Global existence for the full von Kármán System

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Abstract. *We consider here the full system of dynamic Von Kàrmàn equations, taking into account the in-plane acceleration terms, which is a model for the vibrations of a nonlinear elastic plate. We prove global existence and uniqueness of strong solutions for this system with various boundary conditions possibly including feedback terms which are useful for stabilization purposes.*

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1 Introduction

In this paper we give a global existence result for a system of semi-linear evolution equations, with nonlinear boundary conditions, modelling the coupled in-plane and transversal vibrations of a nonlinear plate. We suppose that a part of the boundary is clamped and we give appropriate feedbacks laws for the bending moment and the transverse shear force and twisting moment on the other part of the boundary (in particular our result includes the homogeneous Dirichlet-Neuman boundary conditions). The motivation of the particular form of the feedback laws is given in [14], [15] where the related stabilization problem is studied. In the stationary case the system we consider, proposed in [7], gives the classical von Kármán equations for the equilibrium configurations of a plate. As far as we know, with the exception of [14], [15] all previous papers on global existence and asymptotic behaviour of solutions for related problems neglected the in-plane accelerations (see [1], [3], [7], [17], [18]). This assumption leads, for star shaped domains and suitable boundary conditions, to the existence of an Airy stress function, which considerably simplifies the equations.
In order to state the problem consider $\Omega \subset \mathbf{R}^2$ an open bounded set representing the domain of the plate, $Q = \Omega \times (0,T)$, $\Gamma = \partial\Omega$. We suppose that there exists $\Gamma_0,\ \Gamma_1$ a partition of Γ such that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, $\Gamma_0 \neq \emptyset$, and we shall use the notations $\Sigma_i = \Gamma_i \times (0,T),\ i = 0,1$. The functions $u : \Omega \to \mathbf{R}^2$, $w : \Omega \to \mathbf{R}$ represent the in-plane, respectively the transverse displacement of the plate. Let us introduce the linear plane strain tensor $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u)$ and denote by S the set of all fourth order symmetric tensors on $\mathbf{R}^2$. Define $\mathcal{C} \in \mathcal{L}(S)$ by

$$\mathcal{C}[\epsilon] = \frac{E}{d(1-\mu^2)}[\mu(tr\epsilon)I_{\mathcal{S}} + (1-\mu)\epsilon], \forall \epsilon \in S,$$

where I_S is the identity of S, d is the density, E is the Young's modulus and μ is the Poisson's ratio of the material. We shall also denote by $g',\ g''$ for the time derivatives of a fonction g.

With the notations above the full dynamic von Kármán system with feedbacks acting on a part of the boundary can be written in the following form (cf.[7]):

$$u'' - div\{\mathcal{C}[\epsilon(u) + f(\nabla w)]\} = 0, \text{ in Q}, \tag{1.1}$$

$$w'' - \gamma\Delta w'' + D\Delta^2 w - div\{\mathcal{C}[\epsilon(u) + f(\nabla w)]\nabla w\} = 0, \text{ in Q}, \tag{1.2}$$

$$u = 0, w = \frac{\partial w}{\partial \nu} = 0, \text{ on } \Sigma_0, \tag{1.3}$$

$$\mathcal{C}[\epsilon(u) + f(\nabla w)]\nu = -c_1(m \cdot \nu)u' - c_2(\frac{\partial u_2}{\partial s}, -\frac{\partial u_1}{\partial s}), \text{ on } \Sigma_1 \tag{1.4}$$

$$D[\Delta w + (1-\mu)B_1 w] = -c_1(m \cdot \nu)\frac{\partial w'}{\partial \nu}, \text{ on } \Sigma_1, \tag{1.5}$$

$$D[\frac{\partial \Delta w}{\partial \nu} + (1-\mu)\frac{\partial B_2 w}{\partial s}] - \gamma\frac{\partial w''}{\partial \nu} - \mathcal{C}[\epsilon(u) + f(\nabla w)]\nu \cdot \nabla w =$$

$$= c_1(m \cdot \nu)w' - c_1\frac{\partial}{\partial s}[(m \cdot \nu)\frac{\partial w'}{\partial s}], \text{ on } \Sigma_1 \tag{1.6}$$

$$u(0) = u^0, u'(0) = u^1, w(0) = w^0, w'(0) = w^1, \text{ in } \Omega, \tag{1.7}$$

where $f : \mathbf{R^2} \to S$ is defined by $f(x) = \frac{1}{2}x \otimes x$, D represents the flexural rigidity of the plate, $\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$ and s are the unit outward normal, repectively the unit tangent vector to $\partial\Omega$ and the operators B_1 and B_2 are given by:

$$B_1 w = 2\nu_1\nu_2\frac{\partial^2 w}{\partial x_1 \partial x_2} - \nu_1^2\frac{\partial^2 w}{\partial x_2^2} - \nu_2^2\frac{\partial^2 w}{\partial x_1^2}$$

$$B_2 w = (\nu_1^2 - \nu_2^2)\frac{\partial^2 w}{\partial x_1 \partial x_2} + \nu_1\nu_2(\frac{\partial^2 w}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_1^2}).$$

Moreover by m is a smooth vector field and γ, c_1, c_2 are positive constants. We assume that $E > 0$ and $0 < \mu < 1/2$ in such a way that the tensor $\mathcal{C}$ satisfies the condition:

$$\mathcal{C}[\epsilon] \cdot \epsilon \geq \lambda_0|\epsilon|^2, \forall \epsilon \in S, \text{ where } \lambda_0 > 0. \tag{1.8}$$

Let us also mention that for the one dimensional rod model, as well as in a radial symmetric case, the existence and uniqueness of global finite energy solutions is proved in [9], respectively in [14], [15], where damping at the boundary is also considered. The papers quoted above used the imbedding $H^1(\Omega) \subset C^0(\Omega)$ which fails in the two-dimensional case. This is why, in the bidimensional case then nonlinear terms in (1.1), (1.2) are not lipschitz functions in the energy space.

2 Notations and preliminaries

Let us introduce, following [7] the function spaces

$$W=\{w\in H^2(\Omega)|\ w_{|\Gamma_0}=\frac{\partial w}{\partial\nu}|\Gamma_0=0\};\ V=\{w\in H^1(\Omega)|\ w_{|\Gamma_0}=0\};\ H=L^2(\Omega);$$

$$U=\{u\in[H^1(\Omega)]^2|\ u_{|\Gamma_0}=0\},\mathcal{K}=[L^2(\Omega)]^2,$$
$$\mathcal{S}=L^2(\Omega,S),\ \mathcal{S}_p=H^p(\Omega,S),\ \forall p\neq 0.$$

We denote by $(\cdot,\cdot)$ the scalar product in H or in $\mathcal{K}$ and by $||\cdot||$ the corresponding norm. Consider also the following bilinear and continuous forms

$$e:U\times U\to\mathbf{R},$$
$$e(u,\phi)=(\mathcal{C}[\epsilon(u)],\epsilon(\phi))+$$
$$+c_2\int_\Omega(\frac{\partial u_1}{\partial x_1}\frac{\partial\phi_2}{\partial x_2}+\frac{\partial u_2}{\partial x_2}\frac{\partial\phi_1}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\frac{\partial\phi_2}{\partial x_1}-\frac{\partial u_2}{\partial x_1}\frac{\partial\phi_1}{\partial x_2})dX,\ \forall u,\phi\in U \quad (2.1)$$

$$a:W\times W\to\mathbf{R},$$
$$a(w,\widetilde{w})=D\int_\Omega[\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2\widetilde{w}}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2\widetilde{w}}{\partial x_2^2}+\mu(\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2\widetilde{w}}{\partial x_2^2}+$$
$$+\frac{\partial^2\widetilde{w}}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2})+2(1-\mu)\frac{\partial^2 w}{\partial x_1\partial x_2}\frac{\partial^2\widetilde{w}}{\partial x_1\partial x_2}]dx. \quad (2.2)$$

$$b:W\times W\to\mathbf{R},\ b(w,\psi)=c_1\int_{\Gamma_1}(w\psi+\nabla w\cdot\nabla\psi)d\Gamma_1, \quad (2.3)$$

$$c:V\times V\to\mathbf{R},\ c(v,\psi)=\int_\Omega(w\psi+\gamma\nabla w\cdot\nabla\psi)dX. \quad (2.4)$$

