CONTROLS INSENSITIZING THE NORM OF THE SOLUTION OF A SEMI-LINEAR HEAT EQUATION

Olivier BODART* and Caroline FABRE†

Abstract

We consider here a semilinear heat equation with partially known initial and boundary conditions. The insensitizing problem consists in finding a control function such that some functionnal of the state is locally insensitive to the perturbations of these initial and boundary data. In this paper the insensitizing control of the norm of the observation of the solution in an open subset of the domain is studied under appropriate assumptions on the non linearity and the observation subset. It is shown that the insensitivity conditions are equivalent to a particular non linear exact controllability problem for parabolic equations. Due to the smoothing effects of this type of equations, exact controllability is very hard to achieve and this is why it seems natural to introduce the idea of approximately insensitizing control and then to solve a non linear approximate controllability problem of a special type. That is done using a linearization and fixed point method. Solving the linear problem leads to prove a non trivial uniqueness property which is also used to characterize a particular subset of the admissible controls. This characterization is made thanks to a convex duality theorem and allows to solve a fixed point problem and get the result for the non linear case. Various comments and conclusions are eventually given, with other (approximately) insensitizing problems that can be solved by our methods.

---

*Div. Math. Appliquées, Université de Technologie de Compiègne, B.P. 649, 60206 COMPIEGNE Cedex, FRANCE

†Université Paris-Val de Marne, Département de Mathématiques, Avenue du Général de Gaulle, 94010 CRETEIL Cedex and C.M.A.P., Ecole Polytechnique, 91129 PALAISEAU Cedex

1 Statement Of The Problem. Main Result

Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 1$ with sufficiently smooth boundary Γ and let ω and $\mathcal{O}$ be two open subsets of Ω. Let f be a C^1 and Lipschitz function defined on $\mathbb{R}$. Let $T > 0$, $Q = \Omega \times]0, T[$ and $\Sigma = \Gamma \times]0, T[$. We consider the solution of the semilinear heat equation :

$$\left| \begin{array}{rcll} y' - \Delta y + f(y) & = & \xi + v\chi_\omega & \text{in } Q, \\ y & = & g + \tau_1 \widehat{g} & \text{on } \Sigma, \\ y(\cdot, 0) & = & y^0 + \tau_0 \hat{y}^0 & \text{in } \Omega. \end{array} \right. \qquad (1)$$

where $'$ denotes the derivation with respect to the time variable.

The function $y(x,t)$ is only partially defined in the following way :

- ξ, y^0 and g are given respectively in $L^2(Q)$, $L^2(\Omega)$ and $L^2(\Sigma)$,
- $\widehat{y}^0 \in L^2(\Omega)$ and $\widehat{g} \in L^2(\Sigma)$ are unknown and $\|\widehat{y}^0\|_{L^2(\Omega)} = \|\widehat{g}\|_{L^2(\Sigma)} = 1$,
- $\tau_0 \in \mathbb{R}$ and $\tau_1 \in \mathbb{R}$ are unknown and small,
- $v = v(x,t)$ represents a control term to be determined in $L^2(\omega \times]0, T[)$ and χ_ω is the characteristic function of ω (the subset including the controls supports).

It is known (see [3][5]) that under the above assumptions on f and the data, system (1) possesses a unique solution in $L^2(Q)$. The solution of (1) will be written y or $y(\tau_0, \tau_1)$ or $y(x, t; v, \tau_0, \tau_1)$ when it will be necessary.

The notion of insensitizing control has been introduced by J.L. Lions (see [10]), and its most general formulation is as follows :

Definition 1 : *Let Φ : $L^2(Q) \to \mathbb{R}$ be a differentiable functionnal defined on the set of the solutions of (1). We say that the control v insensitizes $\Phi(y)$ if :*

$$\forall (\widehat{y}^0, \widehat{g}) \in L^2(\Omega) \times L^2(\Sigma) \qquad \left\{ \begin{array}{l} \dfrac{\partial \Phi(y(x,t;v,\tau_0,\tau_1))}{\partial \tau_0}\Big|_{\tau_0=0,\tau_1=0} = 0 \\ \dfrac{\partial \Phi(y(x,t;v,\tau_0,\tau_1))}{\partial \tau_1}\Big|_{\tau_0=0,\tau_1=0} = 0 \end{array} \right. \qquad (2)$$

This definition means that we want the functionnal Φ to be locally insensitive to the perturbations $\tau_0\hat{y}^0$ and $\tau_1\hat{g}$. There are of course many possible choices of functionnals Φ, but the insensitivity condition (2) is of no use unless it is reformulated into a more explicit control problem. This is why it seems reasonnable for Φ to be the square of the L^2-norm of some function of the state. In that case the condition (2) leads to exact controllability problems of non linear systems. That kind of problem is very hard to solve, and very few results are available for the moment (see [16] concerning the wave equation) particularly in the context of parabolic systems (see [11][12]). Hence it is useful to introduce the notion of approximately insentizing controls :

Definition 2 *: Let $\Phi : L^2(Q) \to I\!R$ be a differentiable functionnal defined on the solutions of (1). Let $\varepsilon_1, \varepsilon_2 > 0$, we say that the control v $(\varepsilon_1, \varepsilon_2)$-insensitizes $\Phi(y)$ if :*

$$\forall(\hat{y}^0, \hat{g}) \in L^2(\Omega) \times L^2(\Sigma) \qquad \begin{cases} \left| \dfrac{\partial\Phi(y(x,t;v,\tau_0,\tau_1))}{\partial\tau_0} \Big|_{\tau_0=0,\tau_1=0} \right| \leq \varepsilon_1 \\ \left| \dfrac{\partial\Phi(y(x,t;v,\tau_0,\tau_1))}{\partial\tau_1} \Big|_{\tau_0=0,\tau_1=0} \right| \leq \varepsilon_2 \end{cases} \tag{3}$$

In this paper we study the existence of controls $(\varepsilon_1, \varepsilon_2)$-insensitizing the following functionnal :

$$\Phi(y(\tau_0, \tau_1)) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} y^2(x,t)\, dx\, dt\,. \tag{4}$$

More precisely we prove the following :

Theorem 1 *: If $\omega \cap \mathcal{O} \neq \emptyset$ and if f is a C^1 and Lipschitz function on $I\!R$, then for every $\varepsilon_1 > 0, \varepsilon_2 > 0$ there exists at least a control $v \in L^2(\omega\times]0,T[)$ $(\varepsilon_1, \varepsilon_2)$-insensitizing the functionnal (4). Moreover there exist functions $\varphi^0 \in L^2(\Omega), k \in L^2(\Sigma), a \in L^\infty(Q), b \in L^\infty(Q)$ such that if φ and ψ solve*

$$\left| \begin{array}{rcll} \varphi' - \Delta\varphi + b(x,t)\varphi &=& 0 & \text{in } Q, \\ \varphi &=& k & \text{on } \Sigma, \\ \varphi(\cdot,0) &=& \varphi^0 & \text{in } \Omega, \end{array} \right.$$

$$\left| \begin{array}{rcll} -\psi' - \Delta\psi + a(x,t)\psi &=& \varphi\chi_{\mathcal{O}} & \text{in } Q, \\ \psi &=& 0 & \text{on } \Sigma, \\ \psi(\cdot,T) &=& 0 & \text{in } \Omega, \end{array} \right.$$

the function $v = \psi|_{\omega\times]0,T[}$ is a control satisfying the $(\varepsilon_1, \varepsilon_2)$-insensitivity conditions.

Remark 1 : The open sets ω and $\mathcal{O}$ will be respectively called control and observation set.

$\square$

The proof of that theorem will be done in three steps. First we will reformulate the conditions (2) (resp. (3)) into a non linear exact (resp. approximate) controllability type problem. We will then study the linear case perturbed by a bounded potential and obtain a characterization of the controls. The third part will consist in the proof of theorem 1 using the previous results and a fixed point argument. In a last section we will make a few remarks and conclusions.

