ENTROPY FLUX-SPLITTINGS FOR HYPERBOLIC CONSERVATION LAWS PART I: GENERAL FRAMEWORK

Gui-Qiang Chen[1] and Philippe G. LeFloch[2]

Abstract A general framework is proposed for the derivation and analysis of flux-splittings and the corresponding flux-splitting schemes for systems of conservation laws endowed with a strictly convex entropy. The approach leads to several new properties of the existing flux-splittings and to a method for the construction of entropy flux-splittings for general situations. A large family of genuine entropy flux-splittings is derived for several significant examples: the scalar conservation laws, the p-system, and the Euler system of isentropic gas dynamics. In particular, for the isentropic Euler system, we obtain a family of splittings that satisfy the entropy inequality associated with the mechanical energy. For this system, it is proved that there exists a unique genuine entropy flux-splitting that satisfies all of the entropy inequalities, which is also the unique diagonalizable splitting. This splitting can be also derived by the so-called kinetic formulation. Simple and useful difference schemes are derived from the flux-splittings for hyperbolic systems. Such entropy flux-splitting schemes are shown to satisfy a discrete cell entropy inequality. For the diagonalizable splitting schemes, the principle of bounded invariant regions applies and provides an a priori L^∞ estimate. The convergence of entropy flux-splitting schemes is proved for the 2×2 systems of conservation laws and the isentropic Euler system.

1. Introduction

We are concerned with entropy flux-splittings for nonlinear hyperbolic systems of conservation laws. Flux-splittings are used in order to design the so-called flux-splitting schemes and their high order accurate extensions. Many flux-splittings have been derived in the literature for the compressible Euler equations of gas dynamics and are currently used in fluid dynamics codes. In Steger-Warming [59] and Van Leer [38], the derivation is based on specific properties of the Euler equations (such as the homogeneity), or on searching for the "simplest possible" splitting with a given a priori expression (such as a polynomial function with minimum order). It is still unknown whether these splittings are consistent with the entropy inequality. In [24, 34, 53, 54], flux-splittings

Revised version February 1995.
[1] Department of Mathematics, University of Chicago, Chicago, IL 60637
Current address: Department of Mathematics, Northwestern University, Evanston, IL 60208
E-mail: *gqchen@math.nwu.edu.*
[2] Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau cedex, France.
E-mail: *lefloch@cmapx.polytechnique.fr*
1980 Mathematics Subject Classification (1985 Revision). 35L65, 65M12
Key words and phrases: conservation laws, entropy discontinuous solutions, flux-splittings, difference schemes

are derived from a kinetic formulation of the Euler equations. It is known that some of the latter satisfy the entropy inequality [24, 51, 52, 20]. The present paper is built on this body of works and is directed toward the investigation of fundamental mathematical properties of entropy flux-splittings. The approach is intended to be rather general: Without making a priori assumptions on both the hyperbolic system under consideration and the form of the flux-splittings, we derive necessary and sufficient conditions for the existence of entropy flux-splittings. In this first paper, we explain how some of the classical flux-splittings can be derived in a simple and systematic manner and, in some cases, characterized uniquely. The properties of general flux-splittings, especially of those associated with the isentropic Euler equations of gas dynamics, are studied. We also address the questions of stability and convergence of the corresponding difference schemes. We will return to the Euler equations of gas dynamics in [9].

Consider a system of N conservation laws in one space variable

$$\partial_t u + \partial_x f(u) = 0, \qquad u(t,x) \in \mathcal{O}, \quad t > 0,\, x \in \mathbf{R}, \tag{1.1}$$

together with the Cauchy data

$$u(0,x) = u_0(x), \qquad x \in \mathbf{R}. \tag{1.2}$$

The subset $\mathcal{O} \subset \mathbf{R}^N$ is assumed to be convex and open, the flux $f : \mathcal{O} \to \mathbf{R}^N$ to be Lipschitz continuous at least, and the initial data $u_0 : \mathbf{R} \to \mathcal{O}$ to belong to the space $L^\infty_{\rm loc}$ of all measurable and bounded functions. We assume that the system (1.1) is endowed with a strictly convex entropy pair $(U, F) : \mathcal{O} \to \mathbf{R} \times \mathbf{R}$ satisfying $\nabla F = \nabla U \cdot \nabla f$ by definition. In particular, for each compact subset $K \subset\subset \mathcal{O}$, one has

$$\nabla^2 U \geq C_K I, \tag{1.3}$$

where $C_K > 0$ is a constant depending on K, $\nabla^2 U$ is the Hessian matrix of the scalar-valued function U, I is the $N \times N$ identity matrix, and ∇G is the Jacobian matrix of a vector-valued function G. As is well-known, the existence of an entropy pair implies that (1.1) is hyperbolic but not necessarily strictly hyperbolic. This means that, for each $u \in \mathcal{O}$, the matrix $\nabla f(u)$ admits N real eigenvalues $\lambda_1(u) \leq \lambda_2(u) \leq \cdots \leq \lambda_N(u)$ and a basis of right eigenvectors $r_1(u), r_2(u), \cdots, r_N(u)$.

We recall that solutions to hyperbolic conservation laws generally are not smooth and must be understood in the sense of distributions. Weak solutions in the space $L^\infty_{\rm loc}$ are not uniquely determined by their initial data and the so-called entropy condition is necessary to select the relevant solutions. Several formulations can be considered (cf. [37, 22, 45]). In this paper, we refer to the entropy inequality (cf. [36])

$$\partial_t U(u) + \partial_x F(u) \leq 0 \tag{1.4}$$

in the sense of distributions.