A simple calculation shows that the integration by parts formulas

$$D\int_\Omega(\Delta^2 w)\widetilde{w}dX=a(w,\widetilde{w})-D\int_\Gamma\{[\Delta w+(1-\mu)B_1 w]\frac{\partial\widetilde{w}}{\partial\nu}-$$
$$-[\frac{\partial\Delta w}{\partial\nu}+(1-\mu)\frac{\partial B_2 w}{\partial s}]\widetilde{w}\}d\Gamma, \quad (2.5)$$

$$\int_\Gamma(\frac{\partial u_2}{\partial s}\phi_1-\frac{\partial u_1}{\partial s}\phi_2)d\Gamma=\int_\Omega(\frac{\partial u_1}{\partial x_1}\frac{\partial\phi_2}{\partial x_2}+\frac{\partial u_2}{\partial x_2}\frac{\partial\phi_1}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\frac{\partial\phi_2}{\partial x_1}-\frac{\partial u_2}{\partial x_1}\frac{\partial\phi_1}{\partial x_2})dX. \quad (2.6)$$

hold for $u,\ w,\ \phi_1,\ \phi_2,\ \widetilde{w}$ smooth enough.
Let $\{u, w\}$ be a classical solution of (1.1)-(1.6), i.e.

$$u \in C(0,T;[H^2(\Omega)]^2) \cap C^1(0,T;[H^1(\Omega)]^2),$$

$$w \in C(0,T;H^4(\Omega) \cap C^1(0,T;H^3(\Omega)).$$

If we take the inner product in $\mathcal{K}$ of (1.1) with $\phi \in U$ and of (1.2) with $\psi \in W$, by the use of (1.1)-(1.6), (2.5) and (2.6) we get:

$$[(u',\phi)]' + e(u,\phi) + (\mathcal{C}f(\nabla w),\epsilon(\phi)) + c_1[\int_{\Gamma_1}(m\cdot\nu)u\cdot\phi d\Gamma]' = 0,$$

$$\forall\phi \in U,\ \forall t \in [0,T) \tag{2.7}$$

$$[(w',\psi) + \gamma(\nabla w',\nabla\psi)]' + a(w,\psi) + [b(w,\psi)]' +$$

$$+ (\mathcal{C}[\epsilon(u) + f(\nabla w)]\nabla w,\nabla\psi) = 0,\ \forall\psi \in W,\ \forall t \in [0,T). \tag{2.8}$$

Definition 2.1 *We say that $\{u, w\}$ is a* **strong** *solution of (1.1)-(1.6), if*

$$u \in C(0,T;[H^2(\Omega)]^2 \cap U) \cap C^1(0,T;U) \cap C^2([0,T],\mathcal{K}), \tag{2.9}$$

$$w \in C(0,T;H^3(\Omega) \cap W) \cap C^1(0,T;W) \cap C^2([0,T],V), \tag{2.10}$$

and they satisfy (2.7), (2.8) and (1.7).

Definition 2.2 *We say that $\{u, w\}$ is a* **finite energy (weak)** *solution of (1.1)-(1.6), if:*

$$u \in C(0,T;U) \cap C^1(0,T;\mathcal{K}),\ w \in C(0,T;W) \cap C^1(0,T;V), \tag{2.11}$$

and they satisfy (2.7), (2.8) and (1.7).

The total energy of the plate is given (cf.[8,p.18]) by

$$E(t) = \frac{1}{2}\{||u'(t)||^2 + ||w'(t)||^2 + \gamma||\nabla w'(t)||^2 +$$

$$+ a(w(t),w(t)) + (\mathcal{C}[\epsilon(u(t)) + f(\nabla w(t))],\epsilon(u(t)) + f(\nabla w(t)))\}, \tag{2.12}$$

where $||\cdot||$ and $(\cdot,\cdot)$ represent the norm, respectively the inner product, in $[L^2(\Omega)]^k$, $k=1,2$. For technical reasons (explained in [8], [15]) we shall use the following perturbated energy:

$$\widetilde{E}(t)=E(t)+c_2\int_\Omega(\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}-\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1})dX\,dt. \tag{2.13}$$

It is obvious that, for c_2 small enough we have the inequality

$$C_1E(t)\le\widetilde{E}(t)\le C_2E(t),\ \forall t\ge 0, \tag{2.14}$$

where $C_1,\ C_2$ are positive constants. A simple calculation, based on (2.5), (2.6) (see also [15]) shows that we have the following estimate

Lemma 2.1 *Let $\{u,\,w\}$ be a strong solution of (1.1)-(1.6). Then the inequality*

$$\widetilde{E}(t)\le\widetilde{E}(0),$$

holds for any $t\in[0,T]$.

The following three lemmas give some properties of the bilinear forms e and a, introduced by (2.1), (2.2).

Lemma 2.2 *For any $w\in W$ there exists at most one couple $(y,h)\in V\times H^{\frac{1}{2}}(\Gamma_1)$ such that*

$$a(w,\psi)=c(y,\psi)+\int_{\Gamma_1}h\frac{\partial\psi}{\partial\nu},\ \forall\psi\in W, \tag{2.15}$$

Proof: Consider two couples (y_1,h_1), y_2,h_2 satisfying (2.15) and define

$$\widetilde{V}=\left\{\psi\in W|\frac{\partial\psi}{\partial\nu}_{|\Gamma_1}=0\right\}.$$

If we choose $w\in\widetilde{V}$ in (2.15) we obtain

$$c(y_1,\psi)=c(y_2,\psi),\ \forall\psi\in\widetilde{V}.$$

As we can easily check, $\widetilde{V}$ is dense in V, so the relation above implies that $y_1=y_2$. The equality $h_1=h_2$ follows now easily. $\square$

Lemma 2.3 *Suppose that (2.15) holds for some* $(w, y, h) \in W \times V \times H^{\frac{1}{2}}(\Gamma_1)$. *Then we have the regularity property* $w \in H^3(\Omega)$. *Moreover there exists a constant* $C > 0$ *such that*

$$\|w\|_{H^3(\Omega)} \leq C(\|y\|_{H^1(\Omega)} + \|h\|_{H^{1/2}(\Gamma)}), \tag{2.16}$$

and the relation

$$D[\Delta w + (1-\mu)B_1 w] = h,$$

holds in $H^{1/2}(\Gamma_1)$.

