PARTICLE APPROXIMATION OF THE LINEARISED NAVIER-STOKES PROBLEM WITH NEUMANN BOUNDARY CONDITIONS

S. Mas-Gallic
Centre de Mathématiques Appliquées
Ecole Polytechnique
91128 Palaiseau Cedex, FRANCE

Summary

We present a particle method to solve numerically the linearized Navier-Stokes equation with Neumann boundary conditions on the vorticity. The method is based on the use of a boundary integral equation formulation and the introduction of cut-off functions. As a result of this formulation, the vorticity creation due to the boundary conditions is taken into account by the time evolution of the particle weights. The scheme is conservative in L^1 and positive in L^∞ and L^2 if the cut-off is non negative. The stability of the method is proved with no condition on the viscosity and the convergence of the approximate solution towards the solution of the advection diffusion problem is established in L^∞ and in L^2.

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Introduction

The vortex method was first introduced to compute flows of incompressible ideal fluids by O. Hald [14]. It was then extended to the case of viscous flows by A. Chorin [7], S. Huberson [15], G.H. Cottet-S. Mas-Gallic [9] (where the method is based on a viscous splitting first introduced by J.T. Beale and A. Majda [5]), S. Mas-Gallic-P.-A. Raviart [27], P. Degond-S. Mas-Gallic [11] (this work is related to that of R.A. Gingold and J.J. Monaghan [13]) and the references therein (see also the more recent work of D. Fishelov [12]). A review of the different treatments of viscous terms proposed in the weighted particle method can be found in [10] and [21] while in [6], a comparison between the random walk and the deterministic method is achieved in the case of two-dimensionnal computations. The weighted particle method has been used by A. Leonard and G. Winckelmans to compute three-dimensionnal flows (see [18]).

An extension of the random walk method to the case of a problem in a domain Ω was first given by C. Anderson [2], [3]. In the case of the weighted particle method, a coupling with a finite difference method has been proposed by G.H. Cottet [8] (see also the work of S. Huberson-A. Jollès [16] [17], for example). Then noticing that the vortex method is lagrangean and based on the use of quadrature rules, an approximation of convection-diffusion problems with boundary conditions based on the introduction of boundary integral equation was introduced by S. Mas-Gallic in [22]. A quite different method still based on boundary integral equation formulation was recently introduced by Y. Achdou and O. Pironneau [1].

This paper which is summarized in [24], presents an extension of the method introduced in [22] and [23]. These papers were concerned with the particle approximation of convection-diffusion problem with Dirichlet boundary conditions (see also [20] and [19] for a conservative version of the method) and a stability condition relating the physical viscosity ν to the numerical viscosity ε^2 arised. Though we always have in mind that we consider a slightly viscous fluid, such a condition is not convenient form the numerical point of view. The extension to the case of Neumann boundary conditions proposed in the present paper avoids this difficulty and is furthermore conservative.

We consider the flow of an incompressible viscous fluid around an obstacle Ω, governed by the Navier-Stokes equations. In velocity-vorticity formulation, the problem to be solved is the following system (see [8] and also [23] for a justification of the equivalence with the classical Navier-Stokes formulation),

$$\text{(I.1)} \qquad \begin{cases} \dfrac{\partial \omega}{\partial t} + \text{div } (\mathbf{u}\omega) - \nu\Delta\omega = 0 \quad \text{in } \Omega \times (0,T) \\ \omega(.,0) = \omega_0 \quad \text{on } \Omega \\ -\Delta\mathbf{u} = \nabla \times \omega \quad \text{in } \Omega \times (0,T) \\ \mathbf{u} = 0 \quad \text{on } \partial\Omega \times (0,T) \\ \dfrac{\partial \omega}{\partial n} = -g \quad \text{in } \partial\Omega \times (0,T) \end{cases}$$

where g depends on the velocity $\mathbf{u}$, $\mathbf{n}$ is the normal to $\partial\Omega$ outgoing from Ω and $T > 0$ is fixed. Under this form the system is naturally split into a system which gives the evolution of the vorticity once the velocity is given coupled with an elliptic system which gives the velocity. The velocity is usually computed by a classical finite difference or finite element method and we shall not consider here the coupling of these systems but consider later on that the velocity is given. We focuse our attention on the treatment of the viscous term and the boundary condition on ω.

We thus suppose that a sufficiently smooth vector field $\mathbf{a}$ such that,

$$\text{(I.2)} \qquad \text{div } \mathbf{a}(x,t) = 0, \quad x \in \Omega,\ t > 0, \qquad \mathbf{a}.\mathbf{n}(y,t) = 0, \quad y \in \partial\Omega,\ t > 0,$$

is given. We set $Q_T = \Omega \times (0,T)$ and we consider the following problem

$$\text{(I.3)} \qquad \begin{cases} \dfrac{\partial \omega}{\partial t} + \text{div } (\mathbf{a}\omega) - \nu \Delta \omega = 0 \quad \text{in } Q_T \\ \omega(.,0) = \omega_0 \quad \text{on } \Omega \\ \dfrac{\partial \omega}{\partial n} = -g \quad \text{in } \partial\Omega \times (0,T). \end{cases}$$

The first step in the approximation of the solution of this problem by the weighted particle method is to define an integral operator which, if the cut-off function satisfies some assumptions, approximates the Laplace operator (see [11] for example). Doing this, we replace a second order differential operator by a differential operator of order zero and the boundary conditions can no more be imposed. Introduction of a boundary integral equation formulation allows to define an integral operator which includes the boundary conditions and ensures that the solution of the approximate problem satisfies them. The discretisation of the resulting equations leads to a method in which the creation of vorticity due to the boundary is taken into account by an evolution of the strength of existing particles rather than by a creation of new particles. Let us now describe the integral operator that is meant to to approximate the laplacian with Neumann boundary condition.

We introduce 2 functions η and ζ, a real parameter $\varepsilon > 0$ and set, for any $x \in \Omega$,

$$\text{(I.4)} \qquad \eta_\varepsilon(x) = \frac{1}{\varepsilon^2}\eta(\frac{x}{\varepsilon}), \quad \zeta_\varepsilon(x) = \frac{1}{\varepsilon^2}\zeta(\frac{x}{\varepsilon}).$$

and set

$$\text{(I.5)} \qquad \Delta^\varepsilon(t)\omega(x) = \frac{1}{\varepsilon^2}\Lambda^\varepsilon \omega(x) - 2\int_{\partial\Omega} \zeta_\varepsilon(x-y) g(y,t) d\gamma(y),$$

where

$$\text{(I.6)} \qquad \Lambda^\varepsilon \omega(x) = \int_\Omega (\eta_\varepsilon(x-y) + \eta_\varepsilon(x-\tilde{y}))(\omega(y) - \omega(x))dy,$$

and $\tilde{y}$ is the symmetric point of y with respect to the boundary (the point $\tilde{y}$ will be defined more precisely in the sequel). We shall prove later that, under some assumptions on the functions η and ζ, $\Delta^\varepsilon v$ defines an approximation of Δv for any function v which satisfies the Neumann boundary conditions and moreover that, see the Appendix, the solution ω of problem (I.3) is approximated by the solution of the following integro-differential problem

$$\text{(I.7)} \qquad \begin{cases} \dfrac{\partial \omega^\varepsilon}{\partial t} + \text{div } (\mathbf{a}\omega^\varepsilon) - \nu \Delta^\varepsilon \omega^\varepsilon = 0 \quad \text{in } Q_T \\ \omega^\varepsilon(.,0) = \omega_0 \quad \text{on } \Omega. \end{cases}$$

In order to simplify the notations, we have not made explicit the obvious dependence of Δ^ε with respect to the boundary condition g.

Then, in order to define the numerical method, we introduce two quadrature rules, one on Ω and one on $\partial\Omega$. The first one is defined by a set of weighted points $(x_p^0,\ \alpha_p)_{p\in J}$, where $x_p^0 \in \Omega$ and $\alpha_p > 0$, and the second one by a set of points $(y_j, \beta_j)_{j\in\mathbb{N}}$, where $y_j \in \partial\Omega$ and $\beta_j > 0$. For any functions $\phi \in C^0(\Omega)$, $\psi \in C^0(\partial\Omega)$, we have

$$\int_\Omega \phi(x)dx \approx \sum_{p\in J} \alpha_p \phi(x_p^0), \quad \int_{\partial\Omega} \psi(z)dz \approx \sum_{j\in\mathbb{N}} \beta_j \psi(y_j).$$

The method is lagrangean, the particle evolve in time and the set $(x_p(t))_p$ of their positions is solution of the following ordinary differential system

$$\frac{dx_p}{dt}(t) = \mathbf{a}(x_p(t),t), \quad x_p(0) = x_p^0. \tag{I.8}$$

Then, we define the way the particle weights evolve in time. First we introduce an approximation of Δ^ε, Δ_h^ε which operates on sequences. For any sequence $\bar\omega = (\omega_p)_{p\in J}$,we set

$$\Delta_h^\varepsilon\bar\omega(t) = (\Delta_h^\varepsilon\omega_p(t))_p, \quad \Delta_h^\varepsilon\omega_p(t) = \frac{1}{\varepsilon^2}\Lambda_h^\varepsilon\omega_p(t) - 2\sum_{j\in\mathbb{N}}\beta_j\zeta_\varepsilon(x_p(t)-(y_j,0))g(y_j,t), \tag{I.9}$$

where

$$\Lambda_h^\varepsilon\bar\omega(t) = (\Lambda_h^\varepsilon\omega_p(t))_p, \quad \Lambda_h^\varepsilon\omega_p(t) = \sum_{q\in J}\alpha_q(\eta_\varepsilon(x_p(t)-x_q(t)) + \eta_\varepsilon(x_p(t)-\tilde x_q(t)))(\omega_q-\omega_p). \tag{I.10}$$

The particle method is thus defined as follows ; the solution ω of (I.3) is approximated in the sense of distributions by the following combination of Dirac mesures

$$\omega_h(x,t) = \sum_{p\in J}\alpha_p\omega_p(t)\delta(x-x_p(t)), \tag{I.11}$$

where the positions $(x_p(t))_p$ of the particles are defined by (I.8) and the weights $(\omega_p(t))_p)$ are determined by resolution of the following ordinary differential system

$$\frac{d\omega_p}{dt}(t) - \nu\Delta_h^\varepsilon\omega_p(t) = 0, \quad \omega_p(0) = \omega_0(x_p^0). \tag{I.12}$$

A more classical approximation of ω is defined by regularisation. Let χ be a compactly supported function such that

$$\int_{\mathbb{R}^2}\chi(x)dx = 1, \tag{I.13}$$

and $\chi_\varepsilon(x) = \varepsilon^{-2}\chi(x/\varepsilon)$. The solution ω of the advection-diffusion problem is approximated, in the sense of functions by

$$\omega_h^\varepsilon(x,t) = \sum_{p\in J}\alpha_p\omega_p(t)\chi_\varepsilon(x-x_p(t)). \tag{I.14}$$

and the resolution of the partial differential equation appearing in system (I.3) is replaced by the resolution of a large system of coupled ordinary differential equations for the positions and the weights of the particles

$$\begin{cases} \dfrac{dx_p}{dt}(t) = \mathbf{a}(x_p(t),t), \\ \dfrac{d\omega_p}{dt}(t) - \nu\Delta_h^\varepsilon\omega_p(t) = 0, \\ x_p(0) = x_p^0, \quad \omega_p(0) = \omega_0(x_p^0). \end{cases}$$

Again, in order to simplify the notations, we have not made explicit the obvious dependence of Δ_h^ε with respect to the velocity $\mathbf{a}$ and the boundary condition g.