The stability and convergence of finite difference schemes in conservative form for the Cauchy problem (1.1)-(1.2) have been actively studied in the last decades. We recall that the convergence of the Lax-Friedrichs scheme and the Godunov scheme for a class of systems was first studied by DiPerna [26] and extended in [5, 6] (cf. also [25]). The L^∞ stability of the Godunov scheme for general systems was studied by Hoff [32]. The entropy consistency of a class of schemes for systems was investigated by Tadmor [62, 63]. For high-order methods, see Chen-Liu [10], Coquel-LeFloch [18], and Tadmor [63]. For reference, let us recall that the scalar case (i.e. $N = 1$) has also received a lot of attention for many years. For background on the numerical analysis of scalar problems, we refer the reader to [29, 31, 30, 36, 48, 50, 61, 39] and the references therein. Recent results on one-dimensional problems include [33, 40]. For multi-dimensional problems see [7, 13, 14, 16, 17, 60]. Additional references on the approximation of conservation laws can be found in [15, 11]. To our knowledge, the present paper represents the first attempt to address the convergence of a class of flux-splitting schemes for general hyperbolic systems. Note that our rigorous convergence results will be restricted to systems of two equations, while L^∞ stability and entropy consistency will hold for general systems. Results on L^∞ stability will be restricted to the systems endowed with invariant regions (cf. [12]). The entropy inequality (1.4) will be satisfied at the discrete level and the passage to the limit will be justified.

Consider the class of schemes based on a flux-splitting, i.e., a decomposition of the flux f in the form

$$f(u) = f^+(u) + f^-(u), \tag{1.5}$$

where the functions $f^\pm$ are, at least, locally Lipschitz continuous and $\pm\nabla f^\pm(u)$ have real and nonnegative eigenvalues. Throughout this paper we shall use the notation $\pm\nabla f^\pm(u) \leq 0$ for latter property. A scheme based on (1.5) is obtained by averaging the two Godunov schemes associated with the corresponding systems

$$\partial_t v + \partial_x f^+(v) = 0, \qquad \text{and} \qquad \partial_t w + \partial_x f^-(w) = 0 \tag{1.6}$$

together (cf. Section 3). Since all of the eigenvalues of $\pm\nabla f^\pm$ have the same sign, the Godunov scheme for each of the systems in (1.6) reduces to the upwind scheme. Thus (1.6) yields a simple and efficient method in order to approximate the discontinuous solutions to (1.1). First studies from the mathematical point of view on flux-splittings are due to Engquist-Osher [29] and Harten-Lax-Van Leer [31] (cf. also Van Leer [39]). A related approach was suggested by Brenier [1,2].

The idea of splitting is actually well-spread in the engineering literature. The first flux-splittings were derived for the Euler equations of gas dynamics by Sanders-Prendergast [55], Steger-Warming [59], and Van Leer [38] for the polytropic and perfect gas. Based on a kinetic interpretation of the system of gas dynamics, flux-splittings were derived in [24, 34, 53, 54]. Some of them were extended to the so-called real gases in [57, 65]. A related approach to flux-splitting was proposed by Osher-Solomon in [49]. The latter was extended to reactive gases by Abgrall [3]. Multi-species flows are also considered by Larrouturou [35].

Regarding mathematical analysis of splittings for systems, we refer to Lerat [42] for an investigation of the properties of the Steger-Warming scheme. Perthame [51, 52] investigated the mathematical properties of the the kinetic schemes, especially on the entropy consistency of certain kinetic schemes for gas dynamics. Croisille-Delorme [20] contributed to the derivation and mathematical analysis of the kinetic schemes. The kinetic approximation at the continuous level was addressed in [43, 44]. A related approach to derive flux-splitting was proposed by Bourdel-Delorme-Mazet [4].

In this paper we attempt to provide a systematic framework for the derivation and analysis of flux-splittings and their corresponding flux-splitting schemes. The splitting (1.5) is called an *entropy flux-splitting* if each of the systems in (1.6) is endowed with the same entropy U. This is actually equivalent to saying that the matrices $\nabla^2 U\, \nabla f^\pm$ are symmetric. When $N \geq 2$, the notion depends upon the choice of the entropy U. The flux-splitting (1.5) is called *diagonalizable* if the matrices ∇f and $\nabla f^\pm$ have a common basis of eigenvectors. We say that (1.5) is a *genuine flux-splitting* if $f^\pm(u) = f(u)$ and $f^\mp(u) = 0$ when $\pm\nabla f \geq \pm C$ for $C > 0$ large enough. This condition is weaker than the classical condition used in Steger-Warming [59] and Van Leer [38], in which $f^\pm(u) = f(u)$ and $f^\mp(u) = 0$ when $\pm\nabla f \geq 0$. A genuine splitting leads to an upwind scheme similar to the Godunov scheme (upwind is often an attractive property for numerical computations), while a non-genuine splitting leads to a scheme similar to the Lax-Friedrichs scheme, which generally contains more numerical diffusion than an upwind scheme.

At first glance, these definitions may seem rather restrictive. Actually, one of the main results in this paper is the *existence* of *a family* of entropy flux-splittings for many systems occurring in physics. Diagonalizable splittings can be also constructed for certain systems. Note that, for the diagonalizable splittings, the systems (1.1) and (1.6) have the same Riemann invariants, i.e., the same invariant regions.

Section 2 lists general properties shared by entropy flux-splittings. We observe that any system of conservation laws admits a family of entropy flux-splittings —the Lax-Friedrichs type splittings, a variant of the Lax-Friedrichs scheme (cf. Shu [56]). These splittings, however, are not genuine. A scalar conservation law admits a family of genuine flux-splittings. We recover the Engquist-Osher splitting as an "extremal" case in this family. With regard to the entropy inequality (1.4), we observe that any splitting is actually an *entropy* flux-splitting because any function is an entropy when $N = 1$. In other words, in this special case, the condition $\pm\nabla f^{\pm}(u) \geq 0$ is enough to imply the consistency of the splitting scheme with the entropy inequality (1.4). The result does not hold for systems due to the fact that any entropy must satisfy a set of compatibility relations, which are generally overdetermined when $N \geq 2$. We prove that the p-system of isentropic gas dynamics in Lagrangian coordinates admits a family of entropy flux-splittings. With any entropy flux to the p-system, we can associate an entropy flux-splitting. Note that the p-system does not admit any genuine flux-splitting due to the fact that the two eigenvalues of the p-system have distinct signs.