Proof: Let us first consider the variatonal problem

$$a(w_1, \psi) = \int_{\Gamma_1} h \frac{\partial \psi}{\partial \nu} d\Gamma_1, \forall \psi \in W. \tag{2.17}$$

By Lax Milgram theorem, the equation above admits an unique solution $w_1 \in W$, for any $h \in H^{-\frac{1}{2}}(\Gamma_1)$. Moreover we can define a linear isomorphism

$$L : H^{-\frac{1}{2}}(\Gamma_1) \to W, \ Lh = w_1. \tag{2.18}$$

On the other hand, it is clear that (2.17) is the variatonal form of the regular elliptic boundary value problem

$$\Delta^2 w_1 = 0, \tag{2.19}$$

$$w_1 = \frac{\partial w_1}{\partial \nu} = 0, \text{ on } \Gamma_0, \tag{2.20}$$

$$D[\Delta w_1 + (1-\mu)B_1 w_1] = h, \text{ on } \Gamma_1, \tag{2.21}$$

$$D[\frac{\partial \Delta w_1}{\partial \nu} + (1-\mu)\frac{\partial B_2 w_1}{\partial s}] = 0, \quad \text{on } \Gamma_1. \tag{2.22}$$

By classical elliptic regularity theory, it follows that the solution of (2.19)-(2.22) belongs to $H^4(\Omega)$, for any $h \in H^{\frac{3}{2}}(\Gamma_1)$,so

$$L : H^{\frac{3}{2}}(\Gamma_1) \to H^4(\Omega), \tag{2.23}$$

is also a linear isomorphism.
By interpolating between (2.18) and (2.23) we get that

$$h \in H^{\frac{1}{2}}(\Gamma_1) \Longrightarrow w_1 \in H^3(\Omega). \tag{2.24}$$

We still have to check the regularity of $w_2 = w - w_1$. Let us notice that w_2 is a solution of the variational problem

$$a(w_2, \psi) = c(y, \psi), \ \forall \psi \in W, \tag{2.25}$$

If we consider the operator

$$\widetilde{A} : W \to W^*, < \widetilde{A} w_2, \psi >= a(w_2, \psi), \ \forall \psi \in W,$$

the equation (2.25) can be written in the form

$$\widetilde{A} w_2 = y^* \in V^*, \tag{2.26}$$

where y^* is defined by the relation

$$< y^*, \psi >= c(y, \psi), \ \forall \psi \in V.$$

Let us notice that, by Lax-Milgram theorem, equation (2.26) admits an unique solution $w_2 \in W$, for any $y^* \in W^*$. On the other hand , if $y^* \in L^2(\Omega)$, then (2.26) is equivalent to regular elliptic boundary value problem

$$\Delta^2 w_2 = y^*, \tag{2.27}$$

$$w_2 = \frac{\partial w_2}{\partial \nu} = 0, \text{ on } \Sigma_0, \tag{2.28}$$

$$D[\Delta w_2 + (1-\mu) B_1 w_2] = 0, \text{ on } \Sigma_1, \tag{2.29}$$

$$D[\frac{\partial \Delta w_2}{\partial \nu} + (1-\mu)\frac{\partial B_2 w_2}{\partial s}] = 0, \quad \text{on } \Gamma_1. \tag{2.30}$$

By applying again elliptic regularity we get that for $y^* \in L^2(\Omega)$, we have $w_2 \in H^4(\Omega)$. As V^* is the interpolation space of order $1/2$ between $L^2(\Omega)$ and W^* (cf.[12]) we obtain

$$y^* \in V^* \Longrightarrow w_2 \in H^3(\Omega),$$

By applying (2.24) and the relation above we get that $w \in H^3(\Omega)$. The estimate (2.16) follows from continuous dependence on boundary data for regular elliptic problems. □

Remark. *The results of Lemma 2.3 can be obtained directly by Nirenberg translations method after a localization argument.*

Lemma 2.4 *Suppose that c_2 is small enough, that $u \in U$ and there exist $f \in \mathcal{K}$, $g \in [H^{1/2}(\Gamma_1)]^2$ such that*

$$e(u, \phi) = (f, \phi) + \int_{\Gamma_1} g \cdot \phi d\Gamma_1, \ \forall \phi \in U. \tag{2.31}$$

Then, we have $u \in [H^2(\Omega)]^2$ and there exists a constant $C > 0$ such that

$$\|u\|_{[H^2(\Omega)]^2} \leq C(\|f\|_{\mathcal{K}} + \|g\|_{[H^{1/2}(\Gamma_1)]^2}). \tag{2.32}$$

Proof: Let us notice that (2.31) is the variational form of the problem

$$- div\mathcal{C}[\epsilon(u)] = f, \text{ in } \Omega, \tag{2.33}$$

$$u = 0, \text{ on } \Gamma_0, \tag{2.34}$$

$$\mathcal{C}[\epsilon(u)]\nu = g - c_2 \begin{pmatrix} \frac{\partial u_2}{\partial s} \\ \frac{-\partial u_1}{\partial s} \end{pmatrix}, \text{ on } \Gamma_1. \tag{2.35}$$

In order to prove that (2.33)-(2.35) admits an unique solution $u \in [H^2(\Omega)]^2 \cap U$, we shall apply a simple fixed point method. More precisely, consider a given function $v \in [H^2(\Omega)]^2 \cap U$ and the application $v \rightarrow \widetilde{v}$, where $\widetilde{v}$ is the unique solution of

$$- div\mathcal{C}[\epsilon(\widetilde{v})] = f, \text{ in } \Omega, \tag{2.36}$$

$$\widetilde{v} = 0, \text{ on } \Gamma_0, \tag{2.37}$$

$$\mathcal{C}[\epsilon(\widetilde{v})]\nu = g - c_2 \begin{pmatrix} \frac{\partial v_2}{\partial s} \\ \frac{-\partial v_1}{\partial s} \end{pmatrix}, \text{ on } \Gamma_1. \tag{2.38}$$

By classical elliptic regularity, the problem (2.36)-(2.38) admits an unique solution $\widetilde{v} \in [H^2(\Omega)]^2 \cap U$, depending continuously on the boundary data. It is easy to see that the application $v \rightarrow \widetilde{v}$ is a strict contraction in $[H^2(\Omega)]^2 \cap U$, provided that c_2 is small enough. By Banach fixed point theorem we get existence, regularity and continuous dependence on data for (2.33)-(2.35). $\square$

3 The linearized problem

In this section we shall derive some estimates on solutions of the linearization of (1.1)-(1.6). More precisely we shall give some well-possedness results for the following initial value problems:

$$[(u',\phi)]' + e(u,\phi) + c_1[\int_{\Gamma_1}(m\cdot\nu)u\cdot\phi d\Gamma]' + (f_1(t),\epsilon(\phi)) = 0,$$

$$\forall\phi\in U, \tag{3.1}$$

$$u(0)=u^0, u'(0)=u^1, \text{ in } \Omega \tag{3.2}$$

where

$$f_1(t):\Omega\to S,\ \forall t\in[0,T),$$

and

$$[(w',\psi)+\gamma(\nabla w',\nabla\psi)]'+a(w,\psi)+[b(w,\psi)]'+(f_2(t),\nabla\psi)=0,\ \forall\psi\in W. \tag{3.3}$$

$$w(0)=w^0,\ w'(0)=w^1, \tag{3.4}$$

where,

$$f_2(t):\Omega\to\mathbf{R^2},\ \forall t\geq 0.$$

We begin with the definition of some linear unbounded operators and prove that some of them generate strongly continuous semigroups in appropriate Hilbert spaces.
Let us first define the space:

$$\mathcal{D}(\mathcal{A}_1)=\{\begin{pmatrix} u\\ v\end{pmatrix}\in\{[H^2(\Omega)]^2\cap U\}\times U,\text{ such that}$$

$$\mathcal{C}[\epsilon(u)]\nu + c_1(m\cdot\nu)v + c_2\begin{pmatrix}\dfrac{\partial u_2}{\partial s}\\ -\dfrac{\partial u_1}{\partial s}\end{pmatrix} = 0 \text{ on } \Gamma_1\}, \tag{3.5}$$

and the linear operator

$$\mathcal{A}_1:\mathcal{D}(\mathcal{A}_1)\to U\times\mathcal{K},$$

defined by

$$\mathcal{A}_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ -\mathrm{div}\mathcal{C}[\epsilon(u)] \end{pmatrix}. \tag{3.6}$$

The following Lemma is related to the the results of Theorem 2.2 and Theorem 2.3 of [8].