Before going further let us recall important results that will be constantly used in the sequel and concern the existence and uniqueness of weak solutions for the linear heat equation. The solutions will be found in the following class of Hilbert spaces; if $r > 0$ and $s > 0$ are two real numbers we have :

$$H^{r,s}(Q) = L^2(0,T;H^r(\Omega)) \cap H^s(0,T;L^2(\Omega)), \tag{5}$$

endowed with the natural norm

$$\|u\|_{r,s} = \left(\int_0^T \|u\|^2_{H^r(\Omega)}\,dt + \|u\|^2_{H^s(0,T;L^2(\Omega))}\right)^{1/2}. \tag{6}$$

Moreover in the sequel $|\cdot|$ and $\|\cdot\|$ will respectively denote the norm in $L^2(\Omega)$ and $L^2(\Sigma)$.
The three following results (adapted here from more general theorems to our purposes) can be found in [13]. First let $a \in \mathrm{L}^\infty(Q)$ and $f \in L^2(Q)$. The following system

$$\left|\begin{array}{rcll} u' - \Delta u + a(x,t)u &=& f & \text{in } Q, \\ u &=& 0 & \text{on } \Sigma, \\ u(\cdot,0) &=& 0 & \text{in } \Omega, \end{array}\right.$$

admits a unique weak solution $u \in H^{2,1}(Q) \cap C([0,T];H^1_0(\Omega))$. Moreover there exists a constant $C > 0$ depending on Ω, a, T with $C = O(1+|a|_\infty e^{|a|_\infty})$ such that

$$\|u\|_{1,2} \le C\|f\|_{L^2(Q)}.$$

Let now $a \in \mathrm{L}^\infty(Q)$. Let $g \in L^2(\Sigma)$ and $u^0 \in L^2(\Omega)$. The following system

$$\left| \begin{array}{rcll} u' - \Delta u + a(x,t)u &=& 0 & \text{in } Q, \\ u &=& g & \text{on } \Sigma, \\ u(\cdot,0) &=& u^0 & \text{in } \Omega, \end{array} \right.$$

admits a unique weak solution $u \in H^{\frac{1}{2},\frac{1}{4}}(Q)$. Moreover there exists a constant $C > 0$ depending on Ω, a, T such that

$$\|u\|_{\frac{1}{2},\frac{1}{4}} \leq C\Big(|u_0| + \|g\|\Big).$$

Let finally $a \in \mathrm{L}^\infty(Q)$, $f \in L^2(Q)$ and $u^0 \in L^2(\Omega)$. The following system

$$\left| \begin{array}{rcll} u' - \Delta u + a(x,t)u &=& f & \text{in } Q, \\ u &=& 0 & \text{on } \Sigma, \\ u(\cdot,0) &=& u^0 & \text{in } \Omega, \end{array} \right.$$

admits a unique weak solution $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$.
$\square$

2 Formulation Of The Controllability Problem

The insensitivity and approximate insensitivity conditions obviously have to be reformulated to relate them to the control function v. From now on we will suppose the assumptions of theorem 1 to be valid. We have the

Proposition 1 : *Let $\bar{y}(x,t)$ and $q(x,t)$ be the solutions of the following equations :*

$$\left| \begin{array}{rcll} \bar{y}' - \Delta \bar{y} + f(\bar{y}) &=& \xi + v\chi_\omega & \textit{in } Q, \\ \bar{y} &=& g & \textit{on } \Sigma, \\ \bar{y}(\cdot,0) &=& y^0 & \textit{in } \Omega, \end{array} \right. \tag{7}$$

$$\left| \begin{array}{rcll} -q' - \Delta q + f'(\bar{y})q &=& \bar{y}\chi_{\mathcal{O}} & \textit{in } Q, \\ q &=& 0 & \textit{on } \Sigma, \\ q(\cdot,T) &=& 0 & \textit{in } \Omega, \end{array} \right. \tag{8}$$

where $\chi_{\mathcal{O}}$ is the characteristic function of the observation subset $\mathcal{O}$.

Conditions (2) and (3) are respectively equivalent to :

$$\left\{ \begin{array}{l} q(\cdot,0) = 0 \ \ a.e. \ \ in \ \Omega, \\ \dfrac{\partial q}{\partial \nu} = 0 \ \ a.e. \ \ on \ \Sigma, \end{array} \right. \tag{9}$$

$$\left\{ \begin{array}{l} |q(\cdot,0)| \leq \varepsilon_1, \\ \|\dfrac{\partial q}{\partial \nu}\| \leq \varepsilon_2, \end{array} \right. \tag{10}$$

where $\dfrac{\partial}{\partial \nu}$ *denotes the outward normal derivation.*

Remark 2 : Since the solution $\bar{y}$ of (7) is in $L^2(Q)$ and since $q(\cdot,T) = 0$, we have $q \in H^{2,1}(Q) \cap C([0,T]; H_0^1(\Omega))$ so that conditions (10) make sense.

□

Proof : Since f is C^1 the state y solution of (1) is differentiable with respect to τ_0 and τ_1. As we have defined $\bar{y}$ as $y(\tau_0 = 0, \tau_1 = 0)$, the derivatives of y with respect to τ_0 and τ_1 at $\tau_0 = \tau_1 = 0$, denoted respectively by y_{τ_0} and y_{τ_1}, are the solutions of

$$\left| \begin{array}{rcll} y'_{\tau_0} - \Delta y_{\tau_0} + f'(\bar{y}) y_{\tau_0} & = & 0 & \text{in } Q, \\ y_{\tau_0} & = & 0 & \text{on } \Sigma, \\ y_{\tau_0}(\cdot,0) & = & \widehat{y}^0 & \text{in } \Omega, \end{array} \right. \tag{11}$$

$$\left| \begin{array}{rcll} y'_{\tau_1} - \Delta y_{\tau_1} + f'(\bar{y}) y_{\tau_1} & = & 0 & \text{in } Q, \\ y_{\tau_1} & = & \widehat{g} & \text{on } \Sigma, \\ y_{\tau_1}(\cdot,0) & = & 0 & \text{in } \Omega, \end{array} \right. \tag{12}$$

hence the derivatives of Φ with respect to τ_0 and τ_1 at $\tau_0 = \tau_1 = 0$ are

$$\left. \frac{\partial \Phi}{\partial \tau_0} \right|_{\tau_0=0,\tau_1=0} = \int_0^T \int_{\mathcal{O}} \bar{y} y_{\tau_0} \, dx \, dt \,, \tag{13}$$

$$\left. \frac{\partial \Phi}{\partial \tau_1} \right|_{\tau_0=0,\tau_1=0} = \int_0^T \int_{\mathcal{O}} \bar{y} y_{\tau_1} \, dx \, dt \,. \tag{14}$$

Substituting $\bar{y}$ by the left hand side of (8) in (13)(14) and integrating by parts one easily obtains :

$$\left.\frac{\partial \Phi}{\partial \tau_0}\right|_{\tau_0=0,\tau_1=0} = \int_\Omega q(x,0)\,\widehat{y}^0(x)\,dx\,,$$

$$\left.\frac{\partial \Phi}{\partial \tau_1}\right|_{\tau_0=0,\tau_1=0} = -\int_\Sigma \frac{\partial q}{\partial \nu}\,\widehat{g}(\sigma,t)\,d\sigma\,dt\,,$$

where $d\sigma$ denotes measure over the boundary, which gives immediately (9) and (10) (as $\widehat{y}^o$ and $\widehat{g}$ are of norm 1).

□

We then have to prove the approximate controllability nor only of the final data, but also of the normal derivative of a couple of parabolic systems with one of them of semilinear type. In order to solve it we are first going to study the case of two linear systems with bounded potentials. A fixed point theorem will then be applied to solve the nonlinear problem (10). The fixed point technique for solving controllability of non linear systems has been introduced by E. Zuazua in [16] for the exact controllability of the semilinear wave equation.

3 Approximate Controllability Of A Couple Of Linear Systems With Bounded Potentials

Let $a(x,t)$ and $b(x,t)$ be two functions in $L^\infty(Q)$. We consider the following systems :

$$\left|\begin{array}{rcll} \zeta' - \Delta\zeta + a(x,t)\zeta &=& v\chi_\omega & \text{in } Q, \\ \zeta &=& 0 & \text{on } \Sigma, \\ \zeta(\cdot,0) &=& 0 & \text{in } \Omega, \end{array}\right. \tag{15}$$

$$\left|\begin{array}{rcll} -\eta' - \Delta\eta + b(x,t)\eta &=& \zeta\chi_{\mathcal{O}} & \text{in } Q, \\ \eta &=& 0 & \text{on } \Sigma, \\ \eta(\cdot,T) &=& 0 & \text{in } \Omega, \end{array}\right. \tag{16}$$

In this section we will prove the approximate controllability of these systems, in the sense of the approximately insensitizing conditions, which means that for any $\varepsilon_1, \varepsilon_2 > 0$ and for any couple $\{\eta^d, g^d\} \in L^2(\Omega) \times L^2(\Sigma)$ there exists at least one control v in $L^2(\omega \times]0, T[)$ such that

$$\left\{ \begin{array}{rcl} |\eta(\cdot, 0) - \eta^d| & \leq & \varepsilon_1 \quad , \\ \|\dfrac{\partial \eta}{\partial \nu} - g^d\| & \leq & \varepsilon_2 \quad . \end{array} \right. \tag{17}$$

We will prove this result using the Hahn-Banach theorem. As we are interested in a fixed point argument to solve our primal problem, we will need estimates on the L^2-norm of the controls with respect to the potentials a and b. This is why we will first characterize, for fixed a and b, the control v minimizing the L^2-norm among the admissible controls. That will be done using a convex duality technique (see [4]). This characterization will allow us to estimate the norm of the controls and then to treat the non-linear case. Notice that the steps of our proofs have already been used by C. Fabre, J.P. Puel, E. Zuazua in [5] in order to prove the apporoximate controllability for the semi-linear heat equation. In this paper a linearization and fixed point method was also involved.