In Section 4 the isentropic Euler system for the so-called γ–law gas is fully studied. We prove that this system admits a whole family of entropy flux-splittings for the physical entropy, the mechanical energy. A representative family of splittings, particularly attractive for their simplicity, is presented in Theorem 4.1. Moreover, among these decompositions, there exists a *unique* entropy splitting for *all* of the entropies of the Euler system. Actually we show that two entropies are enough to characterize a unique splitting. It is noteworthy that this splitting can be also characterized as the unique *diagonalizable* splitting. Our results also show that the class of the so-called kinetic splittings contains a single flux-splitting that satisfies all of the entropy inequalities. This later splitting was derived by Perthame [51] in the context of the full 3×3 system of gas dynamics. The results in Section 4 will be extended (cf. Chen-LeFloch [9]) to the full 3×3 system of gas dynamics and to the Euler systems with general equations of state.

The main results of stability and convergence in this paper are as follows. In Section 3 we consider a strictly hyperbolic system of conservation laws and prove that a scheme based on an entropy flux-splitting is consistent with the entropy inequality (1.4) and thus converges to an entropy weak solution. We next focus on a system of two equations endowed with bounded invariant regions [12] and show that the flux-splitting scheme is stable in the L^∞ norm if (1.5) is a diagonalizable splitting. From DiPerna's work [26] based on the compensated compactness method developed by Murat and Tartar [46, 47, 64], it follows that the entropy flux-splitting schemes converge in the strong L^1 norm to an entropy weak solution. For general (i.e. not necessarily entropy satisfying) flux-splittings, we use the notion of nonconservative product by Dal Maso-LeFloch-Murat [23] (cf. also [41]) to derive a discrete cell entropy inequality in nonconservative form. Section 5 is devoted to the proof of convergence of a flux-splitting scheme for the isentropic Euler system. The solutions include the vacuum state, at which point the system is nonstrictly hyperbolic. The proof is based on the compactness framework formulated in [5,6] (cf. also [25]). For the analysis, one has to determine the regions where the systems (1.6) are strictly hyperbolic and genuinely nonlinear, and to prove that the Riemann solutions for both systems in (1.6) exist for arbitrary large initial data and satisfy the entropy inequality (1.4).

Finally, let us recall that the flux-splitting schemes provide some alternative between the Godunov scheme and the Lax-Friedrichs scheme. Such schemes combine the upwind feature of the Godunov scheme without the complicated and costly resolution of Riemann problems. The numerical flux functions of flux-splitting schemes are given by explicit formulas, as is the Lax-Friedrichs flux-function; they contain however less numerical viscosity and so allow for a better approximation of the discontinuities. Furthermore, the numerical flux of a flux-splitting scheme can be designed to be of class C^1 and even, in some cases, of class C^2, while most schemes have only Lipschitz continuous flux functions. This latter property is useful for the computation of steady state solutions using a time-dependent method.

The content of this paper was announced in [8].

2. Definitions and Examples

In this section we introduce several definitions and derive main properties of the entropy flux-splittings. As an illustration to the definitions, the existence of such splittings is checked for the scalar conservation laws and the system of isentropic gas dynamics in Lagrangian coordinates.

2.1. Definitions and Main Properties. We consider the system of conservation laws (1.1) endowed with a strictly convex entropy pair (U, F) satisfying (1.3). Let us recall the following definition, which is classical.

Definition 2.1. *A* flux-splitting *for the system* (1.1) *is a decomposition of the form*

$$f(u) = f^+(u) + f^-(u), \qquad u \in \mathcal{O}, \tag{2.1a}$$

where the functions $f^\pm : \mathcal{O} \to \mathbf{R}^N$ are locally Lipschitz continuous, the Jacobian matrices $\nabla f^\pm$ have real eigenvalues $\lambda_j^\pm, 1 \le j \le N$, and a basis of eigenvectors $r_j^\pm, 1 \le j \le N$, satisfying

$$\lambda_1^-(u) \le \lambda_2^-(u) \le \cdots \le \lambda_N^-(u) \le 0 \le \lambda_1^+(u) \le \lambda_2^+(u) \le \cdots \le \lambda_N^+(u), \quad u \in \mathcal{O}. \tag{2.1b}$$

We will use the notation $\pm\nabla f^\pm \ge 0$ if the matrices $\pm\nabla f^\pm$ have only real and nonnegative eigenvalues as well as a basis formed by eigenvectors. The following definition contains a new concept, which is the basis of this paper. The definition involves the entropy U of the system (1.1) and the following two systems

$$\partial_t v + \partial_x f^+(v) = 0, \quad v \in \mathcal{O}, \tag{2.2a}$$

$$\partial_t w + \partial_x f^-(w) = 0, \quad w \in \mathcal{O}. \tag{2.2b}$$

Note in passing that Definition 2.1 implies that the systems (2.2a) and (2.2b) are hyperbolic. The following definition is concerned with the entropy inequality (1.4).

Definition 2.2. *We say that* (2.1) *is an* entropy flux-splitting *for the entropy U of the system* (1.1) *if the function U is also an entropy for both systems* (2.2*a*) *and* (2.2*b*). *In other words, there must exist entropy flux functions $F^\pm : \mathcal{O} \to \mathbf{R}$ such that all the smooth solutions v and w to* (2.2*a*) *and* (2.2*b*) *satisfy*

$$\partial_t U(v) + \partial_x F^+(v) = 0, \tag{2.3a}$$

$$\partial_t U(w) + \partial_x F^-(w) = 0, \tag{2.3b}$$

respectively.

If the condition in Definition 2.2 holds for an arbitrary decomposition like (2.1a), then the systems (2.2) are necessarily hyperbolic although the sign conditions (2.1b) need not hold. When necessary, we will specify that (2.1) is an entropy splitting for the entropy U. For some systems, we will be able to derive splittings that are entropy splittings for *all* of the entropies. As we will show in Proposition 2.1, the latter property is shared by the diagonalizable splittings.