Lemma 3.1 *Suppose that c_2 is small enough. Than the operator $-\mathcal{A}_1$, defined by (3.6) generates a strongly continuous semigroup in $U \times \mathcal{K}$.*

Proof: Let us consider the bilinear form $(\cdot, \cdot)_1 : U \times K \to \mathbf{R}$ defined by

$$\left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right)_1 = e(u, \tilde{u}) + (v, \tilde{v}). \tag{3.7}$$

It is easy to see that, for c_2 small enough, the above bilinear form defines a scalar product in $U \times \mathcal{K}$, which is equivalent to the usual one. A simple calculation shows that

$$\left(\mathcal{A}_1 \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)_1 = c_1 \int_{\Gamma_1} (m \cdot \nu) v^2 d\Gamma,$$

so $\mathcal{A}_1$ is a linear monotone operator. In order to verify that it is maximal monotone, it is sufficient to show that it is onto. Consider equation

$$\mathcal{A}_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ f \end{pmatrix}, \text{ with } \tilde{f} \in U,\ f \in \mathcal{K}. \tag{3.8}$$

It is obvious that (3.8) implies that $v = -\tilde{f} \in U$, and that u is a solution of the problem:

$$e(u, \phi) = (f, \phi) \tag{3.9}$$

By the use of Lemma 2.4, we obtain that equation (3.9) admits an unique solution $u \in [H^2(\Omega)]^2$. It follows that equation (3.8) admits an unique solution

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(\mathcal{A}_1),$$

for any $f \in U,\ g \in \mathcal{K}$, so $\mathcal{A}_1$ is onto. □

In order to study equation (2.8) we introduce the space

$$\mathcal{D}(A) = \{w \in W \text{ such that it exists } (y, h) \in V \times H^{1/2}(\Gamma_1),$$

$$\text{satisfying } a(w,\psi) = c(y,\psi) + \int_{\Gamma_1} h \frac{\partial \psi}{\partial \nu} d\Gamma,\ \forall \psi \in W\}. \tag{3.10}$$

By Lemma 2.3 we know that $\mathcal{D}(A) \subset H^3(\Omega) \cap W$, that y and h as above are unique and that

$$D[\Delta w + (1-\mu)B_1 w] = h, \text{ on } \Gamma_1. \tag{3.11}$$

This allows us to introduce the operator

$$A : \mathcal{D}(A) \to V,\ Aw = y. \tag{3.12}$$

We also define the operator $B : W \to V$ by:

$$c(By,\psi) = c_1 \int_{\Gamma_1} (m\cdot\nu) y\psi d\Gamma + c_1 \int_{\Gamma_1} (m\cdot\nu) \frac{\partial y}{\partial s}\frac{\partial \psi}{\partial s} d\Gamma,\ \forall \psi \in V, \tag{3.13}$$

where the second integral is taken in the sense of the duality $H^{1/2}(\Gamma_1),\ H^{-1/2}(\Gamma_1)$. Consider now the space:

$$\mathcal{D}(\mathcal{A}_2) = \{ \begin{pmatrix} w \\ y \end{pmatrix} \in \mathcal{D}(A) \times W, \text{ s. t.}$$

$$D[\Delta w + (1-\mu)B_1 w] = -c_1 (m\cdot\nu)\frac{\partial y}{\partial \nu} \text{ on } \Gamma_1\}, \tag{3.14}$$

and define the operator

$$\mathcal{A}_2 \begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} -y \\ Aw + By \end{pmatrix},\ \forall \begin{pmatrix} w \\ y \end{pmatrix} \in \mathcal{D}(\mathcal{A}_2). \tag{3.15}$$

The following lemma partially answers to a regularity question raised in [7] section 4.2.1, and gives the generator of the semigroup associated to equation (2.8).

Lemma 3.2 *The operator*

$$-\mathcal{A}_2 : \mathcal{D}(\mathcal{A}_2) \to W \times V,$$

defined by (3.15) generates a strongly continuous semigroup of contractions in $W \times V$.

Proof: As in the previous lemma we shall prove that $\mathcal{A}_2$ is a maximal monotone operator in the space $W \times V$, endowed with a suitable scalar product. More precisely, we choose the scalar product

$$\left(\begin{pmatrix} w \\ y \end{pmatrix}, \begin{pmatrix} \widetilde{w} \\ \widetilde{y} \end{pmatrix} \right)_2 = a(w, \widetilde{w}) + c(y, \widetilde{y}), \ \forall \begin{pmatrix} w \\ y \end{pmatrix} \in W \times V.$$

A simple calculation, based on (3.11), shows that

$$\left(\mathcal{A}_2 \begin{pmatrix} w \\ y \end{pmatrix}, \begin{pmatrix} w \\ y \end{pmatrix} \right)_2 = c_1 \int_{\Gamma_1} (m \cdot \nu)(y^2 + |\nabla y|^2) d\Gamma_1, \ \forall \begin{pmatrix} w \\ y \end{pmatrix} \in \mathcal{D}(\mathcal{A}_2),$$

so $\mathcal{A}_2$ is a monotone operator. In order to prove that it is maximal monotone we shall verify that $\mathcal{A}_2$ is onto. It is clear that the equation

$$\mathcal{A}_2 \begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \in W \times V, \tag{3.16}$$

is equivalent to the system

$$-y = f \in W, \tag{3.17}$$

$$a(w, \psi) = c_1 \int_{\Gamma_1} (m \cdot \nu) f \psi d\Gamma + c_1 \int_{\Gamma_1} (m \cdot \nu) \frac{\partial f}{\partial s} \frac{\partial \psi}{\partial s} d\Gamma + (g, \psi), \ \forall \psi \in W. \tag{3.18}$$

As we can easily check, the right hand side of (3.18) defines a linear and continuous form on V so, by Lemma 2.3, the unique solution w of (3.18) has the regularity $w \in H^3(\Omega) \cap W$. By using (3.17) we get that

$$\begin{pmatrix} w \\ y \end{pmatrix} \in \mathcal{D}(\mathcal{A}_2),$$

so $\mathcal{A}_2$ is onto. □

We can now prove the existence of finite energy solutions of (3.1)-(3.2).