More precisely what is known in the linear case (see [6][5][8][11] that the sets

$$\Big\{\zeta(\cdot, T) \mid v \text{ spans } L^2(\omega \times]0, T[)\Big\},$$

or

$$\Big\{\zeta(\cdot, T) \mid v \text{ spans } L^\infty(\omega \times]0, T[)\Big\},$$

or

$$\Big\{\zeta(\cdot, T) \mid v \text{ is quasi bang-bang}\Big\},$$

where ζ solves (15), are dense in $L^2(\Omega)$. For the definition of a quasi bang-bang control, see [5].

It is not very difficult to prove that these results remain valid for $\eta(\cdot, 0)$. What is really new here, always for the linear case, is that we prove the density in $L^2(\Sigma)$ of the set spanned by $\partial\eta/\partial\nu$ when v spans $L^2(\omega \times]0, T[)$ (see theorem 2).

Let us first recall the very important unique continuation property proved by J.C. Saut and B. Scheurer in [15]. Let φ be a function of $L^2(Q)$ satisfying :

$$\varphi' + \mathcal{A}\varphi = 0 \qquad \text{in } Q$$

where $\mathcal{A}$ is a second order elliptic operator, i.e.

$$\mathcal{A}u = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u + \sum_{i=1}^{n} b_i(x,t)\frac{\partial}{\partial x_i}u + c(x,t)u,$$

with coefficients satisfying the following assumptions :

- $a_{ij} \in C^1(Q), \quad 1 \leq i,j \leq n,$
- $b_i \in L^\infty_{loc}(Q), \quad 1 \leq i \leq n,$
- $c \in L^\infty(0,T; L^p_{loc}(\Omega)).$

Then let $\mathcal{U}$ be an open subset of Ω. Assume that φ vanishes in the cylinder $\mathcal{U}\times]0,T[$ and that $\varphi \in L^2(0,T; H^2_{loc}(\Omega))$; then φ vanishes in all $\Omega\times]0,T[$. $\square$

Using this property we first prove the following theorem :

Theorem 2 : *Suppose that $\omega \cap \mathcal{O} \neq \emptyset$. The set*

$$\Big\{(\eta(\cdot,0), \partial\eta/\partial\nu) \,|\, v \text{ spans } L^2(\omega\times]0,T[)\Big\}$$

is dense in $L^2(\Omega)\times L^2(\Sigma)$.

Proof : Both systems (15) and (16) admit a unique solution in $H^{2,1}(Q)\cap C([0,T]; H^1_0(\Omega))$. In order to make the proof, we are going to apply the Hahn-Banach theorem. Consider two functions $(\varphi^0, k) \in L^2(\Omega)\times L^2(\Sigma)$ such that

$$\forall v \in L^2(\omega\times]0,T[), \quad \int_\Omega \varphi^0(x)\,\eta(x,0)\,dx - \int_\Sigma k(\sigma,t)\,\frac{\partial\eta}{\partial\nu}(\sigma,t)\,d\sigma\,dt = 0 \quad (18)$$

Once again a transposition technique will be used : let φ and ψ be the solutions of :

$$\left|\begin{array}{rcll} \varphi' - \Delta\varphi + b(x,t)\varphi &=& 0 & \text{in } Q, \\ \varphi &=& k & \text{on } \Sigma, \\ \varphi(\cdot,0) &=& \varphi^0 & \text{in } \Omega, \end{array}\right. \quad (19)$$

$$
\left|\begin{array}{rcll}
-\psi' - \Delta\psi + a(x,t)\psi &=& \varphi\chi_{\mathcal{O}} & \text{in } Q,\\
\psi &=& 0 & \text{on } \Sigma,\\
\psi(\cdot,T) &=& 0 & \text{in } \Omega,
\end{array}\right. \tag{20}
$$

using (15), (16), (19) and (20) we have

$$
\begin{aligned}
\int_{\omega\times]0,T[} v\,\psi\,dx\,dt &= \int_Q (-\psi' - \Delta\psi + a\psi)\,\zeta\,dx\,dt = \int_{\mathcal{O}\times]0,T[} \varphi\,\zeta\,dx\,dt\\
&= \int_Q \varphi\,(-\eta' - \Delta\eta + b\eta)\,dx\,dt\\
&= \int_Q (\varphi' - \Delta\varphi + b\varphi)\,\eta\,dx\,dt + \int_\Omega \varphi^0(x)\,\eta(x,0)\,dx\\
&- \int_\Sigma \frac{\partial\eta}{\partial\nu}(\sigma,t)\,k(\sigma,t)\,d\sigma\,dt\\
&= 0 \qquad \text{from (18) and (19),}
\end{aligned}
$$

so that $\psi = 0$ a.e. in $\omega\times]0,T[$ and so is φ in $(\omega\cap\mathcal{O})\times]0,T[$ (which is assumed to be a non empty open subset of Q). To end the proof we have to show that it implies that $\varphi^0 = 0$ and $k = 0$ which will need three steps :

- prove that $\varphi \in L^2(\delta,T;H^2_{loc}(\Omega))$ for any $\delta > 0$ in order to apply J.C. Saut's and B. Scheurer's uniqueness property to conclude that $\varphi \equiv 0$ in Q,
- deduce that $\varphi^0 = 0$ by a continuity argument,
- deduce that $k = 0$ by a density argument.

Step 1 : We have $\varphi \in H^{\frac{1}{2},\frac{1}{4}}(Q) = L^2(0,T;H^{\frac{1}{2}}(\Omega)) \cap H^{\frac{1}{4}}(0,T;L^2(\Omega))$ but that regularity is not enough for the uniqueness result proved in [15] to be valid. First let us rewrite φ as

$$
\varphi = \varphi_1 + \varphi_2 + \varphi_3,
$$

where φ_1, φ_2 and φ_3 solve :

$$
\left|\begin{array}{rcll}
\varphi_1' - \Delta\varphi_1 &=& -b\,\varphi & \text{in } Q,\\
\varphi_1 &=& 0 & \text{on } \Sigma,\\
\varphi_1(\cdot,0) &=& 0 & \text{in } \Omega,
\end{array}\right.
$$

$$\left|\begin{array}{rcll}\varphi_2' - \Delta\varphi_2 & = & 0 & \text{in } Q,\\ \varphi_2 & = & 0 & \text{on } \Sigma,\\ \varphi_2(\cdot,0) & = & \varphi^0 & \text{in } \Omega,\end{array}\right.$$

$$\left|\begin{array}{rcll}\varphi_3' - \Delta\varphi_3 & = & 0 & \text{in } Q,\\ \varphi_3 & = & k & \text{on } \Sigma,\\ \varphi_3(\cdot,0) & = & 0 & \text{in } \Omega,\end{array}\right. \tag{21}$$

Since $b\varphi \in L^2(Q)$ we have $\varphi_1 \in H^{2,1}(Q)$. Moreover it is known that $\varphi_2 \in L^2(\delta, T; H^2(\Omega) \cap H_0^1(\Omega))$ for any $\delta > 0$. One can now prove that $\varphi_3 \in L^2(0,T; H^2_{loc}(\Omega))$: let us first estimate $\nabla\varphi_3$ to prove that $\varphi_3 \in L^2(0,T; H^2_{loc}(\Omega))$. For that purpose we build a suitable positive function $\rho \in W^{1,\infty}(\Omega)$ with support $\mathcal{U}$ which will be any open subset of Ω that does not intersect the boundary. In the sequel $\mathcal{U}'$ and $\mathcal{U}''$ are two open subsets such that $\mathcal{U}'' \subset \mathcal{U}' \subset \mathcal{U}$ and the function ρ is defined as follows :

$$\rho(x) = \begin{cases} 1 & \text{if } x \in \mathcal{U}'' \\ 0 & \text{if } x \in \Omega - \mathcal{U} \\ d(x, \partial\mathcal{U})^2 & \text{if } x \in \mathcal{U} - \mathcal{U}' \\ C^\infty \text{ junction} & \text{between } \mathcal{U}'' \text{ and } \mathcal{U}' \end{cases}, \tag{22}$$

such that we have :

$$\frac{|\nabla\rho|}{\sqrt{\rho}} \quad \in \ L^\infty(\mathcal{U}). \tag{23}$$

Multipliying (21) by $\rho(x)\varphi_3(x,t)$ and integrating on $\mathcal{U}\times]0,t_0[$, $t_0 > 0$ we get

$$\frac{1}{2}\int_{\mathcal{U}} \rho\,|\varphi_3(t_0)|^2\,dx + \int_{\mathcal{U}\times]0,t_0[} |\nabla\varphi_3|^2\rho\,dx\,dt + \int_{\mathcal{U}\times]0,t_0[} \nabla\varphi_3\cdot\nabla\rho\;\varphi_3\,dx\,dt = 0,$$