Definition 2.3.

(1) *A flux-splitting (2.1) is* diagonalizable *if the three matrices ∇f, ∇f^+, and ∇f^- have a common basis of eigenvectors.*

(2) *A flux-splitting* (2.1) *is* genuine *if, for each compact subset $K \subset\subset \mathcal{O}$, some positive constant C_K, and all u in K,*

$$\begin{aligned} &f^+(u) = f(u), \quad f^-(u) = 0, \qquad \text{when} \quad \lambda_1(u) \geq C_K, \\ &f^+(u) = 0, \quad f^-(u) = f(u), \qquad \text{when} \quad \lambda_N(u) \leq -C_K. \end{aligned} \tag{2.4}$$

Our primary interest concerns the genuine flux-splittings. Note that the notion of genuine splitting is of interest only for systems whose eigenvalues change sign, including the system of gas dynamics in Eulerian coordinates. A system, whose all eigenvalues are either non-negative or non-positive, admits a trivial splitting (Proposition 2.2). A system, whose eigenvalues have a constant sign independent of u but whose eigenvalues have not the same sign, can not admit a genuine splitting. This latter case arises in the system of gas dynamics in Lagrangian coordinates (see Proposition 2.6). Proposition 2.5 will give a full description of the genuine entropy flux-splittings for the scalar case.

The following observation provides an important characterization of the entropy flux-splittings. From now on, we tacitly assume that the flux f and the entropy U admit locally bounded second order derivatives.

Proposition 2.1.

(1) *The decomposition* (2.1) *with $f^\pm \in W^{2,\infty}(\mathcal{O})^N$ is an entropy flux-splitting if and only if*

$$\nabla^2 U \nabla f^+ \ \text{ and } \ \nabla^2 U \nabla f^- \ \text{ are symmetric matrices.} \tag{2.5}$$

(2) *In particular, a diagonalizable splitting is an entropy flux-splitting for any choice of entropy U and the three systems in* (1.1) *and* (2.2) *possess the same set of Riemann invariants.*

This observation is based on the following classical fact: since U is an entropy to the system (1.1) and $\nabla F = \nabla U \nabla f$ is a gradient, the matrix $\nabla^2 U \nabla f$ is symmetric. The latter condition is also a sufficient condition for U to be an entropy. This characterization of the entropy can be also applied to each system in (2.2).

Proposition 2.2.

(1) *Consider the decomposition:*

$$f^+(u) = Af(u) + Bu, \quad f^-(u) = (I - A)f(u) - Bu, \qquad u \in \mathcal{O}, \tag{2.6}$$

where, for all $u \in \mathcal{O}$, the constant $N \times N$ matrices A and B satisfy that

$$\nabla^2 U(u) A \nabla f(u) \quad \text{ and } \quad \nabla^2 U(u) B \quad \text{ are symmetric matrices,} \tag{2.7}$$

and

$$B + A\nabla f(u) \geq 0, \quad B + (A - I)\nabla f(u) \geq 0. \tag{2.8}$$

Then (2.6) – (2.8) *defines an entropy flux-splitting for* (1.1).

(2) *Conditions* (2.7) – (2.8) *are satisfied (at least on any compact subset of $\mathcal{O}$) with the choice*

$$A = \frac{1}{2} I, \quad B = bI, \qquad \text{when } b > 0 \text{ large enough.} \tag{2.9}$$

In that case (2.6) *is a diagonalizable splitting.*

(3) *Consider the decomposition:*

$$f^+(u) = f(u), \qquad f^-(u) = 0, \qquad u \in \mathcal{O}, \tag{2.10}$$

for a system whose characteristic eigenvalues $\lambda_i(u)$ have a constant sign independent of both i and u, say $\lambda_i(u) \geq 0$. Then (2.10) *defines an entropy flux-splitting.*

The splittings in Proposition 2.2 will be refered to as the *Lax-Friedrichs type splittings*. The scheme designed from (2.6) (cf. Section 3) turns out to be a simple extension to the Lax-Friedrichs scheme. This generalization of the Lax-Friedrichs scheme was also considered in Shu [56] for the computation of a phase-transition model.

In contrast to the result in Proposition 2.2, we observe that an arbitrary system need not admit a *genuine* entropy flux-splitting. In this section and Section 3, we will derive such genuine splittings. Also note that a given system admits several distinct splittings. However, it will be shown in Section 4 (also [9]) that the system of gas dynamics admits a class of genuine entropy splittings, in which a unique splitting for all the entropies can be selected. In the general definitions, the splitting is assumed to be Lipschitz continuous only. However, smoother flux-splittings often exist (cf. Section 4).

2.2. Actual Derivation of Flux-Splittings. The actual construction of entropy flux-splittings is based on the characterization in Proposition 2.1. Let us discuss the general splittings first and the diagonalizable ones next.

For clarity, consider first a symmetric system (1.1), that is, the matrix ∇f is symmetric. Then the function $U(u) = \frac{|u|^2}{2}$ is an entropy. Since ∇f is symmetric, there exists a scalar function $\psi : \mathcal{O} \to \mathbf{R}$ such that $f = \nabla\psi$. The entropy inequality (1.4) then becomes

$$\partial_t\big(\frac{|u|^2}{2}\big) + \partial_x\big(u \cdot \nabla\psi(u) - \psi(u)\big) \leq 0.$$

With this choice of entropy, (2.5) is equivalent to saying that each matrix $\nabla f^\pm$ is symmetric. So we can search for $f^\pm$ in the form:

$$f^\pm = \nabla\psi^\pm, \qquad \psi^+ + \psi^- = \psi \qquad \text{with } \psi^\pm : \mathcal{O} \to \mathbf{R}.$$

It is easily checked that the sign conditions (2.1b) are equivalent to saying that ψ^+ and ψ^- are convex and concave functions, respectively. Therefore the question of the construction of flux-splittings is reduced to that of finding a decomposition of the function ψ as the sum of a convex function and a concave function. As we will see, this can be done (geometrically) in the scalar case. However, such a decomposition need not be possible for a system.