In the sequel, we shall assume that $\Omega = \mathbb{R}_+^2$ and thus that $\partial\Omega = \mathbb{R}$ (the case of a more general domain Ω will be treated in a forthcoming paper).

An outline of the paper is as follows. In Section 1, the conservativity of the numerical approximation, which is a physically important property, is proved and the convergence of the method is analysed in L^∞. The convergence of the integral operator Δ^ε towards the diffusion operator Δ when ε goes to 0 is then established under the condition that the functions η and ζ satisfy some moment conditions. Then we consider the discretized integral operator and prove the convergence of the operator Δ_h^ε towards the integral operator when h, h_* and ε go to 0 (h and h_* are respectively the discretisation parameter in $\mathbb{R}_+^2$ and on $\mathbb{R}$), under the conditions that $h << \varepsilon$ and $h_* << \varepsilon$. These two results ensure the consistency of the particle method. The boundedness of the operator Δ_h^ε in the discrete maximum norm is established which proves the existence of a solution of problem (I.8)-(I.12). Then, a positivity result established by S.Mas-Gallic and P.A.Raviart in [26] under the condition that the function η is non negative is recalled and this gives the stability of the method. Finally, the convergence of the sequence $(\omega_p(t))_p$ to the sequence $(\omega(x_p(t),t))_p$ and the convergence of the function ω_h^ε towards ω when h, h_* and ε go to 0 are proved.

Section 2 is devoted to the analysis of the method in l^2 and we follow the same outline as the previous Section. First we analyse the convergence of the Δ^ε towards Δ and Δ_h^ε towards Δ^ε in the space of squarely summable sequences. Then, we prove the boundedness of the discrete operator in this space and, in the case of a non negative cut-off, a stability result based on the positivity of the operator Δ_h^ε. Then, the convergence of the sequence $(\omega_p(t))_p$ to the sequence $(\omega(x_p(t),t))_p$ and the convergence of the function ω_h^ε towards ω when h, h_* and ε go to 0 are proved. Most of the results of this section are identical to those proved in Section 1 but the proofs are mainly different. Notice also that the L^p, $p > 2$, case would be treated in a similar way.

Section 3 is concerned with the analysis of the L^2 convergence of the integral model towards the diffusion model when the parameter ε goes to 0. First, we establish the conservativity of the integral model, the convergence of Δ^ε towards Δ and its boundedness in L^2. Thus, in the case of a non negative cut-off, the stability of the integro-differential system (I.7) is obtained and the convergence of its solution towards the solution of (I.3) finally proved.

1. Analysis of convergence in l^∞

Before coming to the analysis of convergence, let us introduce some notations. The Sobolev spaces $\mathrm{W}^{s,\infty}$ are provided with their classical norms denoted by $||.||_{s,\infty}$. We shall also work in the space $l^\infty(J)$ of bounded sequences provided with the norm

$$\bar{v} = (v_p)_p \longrightarrow ||\bar{v}||_{h,\infty} = \sup_{p\in J} |v_p|.$$

(Indices $j, k, \ldots$ denote simple integers whereas indices $p, q, \ldots$ denote multiindices $p = (p_1, p_2)$, $q = (q_1, q_2), \ldots$).

Let v and f be respectively continuous functions defined on $\mathbb{R}_+^2$ and $\mathbb{R}$. We shall denote by $\bar{P}(t)v$ and $\bar{Q}f$ the associate sequences

$$\bar{P}(t)v = (v(x_p(t)))_p, \quad \bar{Q}f = (f(jh_*))_j.$$

More generaly, sequences will be denoted with an upper bar. Let us assume, from now on that the initial quadrature rules are as follows. Given $h > 0$ and $h_* > 0$, we set

$$x_p^0 = (p_1 + \frac{1}{2}, p_2 + \frac{1}{2})h, \quad \alpha_p = h^2, \ p \in J \qquad y_j = jh_*, \quad \beta_j = h_*, \ j \in \mathbb{N}. \tag{1.1}$$

Assume that the vector field $\mathbf{a}$, the initial condition u_0 and the boundary condition g are smooth enough. Then, integrating equation (I.3i) with respect to x, we get

$$\frac{d}{dt}\int_{\mathbb{R}^2_+} \omega(x,t)dx + \nu \int_{\mathbb{R}} g(x_1,t)dx_1 = 0,$$

which is a conservation relation. A discrete analogue of this relation is satisfied by the scheme :

Proposition 1.1 - *Assume that the function η is even. Then the sequence $\bar{\omega}(t) = (\omega_p(t))_p$, where the $\omega_p(t)$'s are solution of the differential system (I.12), satisfies the following conservation relation*

$$\frac{d}{dt}\sum_{p\in J} h^2\omega_p(t) + 2\nu \sum_{j\in\mathbb{N}} h_* g(jh_*,t) \sum_{p\in J} h^2 \zeta_\varepsilon(x_p(t) - (jh_*,0)) = 0 \tag{1.2}$$

Proof - Summing over p we have for any sequence $\bar{\omega}$

$$\sum_{p\in J} \Delta_h^\varepsilon \omega_p(t) = \frac{1}{\varepsilon^2}\sum_{p\in J} \Lambda_h^\varepsilon \omega_p(t) - 2h_* \sum_{j\in\mathbb{N}}\sum_{p\in J} \zeta_\varepsilon(x_p(t) - (jh_*,0))g(jh_*,t).$$

Since the function η is even, exchanging the roles of p and q we also have

$$\sum_{p\in J} \Lambda_h^\varepsilon \omega_p(t) = \sum_{p,q\in J} h^2(\eta_\varepsilon(x_p(t) - x_q(t)) + \eta_\varepsilon(x_p(t) - \tilde{x}_q(t)))(\omega_q(t) - \omega_p(t)) = 0$$

which yields the result. □

Remark 1 : In the case of a homogeneous boundary condition ($g \equiv 0$), the conservation relation is exactly satisfied, which is no more true in the general case. We have

$$\sum_{p\in J} h^2\zeta_\varepsilon(x_p(t) - (jh_*,0)) \approx \int_{\mathbb{R}^2_+} \zeta(y)dy$$

but since the particle move, we cannot ensure that the value of the sum is exactly equal to $1/2$. □

Let us now come to the convergence analysis. If the functions η and ζ satisfy some conditions, we prove the convergence of the integral operator Δ^ε towards the diffusion operator Δ with a Neumann boundary condition

Proposition 1.2 - *Assume that the functions η and ζ are compactly supported even functions of each variable, such that*

$$\int_{\mathbb{R}^2_+} y_2\eta(y_1, x_2+y_2)dy = \int_{\mathbb{R}} \zeta(y_1,x_2)dy_1, \quad \text{for any } x_2 > 0, \tag{1.3i}$$

$$(1.3ii) \qquad \int_{\mathbb{R}^2} x_i^2 \eta(x)dx = 2, \qquad \text{for } i = 1, 2.$$

There exists a constant $C > 0$, such that if $g \in L^\infty(0,T; W^{2,\infty}(\mathbb{R}))$, for any function $v \in L^\infty(0,T; W^{3,\infty}(\mathbb{R}^2_+))$ which for any $x_1 \in \mathbb{R}$, $t \in (0,T)$, satisfies the boundary condition

$$(1.4) \qquad \frac{\partial v}{\partial x_2}(x_1, 0, t) = g(x_1, t),$$

the following estimate holds for any $t \in (0,T)$,

$$(1.5) \qquad \|(\Delta v - \Delta^\varepsilon v)(.,t)\|_{0,\infty} \leq C\varepsilon(\|v(.,t)\|_{3,\infty} + \|g(.,t)\|_{2,\infty}).$$

Proof - The time dependence of the functions does not play any role here and we will forget it for the moment. We expand the function v around x by means of a Taylor formula and get

$$(1.6) \qquad v(y) = v(x) + \nabla v(x).(y-x) + \frac{1}{2}\nabla^2 v(x)(y-x)^{\otimes 2} + \frac{1}{2}\int_0^1 \nabla^3 v(y + \lambda(x-y))(y-x)^{\otimes 3}\lambda^2 d\lambda.$$

Merging this expansion in the definition of the operator Λ^ε we obtain

$$\Delta^\varepsilon v(x) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2_+} (\eta_\varepsilon(x-y) + \eta_\varepsilon(x-\tilde{y}))\{\nabla v(x).(y-x) + \frac{1}{2}\nabla^2 v(x)(y-x)^{\otimes 2}\}dy -$$

$$-2\int_{\mathbb{R}} \zeta_\varepsilon(x_1 - y_1, x_2) g(y_1) dy_1 + R^\varepsilon(x)$$

where the rest is given by

$$(1.7) \qquad R^\varepsilon(x) = \frac{1}{2\varepsilon^2}\int_{\mathbb{R}^2_+} (\eta_\varepsilon(x-y) + \eta_\varepsilon(x-\tilde{y}))\int_0^1 \nabla^3 v(y + \lambda(x-y))(y-x)^{\otimes 3}\lambda^2 d\lambda dy.$$

Noticing that the functions η and ζ are even we have

$$\int_{\mathbb{R}} x_i \eta(x) dx_i = 0 = \int_{\mathbb{R}} x_i \zeta(x) dx_i, \qquad for\ i = 1, 2,$$

and we deduce

$$\int_{\mathbb{R}^2_+} (\eta(x-y) + \eta(x-\tilde{y}))(y_1 - x_1)dy = 0,$$

$$\int_{\mathbb{R}^2_+} (\eta(x-y) + \eta(x-\tilde{y}))(y_2 - x_2)dy = -2\int_{\mathbb{R}^2_-} \eta(x-y)y_2 dy = 2\int_{\mathbb{R}^2_+} \eta(y_1, x_2 + y_2)y_2 dy,$$

$$\int_{\mathbb{R}^2_+} (\eta(x-y) + \eta(x-\tilde{y}))(y_1 - x_1)(y_2 - x_2)dy = 2\int_{\mathbb{R}^2_+} \eta(y_1, x_2 + y_2)y_2 y_1 dy.$$

Now, the second order moment conditions imposed to η yield

$$\int_{\mathbb{R}^2_+} (\eta(x-y) + \eta(x-\tilde{y}))(y_1 - x_1)^2 dy = 2$$

$$\int_{\mathbb{R}^2_+}(\eta(x-y)+\eta(x-\tilde{y}))(y_2-x_2)^2dy = 2-4x_2\int_{\mathbb{R}^2_+}\eta(x-\tilde{y})y_2dy.$$

According to these properties, we obtain

$$\Delta^\varepsilon v(x) = \Delta v(x)+\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2_+}\eta_\varepsilon(x-\tilde{y})(2y_2\frac{\partial v}{\partial x_2}(x)-2x_2y_2\frac{\partial^2 v}{\partial x_2^2}(x)+2(y_1-x_1)y_2\frac{\partial^2 v}{\partial x_1\partial x_2}(x))dy- \tag{1.8}$$

$$-2\int_{\mathbb{R}}\zeta_\varepsilon(x_1-y_1,x_2)g(y_1)dy_1+R^\varepsilon(x).$$