and the following inequalities come straightforward

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{U}} \rho\,|\varphi_3(t_0)|^2\,dx + \int_{\mathcal{U}\times]0,t_0[} |\nabla\varphi_3|^2\rho\,dx\,dt \\
&\leq \int_{\mathcal{U}\times]0,t_0[} |\nabla\varphi_3|\,|\nabla\rho|\,|\varphi_3|\,dx\,dt \\
&\leq \int_{\mathcal{U}\times]0,t_0[} \sqrt{\rho}|\nabla\varphi_3|\,\frac{|\nabla\rho|}{\sqrt{\rho}}\,|\varphi_3|\,dx\,dt \\
&\leq \frac{1}{2}\int_{\mathcal{U}\times]0,t_0[} \rho|\nabla\varphi_3|^2\,dx\,dt + \frac{1}{2}\int_{\mathcal{U}\times]0,t_0[} |\varphi_3|^2\,\frac{|\nabla\rho|^2}{\rho}\,dx\,dt,
\end{aligned}$$

so that we finally have

$$\int_{\mathcal{U}} \rho\, |\varphi_3(t_0)|^2\, dx + \int_{\mathcal{U}\times]0,t_0[} |\nabla \varphi_3|^2 \rho\, dx\, dt$$
$$\leq \int_{\mathcal{U}\times]0,t_0[} |\varphi_3(t_0)|^2\, \frac{|\nabla \rho|^2}{\rho}\, dx\, dt < +\infty,$$

since $\varphi_3 \in L^2(Q)$ and ρ statisfies (23) hence :

$$\varphi_3 \in L^2(0,T; H^1_{loc}(\Omega)). \tag{24}$$

Now for any function $\mu \in \mathcal{D}(\Omega)$, it is indeed simple to check that the function $\vartheta = \mu\varphi_3$ satisfies

$$\left|\begin{array}{rcll} \vartheta' - \Delta\vartheta &=& 2\nabla\mu\nabla\varphi_3 - \Delta\mu\varphi_3 & \text{in } Q,\\ \vartheta &=& 0 & \text{on } \Sigma,\\ \vartheta(\cdot,0) &=& 0 & \text{in } \Omega, \end{array}\right.$$

and then belongs to $L^2(0,T;H^2(\Omega))$. We can finally conclude that

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 \in L^2(\delta, T; H^2_{loc}(\Omega)) \quad \text{for any } \delta > 0. \tag{25}$$

Applying J.C. Saut's and B. Scheurer's uniqueness theorem one has $\varphi \equiv 0$ in Q. To obtain the density result we need $\varphi^0 = 0$ and $k = 0$.

Step 2 : As the function φ belongs to $L^2(0,T;L^2(\Omega))$ and its time derivative φ' belongs to $L^2(0,T;H^{-2}(\Omega))$ we have :

$$\varphi \in C([0,T]; H^{-1}(\Omega)),$$

and this continuity, combined with (25), allows to say that :

$$\varphi^0 = 0 \quad \text{a.e. in } \Omega. \tag{26}$$

Step 3 : Let $f \in L^2(Q)$ and let $u(x,t)$ be the solution of :

$$\left|\begin{array}{rcll} -u' - \Delta u + b(x,t)u &=& f & \text{in } Q,\\ u &=& 0 & \text{on } \Sigma,\\ u(\cdot,T) &=& 0 & \text{in } \Omega. \end{array}\right. \tag{27}$$

Multiplying (19) by $u(x,t)$ and integrating on Q one easily gets :

$$\int_Q f(x,t)\varphi(x,t)\,dx\,dt = \int_\Sigma \frac{\partial u(\sigma,t)}{\partial\nu} k(\sigma,t)\,d\sigma\,dt = 0,$$

hence

$$k \in E^\perp, \quad \text{where } E = \left\{ \frac{\partial u}{\partial \nu}\,;\, u \text{ solves (27) when } f \text{ spans } L^2(Q) \right\}, \tag{28}$$

so that k will vanish if and only if E is dense in $L^2(\Sigma)$. In order to prove this last point, we introduce the following subset F :

$$F = \{g(\sigma,t) = g_0(t)g_1(\sigma)\,;\, g_0 \in C^\infty(0,T), g_0(T) = 0,\, g_1 \in C^\infty(\Gamma)\}, \tag{29}$$

and we show that $F \subset E \subset L^2(\Sigma)$, F being dense in $L^2(\Sigma)$. The first point is indeed obvious : for any function $g_1 \in C^\infty(\Gamma)$ one can build a function $G_1 \in C^\infty(\bar\Omega)$ such that G_1 vanishes on Γ and $\partial G_1/\partial\nu = g_1$. Then the function $w = g_0(t)G_1(x)$ satisfies (27) (with $f = -w' - \Delta w + bw$) and its normal derivative belongs to F. To prove that $\bar F = L^2(\Sigma)$ one can apply Fubini's theorem to see that $F^\perp = \{0\}$. Hence $E^\perp = \{0\}$ and that allows us to conclude that $k = 0$ and end the proof of the theorem.

□

Remark 3 : in this theorem we have proved a useful uniqueness property that can be summed up in the

Proposition 2 *: Let φ be the solution of (19) and let U be an open non empty subset of Ω. Assume that φ vanishes in $U\times]0,T[$. Then*

$$\varphi^0 \equiv 0, \tag{30}$$
$$k \equiv 0, \tag{31}$$

and thus

$$\varphi \equiv 0 \text{ in } \Omega\times]0,T[. \tag{32}$$

□

The previous theorem proves the approximate controllability in $L^2(\Omega) \times L^2(\Sigma)$ for $\{\eta(\cdot,0), \partial\eta/\partial\nu\}$ where ζ and η solve (15)(16), which means that for a desired state $\{\eta^d, g^d\} \in L^2(\Omega) \times L^2(\Sigma)$ and for given values $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, the set of admissible controls

$$\mathcal{U}_{ad}(T) = \mathcal{U}_{ad}(T;\eta^d, g^d, \varepsilon_1, \varepsilon_2) = \left\{ v \in L^2(\omega\times]0,T[) \,\middle|\, (34)(35) \text{ hold} \right\}, \quad (33)$$

with

$$\begin{aligned} |\eta(\cdot,0) - \eta^d| &\leq \varepsilon_1, & (34)\\ \|\frac{\partial\eta}{\partial\nu} - g^d\| &\leq \varepsilon_2. & (35) \end{aligned}$$

is not empty. In order to solve the non linear problem we will apply a fixed point theorem, and for this we need a characterization of one control with respect to the desired state. The most natural idea seems to follow [9][6] and chose the control of minimum norm in $L^2(\omega\times]0,T[)$. In the next theorem we will apply the convex duality theorem of Fenchel-Rockafellar (see [4]). We will then be able to characterize the minima of the dual problem when the desired state lies in a compact subset of $L^2(\Omega) \times L^2(\Sigma)$.

Theorem 3 : *Let $\eta^d \in L^2(\Omega)$, $g^d \in L^2(\Sigma)$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, let φ and ψ be the solutions of*

$$\left| \begin{array}{rcll} \varphi' - \Delta\varphi + b(x,t)\varphi &=& 0 & \textit{in } Q,\\ \varphi &=& k & \textit{on } \Sigma,\\ \varphi(\cdot,0) &=& \varphi^0 & \textit{in } \Omega, \end{array} \right. \quad (36)$$

$$\left| \begin{array}{rcll} -\psi' - \Delta\psi + a(x,t)\psi &=& \varphi\chi_{\mathcal{O}} & \textit{in } Q,\\ \psi &=& 0 & \textit{on } \Sigma,\\ \psi(\cdot,T) &=& 0 & \textit{in } \Omega, \end{array} \right. \quad (37)$$

and finally let the following cost function defined on $L^2(\Omega) \times L^2(\Sigma)$:

$$\begin{aligned} J(\varphi^0,k) &= \frac{1}{2}\int_{\omega\times]0,T[} \psi^2(x,t)\,dx\,dt + \varepsilon_1|\varphi^0| + \varepsilon_2\|k\| \\ &\quad - \int_\Omega \eta^d(x)\varphi^0(x)\,dx + \int_\Sigma g^d(\sigma,t)k(\sigma,t)\,d\sigma\,dt. \end{aligned} \quad (38)$$

Then

(i) $\displaystyle\min_{v\in\mathcal{U}_{ad}} \frac{1}{2}\int_{\omega\times]0,T[} v^2(x,t)\,dx\,dt = -\min_{(\varphi^0,k)\in L^2(\Omega)\times L^2(\Sigma)} J(\varphi^0,k),$

(ii) the control $\bar v$ *of minimum* L^2*-norm is given by* $\bar v = \bar\psi|_{\omega\times]0,T[}$ *where* $\bar\varphi$ *and* $\bar\psi$ *are the solutions of (36)(37) associated to the minimum* $(\bar\varphi^0,\bar k)$ *of* J*,*

(iii) if a *and* b *describe bounded sets of* $L^\infty(Q)$*, if* η^d *lies in a compact set of* $L^2(\Omega)$ *and if* g^d *lies in a compact set of* $L^2(\Sigma)$ *then the controls obtained above describe a bounded set of* $L^2(\omega\times]0,T[)$*.*

Proof : For given $\eta^d, g^d, \varepsilon_1, \varepsilon_2$ we previously have proved that $\mathcal{U}_{ad}(T)$ is a non-empty set for any $T>0$. As it is obviously a closed convex subset of $L^2(\omega\times]0,T[)$, the problem

$$\min_{v\in\mathcal{U}_{ad}(T)} \quad \frac{1}{2}\|v\|^2_{L^2(\omega\times]0,T[)}, \tag{39}$$

admits a unique solution that we will call $\bar v$.