Lemma 3.3 *Suppose that c_2 is small enough. Than, for any*

$$u^0 \in U, \ u^1 \in \mathcal{K},$$

$$f_1 \in C^1([0, T], \mathcal{S}), \tag{3.19}$$

problem (3.1)-(3.2) admits an unique solution, having the regularity

$$u \in C([0,T];U) \cap C^1([0,T];\mathcal{K}), \tag{3.20}$$

Moreover there exists a positive constant C such that

$$||u||_{L^\infty([0,T];U)} + ||u'||_{L^\infty([0,T];\mathcal{K})} \leq$$

$$\leq C(T||f_1||_{W^{1,\infty}([0,T];\mathcal{S})} + \|f_1(0)\|_{\mathcal{S}}\| + u^0\|_U + \|u^1\|_{\mathcal{K}}) \tag{3.21}$$

Proof: It is clear that problem (3.1)-(3.2) is equivalent to the following evolution equation in $U \times \mathcal{K}$.

$$Z_1' + \mathcal{A}_1 Z_1 + F_1 = 0,, \tag{3.22}$$

$$Z_1(0) = Z_1^0, \tag{3.23}$$

where

$$Z_1(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}, \; F_1(t) = \begin{pmatrix} 0 \\ \widetilde{f}_1(t) \end{pmatrix}$$

with

$$(\widetilde{f}_1(t), \phi) = (f_1(t), \epsilon(\phi)), \; \forall \phi \in U. \tag{3.24}$$

By Lemma 3.1 we know that $\mathcal{A}_1$ generates a strongly continuous semigroup

$$T_1(t) : U \times \mathcal{K} \to U \times \mathcal{K}.$$

Let us notice that, by assumption(3.19) the function F_1 defined by (3.24) has the regularity

$$F_1 \in C^1([0,T], (\mathcal{D}(\mathcal{A}_1))'. \tag{3.25}$$

On the other hand, it is well known that if $\mathcal{A}_1 : \mathcal{D}(\mathcal{A}_1) \to U \times \mathcal{K}$ generates a strongly continuous semigroup, then $\mathcal{A}_1$ can be extended to an operator $\widetilde{\mathcal{A}}_1 : U \times \mathcal{K} \to (\mathcal{D}(\mathcal{A}_1))'$ which generates a strongly continuous semigroup $\widetilde{T}_1(t) : (\mathcal{D}(\mathcal{A}_1))' \to (\mathcal{D}(\mathcal{A}_1))'$, extending $T_1(t)$ (cf. Proposition 2.3.1 from [2]). By applying Theorem I.2.2.1 from [5] it follows that problem (3.36), (3.23) admits an unique solution,

$$Z_1 \in C([0,T], U \times \mathcal{K})| \cap C^1([0,T], [\mathcal{D}(\mathcal{A}_1)]'),$$

which obviously implies the existence and uniqueness result of the Lemma. The estimate (3.21) follows easily from our semigroup setting by applying theorem I.2.2.1 from [5]. □

The next result shows problem (3.1), (3.2) is also well posed in a space of more regular functions, provided that the initial data satisfy a compatibility condition.

Lemma 3.4 *Suppose that c_2 is small enough. Than for any*

$$u^0 \in [H^2(\Omega)]^2 \cap U, \;\; u^1 \in U, \tag{3.26}$$

$$f_1 \in C^2([0,T], \mathcal{S}) \cap C^1([0,T], \mathcal{S}_1), \tag{3.27}$$

satisfying the condition

$$\mathcal{C}[\epsilon(u^0)]\nu + \mathcal{C}[f_1(0)]\nu + c_1(m\cdot\nu)u^1 + c_2(\frac{\partial u_2^0}{\partial s}, -\frac{\partial u_1^0}{\partial s}) = 0, \;\; on\ \Gamma_1, \tag{3.28}$$

problem (3.1)-(3.2) admits an unique solution, having the regularity

$$u \in C([0,T]; [H^2(\Omega)]^2 \cap U) \cap C^1([0,T]; U), \tag{3.29}$$

Moreover there exists a positive constant C such that

$$||u||_{L^\infty([0,T];H^2(\Omega))} + ||u'||_{L^\infty([0,T];U)} \leq C(T||f_1''||_{L^\infty([0,T];\mathcal{S})} +$$

$$+ T||f_1'||_{L^\infty([0,T];\mathcal{S}_1)} + ||f_1(0)||_{\mathcal{S}_1} + ||f_1'(0)||_{\mathcal{S}} + \|u^0\|_{H^2(\Omega)} + \|u^1\|_{\mathcal{K}}) \tag{3.30}$$

Proof: Taking in consideration conditions (3.26) and (3.28) it is clear that it exists $u^2 \in \mathcal{K}$ such that

$$(u^2,\phi) + e(u^0,\phi) + c_1\int_{\Gamma_1}(m\cdot\nu)u^1\cdot\phi d\Gamma + (f_1(0),\epsilon(\phi)) = 0, \;\; \forall\phi\in U. \tag{3.31}$$

Consider now the initial value problem:

$$[(v',\phi)]' + e(v,\phi) + c_1[\int_{\Gamma_1}(m\cdot\nu)v\cdot\phi d\Gamma]' + (f_1'(t),\epsilon(\phi)) = 0, \;\; \forall\phi\in U, \tag{3.32}$$

$$v(0) = u^1, v'(0) = u^2, \text{ in } \Omega \tag{3.33}$$

where u^2 is defined by (3.31). By Lemma 3.3 problem (3.32), (3.33) admits an unique solution having the regularity

$$v \in C([0,T];U) \cap C^1([0,T];\mathcal{K}), \tag{3.34}$$

If we now define

$$u(t) = \int_0^t v(\tau)d\tau + u^0,$$

by integrating (3.32) with respect to time and using (3.28) we obtain that u is a solution of (3.1), (3.2), satisfying the estimate

$$||u'||_{L^\infty([0,T];U)} + ||u''||_{L^\infty([0,T];\mathcal{K})} \leq$$

$$\leq C(T||f_1''||_{W^{1,\infty}([0,T];\mathcal{S})} + \|f_1'(0)\|_{\mathcal{S}} + \|u^1\|_U + \|u^2\|_{\mathcal{K}}). \tag{3.35}$$

On the other hand, by using the fact that $f_1(t) \in \mathcal{S}_1,\ \forall t \geq 0$ and a simple integration by parts the equation (3.1) takes the form

$$e(u(t),\phi) = (-u''(t)+\mathrm{div} f_1(t)),\phi) - \int_{\Gamma_1} [(m\cdot\nu)u'(t)+f_1\nu]\cdot\phi d\Gamma_1,\ \forall\phi \in U. \tag{3.36}$$

By the use of (5.1) and of Lemma 2.4, the relation (3.36) easily gives

$$||u(t)||_{H^2(\Omega)} \leq C(||f_1(t)||_{\mathcal{S}_1} + ||u''(t)||_{\mathcal{K}}). \tag{3.37}$$

The estimate (3.30) is now a direct consequence of (3.35) and (3.37).
□

The counterpart of Lemma 3.4 for (3.3), (3.4) is given by the following result:

Lemma 3.5 *If*

$$w^0 \in H^3(\Omega) \cap W,\ \ w^1 \in W, f_2 \in C^1([0,T],\mathcal{K}), \tag{3.38}$$

satisfying the condition

$$D[\Delta w^0 + (1-\mu)B_1 w^0] = -c_1(m\cdot\nu)\frac{\partial w^1}{\partial\nu},\ \ on\ \Sigma_1, \tag{3.39}$$

then problem (3.3), (3.4) admits an unique solution such that

$$w \in C([0,T],H^3(\Omega)\cap W) \cap C^1([0,T],W) \cap C^2([0,T],V), \tag{3.40}$$

Moreover there exists a constant C such that

$$\|w\|_{L^\infty([0,T],H^3)} + \|w'\|_{L^\infty([0,T],W)} + ||w''||_{L^\infty([0,T],V)} \leq$$
$$\leq C(T\|f_2'\|_{L^\infty([0,T],\mathcal{K})} + ||f_2(0)||_{\mathcal{K}} + \|w^0\|_{H^3} + \|w^1\|_W). \qquad (3.41)$$