Similarly, expanding the functions $\frac{\partial v}{\partial x_2}, \frac{\partial^2 v}{\partial x_2^2}, \frac{\partial^2 v}{\partial x_1\partial x_2}$ around $(x_1,0)$ and g around x_1 yields, by (1.4),

$$\frac{\partial v}{\partial x_2}(x) = g(x_1)+x_2\frac{\partial^2 v}{\partial x_2^2}(x_1,0)+\int_0^1 x_2^2\lambda\frac{\partial^3 v}{\partial x_2^3}(x_1,(1-\lambda)x_2)d\lambda, \tag{1.9}$$

$$\frac{\partial^2 v}{\partial x_2^2}(x) = \frac{\partial^2 v}{\partial x_2^2}(x_1,0)+\int_0^1 x_2\frac{\partial^3 v}{\partial x_2^3}(x_1,(1-\lambda)x_2)d\lambda, \tag{1.10}$$

$$\frac{\partial^2 v}{\partial x_1\partial x_2}(x) = g'(x_1)+\int_0^1 x_2\frac{\partial^3 v}{\partial x_1\partial x_2^2}(x_1,(1-\lambda)x_2)d\lambda, \tag{1.11}$$

$$g(y_1) = g(x_1)+(y_1-x_1)g'(x_1)+\int_0^1(y_1-x_1)^2\lambda g''(y_1+\lambda(x_1-y_1))d\lambda. \tag{1.12}$$

Combining (1.8) with the previous equalities (1.9)-(1.12), we obtain

$$\Delta^\varepsilon v(x) = \Delta v(x)+\frac{2}{\varepsilon^2}(\int_{\mathbb{R}^2_+}y_2\eta_\varepsilon(x-\tilde{y})dy-\varepsilon^2\int_{\mathbb{R}}\zeta_\varepsilon(x_1-y_1,x_2)dy_1)g(x_1)+ \tag{1.13}$$

$$+\frac{2}{\varepsilon^2}(\int_{\mathbb{R}^2_+}y_2(y_1-x_1)\eta_\varepsilon(x-\tilde{y})dy-\varepsilon^2\int_{\mathbb{R}}(y_1-x_1)\zeta_\varepsilon(x_1-y_1,x_2)dy_1)g'(x_1)+Q^\varepsilon(x)$$

where Q^ε is the sum of the rests

$$Q^\varepsilon(x) = Q_1^\varepsilon(x)+Q_2^\varepsilon(x)+Q_3^\varepsilon(x)+R^\varepsilon(x) \tag{1.14}$$

defined as follows

$$Q_1^\varepsilon(x) = \frac{2}{\varepsilon^2}\int_{\mathbb{R}^2_+}x_2^2y_2\eta_\varepsilon(x-\tilde{y})dy\int_0^1(\lambda-1)\frac{\partial^3 v}{\partial x_2^3}(x_1,(1-\lambda)x_2)d\lambda \tag{1.15}$$

$$Q_2^\varepsilon(x) = \frac{2}{\varepsilon^2}\int_{\mathbb{R}^2_+}x_2y_2(y_1-x_1)\eta_\varepsilon(x-\tilde{y})dy\int_0^1\frac{\partial^3 v}{\partial x_1\partial x_2^2}(x_1,(1-\lambda)x_2)d\lambda \tag{1.16}$$

$$Q_3^\varepsilon(x) = 2\int_{\mathbb{R}} (y_1-x_1)^2 \zeta_\varepsilon(x_1-y_1,x_2)\int_0^1 \lambda g''(y_1+\lambda(x_1-y_1))d\lambda dy_1. \tag{1.17}$$

Condition (1.3i) imposed to the functions ζ and η ensures that the coefficient of $g(x_1)$ is equal to 0, while the coefficient of $g'(x_1)$ is also equal to 0 because the functions are even ; we thus get

$$\Delta^\varepsilon v(x) = \Delta v(x) + Q^\varepsilon(x) \tag{1.18}$$

and the result will follow from a bound for Q^ε. Since η is compactly supported, of diameter say d, the only points for which the function is non zero are such that $|x-\tilde{y}| \le \varepsilon d$; this implies

$$|x_1-\tilde{y}_1| \le \varepsilon d, \quad 0 \le x_2 \le \varepsilon d, \quad 0 \le y_2 \le \varepsilon d$$

and using the change of variable $y_1 \to z_1 = (y_1-x_1)/\varepsilon$, $y_2 \to z_2 = y_2/\varepsilon$, we obtain

$$|Q_1^\varepsilon(x)| \le C\varepsilon \int_{\mathbb{R}_+^2} z_2|\eta(z_1,\frac{x_2}{\varepsilon}+z_2)| \int_0^1 |\frac{\partial^3 v}{\partial x_2^3}(x_1,(1-\lambda)x_2)|d\lambda dz \le C\varepsilon\|\frac{\partial^3 v}{\partial x_2^3}\|_{0,\infty}. \tag{1.19}$$

The second term appearing in the rest Q^ε is easily bounded in the same way, we get

$$|Q_2^\varepsilon(x)| \le C\varepsilon\|\frac{\partial^3 v}{\partial x_1 \partial x_2^2}\|_{0,\infty}, \tag{1.20}$$

while the estimate of the third term, follows since ζ is compactly supported

$$|Q_3^\varepsilon(x)| \le C\varepsilon\|g''\|_{0,\infty}. \tag{1.21}$$

Let us now come to the estimate of R^ε, we recall that

$$R^\varepsilon(x) = \frac{1}{2\varepsilon^2}\int_{\mathbb{R}_+^2} (\eta_\varepsilon(x-y)+\eta_\varepsilon(x-\tilde{y}))\int_0^1 \nabla^3 v(y+\lambda(x-y))(y-x)^{\otimes 3}\lambda^2 d\lambda dy,$$

and since the support of the function η is compact, we obtain very easily

$$|R^\varepsilon(x)| \le C\varepsilon\|v\|_{3,\infty}. \tag{1.22}$$

Finally combining (1.18) with inequalities (1.19)-(1.22), we have proved the Proposition. □

Remark 2 :

1. Proposition 1.2 is true even if the functions η and ζ are not compactly supported. Nevertheless, in the sequel we shall assume for convenience that this assumption is satisfied.

2. The compatibility condition (1.3i) allows to calculate a particular function ζ once η is given. Namely, assume that a function $\zeta_1 : \mathbb{R} \to \mathbb{R}$ with non vanishing integral is given. We define a function ζ_2 by the following relation

$$\zeta_2(x_2) = \frac{\int_{\mathbb{R}_+^2} y_2\eta(y_1,x_2+y_2)dy}{\int_{\mathbb{R}}\zeta_1(y_1)dy_1}, \quad \text{for any } x_2 \in \mathbb{R},$$

and $\zeta(x_1,x_2)=\zeta_1(x_1)\zeta_2(x_2)$ satisfies the compatibility condition.

3. If in addition to (1.3i) and (1.3ii) the functions η and ζ satisfy the following relations for any $x_2>0$,

$$\int_{\mathbb{R}^2_+} y_2 y_1^2 \eta(y_1, x_2+y_2)dy = \int_{\mathbb{R}} y_1^2\zeta(y_1,x_2)dy_1, \tag{1.23ii}$$

$$\int_{\mathbb{R}^2_+} y_2^3 \eta(y_1, x_2+y_2)dy = 0, \tag{1.23ii}$$

a higher order approximation is obtained. We can prove that if $g\in L^\infty(0,T;W^{3,\infty}(\mathbb{R}))$, for any function $v\in L^\infty(0,T;W^{4,\infty}(\mathbb{R}^2_+))$ which satisfies the boundary condition (1.4), the following estimate holds for $t\in(0,T)$,

$$\|(\Delta v-\Delta^\varepsilon v)(.,t)\|_{0,\infty}\le C\varepsilon^2(\|v(.,t)\|_{4,\infty}+\|g(.,t)\|_{3,\infty}). \tag{1.24}$$

Notice that in this case the construction of the function ζ is more complicate that in the lower order case.

4. One can think of the following simpler integral operator

$$\Delta^\varepsilon v(x)=\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2_+}\eta_\varepsilon(x-y)(v(y)-v(x))dy-\int_{\mathbb{R}}\zeta_\varepsilon(x_1-y_1,x_2)g(y_1,t)dy_1 \tag{1.25}$$

as another approximation of Δ. Unfortunately, this operator is conservative but does not lead to a consistent approximation of the diffusion operator. $\square$

Let us now focuse our attention on the consistancy of the discrete operator Δ^ε_h and prove its convergence towards the integral operator Δ^ε when the parameters h, h_* and ε go to zero

Proposition 1.3 - *Assume that the functions η and ζ are compactly supported and moreover that $\eta\in W^{2,\infty}(\mathbb{R}^2)$ and $\zeta\in L^\infty(\mathbb{R}_+;W^{m,\infty}(\mathbb{R}))$ for an integer $m\ge 2$. There exists a constant $C>0$ depending on* $\mathbf{a}$ *and T, such that if $g\in L^\infty(0,T;W^{m,\infty}(\mathbb{R}))$, for any function $v\in L^\infty(0,T;W^{2,\infty}(\mathbb{R}^2_+))$ we have for any $t\in(0,T)$*

$$\|\bar P(t)\Delta^\varepsilon(t)v-\Delta^\varepsilon_h(t)\bar P(t)v\|_{h,\infty}\le\frac{C}{\varepsilon}(\frac{h^2}{\varepsilon^2}\|v(.,t)\|_{2,\infty}+\frac{h_*^m}{\varepsilon^m}\|g(.,t)\|_{m,\infty}). \tag{1.26}$$

Proof - The proof is based on the fact that the quadrature rule on R^2_+ is of order 2 and the quadrature rule on R is of "infinite" order. Let $t\to X(t;z,0)$ be the integral curve of the velocity field $\mathbf{a}$ through the point z at time 0. We write, using the change of variable $y\to z=X(0;y,t)$,

$$\Delta^\varepsilon v(x_p(t))-(\Delta^\varepsilon_h\bar P(t)v)_p=\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2_+}(\eta_\varepsilon(x_p(t)-y)+\eta_\varepsilon(x_p(t)-\tilde y))(v(y)-v(x_p(t)))dy$$

$$-\frac{h^2}{\varepsilon^2}\sum_{q\in J}(\eta_\varepsilon(x_p(t)-x_q(t))+\eta_\varepsilon(x_p(t)-\tilde x_q(t)))(v(x_q(t))-v(x_p(t)))+$$

$$+2\int_{\mathbb{R}}\zeta_\varepsilon(x_p(t)-(y_1,0))g(y_1)dy_1-2\sum_{j\in\mathbb{N}}\zeta_\varepsilon(x_p(t)-(jh_*,0))g(jh_*)=$$

$$= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2_+}(\eta_\varepsilon(X(t;x_p^0,0)-X(t;z,0))+\eta_\varepsilon(X(t;x_p^0,0)-\tilde{X}(t;z,0)))(v(X(t;z,0))-v(X(t;x_p^0,0)))dz$$

$$-\frac{h^2}{\varepsilon^2}\sum_{q\in J}(\eta_\varepsilon(X(t;x_p^0,0)-X(t;x_q^0,0))+\eta_\varepsilon(X(t;x_p^0,0)-\tilde{X}(t;x_q^0,0)))(v(X(t;x_q^0,0))-v(X(t;x_p^0,0)))+$$

$$+2\int_{\mathbb{R}}\zeta_\varepsilon(X(t;x_p^0,0)-(y_1,0))g(y_1)dy_1-2\sum_{j\in\mathbb{N}}\zeta_\varepsilon(X(t;x_p^0,0)-(jh_*,0))g(jh_*).$$