The case of non-symmetric systems is slightly more involved. We observe that any system can be put in symmetric form by using the so-called entropy variable. By Definition 2.2, the three systems under consideration have a common entropy variable. By setting $v = \phi(u) = \nabla U(u)$, the system (1.1) takes the form:

$$\partial_t g(v) + \partial_x h(v) = 0, \qquad v(t,x) \in \mathcal{O}', \tag{2.11}$$

where $\mathcal{O}' = \phi(\mathcal{O})$, $g(v) = \phi^{-1}(v)$, and $h(v) = f(\phi^{-1}(v))$. It can be checked that $\nabla_v g$ and $\nabla_v h$ are symmetric matrices. In view of Proposition 2.1, the entropy flux-splittings for the system (1.1) are those of the form: $h(v) = h^+(v) + h^-(v)$ with $\nabla_v h^\pm(v)$ symmetric. Therefore, a similar argument leads to the following result.

Proposition 2.3. *The entropy flux-splittings for a system endowed with a strictly convex entropy U are given by the formula: $f^+(u) = \nabla_v \psi^+(v)$, $\quad f^-(u) = \nabla_v \psi^-(v)$, where $v = \nabla_u U(u)$ and the scalar-valued function $\psi(v)$ is defined by the relation $f(u) = \nabla_v \psi(v)$:*

$$\psi(v) = \psi^+(v) + \psi^-(v), \tag{2.12a}$$

which is any decomposition of the function ψ in the sum of a convex function ψ^+ and a concave function ψ^- in the following sense:

$$\pm \nabla_{vv}\psi^\pm \nabla_{uu} U \ \geq 0. \tag{2.12b}$$

Decompositions of the form (2.12) were derived first by Bourdel-Delorme-Mazet in [4] together with the notion of K-diagonalizable system, which extends the concept of diagonalization. The latter naturally admit, by construction, an entropy flux-splitting in the form (2.12).

We now consider the derivation of *diagonalizable flux-splittings*. Without further restrictions, we set $r_j(u) = r_j^+(u) = r_j^-(u), \quad j = 1, 2, \cdots, N$. The eigenvalues of the systems (1.1) and (2.2) satisfy the relation $\lambda_j(u) = \lambda_j^+(u) + \lambda_j^-(u)$, $u \in \mathcal{O}$, $1 \leq j \leq N$. We also set $P = (r_1, r_2, \cdots, r_N)$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$. In practical computations, we derive the diagonalizable splittings by searching for a decomposition of the diagonal matrix Λ in the form

$$\Lambda = \Lambda^+ + \Lambda^- \qquad \text{with} \quad \Lambda^\pm = \mathrm{diag}(\lambda_1^\pm, \lambda_2^\pm, \cdots, \lambda_N^\pm) \geq 0, \tag{2.13a}$$

$$P\Lambda^\pm P^{-1} = \nabla f^\pm \quad \text{for some functions } f^\pm : \mathbf{O} \to \mathbf{R}^N. \tag{2.13b}$$

Condition (2.13b) is equivalent to saying that the matrix $\nabla\big(P\Lambda^\pm P^{-1}\big)$ is symmetric.

We also need to discuss the positivity conditions (2.1b). Throughout this paper, all matrices under consideration have real eigenvalues and a full set of eigenvectors. This is the case of the Jacobian matrices $\nabla f^\pm$ in the systems (2.2) satisfying the condition in Definition 2.2. In the examples, we have to verify the positivity condition (2.1b). Observe first that the condition (2.1b), $\pm\nabla f^\pm \geq 0$, is *weaker* than the more classical requirement $\pm r^\top \nabla f^\pm r \geq 0$, for all $r \in \mathbf{R}^N$. The latter condition implies the former one, but the two conditions are distinct when applied to *non-symmetric* matrices, which is generally the case here. For further references, we state the following elementary facts:

Proposition 2.4. *(i) If N=2, then the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has non-negative eigenvalues if and only if*

$$tr(A) = a + d \geq 0, \quad det(A) = ad - bc \geq 0. \tag{2.14}$$

(ii) If N=3, then the matrix $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ b_2 & a_2 & d_1 \\ c_2 & d_2 & a_3 \end{pmatrix}$ has non-negative eigenvalues if and only if

$$tr(A) \geq 0, \quad a_1a_2 + a_2a_3 + a_3a_1 + b_1b_2 + c_1c_2 + d_1d_2 \geq 0, \quad det(A) \geq 0. \tag{2.15}$$

By contrast, the condition $\pm r^\top A r \geq 0$, for all $r \in \mathbf{R}^N$, is equivalent to saying (when $N = 2$):

$$\mathrm{tr}(A) = a + d \geq 0, \quad \det(A + A^\top) = 4ad - (b+c)^2 \geq 0. \tag{2.16}$$

Condition (2.16) implies that a and d are non-negative, which need not be the case under solely the condition (2.14).

Finally, we observe that, when a diagonalizable splitting exists, the system (1.1) must satisfy very peculiar properties. For instance, consider the case $N = 2$. In view of Proposition 2.1, the systems (1.1) and (2.2) have a common set of Riemann invariant coordinates, say (w_1, w_2). In that situation, the entropy U has to satisfy simultaneously the following three linear equations:

$$(\lambda_2 - \lambda_1)\partial^2_{w_1 w_2} U + \partial_{w_1}\lambda_2 \partial_{w_2} U + \partial_{w_2}\lambda_1 \partial_{w_1} U = 0, \tag{2.17}$$

$$(\lambda_2^\pm - \lambda_1^\pm)\partial^2_{w_1 w_2} U + \partial_{w_1}\lambda_2^\pm \partial_{w_2} U + \partial_{w_2}\lambda_1^\pm \partial_{w_1} U = 0. \tag{2.18}$$

In other words, the above equations have one common solution at least. In the case of the system of gas dynamics, we will show that the equation (2.17) is actually a *linear combination* of the equations in (2.18).