Proof: It is easy to verify that problem (3.3), (3.4) can be written as an initial value problem in the space $W \times V$. Let us consider the operator $\mathcal{A}_2$ defined by (3.15) and $\widetilde{f}_2 : [0,T] \to V$ defined by

$$c(\widetilde{f}_2(t), \psi) = (f_2(t), \nabla\psi).$$

With these notations, (3.3), (3.4) are equivalent to the abstract problem

$$Z_2' + \mathcal{A}_2 Z_2 + F_2 = 0, \qquad (3.42)$$

$$Z_2(0) = Z_2^0, \qquad (3.43)$$

where

$$Z_2 = \begin{pmatrix} w \\ w' \end{pmatrix}, \ F_2 = \begin{pmatrix} 0 \\ \widetilde{f}_2(t) \end{pmatrix}.$$

As by Lemma 3.2 $\mathcal{A}_2$ generates a strongly continuous semigroup in $W \times V$ and by (3.38) $F_2 \in C^1([0,T], W \times V)$, the conclusion of the present lemma follows from Theorem I.2.2.1 from [2]. □

4 A local existence result

The aim of this section is to prove that strong solutions of (1.1)-(1.6) exist, at least for small time. More precisely the main result of this section is:

Theorem 4.1 *Suppose that c_2 is small enough and that u^0, u^1, w^0, w^1 satisfy the conditions*

$$u^0 \in [H^2(\Omega)]^2 \cap U, \ u^1 \in U, \ w^0 \in H^3(\Omega) \cap W, \ w^1 \in W, \qquad (4.1)$$

$$\mathcal{C}[\epsilon(u^0) + f(\nabla w^0)]\nu + c_1(m\cdot\nu)u^1 - c_2(\frac{\partial u_2^0}{\partial s}, -\frac{\partial u_1^0}{\partial s}) = 0, \ \textit{on } \Gamma_1. \quad (4.2)$$

$$D[\Delta w^0 + (1-\mu)B_1 w^0] = -c_1(m\cdot\nu)\frac{\partial w^1}{\partial \nu}, \ \ on\ \Sigma_1, \tag{4.3}$$

Then there exists $T_0 > 0$ such that problem (1.1)-(1.6) admits an unique strong solution for any $T < T_0$. Moreover one of the following assertions holds true:

(A1) $T_0 = \infty$;

(A2) $\lim_{t\to T_0}(||u(t)||_{H^2} + ||u'(t)||_{H^1} + ||w(t)||_{H^3} + ||w'(t)||_{H^2}) = \infty.$

In order to prove the above result we shall use Lemmas 3.4, 3.5, combined with a fixed point technique. The first step of this approach consists in some estimates on the nonlinear terms arising in (2.7), (2.8). More precisely, consider u, w satisfying (2.9), (2.10) and introduce the applications

$$w \to f_1^w,\ \ f_1^w(t) = \mathcal{C}f(\nabla w(t)), \tag{4.4}$$

$$w \to f_2^{u,w},\ \ f_2^{u,w} = \mathcal{C}[\epsilon(u(t)) + f(\nabla w(t))]\nabla w(t). \tag{4.5}$$

Let us first notice the following regularity property

Lemma 4.1 *For any $u,\ w$ satisfying (2.9), (2.10) we have*

$$f_1^w \in C^2([0,T],\mathcal{S}) \cap C^1([0,T],\mathcal{S}_1),\ \ f_2^{u,w} \in C^1([0,T],\mathcal{K}). \tag{4.6}$$

Proof: Let us notice that, in two space dimensions, the following property holds true (cf. [4]):

$$\left.\begin{array}{c} \text{if } r = \min\{s_1, s_2, s_1+s_2-2\} \geq 0, \\ \text{then pointwise multiplication maps continuously} \\ H^{s_1}(\Omega)\times H^{s_2}(\Omega) \text{ into } H^r(\Omega). \end{array}\right\} \qquad (\mathbf{P1})$$

A simple calculation shows that

$$\frac{d}{dt}(f_1^w(t)) = \mathcal{C}(\nabla w(t)\otimes\nabla w'(t)),$$

$$\frac{d}{dt}(f_2^{u,w}(t)) = \mathcal{C}[\epsilon(u'(t)) + \nabla w\otimes\nabla w'(t)]\nabla w(t) +$$

$$+\mathcal{C}[\epsilon(u(t)+f(\nabla w(t))]\nabla w'(t),$$

$$\frac{d^2}{dt^2}f_1^w(t) = 2(\mathcal{C}[\nabla w(t)\otimes\nabla w''(t)] + \mathcal{C}[\nabla w(t)\otimes\nabla w'(t)],$$

and, by applying (**P1**) we easily see that (4.6) holds true. □

Let us now introduce the Banach spaces

$$Y_1 = C(0,T;H^3(\Omega)\cap W)\cap C^1(0,T;W)\cap C^2([0,T],V),$$

$$Y_2 = C(0,T;[H^2(\Omega)]^2\cap U)\cap C^1(0,T;U),$$

with usual norms. The polynomial character of the nonlinearities and some Sobolev imbedding theorems imply the following result, which is stated without proof.

Lemma 4.2 *The application* $w\to f_!^w$ *(resp.* $(u,w)\to f_2^{u,w}$*) is* C^1 *from* Y_1 *(resp.* $Y_2\times Y_1$*) into* $C^2([0,T],\mathcal{S})\cap C^1([0,T],\mathcal{S}_1)$ *(resp.* $C^1([0,T],\mathcal{K})$*). Moreover the following estimates holds:*

$$||f_1^w||_{W^{2,\infty}([0,T],\mathcal{S})\cap W^{1,\infty}([0,T],\mathcal{S}_1)} \le C||w||_{Y_2}^2,\ \forall w\in Y_1$$

$$||f_2^{u,w}||_{W^{1,\infty}([0,T],\mathcal{K})} \le C(||w||_{Y_2}^3 + ||w||_{Y_2}||u||_{Y_1}),\ \forall (u,w)\in Y_2\times Y_1.$$

Proof of Theorem 4.1: Let us fix $u^0,\ u^1,\ w^0,\ w^1$ satisfying (4.1)-(4.3). For $R>1$ we make the notation

$$Y_R^0 = \left\{ \begin{pmatrix} \tilde{u} \\ \tilde{w} \end{pmatrix} \in Y_2\times Y_1 \text{ s.t. } \begin{pmatrix} u(0) \\ u'(0) \\ w(0) \\ w'(0) \end{pmatrix} = \begin{pmatrix} u^0 \\ u^1 \\ w^0 \\ w^1 \end{pmatrix}, \left\| \begin{pmatrix} u \\ w \end{pmatrix} \right\|_{Y_2\times Y_1} \le R \right\}.$$

Define $P: Y_R^0 \to Y_2\times Y_1$ by:

$$P\begin{pmatrix} \tilde{u} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} u \\ w \end{pmatrix},$$

where u,w satisfy the equations

$$[(u',\phi)]' + e(u,\phi) + c_1[\int_{\Gamma_1}(m\cdot\nu)u\cdot\phi d\Gamma]' + (f_1^{\widetilde{w}}(t),\epsilon(\phi)) = 0,\ \forall\phi\in U, \tag{4.7}$$

$$[(w', \psi)+\gamma(\nabla w', \nabla\psi)]'+a(w, \psi)+[b(w, \psi)]'+(f_2^{\widetilde{u},\widetilde{w}}(t), \nabla\psi) = 0,\ \forall\psi \in W. \tag{4.8}$$

$$u(0) = u^0, u'(0) = u^1, w(0) = w^0, w'(0) = w^1,\ \text{in } \Omega, \tag{4.9}$$