(i) Let L be the following continuous linear operator :

$$L\ :\left\{\begin{array}{rcl} L^2(\omega\times]0,T[) & \longrightarrow & L^2(\Omega)\times L^2(\Sigma) \\ v & \longmapsto & \{\eta(\cdot,0)\,,\,-\dfrac{\partial\eta}{\partial\nu}\} \end{array}\right. \quad , \tag{40}$$

where ζ and η are the solutions of (15)(16). Let F and G be the functions respectively defined on $L^2(\omega\times]0,T[)$ and $L^2(\Omega)\times L^2(\Sigma)$ by :

$$F(v) = \frac{1}{2}\int_{\omega\times]0,T[} v^2(x,t)\,dx\,dt, \tag{41}$$

$$G(\varphi^0,k) = \left\{\begin{array}{cl} 0 & \text{if } |\varphi^0-\eta^d|\le\varepsilon_1 \text{ and } \|k+g^d\|\le\varepsilon_2, \\ +\infty & \text{otherwise,} \end{array}\right. \tag{42}$$

the minimization problem (39) is then equivalent to the unconstrained problem

$$\min_{v\in L^2(\omega\times]0,T[)} F(v)+G(Lv). \tag{43}$$

In order to apply the Fenchel-Rockafellar theory we have to determine L^*, F^* and G^*. For $\{\varphi^0,k\}\in L^2(\Omega)\times L^2(\Sigma)$ let (φ,ψ) be the solutions of (36)(37). Multiplying (15) by ψ, (16) by φ, integrating on Q and adding the two expressions one easily gets :

$$\int_{\omega\times]0,T[} v(x,t)\psi(x,t)\,dx\,dt = \int_\Omega \varphi^0(x)\eta(x,0)\,dx - \int_\Sigma \frac{\partial\eta(\sigma,t)}{\partial\nu}k(\sigma,t)d\sigma dt,$$

i.e.

$$(v\,,\,\psi)_{L^2(\omega\times]0,T[)} = \Big(Lv\,,\,\{\varphi^0,k\}\Big)_{L^2(\Omega)\times L^2(\Sigma)} \quad ,$$

so that the adjoint operator of L is given by

$$L^*\{\varphi^0,k\} = \psi|_{\omega\times]0,T[} \quad . \tag{44}$$

Furthermore it is well known that the conjugate function of F is given by

$$F^*(v) = F(v). \tag{45}$$

Eventually for any $\{\varphi^0,k\} \in L^2(\Omega)\times L^2(\Sigma)$ one has (with $\bar{B}$ denoting the closed ball either in $L^2(\Omega)$ or in $L^2(\Sigma)$) :

$$\begin{aligned} G^*(\varphi^0,k) &= \sup_{\{\hat\varphi^0,\widehat{k}\}\in L^2(\Omega)\times L^2(\Sigma)}\Big[(\varphi^0,\hat\varphi^0)_{L^2(\Omega)} + (k,\widehat{k})_{L^2(\Sigma)} \\ &\quad -G(\hat\varphi^0,\widehat{k})\Big] \\ &= \sup_{\{\hat\varphi^0,\widehat{k}\}\in(\varepsilon_1\bar{B})\times(\varepsilon_2\bar{B})}\Big[(\varphi^0,\hat\varphi^0+\eta^d)_{L^2(\Omega)} \\ &\quad +(k,\widehat{k}-g^d)_{L^2(\Sigma)}\Big] \\ &= \varepsilon_1|\varphi^0| + \varepsilon_2\|k\| + (\varphi^0,\eta^d)_{L^2(\Omega)} - (k,g^d)_{L^2(\Sigma)} \quad , \end{aligned} \tag{46}$$

hence the dual problem of (43) is

$$\begin{aligned} &-\inf_{\{\varphi^0,k\}\in L^2(\Omega)\times L^2(\Sigma)}\Big(F^*(L^*\{\varphi^0,k\}) + G^*(-\varphi^0,-k)\Big) \\ = \;&-\inf_{\{\varphi^0,k\}\in L^2(\Omega)\times L^2(\Sigma)}\frac{1}{2}\int_{\omega\times]0,T[}\psi^2(x,t)\,dx\,dt + \varepsilon_1|\varphi^0| \\ &+\varepsilon_2\|k\| - \int_\Omega \eta^d(x)\varphi^0(x)\,dx + \int_\Sigma g^d(\sigma,t)k(\sigma,t)\,d\sigma\,dt, \end{aligned} \tag{47}$$

from (44)(45)(46) and the proof of the first point is done.

Remark 4 : Since $J(0,0)=0$ it is obvious that

$$J(\bar\varphi^0,\bar{k}) \le 0. \tag{48}$$

□

(ii) The second part of the theorem is straightforward using the relationship between primal and dual variables in the two previous problems. Let $\bar{v}$ be the optimum of (43) and $(\bar{\varphi}^0, \bar{k})$ be the optimum of J. The cost function $F(v) + G(Lv)$ is convex l.s.c. and never takes the value $-\infty$ hence we are allowed to write (see [4])

$$L^*\{\bar{\varphi}^0, \bar{k}\} = \frac{\partial F(\bar{v})}{\partial v},$$

which gives immediatly

$$\bar{v} = \bar{\psi}|_{\omega\times]0,T[} \quad . \tag{49}$$

(iii) To prove the third point we argue by contradiction ; let us suppose the existence of sequences (a_n), (b_n), (η_n^d), (g_n^d) such that

$$\begin{array}{rcll} a_n & \rightharpoonup & a & \text{weak-* in } L^\infty(Q), \\ b_n & \rightharpoonup & b & \text{weak-* in } L^\infty(Q), \\ \eta_n^d & \to & \eta^d & \text{strongly in } L^2(\Omega), \\ g_n^d & \to & g_n^d & \text{strongly in } L^2(\Sigma), \\ \lambda_n = |\bar{\varphi}_n^0| + \|\bar{k}_n\| & \to & +\infty, & \end{array} \tag{50}$$

where $(\bar{\varphi}_n^0, \bar{k}_n)$ are the corresponding minima of $J(\bar{\varphi}_n^0, \bar{k}_n; a_n, b_n, \eta_n^d, g_n)$. As remarked above, we have $J(\bar{\varphi}_n^0, \bar{k}_n) \leq 0$. Let us prove that, with the assumptions (50), we have in fact (after extraction of a subsequence)

$$J(\bar{\varphi}_n^0, \bar{k}_n) \longrightarrow +\infty \text{ when } n \to +\infty. \tag{51}$$

For a given integer n the solutions of (36)(37) associated to $\bar{\varphi}_n^0$,$\bar{k}_n$,a_n, b_n, η_n^d, g_n^d will be denoted by (φ_n, ψ_n). Let $\tilde{\varphi}_n^0 = \bar{\varphi}_n^0/\lambda_n$ and $\tilde{k}_n = \bar{k}_n/\lambda_n$; these functions are bounded in $L^2(\Omega)$ and $L^2(\Sigma)$ and we can extract subsequences (still called $\tilde{\varphi}_n^0$ and $\tilde{k}_n$) weakly converging towards $\tilde{\varphi}^0 \in L^2(\Omega)$ and $\tilde{k} \in L^2(\Sigma)$ respectively. Furthermore $(\tilde{\varphi}_n, \tilde{\psi}_n) = (\frac{\varphi_n}{\lambda_n}, \frac{\varphi_n}{\lambda_n})$ solve (36)(37) with initial and boundary data $\tilde{\varphi}_n^0$ and $\tilde{k}_n$, and they are bounded respectively in $H^{\frac{1}{2},\frac{1}{4}}(Q)$ and in $H^{1,2}(Q)$. As those spaces are compactly imbedded in $L^2(Q)$ we have

$$\tilde{\varphi}_n \longrightarrow \tilde{\varphi} \in L^2(Q) \text{ strongly}, \tag{52}$$

$$\tilde{\psi}_n \longrightarrow \tilde{\psi} \in L^2(Q) \text{ strongly}, \tag{53}$$

and then $\widetilde{\varphi}$ and $\widetilde{\psi}$ are the solutions of (36)(37) associated to $\widetilde{\varphi}^0$,$\widetilde{k}$,a, b, η^d, g^d.