Now,we recall the following result whose proof can be found in [28] : there exists a constant $C>0$ depending on $\mathbf{a}$ and T, such that for any function $\psi\in W^{m,1}(\mathbb{R})$,

$$|\int_{\mathbb{R}}\psi(y)dy-h_*\sum_{j\in\mathbb{N}}\psi(jh_*)|\le Ch_*^m\|\psi\|_{m,1} \tag{1.27}$$

and the following slightly modified version of it (see S. Mas-Gallic- F. Poupaud [25] for example) : there exists a constant $C=C(\mathbf{a},T)>0$, such that for any function $\phi\in W^{2,1}(\mathbb{R}^2_+)$,

$$|\int_{\mathbb{R}^2_+}\phi(y)dy-h^2\sum_{p\in J}\phi(x_p^0)|\le Ch^2\|\phi\|_{2,1}. \tag{1.28}$$

We apply these results respectively to the functions $z\rightarrow\phi(z)=\eta_\varepsilon(x_p-X(t;z,0))(v(X(t;z,0))-v(x_p))$, $z\rightarrow\phi(z)=\eta_\varepsilon(x_p-\tilde{X}(t;z,0)))(v(X(t;z,0))-v(x_p))$ and $y_1\rightarrow\psi(y_1)=\zeta_\varepsilon(x_p-(y_1,0))g(y_1)$. We thus get, for the first term, arguing as P. Degond-S. Mas-Gallic in [11]

$$|\int_{\mathbb{R}^2_+}\eta_\varepsilon(x_p(t)-y)(v(y)-v(x_p(t)))dy-h^2\sum_{q\in J}\eta_\varepsilon(x_p(t)-x_q(t))(v(x_q(t))-v(x_p(t)))|\le C\frac{h^2}{\varepsilon}\|v\|_{2,\infty}.$$

The estimate of the second term is analogous, finally the last term gives

$$|\int_{\mathbb{R}}\zeta_\varepsilon(x_p-(y_1,0))g(y_1)dy_1-\sum_{j\in\mathbb{N}}\zeta_\varepsilon(x_p-(jh_*,0))g(jh_*)|\le Ch_*^m\|\zeta_\varepsilon\|_{L^\infty(W^{m,1})}\|g\|_{m,\infty}$$

and since

$$\|\zeta_\varepsilon\|_{L^\infty(W^{m,1})}\|\le\frac{C}{\varepsilon^{m+1}}, \tag{1.29}$$

inequality (1.26) is proved. □

The following result of stability for the discrete operator Δ_h^ε ensures the existence of a solution system (I.8), (I.12).

Lemma 1.4 - *Assume that $\bar{g}(t)=(g(jh_*,t))_j\in l^\infty$ for any $t\in(0,T)$. Then, there exists a constant $C>0$ depending on* $\mathbf{a}$ *and T, such that for any $\bar{v}\in l^\infty$ and any $t\in(0,T)$*

$$\|\Delta_h^\varepsilon(t)\bar{v}\|_{h,\infty}\le\frac{C}{\varepsilon^2}(\|\bar{v}\|_{h,\infty}+\varepsilon\|\bar{g}(t)\|_{h_*,\infty}). \tag{1.30}$$

This result follows from the compactness of the support of η and ζ. $\square$

Now that the existence of a solution of the scheme is clear, we recall a stability result established by S. Mas-Gallic and P.A. Raviart in [27]. This result follows from a positivity property of the operator $-\Delta_h^\varepsilon$. We consider the following ordinary differential system of equations

$$\frac{dv_p}{dt}(t) + \sum_{q\in J} a_{p,q} v_q(t) = f_p(t), \quad v_p(0) = v_p^0, \tag{1.31}$$

and we have

Lemma 1.5 - *Assume that the coefficients $(a_{p,q})$ are such that for any p,*

$$a_{p,q} \le 0, \text{ for } q \ne p, \quad \sum_{q\in J} a_{p,q} \ge 0, \quad \sum_{q\in J} |a_{p,q}| \le C, \tag{1.32}$$

$$0 \le v_p^0 \le M, \quad 0 \le f_p(t) \le G(t), \text{ for any } p. \tag{1.33}$$

Then, for any $t \in (0,T)$ the differential system (I.8)-(I.12) has one solution $\bar{v}(t) = (v_p(t))_p$ in l^∞ and this solution is such that

$$0 \le v_p(t) \le M + \int_0^t G(\tau)d\tau, \text{ for any } p \tag{1.34}$$

Remark 3 : This result establishes, in the case of a non negative function η, the stability of the scheme with no assumption on the physical viscosity ν or the numerical parameter ε. Nevertheless, in the general case, of a function η which does not have a constant sign, the stability condition for the scheme is the existence of a constant $c_s > 0$, such that

$$\nu \le c_s \varepsilon^2. \tag{1.35}$$

$\square$

Since the stability and the consistency of the scheme have been obtained, we are now able to prove its convergence. The first result is the convergence of the scheme on the particles. We recall that by ω we denote the solution of problem (I.3) and by $\bar{P}(t)\omega$ the sequence of its value on the particles. The sequence $\bar{\omega}(t) = (\omega_p(t))_p$ is such that the ω_p's are solution of (I.12).

Theorem 1.6 - *Assume that $\eta \in W^{2,\infty}(\mathbb{R}^2)$ and that $\zeta \in L^\infty(\mathbb{R}_+, W^{m,\infty}(\mathbb{R}))$ with $m \ge 2$. Assume moreover that η is non negative and that they satisfy the moment conditions (1.3i) and (1.3ii). There exists a constant $C > 0$ depending on* **a** *and T, such that if $g \in L^\infty(0,T;W^{m,\infty}(\mathbb{R}))$ and if $\omega \in L^\infty(0,T;W^{3,\infty}(\mathbb{R}_+^2))$ the following estimate holds for any $t \in (0,T)$*

$$||\bar{P}(t)\omega(t) - \bar{\omega}(t)||_{h,\infty} \le C\nu\{(\varepsilon + \frac{h^2}{\varepsilon^3})||\omega||_{L^\infty(0,T;W^{3,\infty}(\mathbb{R}_+^2))} + (\varepsilon + \frac{h_*^m}{\varepsilon^{m+1}})||g||_{L^\infty(0,T;W^{m,\infty}(\mathbb{R}))}\}. \tag{1.36}$$

Proof - We set

$$a_{p,q} = -\nu \frac{h^2}{\varepsilon^2}(\eta_\varepsilon(x_p - x_q) + \eta_\varepsilon(x_p - \tilde{x}_q)), \text{ for } q \neq p$$

$$a_{p,p} = \nu \frac{h^2}{\varepsilon^2} \sum_{q \in J, q \neq p} (\eta_\varepsilon(x_p - x_q) + \eta_\varepsilon(x_p - \tilde{x}_q))$$

and $\bar{e}(t) = (e_p(t))_p = (\bar{P}(t)\omega - \bar{\omega})(t)$. We have $e_p(t) = \omega_p(t) - \omega(x_p(t), t)$ and since ω is solution of (I.3), we obtain

$$\frac{de_p}{dt}(t) + \sum_{q \in J} a_{p,q} e_q(t) = \nu \psi_p(t), \quad e_p(0) = 0, \tag{1.37}$$

where $\psi_p(t) = \Delta\omega(x_p(t), t) - (\Delta_h^\varepsilon \bar{P}(t)\omega)_p(t)$. We consider the sequence $\bar{e}^+ = (e_p^+(t))_p$ defined by

$$e_p^+(t) = e_p(t) + \nu \int_0^t \|\bar{\psi}(\tau)\|_{h,\infty} d\tau.$$

This sequence is solution of (1.37) with a right-hand side equal to

$$\nu(\bar{\psi}(t) + \|\bar{\psi}(\tau)\|_{h,\infty}).$$

The hypotheses of Lemma 1.5 are satisfied and inequality (1.34) leads, for any p, to

$$e_p^+(t) \geq 0.$$

Next we consider the sequence $\bar{e}^-(t) = (e_p^-(t))_p$ where

$$e_p^-(t) = e_p(t) - \nu \int_0^t \|\bar{\psi}(\tau)\|_{h,\infty} d\tau$$

and we apply again lemma 1.5 and find

$$e_p^-(t) \leq 0.$$

Combining the two previous inequalities, this gives the estimate of $\bar{e}(t)$

$$\|\bar{e}(t)\|_{h,\infty} \leq \nu \int_0^t \|\bar{\psi}(\tau)\|_{h,\infty} d\tau. \tag{1.38}$$

The result will be proved as soon as $\bar{\psi}$ is bounded. We write

$$\|\bar{\psi}(t)\|_{h,\infty} \leq \|\Delta\omega - \Delta^\varepsilon\omega\|_{0,\infty} + \sup_{\tau \in (0,T)} \|\bar{P}(\tau)\Delta^\varepsilon\omega(\tau) - \Delta_h^\varepsilon \bar{P}(\tau)\omega(\tau)\|_{h,\infty} \tag{1.39}$$

and application of Proposition 1.2 and 1.3 gives the estimate of $\bar{\psi}$ we are looking for. Finally, combination of this estimate with inequalities (1.37)-(1.39) yields (1.36). □

The convergence of the scheme on the particles and in the sense of sequences is proved and now we prove the convergence of the scheme in the more classical sense of functions, thanks to the regularization and the definition (I.14). We define an operator $\Pi_h(t)$ which associates a particle distribution to any sufficiently smooth function by setting

$$\Pi_h(t)v(x) = \sum_{p\in J} h^2 v(x_p(t))\delta(x - x_p(t))$$

and by regularisation we also define

$$\Pi_h^\varepsilon(t)v(x) = \sum_{p\in J} h^2 v(x_p(t))\chi_\varepsilon(x - x_p(t)). \tag{1.40}$$

Now, we have the

Theorem 1.7 - *Assume that hypotheses of theorem 1.6 are satisfied and that the function $\chi \in W^{n,1}(\mathbb{R}^2)$, $n \geq 2$ satisfies condition (I.13) and the following assumptions*

$$\int_{\mathbb{R}^2} \chi(x) x^\alpha dx = 0, \quad \int_{\mathbb{R}^2} \chi(x)|x|^{r+1} dx < +\infty, \tag{1.41}$$

for any $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| \leq r$ for an integer r. There exists a constant $C > 0$ depending on **a** *and T, such that, if $\omega \in L^\infty(0,T; W^{s,\infty}(\mathbb{R}^2_+))$, where s=max(r,n,3), the following estimate holds for any $t \in (0,T)$,*

$$\|(\omega_h^\varepsilon - \omega)(.,t)\|_{0,\infty} \leq C\{(\varepsilon^r + \frac{h^n}{\varepsilon^n} + \nu\varepsilon + \nu\frac{h^2}{\varepsilon^3})\|\omega\|_{L^\infty(0,T;W^{s,\infty}(\mathbb{R}^2_+))} + \nu(\varepsilon + \frac{h_*^m}{\varepsilon^{m+1}})\|g\|_{L^\infty(0,T;W^{m,\infty}(\mathbb{R}))}\} \tag{1.42}$$