By applying Lemmas 3.4, 3.5 and 4.2 we see that P is well defined and that there exist two positive constants $C_1,\ C_2$ such that

$$\left\| \begin{pmatrix} u \\ w \end{pmatrix} \right\|_{Y_2\times Y_1} \leq C_1TR^3 + C_2. \tag{4.10}$$

Inequality (4.10) implies that, if we choose R great enough and T sufficiently small, the application P maps Y_R^0 into itself. On the other hand from Lemma 4.2, combined with Lemmas 3.4, 3.5, we obtain that, for T small enough P is a strict contraction . By applying Banach's fixed point theorem we obtain the local existence result. Moreover, by a classical continuation argument we obtain that one of the assertions (A1) or (A2) holds true. □

5 Global existence and uniqueness of strong solutions

The main result of this section is

Theorem 5.1 *Suppose that c_2, u^0, u^1, w^0, w^1 satisfy the assumptions of Theorem 4.1. Then there exists a unique global strong solution u, w of (1.1)-(1.7).*

We shall easily check that Theorem 5.1 is a straightforward consequence of Theorem 4.1 and of the following estimates:

Lemma 5.1 *Suppose that (u, w) is a strong solution of (1.1)-(1.7) defined on $[0, T)$. Then there exists a constant M_T depending on T such that*

$$||u''(t)||^2+||u'(t)||^2_{H^1}+||w''(t)||^2_{H^1}+||w'(t)||^2_{H^2} \leq M_T,\ \forall t \in [0, T), \tag{5.1}$$

Lemma 5.2 *Suppose that (u, w) is a strong solution of (1.1)-(1.7) defined on $[0, T)$. Then there exists a constant M_T depending on T such that*

$$||u(t)||_{H^2}^2 + ||w(t)||_{H^3}^2 \leq M_T, \ \forall t \in [0, T) \tag{5.2}$$

Proof: of Theorem 5.1: It is sufficient to notice that, by Lemmas 5.1 and 5.2 the assertion (A2) in Theorem 4.1 cannot hold true. As a consequence we obtain that (A1) takes place so any local solution can be extended to a global one. □

In the last part of this section we shall prove Lemmas 5.1 and 5.2. Let us denote by C a generic positive constant.

Proof of Lemma 5.1: We notice that our regularity assumptions allow us to to apply Lemma 2.1 which implies that there exists a constant $M_1 > 0$ such that:

$$||u'||^2 + ||w'||_{H^1}^2 + ||u||_{H^1}^2 + ||w||_{H^2}^2 + ||\sigma(u, w)||^2 \leq M_1 \tag{5.3}$$

for all $t \in [0, T)$, where we used the notation

$$\sigma(u, w) = \mathcal{C}[\epsilon(u) + f(\nabla w)]. \tag{5.4}$$

If we write $v = u'$ and $y = w'$, by taking the derivative of (2.7)-(2.8) with respect to time we obtain:

$$(v'', \phi) + (\widehat{\sigma}(v, w, y), \epsilon(\phi)) + c_2\widetilde{e}(v, \phi) + c_1 \int_{\Gamma_1} (m \cdot \nu)(u' \cdot \phi) = 0, \ \forall \phi \in U, \tag{5.5}$$

$$(y'', \psi) + \gamma(\nabla y'', \nabla \psi) + a(y, \psi) +$$

$$(\widehat{\sigma}(v, w, y)\nabla w, \nabla \psi) + (\sigma(u, w)\nabla y, \nabla \psi) + b(y', \psi) = 0, \ \forall \psi \in W, \tag{5.6}$$

where

$$\widehat{\sigma}(v, w, y) = \mathcal{C}[\epsilon(v) + \nabla w \otimes \nabla y], \tag{5.7}$$

$$\widetilde{e}(v, \phi) = \int_\Omega (\frac{\partial v_1}{\partial x_1}\frac{\partial \phi_2}{\partial x_2} + \frac{\partial v_2}{\partial x_2}\frac{\partial \phi_1}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\frac{\partial \phi_2}{\partial x_1} - \frac{\partial v_2}{\partial x_1}\frac{\partial \phi_1}{\partial x_2})dx.$$

If we choose $\phi = v'$ in (5.5) and $\psi = w'$ in (5.6) we obtain:

$$\frac{1}{2}\frac{d}{dt}(||v'||^2 + ||y'||^2 + \gamma||\nabla y'||^2 + a(y, y) + (\widehat{\sigma}(v, w, y), \mathcal{C}^{-1}(\widehat{\sigma}(v, w, y))) +$$

$$+c_2\widetilde{e}(v,v) + (\sigma(u,w), \nabla y \otimes \nabla y)) - \frac{3}{2}(\widehat{\sigma}, \nabla y \otimes \nabla y) \leq 0,\ \forall t \in [0,T).$$

The inequality above can be written in the form:

$$\frac{d}{dt}Q(u,v,w,y) \leq \Psi(v,w,y), \tag{5.8}$$

where

$$Q(u,v,w,y) = \frac{1}{2}(||v'||^2 + ||y'||^2 + \gamma||\nabla y'||^2 + c_2\widetilde{e}(v,v) +$$

$$+ a(y,y) + (\widehat{\sigma}(v,w,y), \mathcal{C}^{-1}(\widehat{\sigma}(v,w,y))) + (\sigma(u,w), \nabla y \otimes \nabla y)), \tag{5.9}$$

and

$$\Psi(v,w,y) = \frac{3}{2}(\widehat{\sigma}(v,w,y), \nabla y \otimes \nabla y), \tag{5.10}$$

From the inequalities of Schwarz and Gagliardo-Nirenberg we have

$$||\nabla y \otimes \nabla y|| \leq C_2||\nabla y||^2_{L^4(\Omega)} \leq C_3||\nabla y||\ ||y||_{H^2},$$

where $C_2,\ C_3$ are positive constants. The inequality above and Lemma 2.1 imply, that

$$||\nabla y \otimes \nabla y|| \leq C||y||_{H^2}. \tag{5.11}$$

As a consequence of (5.9) and (5.11) we see that there exist positive constants $b_1,\ b_2,\ b_3,\ b_4$ such that

$$b_1(||v'||^2 + ||y'||^2 + \gamma||\nabla y'||^2 + (\widehat{\sigma}, \mathcal{C}^{-1}\widehat{\sigma}) + ||y||^2_{H^2}) - b_2 \leq Q(u,v,w,y) \leq$$

$$\leq b_3(||v'||^2 + ||y'||^2 + \gamma||\nabla y'||^2 + (\widehat{\sigma}, \mathcal{C}^{-1}\widehat{\sigma}) + ||y||^2_{H^2}) + b_4. \tag{5.12}$$

On the other hand from (5.10) and (5.11) we obtain

$$|\Psi(v,w,y)| \leq C(||\widehat{\sigma}||^2 + ||y||^2_{H^2}). \tag{5.13}$$

Inequalities (5.8), (5.12) and (5.13) imply, with the use of Gronwall's Lemma that

$$||v'||^2 + ||y'||^2 + \gamma||\nabla y'||^2 + ||\widehat{\sigma}||^2 + ||y||^2_{H^2} \leq C,\ \forall t \in [0,T). \tag{5.14}$$

As from (5.7) we have

$$\epsilon(v) = \mathcal{C}^{-1}[\widehat{\sigma} - \nabla y \otimes \nabla w)],$$

using Lemma 2.1, (5.14) and Korn's inequality we obtain

$$||\nabla v(t)|| \leq C, \ \forall t \in [0,T). \tag{5.15}$$

The relations (5.14) and (5.15) obviously imply the conclusion of Lemma 5.1.

