# The exponential of the square module of a diffusion


Jacques NEVEU
CMAP Ecole Polytechnique*


April 1993

## 1 Introduction

A natural proof is given of a useful bound of the expectation of the exponential of the square modulus of a multi-dimensional diffusion ; it is optimal in the sense that the Ornstein-Uhlenbeck process attains it. It is not to be considered as a really new result but I have not been able to find a proper reference giving this bound.

## 2 The Bound

**Theorem 2.1** *Let $(X_t, t \in R^+)$ be the solution of the Itô s.d.e.on $R^d$ starting at $X_0 = x_0$:*

$$dX_t \;=\; b(X_t)\,dt \;+\; \sigma(X_t)\;dB_t \tag{1}$$

*where $b : R^d \to R^d, \sigma : R^d \to L(R^{d'}, R^d)$ are lipschitzian (we denote by $L(R^{d'}, R^d)$ the normed space of linear mappings from the euclidian space $R^{d'}$ into the euclidian space $R^d$) and where $(B_t, t \in R^+)$ is a d'-dimensional brownian motion with covariance $tI_{d'}$. Assume that*

$$x.b(x) \le [b_0 + b_1 \|x\|^2]/2 \;\;,\;\; \|\sigma(x)\| \le \sigma_0 \tag{2}$$

*for three positive constants $b_0, b_1, \sigma_0$ and that on $R^d$*

$$Tr[\sigma(x)\sigma(x)^T] \le \sigma_2^2 \tag{3}$$

*for a positive constant $\sigma_2$ ($\sigma_2 = \sigma_0\, d^{1/2}$ is allways suitable). Then for every $\epsilon_0 > 0$*

$$\log I\!\!E(\exp(\epsilon(t)\|X_t\|^2/2)) \;-\; \epsilon_0\|x_0\|^2/2 \;\le\; [(b_0 + \sigma_2^2)/2\sigma_0^2]\;\log[1 + \epsilon_0\sigma_0^2\beta(t)] \tag{4}$$

*provided*

$$\beta(t) := [1 - \exp(-b_1 t)]/b_1 \;\;(:=\; t \; if\, b_1 = 0)$$

*and provided the decreasing function $\epsilon(.)$ is defined by*

$$\epsilon(t) := \epsilon_0 \exp(-b_1 t)\;/\;[1 + \epsilon_0\sigma_0^2\beta(t)] \tag{5}$$

Proof. For every $\epsilon > 0$, Itô's formula shows that

$$d(\exp[\epsilon\|X_t\|^2/2]) = \exp[\epsilon\|X_t\|^2/2]\;\;[\epsilon X_t.dX_t + (\epsilon/2)Tr[\sigma(X_t)\sigma(X_t)^T]dt + (\epsilon^2/2)\|\sigma(X_t)^T X_t\|^2 dt] \tag{6}$$


*F-91128 Palaiseau Cedex.Tel: (1) 69 33 34 20; Fax:(1) 69 33 30 11; e-mail: neveu@paris.polytechnique.fr

so that provided the suitable integrability conditions are satisfied (see below) to pass to expectations, we have

$$\begin{aligned}(d/dt)\,\mathbb{E}\{\exp[\epsilon\|X_t\|^2/2]\} &= \mathbb{E}\{\exp[\epsilon\|X_t\|^2/2]\ \ [\epsilon X_t.b(X_t)+(\epsilon/2)Tr[\sigma(X_t)\sigma(X_t)^T]+(\epsilon^2/2)\|\sigma(X_t)^T X_t\|^2]\} \\ &\le (\epsilon/2)\,\mathbb{E}\{\exp[\epsilon\|X_t\|^2/2]\ \ [b_0+\sigma_2^2+(b_1+\epsilon\sigma_0^2)\|X_t\|^2]]\} \qquad (7)\end{aligned}$$

using the hypotheses on the mappings $b(.)$ and $\sigma(.)$.

Let

$$\Phi(t,\epsilon) := \log \mathbb{E}\{\exp[\epsilon\|X_t\|^2/2]\}. \qquad (8)$$

The last member of (7) introduces the expectation $\mathbb{E}\{\exp[\epsilon\|X_t\|^2/2]\ \|X_t\|^2/2\}$ which is the $\epsilon$-derivative of $\exp[\Phi(t,\epsilon)]$ provided this differentiation is licite. Hence with this proviso the inequality (7) may be rewritten in the simple form

$$\partial\Phi(t,\epsilon)/\partial t\ -\ \epsilon(b_1+\epsilon\sigma_0^2)/2\ \partial\Phi(t,\epsilon)/\partial\epsilon\ \le\ \epsilon\ (b_0+\sigma_2^2)\ /2. \qquad (9)$$

But the function $\epsilon(.)$ of the Theorem is the explicit solution of the ordinary differential equation

$$d\epsilon(t)/dt = -\epsilon(t)\ [b_1+\epsilon(t)\sigma_0^2]$$

which is equal to $\epsilon_0$ at the origin. Hence

$$d\Phi(t,\epsilon(t))/dt\ \le\ (b_0+\sigma_2^2)/2\ \ \epsilon(t)$$

and by integration

$$\Phi(t,\epsilon(t))\ \le\ \epsilon_0\|x_0\|^2\ +\ (b_0+\sigma_2^2)/2\int_0^t \epsilon(s)\ ds$$

But the definition of the function $\epsilon(.)$ implies that

$$\int_0^t \epsilon(s)\ ds = (1/\sigma_0^2)\ \log[1+\epsilon_0\sigma_0^2\beta(t)]$$

and the formula of the theorem is thus proved.