Now if $J_n = J(\bar{\varphi}_n^0, \bar{k}_n; a_n, b_n, \eta_n^d, g_n^d)$ we have

$$J_n = \frac{\lambda_n^2}{2} \int_{\omega\times]0,T[} \widetilde{\psi}_n(x,t)^2\, dx\, dt + \lambda_n \alpha_n,$$

where

$$\alpha_n = \Big(\varepsilon_1 |\widetilde{\varphi}_n^0| + \varepsilon_2 \|\widetilde{k}_n\| - (\eta_n^d, \widetilde{\varphi}_n^0)_{L^2(\Omega)} + (g_n^d, \widetilde{k}_n)_{L^2(\Sigma)}\Big).$$

Dividing J_n by λ_n^2 one gets

$$\frac{1}{2} \int_{\omega\times]0,T[} \widetilde{\psi}_n(x,t)^2\, dx\, dt + \frac{\alpha_n}{\lambda_n} \leq 0,$$

but α_n is a bounded real sequence and $\lambda_n \to 0$ hence one has :

$$\int_{\omega\times]0,T[} \widetilde{\psi}_n(x,t)^2\, dx\, dt \longrightarrow 0 \quad \text{when } n \to +\infty.$$

Thus (53) implies that $\widetilde{\psi}$ vanishes in $\omega\times]0,T[$ and $\widetilde{\varphi} = -\widetilde{\psi}' - \Delta\widetilde{\psi} + a\widetilde{\psi}$ vanishes in $(\omega \cap \mathcal{O})\times]0,T[$. Then proposition 2 implies that

$$\begin{cases} \widetilde{\varphi}^0 &= 0, \\ \widetilde{k} &= 0, \end{cases} \tag{54}$$

i.e.

$$\widetilde{\varphi}_n^0 \rightharpoonup 0 \quad \text{and} \quad \widetilde{k}_n \rightharpoonup 0, \tag{55}$$

weakly in $L^2(\Omega)$ and $L^2(\Sigma)$ respectively.

Now we have

$$\begin{aligned} J_n &\geq \lambda_n \Big(\varepsilon_1 |\widetilde{\varphi}_n^0| + \varepsilon_2 \|\widetilde{k}_n\| - (\eta_n^d, \widetilde{\varphi}_n^0)_{L^2(\Omega)} + (g_n^d, \widetilde{k}_n)_{L^2(\Sigma)}\Big) \\ &\geq \lambda_n \Big(\min(\varepsilon_1, \varepsilon_2) \left(|\widetilde{\varphi}_n^0| + \|\widetilde{k}_n\|\right) - (\eta_n^d, \widetilde{\varphi}_n^0)_{L^2(\Omega)} + (g_n^d, \widetilde{k}_n)_{L^2(\Sigma)}\Big), \end{aligned}$$

and since $|\widetilde{\varphi}_n^0| + \|\widetilde{k}_n\| = 1$ from the definition of λ_n we have

$$\frac{J_n}{\lambda_n} \geq \min(\varepsilon_1, \varepsilon_2) - (\eta_n^d, \widetilde{\varphi}_n^0)_{L^2(\Omega)} + (g_n^d, \widetilde{k}_n)_{L^2(\Sigma)}, \tag{56}$$

so that using (50)(55) and letting n go towards infinity one proves (51). The optimal couple $(\bar{\varphi}^0, \bar{k})$ thus takes its values in a bounded subset of $L^2(\Omega) \times L^2(\Sigma)$, and the control function $\bar{v}$ takes its values in a bounded subset of $L^2(\omega \times]0, T[)$.

$\square$

Remark 5 : Using the compact imbedding of the set of the solutions of (20) in $L^2(Q)$ one gets that the control $\bar{v}$ lies in a compact subset of $L^2(\omega \times]0, T[)$.

$\square$

Remark 6 : The basic argument used to prove the density of the set $\{\{\eta(\cdot, 0), \partial\eta/\partial\nu\} \mid v \text{ spans } L^2(\omega \times]0, T[)\}$ in $L^2(\Omega) \times L^2(\Sigma)$ is the unique continuation property given by proposition 2. That remains the main argument used to characterize the minima of the dual cost function.

$\square$

4 The Nonlinear Case

In order to solve the nonlinear problem let us consider the following function :

$$F(s) = \frac{f(s) - f(0)}{s}, \tag{57}$$

and we now can prove theorem 1.

Proof : Since f is C^1 and Lipschitz, $F(\cdot)$ and $f'(\cdot)$ are both continuous and bounded functions. Let $z \in L^2(Q)$ be a given function and $\zeta, \eta, \zeta_0, \eta_0$ be the solutions of the following linear systems :

$$\left| \begin{array}{rcll} \zeta' - \Delta\zeta + F(z)\zeta & = & v\chi_\omega & \text{dans } Q, \\ \zeta & = & 0 & \text{sur } \Sigma, \\ \zeta(\cdot, 0) & = & 0 & \text{dans } \Omega, \end{array} \right. \tag{58}$$

$$\left| \begin{array}{rcll} -\eta' - \Delta\eta + f'(z)\eta & = & \zeta\chi_{\mathcal{O}} & \text{dans } Q, \\ \eta & = & 0 & \text{sur } \Sigma, \\ \eta(\cdot, T) & = & 0 & \text{dans } \Omega, \end{array} \right. \tag{59}$$

$$\left|\begin{array}{rcll}\zeta_0' - \Delta\zeta_0 + F(z)\zeta_0 &=& -f(0) + \xi & \text{dans } Q,\\ \zeta_0 &=& g & \text{sur } \Sigma,\\ \zeta_0(\cdot,0) &=& y^0 & \text{dans } \Omega,\end{array}\right. \tag{60}$$

$$\left|\begin{array}{rcll}-\eta_0' - \Delta\eta_0 + f'(z)\eta_0 &=& \zeta_0\chi_{\mathcal{O}} & \text{dans } Q,\\ \eta_0 &=& 0 & \text{sur } \Sigma,\\ \eta_0(\cdot,T) &=& 0 & \text{dans } \Omega,\end{array}\right. \tag{61}$$

Due to the theorem 2 one can affirm the existence of a control function $v \in L^2(\omega\times]0,T[)$ such that

$$\left\{\begin{array}{rcl}|\eta(\cdot,0) + \eta_0(\cdot,0)| &\leq& \varepsilon_1,\\ \|\dfrac{\partial\eta}{\partial\nu} + \dfrac{\partial\eta_0}{\partial\nu}\| &\leq& \varepsilon_2,\end{array}\right. \tag{62}$$

for any given $\varepsilon_1, \varepsilon_2 > 0$. Any suitable control can be chosen, in particular the control of minimum norm in $L^2(\omega\times]0,T[)$.
Let us then consider

$$\left\{\begin{array}{rcl}y(x,t;z) &=& \zeta(x,t;z,\bar{v}(z)) + \zeta_0(x,t;z,\bar{v}(z)),\\ q(x,t;z) &=& \eta(x,t;z,\bar{v}(z)) + \eta_0(x,t;z,\bar{v}(z)),\end{array}\right. \tag{63}$$

where $\bar{v}$ is the optimal control characterized by theorem 3 such that (62) holds, and let Λ be the following nonlinear operator :

$$\Lambda \ : \left\{\begin{array}{rcl}L^2(Q) &\longrightarrow& L^2(Q)\\ z &\longmapsto& y(\cdot,\cdot;z)\end{array}\right. \tag{64}$$

It is easy to check that if Λ admits a fixed point $y(\cdot,\cdot;y)$ the functions $y(\cdot,\cdot;y)$ and $q(\cdot,\cdot;y)$ solve (7)(8) and the approximate sensitivity conditions (10) are satisfied.