Proof of Lemma 5.2 Let us notice that (2.7) can be considered as the variational formulation of the following elliptic boundary value problem for u

$$-\,div\{\mathcal{C}[\epsilon(u)]\} = -u'' + div\{\mathcal{C}[f(\nabla w)]\}, \text{ in } \Omega, \tag{5.16}$$

$$u_{|\Gamma_0} = 0, \tag{5.17}$$

$$\mathcal{C}[\epsilon(u)]\nu = -\mathcal{C}[f(\nabla w)]\nu - c_1(m\cdot\nu)u'c_2\begin{pmatrix}\frac{\partial u_2}{\partial s}\\ -\frac{\partial u_1}{\partial s}\end{pmatrix}, \text{ on } \Gamma_1 \tag{5.18}$$

On the other hand, by Lemma 5.1, Lemma 2.1 and the property (**P1**) we get that $\operatorname{div}[\mathcal{C}f(\nabla w)]$ is bounded in $L^\infty([0,T],\mathcal{S}_{-\epsilon})$, and the right hand side of (5.18) is bounded in $L^\infty([0,T],H^{1/2-\epsilon}(\Gamma_1))$, $\forall\epsilon\in(0,\frac{1}{2})$. By using classical elliptic regularity theory, combined with an interpolation argument, we obtain that u is bounded in $L^\infty([0,T],[H^{2-\epsilon}(\Omega)]^2)$, so it exists a constant $C>0$ such that

$$||\sigma(u(t),w(t))||_{H^{1-\epsilon}(\Omega)} \leq C, \ \forall t\in[0,T). \tag{5.19}$$

From Lemma 2.1 and (5.19) we obtain

$$||\sigma(t)\nabla w(t)|| \leq C||\sigma(t)||_{L^4}||\nabla w(t)||_{L^4} \leq C||\sigma(t)||_{H^{1-\epsilon}}||\nabla w(t)||_{H^1}. \tag{5.20}$$

On the other hand it is clear that w is a solution of the elliptic problem

$$a(w,\psi) = -(\sigma\nabla w,\nabla\psi) - (w'',\psi) - \gamma(\nabla w'',\nabla\psi) -$$

$$-\int_{\Gamma_1}(w\psi + \frac{\partial w}{\partial s}\frac{\partial \psi}{\partial s})d\Gamma_1 - \int_{\Gamma_1}\frac{\partial w}{\partial \nu}\frac{\partial \psi}{\partial \nu}d\Gamma_1, \forall\psi\in W, \ . \tag{5.21}$$

As, by Lemma 5.1, (5.19), (5.20) the first four terms in the right hand side of (5.21) define a linear continuous form on V, from Lemma 2.3 we get that $w(t)$ is bounded in $H^3(\Omega)$ if $t\in[0,T)$. In order to complete the proof we notice that the H^3 boundedness of $w(t)$ implies,

by applying 2.4, that that u is bounded in $[H^2(\Omega)]^2$. $\square$
The next result asserts the existence of finite energy solutions of (1.1)-(1.7). As in the case of "classical" von Kármán equations (cf. [10]), we are not able to show that this solution is unique.

Theorem 5.2 *Suppose that*

$$u^0 \in U,\ u^1 \in K,\ w^0 \in W,\ w^1 \in V. \tag{5.22}$$

Then there exists a global finite energy solution u, w of (1.1)- (1.7).

Proof: In order to prove the global existence above we shall approach the initial data by more regular functions and then we shall use Theorem 5.1 and a compactness argument. More precisely, let us consider the sequences

$$(u^0)_n \subset [H^2(\Omega)]^2 \cap U,\ (u^1)_n \subset U,$$
$$(w^0)_n \subset H^3(\Omega) \cap W,\ (w^1)_n \subset W,$$

such that

$$u_n^0 \to u^0,\ \text{in } U,\ u_n^1 \to u^1,\ \text{in } \mathcal{K},$$
$$w_n^0 \to w^0,\ \text{in } W,\ w_n^1 \to w^1,\ \text{in } V.$$

We introduce now the following initial and boundary value problem:

$$[(u_n', \phi)]' + e(u_n, \phi) + (\mathcal{C}f(\nabla w_n), \epsilon(\phi)) + c_1[\int_{\Gamma_1} (m \cdot \nu) u_n \cdot \phi d\Gamma]' = 0,$$

$$\forall \phi \in U,\ \forall t \in [0, T) \tag{5.23}$$

$$[(w_n', \psi) + \gamma(\nabla w_n', \nabla \psi)]' + a(w_n, \psi) + [b(w_n, \psi)]' +$$

$$+ (\mathcal{C}[\epsilon(u_n) + f(\nabla w_n)]\nabla w_n, \nabla \psi) = 0,\ \forall \psi \in W,\ \forall t \in [0, T). \tag{5.24}$$

$$u_n(0) = u_n^0, u_n'(0) = u_n^1, w_n(0) = w_n^0, w_n'(0) = w_n^1,\ \text{in } \Omega, \tag{5.25}$$

By Theorem 5.1 problem (5.23)-(5.25) admits a global strong solution

$$u_n \in C([0, T], [H^2(\Omega)]^2) \cap C^1([0, T], U),$$

$$w_n \in C([0, T], H^3(\Omega)) \cap C^1([0, T], W).$$

Moreover it is obvious that this solution satisfies the energy estimate of Lemma 2.1, so we have

$$||u_n'(t)||^2 + ||w_n'(t)||^2 + ||\nabla w_n'(t)||^2 + ||\nabla u_n(t)||^2 + ||w_n(t)||^2_{H^2} +$$
$$+ ||\sigma(u_n(t), w_n(t))||^2 \leq C, \ \forall t \in [0, T), \tag{5.26}$$

where C is a positive constant. From (5.26), by passing to a subsequence, we obtain the existence of $u, \ w$ satisfying (2.11) such that:

$$u_n \to u, \text{ in } L^\infty([0,T], U), \text{weak } *, \tag{5.27}$$
$$u_n' \to u', \text{ in } L^\infty([0,T], \mathcal{K}), \text{weak } *, \tag{5.28}$$
$$w_n \to w, \text{ in } L^\infty([0,T], W), \text{weak } *, \tag{5.29}$$
$$w_n' \to w', \text{ in } L^\infty([0,T], V), \text{weak } *. \tag{5.30}$$

By using a compactness result proved in [16], from (5.29)-(5.30) we obtain that:

$$\nabla w_n \to \nabla w, \text{dans } L^\infty([0,T], H^{1-\epsilon}(\Omega)), \text{ strongly}, \forall \epsilon > 0. \tag{5.31}$$

As, in two space dimensions, we have the imbedding $H^{1-\epsilon}(\Omega) \subset L^{2/\epsilon}(\Omega)$ the relation (5.31) implies that, for ϵ small, we have:

$$\nabla w_n \to \nabla w, \text{dans } L^\infty([0,T], L^6(\Omega)), \text{ strongly }. \tag{5.32}$$

On the other hand from (5.27) we obtain that

$$\epsilon_{ij}(u_n) \to \epsilon_{ij}(u), \text{dans } L^\infty([0,T], L^2(\Omega)]), \text{weak } *. \tag{5.33}$$

From (5.27)-(5.33) we easily get that the nonlinear terms in (5.23)-(5.24) pass to the limit, so u, w is a finite energy solution of (1.1)-(1.7).

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