2.3. Examples of Entropy Flux-Splittings. We now turn to the study of two significant examples: the scalar conservation laws and the p-system. In both cases, we establish the existence of a large family of genuine entropy flux-splittings. We first treat the scalar case. A genuine splitting by definition satisfies

$$\begin{cases} \frac{df^+}{du}(u) = f'(u), & \text{if} \quad f'(u) \geq \lambda_0, \\ \frac{df^-}{du}(u) = f'(u), & \text{if} \quad f'(u) \leq -\lambda_0, \end{cases} \tag{2.19}$$

for some nonnegative constant λ_0. Non-genuine splittings correspond to the limiting value $\lambda_0 = \infty$. With the choice $\lambda_0 = 0$, (2.19) selects a *unique* splitting determined by

$$\frac{df^\pm}{du}(u) = \frac{1}{2}(f'(u) \pm |f'(u)|),$$

that is,

$$f^\pm(u) = \frac{1}{2}\int_0^u (f'(\bar{u}) \pm |f'(\bar{u})|) d\bar{u} + \frac{1}{2} f(0) \pm a. \tag{2.20}$$

We can drop the parameter a since the flux-functions can be defined up to a constant. When the function f is convex and achieves its minimum at the point $u = u_*$, we find

$$f^+(u) = \begin{cases} f(u_*)/2, & \text{for } u \leq u_*, \\ f(u) - f(u_*)/2, & \text{for } u \geq u_*. \end{cases} \tag{2.21}$$

Formulas (2.20)-(2.21), which correspond to $\lambda_0 = 0$, are due to Engquist-Osher [29] and lead to the so-called Engquist-Osher scheme. Definition 2.2 is easily seen to provide no restriction on the splittings when $N = 1$.

For arbitrary values of λ_0, we state the following result.

Proposition 2.5. *The flux-splittings of class $W^{1,\infty}$ for a scalar conservation law have the form $f^+(u) = g(u)$, $f^+(u) = f(u) - g(u)$, where g is any function satisfying*

$$g \in W^{1,\infty}_{loc} \quad \textit{with } g' \geq \max(0, f') \textit{ and } \lim_{u \to -\infty} g(u) = 0.$$

Any such splitting is an entropy flux-splitting. Moreover it is a genuine splitting if and only if, for some constant $\lambda_0 \geq 0$,

$$g'(u) = \max(0, f'(u)) \quad \textit{when} \quad |f'(u)| \geq \lambda_0.$$

Furthermore, a scalar conservation law admits a unique *genuine entropy flux-splitting satisfying the latter condition with the extremal value of the parameter* λ_0*, i.e.* $\lambda_0 = 0$.

Geometrically, it is not hard to determine the function g satisfying the conditions in Proposition 2.5. Consider the graph of the function f'. The graph of the function g' must lie above both the u-axis and the graph of f'. Moreover, the function $g'(u)$ coincides with $f'(u)$ at those points u where $f'(u)$ is larger than λ_0, and coincides with the u-axis when $-f'(u)$ is larger than λ_0. Since g' is defined almost everywhere, the function g' can be constructed to be piecewise continuous only. The normalization $\lim_{u\to-\infty} g(u) = 0$ is chosen for definiteness. Finally, we observe that the numerical viscosity of the difference scheme derived from the flux-splittings in Proposition 2.5 is minimal for the choice $\lambda_0 = 0$. This can be checked from the definition in [61].

Next we treat the case of the p-system, which is the isentropic system of gas dynamics in Lagrangian coordinates:

$$\partial_t w - \partial_x v = 0, \qquad \partial_t v + \partial_x p(w) = 0, \tag{2.22}$$

where $w > 0$ and v represent the specific volume and the velocity of the gas, respectively. The pressure $p = p(w)$ is a given function satisfying $p \in W^{2,\infty}_{loc}(\mathbf{R})$ and $dp/dw < 0$, which implies that (2.22) is strictly hyperbolic. We do not assume that the system is genuinely nonlinear. The system (2.22) has the form (1.1) with $u = (w, v)$ and $f(u) = (-v, p(w))$. The eigenvalues and a set of eigenvectors are given by $\lambda_1(u) = -c(w) < 0 < \lambda_2(u) = c(w)$, $r_1(u) = (1, c(w))$, $r_2(u) = (1, -c(w))$, where $c(w) = \sqrt{-p'(w)}$ is the sound speed. We recall that (2.22) admits an infinite set of entropies U, which are governed by the linear hyperbolic equation:

$$\partial^2_{ww} U + p'(w)\partial^2_{vv} U = 0. \tag{2.23}$$

The latter can be solved locally (at least) in the phase plane. Among these entropy pairs are the mechanical energy-energy flux pair $(U_*, F_*) = (v^2/2 - \int_0^w p(s)\, ds, v\, p(w))$, and the "dual" mechanical entropy pair $(U_{**}, F_{**}) = (wv, -v^2/2 + wp(w) - \int_0^w p(s)\, ds)$.

Proposition 2.6. *Consider the flux-splittings of class* $W^{2,\infty}$ *for the p-system* (2.34).