Proof - We write

$$\|(\omega_h^\varepsilon - \omega)(.,t)\|_{0,\infty} \leq \|\Pi_h^\varepsilon(t)\omega(.,t) - \omega(.,t)\|_{0,\infty} + \|\omega_h^\varepsilon(.,t) - \Pi_h^\varepsilon(t)\omega(.,t)\|_{0,\infty},$$

By application of a result proved in [26], we have

$$\|\omega(.,t) - \Pi_h^\varepsilon(t)\omega(.,t)\|_{0,\infty} \leq C(\varepsilon^r + \frac{h^n}{\varepsilon^n})\|\omega(.,t)\|_{s,\infty},$$

and this gives the desired estimate of the first term. In order to bound the second term we write thanks to the compactness of the support of χ

$$|(\omega_h^\varepsilon - \Pi_h^\varepsilon(t)\omega)(x,t)| = |\sum_{p\in J} h^2(\omega_p(t) - \omega(x_p(t),t))\chi_\varepsilon(x - x_p(t))| \leq C\|\bar{\omega}(t) - \bar{P}(t)\omega(.,t)\|_{h,\infty},$$

with $\bar{\omega}(t) = (\omega_p(t))_p$ and application of Theorem 1.6 yields inequality (1.42). $\square$

2. Analysis of convergence in l^2

The forthcoming section is devoted to the study of the convergence of the method in the space l^2, of squarely summable sequences. Let us first introduce some notations. The classical norm of the Sobolev space H^s is denoted by $\|.\|_{s,2}$ and the space $l^2(J)$ is provided with the following norm which depends obviously on the discretisation parameter h

$$\bar{v} = (v_p)_p \to \|\bar{v}\|_{h,2} = \{\sum_{p\in J} h^2 v_p^2\}^{1/2}.$$

The convergence of the integral operator Δ^ε towards the diffusion operator Δ with Neumann boundary condition in l^2 when ε goes to 0 is proved by

Proposition 2.1 - *Assume that η and ζ are compactly supported even functions of each variable, such that conditions (1.3i) and (1.3ii), that we recall here*

$$(1.3i) \qquad \int_{\mathbb{R}^2_+} y_2\eta(y_1, x_2+y_2)dy = \int_{\mathbb{R}} \zeta(y_1, x_2)dy_1, \quad \text{for any } x_2 > 0,$$

$$(1.3ii) \qquad \int_{\mathbb{R}^2} x_i^2\eta(x)dx = 2, \quad \text{for } i = 1, 2,$$

are satisfied. If $g \in L^\infty(0, T; H^2(\mathbb{R}))$, there exists a constant $C > 0$ depending on **a** *and T, such that for any function $v \in L^\infty(0, T; H^3(\mathbb{R}^2_+))$ satisfying the boundary condition*

$$(1.4) \qquad \frac{\partial v}{\partial x_2}(x_1, 0, t) = g(x_1, t), \quad \text{for any } x_1 \in \mathbb{R},\ t \in (0, T)$$

the following estimate holds

$$(2.1) \qquad \|\bar{P}(t)(\Delta v - \Delta^\varepsilon(t)v)\|_{h,2} \le C\varepsilon(\|v(.,t)\|_{3,2} + \varepsilon^{1/2}\|g(.,t)\|_{2,2}).$$

Proof - Here again, the time dependence of the functions does not play any role here and we will forget about it for the moment. We write

$$\|\bar{P}(t)(\Delta v - \Delta^\varepsilon(t)v)\|_{h,2} = \{\sum_{p\in J} h^2(\Delta v(x_p(t)) - \Delta^\varepsilon(t)v(x_p(t)))^2\}^{1/2},$$

and we argue as in the proof of Proposition 1.2. The functions v, $\dfrac{\partial v}{\partial x_2}, \dfrac{\partial^2 v}{\partial x_2^2}, \dfrac{\partial^2 v}{\partial x_1 \partial x_2}$ are expanded around $(x_1, 0)$ and g around x_1. Keeping the notations of the previous Section, we thus have

$$(2.2) \qquad \|\bar{P}(\Delta v - \Delta^\varepsilon v)\|_{h,2}^2 \le C(\sum_{p\in J} h^2|Q_1^\varepsilon(x_p)|^2 + \sum_{p\in J} h^2|Q_2^\varepsilon(x_p)|^2 + \sum_{p\in J} h^2|Q_3^\varepsilon(x_p)|^2 + \sum_{p\in J} h^2|R^\varepsilon(x_p)|^2).$$

where Q_1^ε, Q_2^ε, Q_2^ε and R^ε are given respectively by (1.13), (1.14), (1.15) and (1.7). Let us now find bounds for the rests. We write, by application of Cauchy-Schwarz inequality

$$|Q_1^\varepsilon(x_p)|^2 \le \frac{4}{\varepsilon^4}x_{p,2}^4 \int_{\mathbb{R}^2_+} y_2^2|\eta_\varepsilon(x_p - \tilde{y})|dy \int_{\mathbb{R}^2_+} |\eta_\varepsilon(x_p - \tilde{y})|| \int_0^1 (\lambda - 1)\frac{\partial^3 v}{\partial x_2^3}(x_{p,1}, \lambda x_{p,2})d\lambda|^2 dy.$$

Now, the compactness of the support of η, yields

$$|Q_1^\varepsilon(x_p)|^2 \le C\varepsilon^2 \int_{\mathbb{R}^2_+} |\eta_\varepsilon(x_p - \tilde{y})|| \int_0^1 (\lambda - 1)\frac{\partial^3 v}{\partial x_2^3}(x_{p,1}, \lambda x_{p,2})d\lambda|^2 dy$$

and

$$\sum_{p\in J} h^2|\eta_\varepsilon(x_p - \tilde{y})| \le C.$$

A second application of Cauchy-Schwarz inequality gives finally

$$\sum_{p\in J} h^2|Q_1^\varepsilon(x_p)|^2 \leq C\varepsilon^2\|v\|_{3,2}^2. \tag{2.3}$$

We have established the bound for the term Q_1^ε and since the arguments for the other terms are similar, we shall skip their proofs. By use of identical arguments, we obtain

$$\sum_{p\in J} h^2|Q_2^\varepsilon(x_p)|^2 \leq C\varepsilon^2\|v\|_{3,2}^2, \tag{2.4}$$

$$\sum_{p\in J} h^2|Q_3^\varepsilon(x_p)|^2 \leq C\varepsilon^3\|g''\|_{0,2}^2, \tag{2.5}$$

and

$$\sum_{p\in J} h^2|R^\varepsilon(x_p)|^2 \leq C\varepsilon^2\|v\|_{3,2}^2, \tag{2.6}$$

which completes the proof combination (2.2) with (2.3)-(2.6). □

We focuse now our attention on the consistancy of the discrete operator Δ_h^ε and prove its convergence towards the integral operator Δ^ε when the parameters h, h_* and ε go to zero

Proposition 2.2 - *Assume that the functions η and ζ are compactly supported and that there exists an integer $m \geq 2$ such that $\zeta \in L^\infty(\mathbb{R}_+; W^{m,\infty}(\mathbb{R}))$, $\eta \in W^{2,\infty}(\mathbb{R}^2)$ and $g \in L^\infty(0,T; H^m(\mathbb{R}))$. Then there exists a constant $C > 0$ depending on $\mathbf{a}$ and T, such that for any function $v \in H^2(\mathbb{R}_+^2)$,*

$$\|(\bar{P}(t)\Delta^\varepsilon(t) - \Delta_h^\varepsilon(t)\bar{P}(t))v\|_{h,2} \leq \frac{C}{\varepsilon}\left(\frac{h^2}{\varepsilon^2}\|v\|_{2,2} + \varepsilon^{1/2}\frac{h_*^m}{\varepsilon^m}\|g\|_{m,2}\right). \tag{2.7}$$

Proof - The proof mimics that of Proposition 1.3. We need to estimate

$$\{\sum_{p\in J} h^2|\Delta^\varepsilon v(x_p(t)) - (\Delta_h^\varepsilon v)_p|^2\}^{1/2}.$$

We write

$$\Delta^\varepsilon v(x_p(t)) - (\Delta_h^\varepsilon v)_p = \frac{1}{\varepsilon^2}A^\varepsilon(x_p(t)) + 2B^\varepsilon(x_p(t))$$

where

$$A^\varepsilon(x_p(t)) = \int_{\mathbb{R}_+^2} \eta_\varepsilon(x_p(t)-y)(v(y)-v(x_p(t)))dy - h^2\sum_{q\in J}\eta_\varepsilon(x_p(t)-x_q(t))(v(x_q(t))-v(x_p(t)))$$

and

$$B^\varepsilon(x_p(t)) = \int_{\mathbb{R}} \zeta_\varepsilon(x_p(t)-(y_1,0))g(y_1)dy_1 - \sum_{j\in\mathbb{N}}\zeta_\varepsilon(x_p(t)-(jh_*,0))g(jh_*).$$

We apply a result established in [26] to A^ε and obtain since the velocity $\mathbf{a}$ is smooth enough.

$$|A^\varepsilon(x_p(t))| \leq Ch^2 \|\eta_\varepsilon(x_p(t) - X(t;.,0))(v(X(t;.,0)) - v(x_p(t)))\|_{2,1}.$$

Since the change of variables $z \to X(t;z,0)$ is smooth and has bounded derivatives, we get

$$\begin{aligned}\sum_{p\in J} h^2 |A^\varepsilon(x_p(t))|^2 \leq Ch^6 \sum_{p\in J} \{ (\int_{\mathbb{R}^2_+} &|\eta_\varepsilon(x_p(t) - X(t;z,0))||\nabla^2 v(X(t;z,0))|dz)^2 \\ + (\int_{\mathbb{R}^2_+} &|\nabla \eta_\varepsilon(x_p(t) - X(t;z,0))||\nabla v(X(t;z,0))|dz)^2 \\ + (\int_{\mathbb{R}^2_+} &|\nabla^2 \eta_\varepsilon(x_p(t) - X(t;z,0))||(v(X(t;z,0)) - v(x_p))|dz)^2 \}.\end{aligned}$$

Let us bound the first term. By application of Cauchy-Schwarz inequality, we have

$$(\int_{\mathbb{R}^2_+} |\eta_\varepsilon(x_p(t) - X(t;z,0))||\nabla^2 v(X(t;z,0))|dz)^2 \leq$$

$$\leq \int_{\mathbb{R}^2_+} |\eta_\varepsilon(x_p(t) - X(t;z,0))|dz \int_{\mathbb{R}^2_+} |\eta_\varepsilon(x_p(t) - X(t;z,0))||\nabla^2 v(X(t;z,0))|^2 dz.$$

The compactness of the support of η yields

$$\sum_{p\in J} |\eta_\varepsilon(x_p(t) - X(t;z,0))| \leq \frac{C}{h^2} \tag{2.9}$$

and we obtain

$$\sum_{p\in J} (\int_{\mathbb{R}^2_+} |\eta_\varepsilon(x_p(t) - X(t;z,0))||\nabla^2 v(X(t;z,0))|dz)^2 \leq \frac{C}{h^2} \|\nabla^2 v\|^2_{0,2}. \tag{2.10}$$