Since $F(z)$ and $f'(z)$ are bounded in $L^\infty(Q)$ when z spans $L^2(Q)$ and since the solutions of (58)(59) depend continuously on their data the operator Λ is continuous. Let us now show that its range is relatively compact in $L^2(Q)$. When z spans $L^2(Q)$ the function η_0 spans a bounded subset of $H^{2,1}(Q)$. Moreover one has

$$\left.\begin{array}{r}\eta_0 \in L^2(0,T;H^2(\Omega))\\ \eta_0' \in L^2(0,T;L^2(\Omega))\end{array}\right\} \Longrightarrow \eta_0(\cdot,t) \in C([0,T];H^1(\Omega)),$$

hence $\eta_0(\cdot, 0)$ spans a bounded subset of $H_0^1(\Omega)$, relatively compact in $L^2(\Omega)$. In a same manner η_0 spans a bounded subset of $H^{2,1}(Q)$, relatively compact in $L^2(0, T; H^{2-\alpha}(\Omega)$ for any $\alpha > 0$. Due to the continuity of the trace $\partial\eta_0/\partial\nu$ spans a relatively compact set of $L^2(0, T; H^{\frac{1}{2}}(\Gamma))$, then relatively compact in $L^2(\Sigma)$. Thus there exist compact subsets $\mathcal{K}_1 \subset L^2(\Omega)$ and $\mathcal{K}_2 \subset L^2(\Sigma)$ such that all the possible desired states $\{\eta_0(\cdot, 0), \partial\eta_0/\partial\nu\}$ belong to $\mathcal{K}_1 \times \mathcal{K}_2$.

When z spans $L^2(Q)$, as $\{\eta_0(\cdot, 0), \partial\eta_0/\partial\nu\}$ lies in a compact subset of $L^2(\Omega)\times L^2(\Sigma)$, the control $\bar{v}$ obtained from the minimization of the cost function $J(\cdot, \cdot; F(z), f'(z), -\eta_0(\cdot, 0), -\partial\eta_0/\partial\nu)$ spans a bounded set of $L^2(\omega\times]0, T[)$. Then $\zeta = \zeta(z)$ lies in a compact subset of $L^2(Q)$, which proves that the range of Λ is compact in $L^2(Q)$.

Schauder's theorem ensures the existence of at least one fixed point for Λ and the problem of approximately insensitizing control is solved.

□

5 Comments and Conclusions

Remark 7 : The result remains valid when $-\Delta$ is replaced by an elliptic operator of the second order with coefficients satisfying the assumptions of J.C. Saut's and B. Scheurer's theorem.

□

Remark 8 : What has been actually proved is the density of the set $\{(q(\cdot, 0), \partial q/\partial\nu)\,;\, v \text{ spans } L^2(\omega\times]0, T[)\}$ in $L^2(\Omega) \times L^2(\Sigma)$ with $\bar{y}$ and q solving (7)(8).

□

Remark 9 : It is important to see that the insensitizing, or approximately insensitizing conditions, on the functionnal Φ are easy to rewrite into a more usual control problem because Φ is actually the square of an hilbertian norm of a function of the state variable y. If Φ has no longer this form it seems very hard to treat the problem by usual control methods.

□

Remark 10 : From the algorithmic point of view our method is not constructive. As a matter of fact, Schauder's theorem does not give any computational way to get the control (moreover the control is not necessarily unique). Numerical computations could then have to be done by minimizing the "classical" optimal control cost function :

$$\tilde{J}(v) = \frac{1}{2}\int_{\Omega} q(x,0)^2\,dx + \frac{1}{2}\int_{\Sigma} \frac{\partial q(\sigma,t)}{\partial \nu}^2\,d\sigma\,dt + \frac{\varepsilon}{2}\int_{\omega\times]0,T[} v(x,t)^2\,dx\,dt$$

where $\bar{y}$ and q solve (7)(8). For such a minimization problem the optimality system is straightforward to compute (see [8] for example) and provides the programmer with an algorithm.

□

Quasi Bang-Bang Control : In [6] and [7] it is shown that one can obtain the approximate controllability for the heat equation with controls belonging to $L^\infty(\omega\times]0,T[)$. Moreover this controls are shown to be of quasi bang-bang type. It is possible to do the same thing for the approximately insensitizing controls; therefore when solving the linear problem one has to study directly a slight variation of the dual minimization problem :

$$\begin{aligned} \min_{L^2(\Omega)\times L^2(\Sigma)} J_\infty(\varphi^0,k) &= \frac{1}{2}\Big(\int_{\omega\times]0,T[} |\psi(x,t)|\,dx\,dt\Big)^2 + \varepsilon_1|\varphi^0| + \varepsilon_2\|k\| \\ &\quad - \int_{\Omega}\eta^d(x)\varphi^0(x)\,dx + \int_{\Sigma} g^d(\sigma,t)k(\sigma,t)\,d\sigma\,dt. \end{aligned} \tag{65}$$

where ψ is given as in theorem 3.

But this new problem is now regarded as a primal problem, and at the optimal point (φ^{0*},k^*) of $J_\infty(\cdot,\cdot)$ the control given by $v_\infty = \psi^*$ on $\omega\times]0,T[$ satisfies the approximate controllability conditions for the couple (ζ,η) (where these functions are given as in theorem 2). The dual problem of this new primal problem can then shown to be :

$$-\min_{v\in U_{ad}} \frac{1}{2}\|v\|_{L^\infty(\omega\times]0,T[} \tag{66}$$

hence v_∞ is a control of minimum L^∞ norm. The study of (65) is slightly more difficult than in theorem 3 because of the non-differentiability of the cost function. One can then compute the sub-differential set of J_∞ to obtain optimality conditions. Then the conclusion can no longer be done by Schauder's theorem, and Kakutani's fixed point theorem is used. One can also study a regularized form of the cost function, say $J_{\infty,\alpha}(\varphi^0, k)$, where α is some small parameter and then pass to the limit. Details will be given in [1].

Other Observations And Controls : It has also to be pointed out that the method used is quite versatile and can be applied to other insensitizing problems, provided a uniqueness property similar to J.C. Saut's and B. Sheurer's theorem is available when solving the linear case. As we have seen before, a uniqueness result is needed not only to prove the approximate controllability of the couple of linear systems, but also to characterize the controls of minimum $L^2(\omega\times]0,T[)$ norm. One can for instance think of a boundary control with boundary observation, initial and boundary perturbations. That can be summed up in the following theorem :

Theorem 4 *: Let $\Omega \subset I\!R^n$ be a bounded open set with C^2 boundary Γ. Let Γ_0 and Γ_1 be two subsets of Γ. Let $y = y(x,t;v,\tau_0,\tau_1,\hat{y}^0,\widehat{g})$ be the solution of :*

$$\left|\begin{array}{rcll} y' - \Delta y + f(y) &=& \xi & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} &=& \left|\begin{array}{l} v \\ g + \tau_1 \widehat{g} \end{array}\right. & \begin{array}{l} \text{on } \Gamma_0\times]0,T[, \\ \text{on } (\Gamma\setminus\Gamma_0)\times]0,T[, \end{array} \\ y(\cdot,0) &=& y^0 + \tau\hat{y}^0 & \text{in } \Omega, \end{array}\right. \tag{67}$$

with assumptions on $\xi, v, g, \widehat{g}, y^0, \hat{y}^0, \tau_0, \tau_1$ analogous to those made before. Let Φ be the following functionnal :

$$\Phi(y) = \frac{1}{2}\int_{\Gamma_1\times]0,T[} y^2(\sigma,t)\,d\sigma\,dt. \tag{68}$$

If $\Gamma_0 \cap \Gamma_1 \neq \emptyset$ and if f is a C^1 and Lipschitz function then for any $\varepsilon_1, \varepsilon_2 > 0$ there exists a control $v_{\varepsilon_1,\varepsilon_2} \in L^2(\Gamma_0\times]0,T[)$ which $\varepsilon_1,\varepsilon_2$-insensitizes Φ. As before one can also find a control in $L^\infty(\Gamma_0\times]0,T[)$.

The proof of theorem 4 is straightforward following the same program as for theorem 1. What is involved here is Cauchy's uniqueness theorem : let $a \in L^\infty(Q)$ and φ be a function such that

$$\left|\begin{array}{rcll} \varphi' - \Delta\varphi + a(x,t)\varphi &=& 0 & \text{in } Q, \\ \varphi &=& 0 & \text{on } \Gamma^* \times]0,T[, \\ \dfrac{\partial \varphi}{\partial \nu} &=& 0 & \text{on } \Gamma^* \times]0,T[, \end{array}\right.$$

where Γ^* is a non empty open subset of Γ. Then φ vanishes identically in Q. □

Other Assumptions : Looking at the expressions (13)(14) could lead to think that if y is "small" enough in $\mathcal{O}\times]0,T[$, then the approximate insensitivity conditions are satisfied. One could then try to prove the density of the set described by the trajectories $y|_{\mathcal{O}\times]0,T[}$ when v spans $L^2(\omega\times]0,T[)$ in $L^2(\mathcal{O}\times]0,T[)$ and apply the same program as before. However when applying Hahn-Banach's theorem for the linear case it is necessary to obtain the following uniqueness property : let $h \in L^2(\mathcal{O}\times]0,T[)$ and let θ be the solution of :

$$\left|\begin{array}{rcll} -\theta' - \Delta\theta + a(x,t)\theta &=& h\chi_{\mathcal{O}} & \text{in } Q, \\ \theta &=& 0 & \text{on } \Gamma^* \times]0,T[, \\ \theta(\cdot,T) &=& 0 & \text{in } \Omega, \end{array}\right. \tag{69}$$

where $a \in L^\infty(Q)$. Then if θ vanishes in $\omega\times]0,T[$ the function h should be equal to 0 which is true if and only if $\mathcal{O}$ is strictly included in ω. It is easy to build a counter example if it is not the case. For example a function equal to any element of $\mathcal{D}((\mathcal{O} \setminus (\omega \cap \mathcal{O}))\times]0,T[)$ and equal to 0 everywhere else vanishes in $\omega\times]0,T[$ and does not vanish in all $\mathcal{O}\times]0,T[$. Thus the condition on the observation and control sets is much more restrictive than the one we have previously obtained. □

Remark 11 : The control and observation sets, in internal or boundary control, have to be of non empty intersection for the assumptions of the uniqueness theorems to be valid. Theses assumptions are similar to the "collocated actuators and sensors" that are classicaly met in automatic control.