(i) The splittings for the mechanical energy U_* *have the form:*

$$f^+(w,v) = \begin{pmatrix} c(w)^{-2}\partial_w G(w,v) \\ \partial_v G(w,v) \end{pmatrix}, \quad f^-(w,v) = \begin{pmatrix} -v - c(w)^{-2}\partial_w G(w,v) \\ p(w) - \partial_v G(w,v) \end{pmatrix}, \tag{2.24}$$

where G *is any function of class* $W^{2,\infty}$ *satisfying the positivity conditions:*

$$\begin{cases} \partial_w\left(c^{-2}\partial_w G\right) + \partial^2_{vv} G \geq 0, \\ c^2\partial_w\left(c^{-2}\partial_w G\right)\partial^2_{vv} G - \max\left(|\partial^2_{wv} G|^2, |c^2 + \partial^2_{wv} G|^2\right) \leq 0. \end{cases} \tag{2.25}$$

(ii) The class of entropy splittings for both the mechanical entropy U_* *and the dual mechanical entropy* U_{**} *coincides with the class of entropy splittings for* all *of the entropies of* (2.22)*. Such splittings are described by* (2.24) – (2.25)*, where* G *is any entropy flux, i.e. a solution to the linear hyperbolic equation:*

$$\partial_w\left(c^{-2}\partial_w G\right) - \partial^2_{vv} G = 0. \tag{2.26}$$

(iii) The class of diagonalizable splittings coincides with the class of splittings described in (ii).

Proof. Using the entropy variables $\nabla U_* = (-p(w), v) = (c(w)^2, v)$, it is not difficult to check the statement (i) from Proposition 2.3. In order to prove (ii), we use the characterization (2.5). The entropy flux-splitting (2.24) is consistent with both the entropy U_* and another entropy U if and only if

$$U_{wv}\big(c^{-2}G_w\big)_w + U_{vv}UG_{vw} = U_{ww}c^{-2}G_{vw} + U_{vw}G_{vv}.$$

Using the entropy equation (2.23), this is seen to be equivalent to the equation

$$U_{wv}\big\{G_{vv} - \big(c^{-2}G_w\big)_w\big\} = 0. \tag{2.27}$$

Clearly, it suffices to pick up one additional entropy such that $U_{wv} \neq 0$ in order to deduce (2.26) from (2.27). This is done by choosing $U = U_{**}$ for instance.

We now discuss the derivation of the diagonalizable splittings. With the notation introduced in Section 2.2, condition (2.13b) here reads

$$P\Lambda^{\pm}P^{-1} = \frac{1}{2c}\begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix}\begin{pmatrix} \lambda_1^{\pm} & 0 \\ 0 & \lambda_2^{\pm} \end{pmatrix}\begin{pmatrix} c & 1 \\ c & -1 \end{pmatrix} = \nabla f^{\pm}$$

for some functions f^+ and f^-, that is,

$$\frac{1}{2c}\begin{pmatrix} c(\lambda_1^{\pm} + \lambda_2^{\pm}) & (\lambda_1^{\pm} - \lambda_2^{\pm}) \\ c^2(\lambda_1^{\pm} - \lambda_2^{\pm}) & c(\lambda_1^{\pm} + \lambda_2^{\pm}) \end{pmatrix} = \nabla f^{\pm}.$$

The coefficients of the matrix above are Lipschitz continuous at least, so the latter condition holds if and only if

$$\begin{cases} \partial_v\left(\lambda_1^{\pm} + \lambda_2^{\pm}\right) - \partial_w\left(\frac{1}{c}(\lambda_1^{\pm} - \lambda_2^{\pm})\right) = 0, \\ \partial_v\left(c(\lambda_1^{\pm} - \lambda_2^{\pm})\right) - \partial_w\left(\lambda_1^{\pm} + \lambda_2^{\pm}\right) = 0. \end{cases} \tag{2.28}$$

For convenience, we define auxilliary unknown functions $\alpha^{\pm}$ and $\beta^{\pm}$ by $\lambda_2^{\pm} - \lambda_1^{\pm} = \alpha^{\pm}c$ and $\lambda_1^{\pm} + \lambda_2^{\pm} = \beta^{\pm}$. Then (2.28) takes the form: $\partial_w\alpha^{\pm} + \partial_v\beta^{\pm} = 0$ and $c(w)^2\partial_v\alpha^{\pm} + \partial_w\beta^{\pm} = 0$. This leads to a second-order linear hyperbolic equation for $\alpha^{\pm}$

$$\partial^2_{ww}\alpha^{\pm} - \partial^2_{vv}\big(c(w)^2\alpha^{\pm}\big) = 0, \tag{2.29}$$

which is the entropy equation for (2.22)(cf. (2.23)). On the other hand, the gradient matrix of $f^{\pm}$ is computed in term of the functions $\alpha^{\pm}$ and $\beta^{\pm}$:

$$\nabla f^{\pm} = \frac{1}{2}\begin{pmatrix} \beta\pm & -\alpha^{\pm} \\ -c^2\alpha^{\pm} & \beta^{\pm} \end{pmatrix}.$$

Thus there exists a pair of functions (k, l) such that $f^+(w, v) = \big(k(w, v), -l(w, v)\big)^{\mathrm{T}}$. That is, there exist k and l such that

$$\partial_w k = \beta^+/2, \qquad \partial_v k = \alpha^+/2,$$
$$\partial_w l = c^2\alpha^+/2, \qquad \partial_v l = -\beta^+/2,$$

which are consistent relations since $\partial_w\alpha^{\pm} + \partial_v\beta^{\pm} = 0$. Moreover, in view of the expression of the gradient of f^+, one has $\partial_v l = -\partial_w k$, $\partial_w l = -c^2\partial_v k$, which means that (k, l) is an entropy pair. Therefore, it follows that f^+ takes the form (2.24). The proof of Proposition 2.6 is completed.

Note that Proposition 2.6 does not assume the genuinely nonlinearity of system (2.22). Also observe also that one can derive the Lax-Friedrichs type splittings (2.6) with $A = aI$ and $B = abI$, a and b constants, with the choice:

$$G(w, v) = a\big(F_*(w, v) - bF_{**}(w, v)\big) = \frac{b}{2}v^2 + avp(w) - b\big(W_*(w) + wp(w)\big),$$

and thus

$$f^+(u) = \begin{pmatrix} bw - av \\ bv + ap(w) \end{pmatrix}, \quad f^-(u) = -\begin{pmatrix} bw + (1-a)v \\ bv - (1-a)p(w) \end{pmatrix}.$$

Condition (2.25) holds if $b \ge \max\big(|a|, |1-a|\big)\max c(w)$, which is satisfied on sets of the form $\{(w,v) | c(w) \le M\}$ with $M > 0$ for $b > 0$ large enough.