The second term is bounded by similar arguments but due to the derivative of η_ε, there is a power of ε and we get

$$\sum_{p\in J} (\int_{\mathbb{R}^2_+} |\nabla \eta_\varepsilon(x_p(t) - X(t;z,0))||\nabla v(X(t;z,0))|dz)^2 \leq \frac{C}{h^2\varepsilon^2} \|\nabla v\|^2_{0,2}. \tag{2.11}$$

Let us now bound the term with no derivative. First, we apply the finite growth formula, we have

$$v(X(t;z,0)) - v(x_p) = \int_0^1 \nabla v((1-\lambda)x_p + \lambda X(t;z,0)).(X(t;z,0) - x_p)d\lambda$$

then we argue as previously ; the Cauchy-Schwarz inequality and an equivalent of (2.9) yield, since $|X(t;z,0) - x_p \leq C\varepsilon|$,

$$\sum_{p\in J} (\int_{\mathbb{R}^2_+} |\nabla^2 \eta_\varepsilon(x_p(t) - X(t;z,0))||(v(X(t;z,0)) - v(x_p))|dz)^2 \leq \frac{C}{h^2\varepsilon^2} \|\nabla v\|^2_{0,2}. \tag{2.12}$$

The bound

$$\sum_{p\in J} h^2|A^\varepsilon(x_p)|^2 \le C\frac{h^4}{\varepsilon^4}\|v\|_{2,2}^2 \tag{2.13}$$

follows by combination of (2.10)-(2.12). Let us now bound B^ε term. Again, we apply a result proved in [26] and get

$$|B^\varepsilon(x_p)| \le C h_*^m \|\zeta_\varepsilon(x_p-(.,0))g\|_{m,1},$$

which yields by application of Cauchy-Schwarz inequality

$$\sum_{p\in J} h^2|B^\varepsilon(x_p(t))|^2 \le C h^2 h_*^{2m}\sum_{k=0}^m\sum_{p\in J}\Big(\int_{\mathbb{R}}|\frac{\partial^k\zeta_\varepsilon}{\partial x_1^k}(x_p-(y_1,0))||g^{(m-k)}(y_1)|dy_1\Big)^2$$

The term which contains the highest derivative of ζ_ε will give the power of ε. We have, applying again the Cauchy-Schwarz inequality

$$\sum_{p\in J}\Big(\int_{\mathbb{R}}|\frac{\partial^m\zeta_\varepsilon}{\partial x_1^m}(x_p-(y_1,0))||g(y_1)|dy_1\Big)^2 \le C\|\frac{\partial^m\zeta_\varepsilon}{\partial x_1^m}\|_{L^\infty(W^{m,1})}\sum_{p\in J}\int_{\mathbb{R}}|\frac{\partial^m\zeta_\varepsilon}{\partial x_1^m}(x_p-(y_1,0))||g(y_1)|^2dy_1.$$

Since the function ζ is bounded and compactly supported, we have

$$\sum_{p\in J}\int_{\mathbb{R}}|\frac{\partial^m\zeta_\varepsilon}{\partial x_1^m}(x_p-(y_1,0))||g(y_1)|^2dy_1 \le \frac{C}{h^2\varepsilon^{2m+1}}\|g\|_{0,2}^2.$$

We apply identical arguments to the other terms to get

$$\sum_{p\in J} h^2|B^\varepsilon(x_p)|^2 \le C\frac{h_*^{2m}}{\varepsilon^{2m+1}}\|g\|_{m,2}^2 \tag{2.14}$$

and the proof is complete by combination of (2.13) and (2.14). □

We study now the stability of the discrete operator

Lemma 2.3 - *Assume that $\bar g = (g(jh_*))_j \in l^2$. Then, there exists a constant $C>0$ depending on* **a** *and T, such that for any sequence $\bar v \in l^2$,*

$$\|\Delta_h^\varepsilon \bar v(t)\|_{h,2} \le \frac{C}{\varepsilon^2}(\|\bar v\|_{h,2} + \varepsilon^{3/2}\|\bar g\|_{h_*,2}). \tag{2.15}$$

Proof - Application of Cauchy-Schwarz inequality gives first

$$\|\Delta_h^\varepsilon \bar v(t)\|_{h,2}^2 = \sum_{p\in J} h^2\Big|\frac{1}{\varepsilon^2}\sum_{q\in J} h^2(\eta_\varepsilon(x_p-x_q)+\eta_\varepsilon(x_p-\tilde x_q))(v_q-v_p) - 2h_*\sum_{j\in\mathbb{N}}\zeta_\varepsilon(x_p-(jh_*,0))g(jh_*)\Big|^2$$

$$\le \frac{C}{\varepsilon^4}\sum_{p\in J} h^2\Big|\sum_{q\in J} h^2(\eta_\varepsilon(x_p-x_q)+\eta_\varepsilon(x_p-\tilde x_q))(v_q-v_p)\Big|^2 + C\sum_{p\in J} h^2\Big|\sum_{j\in\mathbb{N}} h_*\zeta_\varepsilon(x_p-(jh_*,0))g(jh_*)\Big|^2$$

and also

$$|\sum_{q\in J} h^2(\eta_\varepsilon(x_p-x_q)+\eta_\varepsilon(x_p-\tilde{x}_q))(v_q-v_p)|^2 \le$$

$$\le (\sum_{q\in J} h^2|\eta_\varepsilon(x_p-x_q)+\eta_\varepsilon(x_p-\tilde{x}_q)|(v_q-v_p)^2)(\sum_{q\in J} h^2|\eta_\varepsilon(x_p-x_q)+\eta_\varepsilon(x_p-\tilde{x}_q)|).$$

Since η is compactly supported there exists a constant $C=C(\eta)>0$ such that

$$\sum_{q\in J} h^2|\eta_\varepsilon(x_p-x_q)+\eta_\varepsilon(x_p-\tilde{x}_q)| \le C$$

and Cauchy-Schwarz inequality yields then

$$\sum_{q\in J} h^2|\eta_\varepsilon(x_p-x_q)+\eta_\varepsilon(x_p-\tilde{x}_q)|(v_q-v_p)^2 \le C\sum_{q\in J} h^2(|\eta_\varepsilon(x_p-x_q)|+|\eta_\varepsilon(x_p-\tilde{x}_q)|)(v_q^2+v_p^2).$$

Summing over indices p we have thus

$$\sum_{p\in J} h^2|\sum_{q\in J} h^2(\eta_\varepsilon(x_p-x_q)+\eta_\varepsilon(x_p-\tilde{x}_q))(v_q-v_p)|^2 \le C\sum_{p\in J} h^2 v_p^2.$$

The term in which g appears is treated in a similar way, we obtain by application of Cauchy-Schwarz inequality

$$|\sum_{j\in\mathbb{N}} h_*\zeta_\varepsilon(x_p-(jh_*,0))g(jh_*)|^2 \le (\sum_{j\in\mathbb{N}} h_*|\zeta_\varepsilon(x_p-(jh_*,0))|)(\sum_{j\in\mathbb{N}} h_*|\zeta_\varepsilon(x_p-(jh_*,0))||g(jh_*)|^2) \tag{2.16}$$

The compactness of the support of ζ yields

$$\sum_{j\in\mathbb{N}} h_*|\zeta_\varepsilon(x_p-(jh_*,0))| \le \frac{C}{\varepsilon}, \tag{2.17}$$

and

$$\sum_{p\in J} h^2|\zeta_\varepsilon(x_p-(jh_*,0))| \le C, \tag{2.18}$$

where the different constants $C>0$ depend on ζ and η. Finally, by combination of (2.16)-(2.18), we have proved that

$$\sum_{p\in J} h^2(\sum_{j\in\mathbb{N}} h_*\zeta_\varepsilon(x_p-(jh_*,0))g(jh_*))^2 \le \frac{C}{\varepsilon}\sum_{j\in\mathbb{N}} h_*|g(jh_*)|^2$$

and this concludes the proof. □

The previous result ensures the existence of a solution for the particle scheme but the estimate depends on the parameter ε. Let us now prove, in the case of a non negative cut-off, a stability result for the particle scheme. This follows from a positivity property of the operator $-\Delta_h^\varepsilon$ and leads to an estimate which does not depend on ε.

Lemma 2.4 - *Assume that η is non negative . There exists a constant $C > 0$, such that for any $\bar{\omega}^0 \in l^2$, if we denote by $\bar{v}(t)$ the solution of (I.9), (I.10), (I.12) for $\bar{g} = 0$, the following estimate holds*

$$\|\bar{v}(t)\|_{h,2} \leq C\|\bar{\omega}^0\|_{h,2} \tag{2.19}$$

Proof - We multiply equation (I.12) by v_p and sum over p. It comes

$$\frac{1}{2}\frac{d}{dt}(\sum_{p\in J} v_p^2) - \nu \sum_{p\in J} v_p \Delta_h^\varepsilon v_p = 0.$$

Since $g = 0$ and the function η is even, by definition of Δ_h^ε, we have

$$\sum_{p\in J} v_p \Delta_h^\varepsilon v_p = \frac{1}{\varepsilon^2}\sum_{p\in J} v_p \Lambda_h^\varepsilon v_p = \frac{h^2}{\varepsilon^2}\sum_{p,q\in J}(\eta_\varepsilon(x_p - x_q) + \eta_\varepsilon(x_p - \tilde{x}_q))(v_q - v_p)v_p$$

$$= -\frac{h^2}{2\varepsilon^2}\sum_{p,q\in J}(\eta_\varepsilon(x_p - x_q) + \eta_\varepsilon(x_p - \tilde{x}_q))(v_q - v_p)^2 \leq 0.$$

Inequality (2.19) follows then from direct application of Gronwall's lemma. □

Any sequence $\bar{v} = (v_p)_p$ is associated with a function v_h^ε obtained by regularisation as follows

$$v_h^\varepsilon(x) = \sum_{p\in J} h^2 v_p \zeta_\varepsilon(x - x_p) \tag{2.20}$$

and we prove now a bound for the operator $\bar{v} \longrightarrow v_h^\varepsilon$.