To make the preceding proof rigorous, consider the sequence of random functions $f_n(\exp[\epsilon\|X_t\|^2/2])$ defined from the bounded functions $f_n(u) := nu/(n+u)$ on $R^+$. Itô's formula shows as before that

$$\begin{aligned} df_n(\exp[\epsilon\|X_t\|^2/2]]) &= f_n'(\exp[\epsilon\|X_t\|^2/2])\ \ d(\exp[\epsilon\|X_t\|^2/2]) \\ &+\epsilon^2/2\ f"_n(\exp[\epsilon\|X_t\|^2/2])\ \exp(\epsilon\|X_t\|^2)\ \|\sigma(X_t)^T X_t\|^2\ dt\end{aligned}$$

with $f_n'(u) = n^2/(n+u)^2$ and $f"_n(u) = -2n^2/(n+u)^3$.But now since $u^2 f_n'(u)$ is bounded on $R^+$, the local martingale component of the second member of the last formula is uniformly bounded; being null at the origin this martingale has expectation zero. Furthermore since $f_n(u),\ u^3 f"_n(u),\ v^2\exp(-\epsilon v^2/2)$ are bounded on $R^+$, the other terms of the above equality are also bounded and passing to expectations in this formula is completely justified. Using next the hypotheses on the mappings $b()$ and $\sigma()$ and the fact that $f_n'\ge 0$ and that $f"_n \le 0$ on $R^+$, we are then led to the inequality

$$(d/dt)\mathbb{E}[f_n(\exp[\epsilon\|X_t\|^2/2])$$

$$\le\ \epsilon/2\ \mathbb{E}[f_n'(\exp[\epsilon\|X_t\|^2/2])\ \exp[\epsilon\|X_t\|^2/2]\ [b_0+\sigma_2^2+(b_1+\sigma_0^2)\|X_t\|^2]$$

Finally the differential formula

$$(\partial/\partial\epsilon)\mathbb{E}[f_n(\exp[\epsilon\|X_t\|^2/2])] = \mathbb{E}[f_n'(\exp[\epsilon\|X_t\|^2/2])\ \exp[\epsilon\|X_t\|^2/2]\ \|X_t\|^2/2]$$

is exact as the argument of the second expectation is uniformly bounded in $\epsilon$. This implies that

$$[\partial/\partial t\ -\ \epsilon(b_1+\sigma_0^2)\partial/\partial\epsilon]\ \mathbb{E}[f_n(\exp[\epsilon\|X_t\|^2/2])] \le \epsilon(b_0+\sigma_2^2)/2\ \ \mathbb{E}[f_n(\exp[\epsilon\|X_t\|^2/2])]$$

taking also into account the fact that $uf_n'(u) \leq f(u)$ on $R^+$. A simple integration of this inequality and the increasing passage to the limit as $n \uparrow \infty$ end our rigorous proof.$\square$

REMARKS.

1. Another formulation of the theorem is as follows. For a fixed $t > 0$ and given an $\eta > 0$, let us choose $\epsilon_0$ so that $\epsilon(t) = \eta$; according to (5), this is possible if

$$\eta\sigma_0^2\gamma(t) < 1 \quad \text{where} \quad \gamma(t) := [\exp(+b_1 t) - 1]/b_1. \tag{10}$$

Then (4) may be rewritten :

$$\begin{aligned} \log \mathbb{E}\{\exp(\eta\|X_t\|^2/2)\} \;&-\; \eta\exp(b_1 t)\,[1 \;-\; \eta\sigma_0^2\gamma(t)]\;\|x_0\|^2/2 \\ &\leq\; [b_0 + \sigma_2^2]/2\sigma_0^2\; \log[1/[1 - \eta\sigma_0^2\gamma(t)]] \end{aligned} \tag{11}$$

2. Consider the particular case of the simplest d-dimensionnal Ornstein- Uhlenbeck process for which $b(x) = b_1 x/2$ and $\sigma(x) = \sigma_0 I_d$ and $d' = d$; here $b_0 = 0$, $\sigma_2^2 = 2d\sigma_0^2$. The random function $X_t$ has here independent increments and is gaussian with mean and variance given by

$$\mathbb{E}(X_t) = \exp(b_1 t/2)\; x_0 \;\;,\;\; Var(X_t) = \sigma_0^2\gamma(t) I_d.$$

Hence by an elementary formula on gaussian random variables :

$$\begin{gathered} log\mathbb{E}[\exp(\eta\|X_t\|^2/2)] \\ = \; -d/2\; \log[1 - \eta\sigma_0^2\gamma(t)] \;+\; \eta\exp(b_1 t)[1 - \eta\sigma_0^2\gamma(t)]^{-1}\|x_0\|^2/2 \end{gathered}$$

at least if $\eta\sigma_0^2\gamma(t) < 1$. This coincides with (11) with an equality sign.

3. Let $(\mathcal{F}_t, t \geq 0)$ be a filtration and suppose that $(B_t, t \geq 0)$ is a brownian motion with respect to this filtration, then our proof shows that

$$\exp[\epsilon(t)\|X_t^2\|^2/2] \quad [1 \;+\; \epsilon_0\sigma_0^2\beta(t)]^{-[b_0+\sigma_2^2]/2\sigma_0^2}.$$

is a positive supermartingale.