbol{R}$, one gets

$$\big[\|\phi(\tau)\|^p_{L^p(\boldsymbol{R})}\big]_s^t - \int_s^t \int_{\boldsymbol{R}} \partial_x a^\epsilon\, \phi^p\, dx\, d\tau \leq \int_s^t \int_{\boldsymbol{R}} p\phi^{p-1}(a^\epsilon - a)\, \partial_x \phi\, dx\, dz.$$

Taking into account (6.13), one has

$$\big[\|\phi(\tau)\|^p_{L^p(\boldsymbol{R})}\big]_s^t \leq \int_s^t \int_{\boldsymbol{R}} \frac{E}{\tau}\, \phi^p\, dx\, d\tau + \int_s^t \int_{\boldsymbol{R}} p\, \phi^{p-1}(a^\epsilon - a)\, \partial_x \phi\, dx\, d\tau$$

Letting $\epsilon \to 0^+$ and using Gronwall's inequality give

$$\|\phi(t)\|_{L^p(\boldsymbol{R})} \leq \|\phi(s)\|_{L^p(\boldsymbol{R})} \frac{t^{E/p}}{s^{E/p}}. \tag{6.14}$$

In view of (6.10) and the boundedness of u_i, one has

$$\|\phi(s)\|_{L^\infty(\boldsymbol{R})} \leq O(1)s,$$

which implies that the right-hand side of (6.14) goes to zero as $s \to 0^+$ provided that we choose p large enough. Therefore

$$\|\phi(t)\|_{L^p(\boldsymbol{R})} = 0 \qquad \text{for all } 0 < t < T,$$

which shows that $u_1 \equiv u_2$ almost everywhere. The proof of Theorem 2.2 is complete.

Finally we consider the Cauchy problem for the system

$$
\begin{aligned}
&u_t + v_x = 0, \\
&\qquad\qquad\qquad\qquad t > 0,\ x \in \boldsymbol{R} \\
&v_t + (f(v) + u)_x = 0,
\end{aligned}
\tag{6.15}
$$

with initial data

$$(u, v)(0, x) = (u_0, v_0), \tag{6.16}$$

where f is a function of class C^4 such that $f'' > 0$ and $\big(-1/f''\big)'' \geq 0\,.$

This is a symmetric system with

$$B = I, \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & f'(v) \end{pmatrix}. \tag{6.17}$$

It can be checked that (6.15) is a strictly hyperbolic and genuinely nonlinear system. It is also clear that system (6.15) belongs to the class A (see (2.17)). The Wave Entropy Condition (2.16) reads here :

$$
\begin{aligned}
&\text{There exists a set } E \subset \boldsymbol{R}_+ \text{ which has only left accumulation} \\
&\text{points and such that for each } t_* \text{ in } E \text{ one has} \\
&\partial_x f'(v) \leq \frac{\alpha}{(t - t_*)} + K \quad \text{in} \quad (t_*, t^*) \times \boldsymbol{R}, \\
&\text{where } K \text{ and } \alpha \text{ are non-negative constants and} \\
&\qquad t^* = \min(E \cap (t_*, \infty)\,).
\end{aligned}
\tag{6.18}
$$

We assume that α belongs to the interval $[0, 3[$.

Consequently, Theorem 2.3 applies to system (6.15). We obtain uniqueness for the solution to (6.15), (6.16), (6.18) in the class of $L^\infty([0,\infty)\,,\,BV(\boldsymbol{R})\,)$. Note that for system (6.15), the Lax entropy condition across shock discontinuities is equivalent to

$$v_x \leq 0.$$

In other words, (6.18) is automatically satisfied across shock discontinuities. For solutions having centered rarefaction waves, (6.18) imposes a restriction on the range of values of the solution, i.e. $\sqrt{f'(v)^2 + 4}\big(\sqrt{f'(v)^2 + 4} \pm f'(v)\big) < 6$.

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