□

Nominal And Perturbed State : Let us consider here $\bar{y}(x,t)$ as a nominal state and $y(x,t)$ as a perturbed state. On can have the idea of (approximately) controlling the final state $\bar{y}(\cdot,T)$ while keeping the approximate insensitivity of Φ. However one cannot expect to control $\bar{y}(\cdot,T)$ in $L^2(\Omega)$. Actually $\bar{y}$ is defined by a fixed point technique as the sum of the solutions ζ and ζ_0 of (15)(60). Then we have $\bar{y} \in C([0,T];H^{-1}(\Omega))$, hence it is in $H^{-1}(\Omega)$ that we will control $\bar{y}(\cdot,T)$. We then have the

Theorem 5 *: Under the assumptions of theorem 1, for any $y^d \in H^{-1}(\Omega)$ and every $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$ there exists at least a control $v \in L^2(\omega\times]0,T[)$ $(\varepsilon_1,\varepsilon_2)$-insensitizing the functionnal Φ defined by (4) such that :*

$$\|\bar{y}(\cdot,T) - y^d\|_{H^{-1}(\Omega)} \leq \varepsilon_3 \tag{70}$$

The new conditions will obviously be the approximate insensitivity conditions (10) plus the condition (70). Only the proof of the density of the set $\{(\eta(\cdot,0), \partial\eta/\partial\nu), \bar{y}(\cdot,T)\}$ in $L^2(\Omega) \times L^2(\Sigma) \times H^{-1}(\Omega)$ will be detailed. All the rest is quite similar to the proof of theorem 1.

Proposition 3 *: Let ζ and η solve (15)(16), the set described by the triplet $\Big(\eta(\cdot,0), \partial\eta/\partial\nu), \zeta(\cdot,T)\Big)$ when v spans $L^2(\omega\times]0,T[)$ is dense in $L^2(\Omega) \times L^2(\Sigma) \times H^{-1}(\Omega)$.*

Proof : Let us consider three functions $(\varphi^0, k, \varphi^1) \in L^2(\Omega)\times L^2(\Sigma)\times H_0^1(\Omega)$ such that

$$\forall v \in L^2(\omega\times]0,T[), \qquad \begin{array}{l} \displaystyle\int_\Omega \varphi^0(x)\,\eta(x,0)\,dx - \int_\Sigma k(\sigma,t)\,\frac{\partial\eta}{\partial\nu}(\sigma,t)\,d\sigma\,dt \\ +\Big\langle \zeta(\cdot,T)\,,\,\varphi^1\Big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0, \end{array} \tag{71}$$

Let then φ and ψ be the solutions of :

$$\left|\begin{array}{rcll} \varphi' - \Delta\varphi + b(x,t)\varphi &=& 0 & \text{in } Q, \\ \varphi &=& k & \text{on } \Sigma, \\ \varphi(\cdot,0) &=& \varphi^0 & \text{in } \Omega, \end{array}\right. \tag{72}$$

$$\left|\begin{array}{rcll} -\psi' - \Delta\psi + a(x,t)\psi &=& \varphi\chi_{\mathcal{O}} & \text{in } Q, \\ \psi &=& 0 & \text{on } \Sigma, \\ \psi(\cdot,T) &=& \varphi^1 & \text{in } \Omega, \end{array}\right. \tag{73}$$

using (15), (16), (72) and (73) we have

$$\begin{aligned}
\int_{\omega\times]0,T[} v\,\psi\,dx\,dt &= \int_Q (-\psi' - \Delta\psi + a\psi)\,\zeta\,dx\,dt = \int_{\mathcal{O}\times]0,T[} \varphi\,\zeta\,dx\,dt \\
&= \int_Q \varphi\,(-\eta' - \Delta\eta + b\eta)\,dx\,dt \\
&= \int_Q (\varphi' - \Delta\varphi + b\varphi)\,\eta\,dx\,dt + \int_\Omega \varphi^0(x)\,\eta(x,0)\,dx \\
&- \int_\Sigma \frac{\partial\eta}{\partial\nu}(\sigma,t)\,k(\sigma,t)\,d\sigma\,dt + \left\langle \zeta(\cdot,T)\,,\,\varphi^1\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\
&= 0 \qquad \text{from (71) and (72),}
\end{aligned}$$

so that $\psi = 0$ a.e. in $\omega\times]0,T[$ and so is φ in $\omega\cap\mathcal{O}\times]0,T[$. Then $\varphi^0 \equiv 0$ and $k \equiv 0$ from proposition 2, and ψ satisfies :

$$\left|\begin{array}{rcll}
-\psi' - \Delta\psi + a(x,t)\psi &=& 0 & \text{in } Q, \\
\psi &=& 0 & \text{on } \Sigma, \\
\psi(\cdot,T) &=& \varphi^1 & \text{in } \Omega, \\
\psi &=& 0 & \text{a.e. in } \omega\cap\mathcal{O}\times]0,T[,
\end{array}\right.$$

hence $\varphi^1 \equiv 0$ due to J.C. Saut's and B. Scheurer's theorem, and that ends the proof of the proposition.

$\square$

References

[1] O. Bodart. *Sentinelles et contrôles insensibilisants.* PhD thesis, Université de Technologie de Compiègne, 1994.

[2] O. Bodart and C. Fabre. Controle insensibilisant la norme de la solution d'une équation de la chaleur semi-linéaire. *C.R. Acad. Sci., t. 316, Série I*, pages 789–794, 1993.

[3] T. Cazenave and A. Haraux. *Introduction aux problèmes d'évolution semi-linéaires.* Ellipses, coll. S.M.A.I., Paris, 1990.

[4] I. Ekeland and R. Temam. *Analyse convexe et problèmes variationnels.* Dunod Gauthier-Villars, coll. Etudes Mathématiques, Paris, 1974.

[5] C. Fabre, J.P. Puel, and E. Zuazua. Approximate controllability for the semilinear heat equation. *to appear in the Proc. A of the Royal Society of Edimburgh.*

[6] C. Fabre, J.P. Puel, and E. Zuazua. Controlabilité approchée de l'équation de la chaleur semi-linéaire. *C.R. Acad. Sci., t. 315, Série I*, pages 807–812, 1992.

[7] C. Fabre, J.P. Puel, and E. Zuazua. Controlabilité approchée de l'équation de la chaleur linéaire avec des contrôles de norme l^∞ minimale. *C.R. Acad. Sci., t. 316, Série I*, pages 679–684, 1993.

[8] J.L. Lions. *Contrôle optimal des systèmes gouvernés par des équations aux dérivées partielles.* Dunod Gauthier-Villars, coll. Etudes Mathématiques, Paris, 1968.

[9] J.L. Lions. *Contrôlabilité exacte, stabilisation et perturbations des systèmes distribués.* Masson, coll. R.M.A., Paris, 1986.

[10] J.L. Lions. Remarques préliminaires sur le contrôle des systèmes à données incomplètes. In *Actas del Congrecio de Equaciones Differenciales Y Applicaciones (CEDYA)*, pages 43–54, Malaga, 1989. Universidad de Malaga.

[11] J.L. Lions. Exact controllability of distributed systems ; some trends and some remarks. *App. Ind. Math.*, pages 59–84, 1991.

[12] J.L. Lions. Remarks on approximate controllability. *Israel Journal of Mathematics*, 1992.

[13] J.L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications.* Dunod, Paris, 1968.

[14] S. Mizohata. Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques. *Mem. Coll. Sci. Univ. Kyoto, Ser. A31*, 3:219–239, 1958.

[15] J.C. Saut and B. Scheurer. Unique continuation for some evolution equations. *J. Diff. Equations*, 66:118–139, 1987.

[16] E. Zuazua. Exact boundary controllability for the semilinear wave equation. In H. Brézis and J.L. Lions, editors, *Non-linear differential equations and their applications (Séminaire du Collège de France)*, volume 10, pages 357–391. Research Notes in Math., Pitman, 1991.