Lemma 2.5 - *Assume that ζ is compactly supported. There exists a constant $C = C(\zeta) > 0$ such that for any sequence $\bar{v} = (v_p)_p \in l^2$, the following estimate holds*

$$\|v_h^\varepsilon\|_{0,2} \leq C\|\bar{v}\|_{h,2}. \tag{2.21}$$

Proof - We have by application of Cauchy-Schwarz inequality

$$\|v_h^\varepsilon\|_{0,2}^2 = \int_{\mathbb{R}_+^2} |\sum_{p\in J} h^2 v_p \zeta_\varepsilon(x - x_p)|^2 dx \leq C\int_{\mathbb{R}_+^2}\sum_{p\in J} h^2 v_p^2 |\zeta_\varepsilon(x - x_p)| \sum_{p\in J} h^2 |\zeta_\varepsilon(x - x_p)| dx.$$

Now, by compactness of the support of ζ, there exists a constant $C > 0$, such that

$$\sum_{p\in J} h^2 |\zeta_\varepsilon(x - x_p)| \leq C \tag{2.22}$$

and (2.21) follows easily. □

We have proved the stability and the consistency of the scheme, the next step is then to establish its convergence in the space of sequences

Theorem 2.6 - *Assume that $\eta \in W^{2,\infty}(\mathbb{R}^2)$ and $\zeta \in L^\infty(\mathbb{R}, W^{m,\infty}(\mathbb{R}))$ are compactly supported functions. Assume also that they satisfy the moment conditions (1.3i and (1.3ii) and that η is non negative. There exists a constant $C > 0$ depending on* **a** *and T, such that if $g \in H^m(\mathbb{R})$ and $\omega \in H^3(\mathbb{R}^2_+)$ the following estimate holds for any $t \in (0,T)$*

$$(2.23) \qquad \|\bar{P}(t)\omega - \bar{\omega}(t)\|_{h,2} \leq C\nu\{(\varepsilon + \frac{h^2}{\varepsilon^3})\|\omega\|_{L^\infty(0,T;H^3(\mathbb{R}^2_+))} + (\varepsilon^{3/2} + \frac{h_*^m}{\varepsilon^{m+1/2}})\|g\|_{L^\infty(0,T;H^m(\mathbb{R}))}\}.$$

Proof - We set $\bar{e}(t) = (e_p(t))_p$, $e_p(t) = \omega_p(t) - \omega(x_p(t),t)$ and we have

$$\frac{de_p}{dt} - \nu\Delta_h^\varepsilon e_p = \nu(\Delta\omega(x_p) - \Delta_h^\varepsilon \bar{P}(t)\omega(x_p)).$$

Application of Lemma 2.4 with a right-hand side yields since $\eta \geq 0$,

$$\|\bar{e}\|_{h,2} \leq \nu \int_0^t \|\bar{P}(t)\Delta\omega - \Delta_h^\varepsilon \bar{P}(t)\omega\|_{h,2} ds$$

and Propositions 2.1 and 2.2 allow to conclude. □

We come now to the classical approximation result and prove the convergence of the particle scheme in the space of functions L^2

Theorem 2.7 - *Assume that hypotheses of theorem 2.6 are satisfied and that $\chi \in W^{n,1}(\mathbb{R}^2)$ satisfies the moment conditions (1.41). Then there exists a constant $C > 0$ depending on* **a** *and T, such that if $\omega \in H^s(\mathbb{R}^2_+)$, with $s = max(r,n,3)$, the following estimate holds for any $t \in (0,T)$*

$$(2.24) \quad \|(\omega_h^\varepsilon - \omega)(.,t)\|_{0,2} \leq C\{(\varepsilon^r + \frac{h^n}{\varepsilon^n} + \nu\varepsilon + \nu\frac{h^2}{\varepsilon^3})\|\omega\|_{L^\infty(0,T;H^s(\mathbb{R}^2_+))} + \nu(\varepsilon^{3/2} + \frac{h_*^m}{\varepsilon^{m+1/2}})\|g\|_{L^\infty(0,T;H^m(\mathbb{R}))})\}$$

Proof - We argue as in the proof of Theorem 1.7. We write

$$\|\omega_h^\varepsilon - \omega\|_{0,2} \leq \|\Pi_h^\varepsilon\omega - \omega\|_{0,2} + \|\omega_h^\varepsilon - \Pi_h^\varepsilon\omega\|_{0,2},$$

where $\Pi_h^\varepsilon\omega$ is defined by (1.40). By application of a result proved in [26], we have

$$\|\omega - \Pi_h^\varepsilon\omega\|_{0,2} \leq C(\varepsilon^r + \frac{h^n}{\varepsilon^n})\|\omega\|_{s,2}.$$

and this gives the estimate of the first term. For the second one

$$(\omega_h^\varepsilon - \Pi_h^\varepsilon\omega)(x) = \sum_{p\in J} h^2(\omega_p - \omega(x_p))\chi_\varepsilon(x - x_p)$$

we apply Lemma 2.5, obtain

$$\|\omega_h^\varepsilon - \Pi_h^\varepsilon\omega\|_{0,2} \leq C\|\bar{\omega} - \bar{P}(t)\omega\|_{h,2}$$

and Theorem 2.6 yields then inequality (2.24). □

3. Appendix : Analysis of convergence in L^2 of the integro-differential model

This appendix is devoted to the study of the approximation of the diffusion operator Δ with Neumann boundary conditions by the integral operator Δ^ε in the L^2 space. The first result is concerned with the conservation relation. The integro-differential system satisfies the same conservation relation as the advection-diffusion problem.

Proposition 3.1 - *If the function ζ satisfies the following condition*

$$\int_{\mathbb{R}^2_+} \zeta(y)dy = \frac{1}{2} \tag{3.1}$$

then the following relation is satisfied by any solution $\omega^\varepsilon \in L^\infty(0,T;L^1(\mathbb{R}^2_+))$ of (I.7)

$$\frac{d}{dt}\int_{\mathbb{R}^2_+} \omega^\varepsilon(x,t)dx + \nu\int_{\mathbb{R}} g(x_1,t)dx_1 = 0. \tag{3.2}$$

Proof - Noticing that, for any function $v \in L^1(\mathbb{R}^2_+)$ we have

$$\int_{\mathbb{R}^2_+} \Delta^\varepsilon v(x)dx = -\nu\int_{\mathbb{R}} g(x_1,t)dx_1,$$

the proof of inequality (3.2) follows by integration of equation (I.7) on $\mathbb{R}^2_+$. □

Proposition 3.2 - *Assume that η and ζ are compactly supported even functions of each variable which satisfy the moment conditions (1.3i) and (1.3ii). There exists a constant $C > 0$ depending on* **a** *and T, such that if $g \in H^2(\mathbb{R})$, for any function $v \in H^3(\mathbb{R}^2_+)$ which satisfies the boundary condition (1.4), the following estimate holds*

$$||\Delta v - \Delta^\varepsilon v||_{0,2} \leq C\varepsilon(||v||_{3,2} + \varepsilon^{1/2}||g||_{2,2}) \tag{3.3}$$

Proof - Similar estimates have been proved in the previous Sections (inequalities (1.5) and (2.1)). The proof of this estimate will follow from arguments analogous to those used in the proof of Propositions 1.2 and 2.1 but the technics are more complicate. By expansion of the function, we find (1.6), (1.8)-(1.11). We keep the notations previously introduced. The main point is now to find a bound of the rest Q^ε and we start with R^ε. By application of Cauchy-Schwarz inequality we have

$$|R^\varepsilon(x)|^2 \leq \frac{1}{4\varepsilon^4}\int_{\mathbb{R}^2_+} |\eta_\varepsilon(x-y)+\eta_\varepsilon(x-\tilde{y})||y-x|^6 dy \int_{\mathbb{R}^2_+} |\eta_\varepsilon(x-y)+\eta_\varepsilon(x-\tilde{y})|| \int_0^1 \nabla^3 v(y+\lambda(x-y))\lambda^2 d\lambda|^2 dy.$$

We use first the change of variables $y \to z = (y-x)/\varepsilon$ and $y \to z = (x-\tilde{y})/\varepsilon$ and obtain since the support of η is compact

$$\int_{\mathbb{R}^2_+} |\eta_\varepsilon(x-y)+\eta_\varepsilon(x-\tilde{y})||y-x|^6 dy \leq C\varepsilon^6 \int_{\mathbb{R}^2} |\eta(z)|(1+|z|^6)dz.$$

Then, we set

$$I_1 = \int_0^1 \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} |\eta_\varepsilon(x-y)||\nabla^3 v(y+\lambda(x-y))|^2 dydxd\lambda$$

and

$$I_2=\int_0^1\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+}|\eta_\varepsilon(x-\tilde{y})||\nabla^3 v(y+\lambda(x-y))|^2dydxd\lambda,$$

and we have by Cauchy-Schwarz inequality

$$\int_{\mathbb{R}^2_+}|R^\varepsilon(x)|^2dx\leq C\varepsilon^2(I_1+I_2).$$

Let us now estimate I_1 and I_2. By the successive change of variables $y\to z=y+\lambda(x-y)$ and $x\to y=x-z$, we obtain

$$I_1=\int_0^1\int_{\mathbb{R}^2_+}\int_{\mathbb{R}}\int_{\lambda x_2}^{+\infty}|\eta_\varepsilon(\frac{x-z}{1-\lambda})||\nabla^3 v(z)|^2\frac{dz}{(1-\lambda)^2}dxd\lambda\leq\int_0^1\int_{\mathbb{R}^2}|\eta_{\varepsilon(1-\lambda)}(y)|dy\int_{\mathbb{R}^2_+}|\nabla^3 v(z)|^2dzd\lambda$$

which yields

$$I_1\leq\|\nabla^3 v\|^2_{0,2}.$$

Using similar arguments for the second integral, we have

$$I_2=\int_0^1\int_{\mathbb{R}^2_+}\int_{\mathbb{R}}\int_{\lambda x_2}^{+\infty}|\eta_\varepsilon(\frac{x_1-z_1}{1-\lambda},x_2+\frac{z_2-\lambda x_2}{1-\lambda})||\nabla^3 v(z)|^2\frac{dz}{(1-\lambda)^2}dxd\lambda$$

$$=\int_0^1\int_{\mathbb{R}^2_+}|\nabla^3 v(z)|^2\int_{\mathbb{R}}\int_0^{\frac{z_2}{\lambda}}|\eta_{\varepsilon(1-\lambda)}(x_1-z_1,(1-2\lambda)x_2+z_2)|dxdzd\lambda.$$

The estimate of the rest R^ε will thus follow from the bound of the integral

$$I_3(z)=\int_0^1\int_{\mathbb{R}}\int_0^{\frac{z_2}{\lambda}}|\eta_{\varepsilon(1-\lambda)}(x_1-z_1,(1-2\lambda)x_2+z_2)|dxd\lambda,$$

which is split into two parts as follows

$$I_3(z)=\int_0^{1/2}\int_{\mathbb{R}}\int_0^{\frac{z_2}{\lambda}}|\eta_{\varepsilon(1-\lambda)}(x_1-z_1,(1-2\lambda)x_2+z_2)|dxd\lambda+\int_{1/2}^1\int_{\mathbb{R}}\int_0^{\frac{z_2}{\lambda}}|\eta_{\varepsilon(1-\lambda)}(x_1-z_1,(1-2\lambda)x_2+z_2)|dxd\lambda.$$

We focuse our attention on the bound of the first part

$$I_3^1(z)=\int_0^{1/2}\int_{\mathbb{R}}\int_0^{\frac{z_2}{\lambda}}|\eta_{\varepsilon(1-\lambda)}(x_1-z_1,(1-2\lambda)x_2+z_2)|dxd\lambda,$$

and the bound of the second one is obtained by similar arguments. The change of variable $x\to y=(x_1-z_1,(1-2\lambda)x_2+z_2)$ yields

$$I_3^1(z)=\int_0^{1/2}\int_{\mathbb{R}}\int_{z_2}^{\frac{1-\lambda}{\lambda}z_2}|\eta_{\varepsilon(1-\lambda)}(y)|\frac{dy}{1-2\lambda}d\lambda.$$

Again, we split the integral into two terms and write for the first one

$$\int_0^{1/4}\int_{\mathbb{R}}\int_{z_2}^{\frac{1-\lambda}{\lambda}z_2}|\eta_{\varepsilon(1-\lambda)}(y)|\frac{dy}{1-2\lambda}d\lambda \le \int_0^{1/4}\frac{d\lambda}{1-2\lambda}\int_{\mathbb{R}^2_+}|\eta_{\varepsilon(1-\lambda)}(y)|dy \le \frac{1}{2}\|\eta\|_{0,1}.$$

For the second one, we have, since the function η is compactly supported

$$\int_{1/4}^{1/2}\int_{\mathbb{R}}\int_{z_2}^{\frac{1-\lambda}{\lambda}z_2}|\eta_{\varepsilon(1-\lambda)}(y)|\frac{dy}{1-2\lambda}d\lambda \le \int_{1/4}^{1/2}\frac{C\varepsilon(1-\lambda)}{1-2\lambda}\frac{1}{\varepsilon(1-\lambda)}(\frac{1-\lambda}{\lambda}-1)\sup_{y_2>0}\int_{\mathbb{R}}|\eta(y)|dy_1 d\lambda \le$$

$$\le C\|\eta\|_{L^\infty(\mathbb{R}_+;L^1(\mathbb{R}))}$$

The integral I_3 is thus bounded by a constant which does not depend on z. Summarizing, we have finally proved that

$$\int_{\mathbb{R}^2_+}|R^\varepsilon(x)|^2 \le C\varepsilon^2\|v\|^2_{3,2}.$$

In order to estimate the first term of Q^ε, we write

$$|\frac{2}{\varepsilon^2}\int_{\mathbb{R}^2_+}x_2y_2\eta_\varepsilon(x-\tilde{y})\int_0^1(1-\lambda)x_2\frac{\partial^3 v}{\partial x_2^3}(x_1,\lambda x_2)d\lambda dy|^2 = \frac{4}{\varepsilon^4}x_2^4|\int_{\mathbb{R}^2_+}y_2\eta_\varepsilon(x-\tilde{y})dy|^2|\int_0^1(1-\lambda)x_2\frac{\partial^3 v}{\partial x_2^3}(x_1,\lambda x_2)d\lambda|^2 \le$$

$$\le C\varepsilon^2|\int_{\mathbb{R}^2_+}\eta(z)dz|^2|\int_0^1(1-\lambda)x_2\frac{\partial^3 v}{\partial x_2^3}(x_1,\lambda x_2)d\lambda|^2,$$

where we have again used the fact that the function η is compactly supported. Integrating with respect to x we finally have

$$|\frac{2}{\varepsilon^2}\int_{\mathbb{R}^2_+}x_2y_2\eta_\varepsilon(x-\tilde{y})\int_0^1(1-\lambda)x_2\frac{\partial^3 v}{\partial x_2^3}(x_1,\lambda x_2)d\lambda dy|^2 dx \le C\varepsilon^2\|v\|^2_{3,2}.$$

The second term is bounded in the same way ; let us estimate the term in g. Using the change of variables $y_1 \to z_1 = (y_1-x_1)/\varepsilon$ and $x_1 \to u_1 = (1-\lambda)x_1 + \lambda y_1$, we have

$$\int_{\mathbb{R}^2_+}|2\int_{\mathbb{R}}(y_1-x_1)^2\zeta_\varepsilon(x_1-y_1,x_2)\int_0^1\lambda g''(x_1+\lambda(y_1-x_1))d\lambda dy_1|^2 dx =$$

$$= 4\varepsilon^2\int_{\mathbb{R}^2_+}|\int_{\mathbb{R}}z_1^2\zeta(z_1,\frac{x_2}{\varepsilon})\int_0^1\lambda g''(u_1)d\lambda dz_1|^2 du_1 dx_2 \le C\varepsilon^3\int_{\mathbb{R}}|g''(u_1)|^2 du_1$$

and this last inequality allows to conclude the proof. □

Remark 4 : As noticed in remark 2, if $g \in H^3(\mathbb{R})$, if the functions η and ζ satisfy the additional condition (1.17), there exists a constant $C>0$ depending on **a** and T, such that for any function $v \in H^4(\mathbb{R}^2_+)$ which satisfies the boundary condition (1.4), the following estimate holds

$$\|\Delta v - \Delta^\varepsilon v\|_{0,2} \le C\varepsilon^2(\|v\|_{4,2} + \varepsilon^{1/2}\|g\|_{3,2}). \tag{3.4}$$

□

In order to ensure the existence of a solution of problem (I.7), we establish now the stability of the integral operator

Lemma 3.3 - *Assume that $\eta \in L^1$, $g \in L^2$. Then there exists a constant $C > 0$ depending on* **a** *and T, such that for any $v \in L^2$,*

$$||\Delta^\varepsilon v||_{0,2} \leq \frac{C}{\varepsilon^2}(||v||_{0,2} + \varepsilon^{3/2}||g||_{0,2}). \tag{3.5}$$

Proof - We recall that

$$\Delta^\varepsilon v(x) = \frac{1}{\varepsilon^2}\Lambda^\varepsilon v(x) - 2\int_{\mathbb{R}} \zeta_\varepsilon(x_1 - y_1, x_2) g(y_1, t) dy_1, \tag{I.5}$$

and a first application of Cauchy-Schwarz inequality gives

$$|\int_{\mathbb{R}^2_+} |\Delta^\varepsilon v(x)|^2 dx \leq 2(\frac{1}{\varepsilon^4}\int_{\mathbb{R}^2_+} |\Lambda^\varepsilon v(x)|^2 dx + 4\int_{\mathbb{R}^2_+} |\int_{\mathbb{R}} \zeta_\varepsilon(x_1 - y_1, x_2) g(y_1, t) dy_1|^2 dx.$$

By several applications of Cauchy-Schwarz inequality, we have

$$|\int_{\mathbb{R}^2_+} (\eta_\varepsilon(x-y)+\eta_\varepsilon(x-\tilde{y}))(v(y)-v(x))dy|^2 \leq 2(|\int_{\mathbb{R}^2_+} \eta_\varepsilon(x-y)(v(y)-v(x))dy|^2 + |\int_{\mathbb{R}^2_+} \eta_\varepsilon(x-\tilde{y})(v(y)-v(x))dy|^2)$$

$$\leq 2(\int_{\mathbb{R}^2} |\eta_\varepsilon(y)| dy \int_{\mathbb{R}^2_+} |\eta_\varepsilon(x-y)||v(y)-v(x)|^2 dy + \int_{\mathbb{R}^2} |\eta_\varepsilon(y)| dy \int_{\mathbb{R}^2_+} |\eta_\varepsilon(x-\tilde{y})||v(y)-v(x)|^2 dy).$$

We thus obtain by the change of variable $y \to z = (x-y)/\varepsilon$

$$\int_{\mathbb{R}^2_+} |\Lambda^\varepsilon v(x)|^2 dx \leq 2||\eta||_{0,1}(\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} |\eta_\varepsilon(x-y)||v(y)-v(x)|^2 dy dx + \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} |\eta_\varepsilon(x-\tilde{y})||v(y)-v(x)|^2 dy dx). \tag{3.6}$$

Applying again the Cauchy-Schwarz inequality, we also have

$$\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} |\eta_\varepsilon(x-y)||v(y)-v(x))|^2 dy dx \leq 2\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} |\eta_\varepsilon(x-y)||v(y)|^2 dy dx,$$

and the change of variable $x \to z = (x-y)/\varepsilon$ yields

$$\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} |\eta_\varepsilon(x-y)||v(y)-v(x))|^2 dy dx \leq 2||\eta||_{0,1}||v||^2_{0,2}.$$

The estimate of the other term is analogous and we obtain by combination of this inequality with (3.6)

$$||\Lambda^\varepsilon v||^2_{0,2} \leq 8||\eta||^2_{0,1}||v||^2_{0,2}.$$

Using similar arguments, which are Cauchy-Schwarz inequalities and changes of variables, we bound the term which depends on g but since the domain of integration has not the same dimension, a power of ε appears. □

We prove now that in the case of a non negative cut-off, the operator $-\Delta^\varepsilon$ is positive which ensures the stability of the integro-differential system (I.7)

Lemma 3.4 - *Assume that the function η is non negative. There exists a constant $C > 0$ depending on* $\mathbf{a}$ *and T, such that for any $\omega_0 \in L^2$, if we denote by v the solution of (I.5)-(I.7) for $g = 0$, the following estimate holds for any $t \in (0, T)$*

$$\|v(.,t)\|_{0,2} \leq C\|\omega_0\|_{0,2}. \tag{3.7}$$

Proof - Since $g = 0$, we have for any function v in L^2

$$\Delta^\varepsilon v(x) = \frac{1}{\varepsilon^2}\Lambda^\varepsilon v(x) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2_+}(\eta_\varepsilon(x-y) + \eta_\varepsilon(x-\tilde{y}))(v(y) - v(x))dy.$$

Then, we notice that, since $\eta \geq 0$ is even,

$$\int_{\mathbb{R}^2_+}\Lambda^\varepsilon v(x)v(x)dx = \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+}(\eta_\varepsilon(x-y) + \eta_\varepsilon(x-\tilde{y}))(v(y) - v(x))v(x)dydx,$$

$$= -\frac{1}{2}\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+}(\eta_\varepsilon(x-y) + \eta_\varepsilon(x-\tilde{y}))(v(y) - v(x))^2 dydx \leq 0. \tag{3.8}$$

By integration by parts and conditions (I.2) imposed to $\mathbf{a}$, we also have

$$\int_{\mathbb{R}^2_+} div(\mathbf{a}v)(x)v(x)dx = -\frac{1}{2}\int_{\mathbb{R}^2_+}\mathbf{a}\nabla(|v(x)|^2)dx = 0$$

Now, by definition of v, we have

$$\frac{\partial v}{\partial t} + div(\mathbf{a}v) - \frac{\nu}{\varepsilon^2}\Lambda^\varepsilon v = 0,$$

and multiplying the equation by v and integrating over $\mathbb{R}^2_+$ we obtain, thanks to (3.8) and the previous identity

$$\frac{d}{dt}\|v(.,t)\|^2_{0,2} \leq 0.$$

The conclusion follows then from application of Gronwall's lemma. □

Let us now come to the convergence of the solution ω^ε of the integro-differential (1.7) towards the solution ω of the advection-diffusion system

Theorem 3.5 - *Assume that the function η is non negative and that η and ζ are compactly supported even functions of each variable which satisfy the following moment conditions*

$$\int_{\mathbb{R}^2_+} y_2\eta(y_1, x_2 + y_2)dy = \int_{\mathbb{R}}\zeta(y_1, x_2)dy_1, \quad \text{for any } x_2 > 0, \tag{1.3i}$$

$$\int_{\mathbb{R}^2} x_i^2\eta(x)dx = 2, \quad \text{for } i = 1, 2. \tag{1.3ii}$$

Then, there exists a constant $C > 0$ *depending on* $\mathbf{a}$ *and* T*, such that if* $g \in L^\infty(0,T;H^2(\mathbb{R}))$ *and* $\omega \in L^\infty(0,T;H^3(\mathbb{R}^2_+))$

$$\|\omega - \omega^\varepsilon\|_{0,2} \leq C\nu\varepsilon(\|\omega\|_{3,2} + \varepsilon^{1/2}\|g\|_{2,2}). \tag{3.9}$$

Proof - We set $e^\varepsilon = \omega - \omega^\varepsilon$, where ω and ω^ε are respectively solution of (I.3) and of (I.5)-(I.7), and we have

$$\frac{\partial e^\varepsilon}{\partial t} + div(\mathbf{a}e^\varepsilon) - \frac{\nu}{\varepsilon^2}\Delta^\varepsilon e^\varepsilon = \frac{\nu}{\varepsilon^2}(\Delta\omega - \Delta^\varepsilon\omega).$$

Inequality (3.9) follows then from application of lemma 3.4 and proposition 3.2. □

This last result concludes the analysis and the paper.

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