3. Convergence of Entropy Flux-Splittings Schemes

In this section we consider the finite difference scheme derived from a flux-splitting for (1.1). The convergence of the scheme is investigated in the case of entropy flux-splittings (cf. Theorem 3.1). In particular, discrete cell entropy inequalities are derived therein. On the other hand, the entropy consistency of general flux-splitting is tackled in Theorem 3.2.

Let $\tau > 0$ and $h > 0$ be the time and space mesh lengths, respectively. For all integers $n \ge 0$ and j, set $\lambda = \tau/h$, $x_{j+1/2} = (j+1/2)h$, and $t_n = n\tau$. We consider finite difference schemes in conservative form for (1.1)–(1.4):

$$u_j^{n+1} = u_j^n - \lambda(g_{j+1/2}^n - g_{j-1/2}^n), \quad g_{j+1/2}^n = g(u_j^n, u_{j+1}^n), \tag{3.1}$$

for all integers $n \ge 0$ and j with u_j^0 given by

$$u_j^0 = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx. \tag{3.2}$$

The numerical flux $g : \mathcal{O} \times \mathcal{O} \to \mathbf{R}^N$ is a locally Lipschitz continuous function. We always assume that g is consistent with the exact flux f, that is, $g(u,u) = f(u)$ for all u in $\mathcal{O}$. By definition, for a flux-splitting scheme, one has

$$g(v,w) = f^+(v) + f^-(w), \qquad \text{for all } v \text{ and } w \text{ in } \mathcal{O}, \tag{3.3}$$

where $f = f^+ + f^-$ is a flux-splitting for (1.1).

We point out that the scheme (3.1)–(3.3) can be decomposed in the form $u_j^{n+1} = \frac{1}{2}(u_{j-1/2}^+ + u_{j+1/2}^-)$ with

$$u_{j\pm1/2}^\pm = u_j^n \pm 2\lambda(f^\pm(u_j^n) - f^+(u_{j\pm1}^n)),$$

for all integers $n \ge 0$ and j. Note that $u_{j-1/2}^+$ and $u_{j+1/2}^-$ are the approximations obtained from the initial data u_j^n by applying the upwind scheme to the systems (2.2a) and (2.2b), respectively. The stability of the scheme requires λ to satisfy a CFL condition:

$$2\lambda \max_{\substack{1 \le i \le N \\ u \in K}} \big(|\lambda_i^+(u)|, |\lambda_i^-(u)|\big) < 1.$$

Decompositions of the form (3.4)-(3.5) were introduced by Tadmor [61] for studying the numerical viscosity of E-schemes and used by Coquel-LeFloch [17] for deriving sharp entropy dissipation estimates.

We denote by $R^\pm(u_l, u_r)$ the solution to the Riemann problem for systems (2.2) with the initial data:

$$u^\pm(u_l, u_r)(0,x) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0, \end{cases}$$

respectively, where u_l and u_r are given in $\mathcal{O}$. It is assumed that, for arbitrary initial data in the set $\mathcal{O}$, the Riemann problem admits a unique solution satisfying the entropy inequality (1.4) [37]. Throughout this section, the functions $f^\pm$ are assumed to be at least of class $W^{2,\infty}$. For definiteness, we assume that, for each u, the characteristic eigenvalues of the gradients $\nabla f^\pm(u)$ are either genuinely nonlinear or linearly degenerate and, moreover,

$$\lambda_j^\pm(u) = \lambda_j^\pm(u) \Longrightarrow \nabla\lambda_j^\pm(u)\cdot r_j^\pm(u) = 0. \tag{3.4}$$

The latter means that, if strict hyperbolicity fails at u, then the corresponding characteristic fields are linearly degenerate at that point. This is indeed a sufficient condition for the Riemann problem to have a unique solution in the class of piecewise smooth solutions composed of constant states separated by shock discontinuities (satisfying the Lax entropy condition), rarefaction waves, or contact discontinuities. We refer to Lax [37] for an explicit construction. In view of (2.1b), the Riemann solution $R^+(u_l,u_r)$ ($R^-(u_l,u_r)$) contains only non-negative (non-positive) wave speeds. In particular, we will use

$$f^+(R^+(u_l,u_r)(t,x=0)) = f^+(u_l), \qquad f^-(R^-(u_l,u_r)(t,x=0)) = f^-(u_r). \tag{3.5}$$

The assumption (3.4) will be satisfied in the applications (cf., for instance, Section 4).

We now define the approximate solutions u^h, u^h_+, and u^h_- from the flux-splitting scheme. Suppose that $u^h(t)$ has been defined for all $t \le t_n$. Then we define $u^h(t)$ on the cell $\big\{(t,x) : x_j < x < x_{j+1},\ t_n < t < t_{n+1}\big\}$ by

$$u^h(t,x) = \frac{1}{2}\big(u^h_+(t,x) + u^h_+(t,x)\big), \tag{3.6}$$

where u^h_+ and u^h_- are the solutions to the Riemann problems for (2.2) with initial data

$$u^h_\pm(t_n,x) = \begin{cases} u_j^n, & x < x_{j+1/2}, \\ u_{j+1}^n, & x > x_{j+1/2}, \end{cases}$$

In view of (3.5), we have $u_j^{n+1} = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u^h(t_{n+1}-0,x)\,dx$. This completes the construction of the flux-splitting approximation.

The following result summarize our convergence results.

Theorem 3.1. *Consider a scheme constructed from an entropy flux-splitting.*

(1) *Then the scheme satisfies discrete entropy inequalities:*

$$U(u_j^{n+1}) - U(u_j^n) + \lambda\big(F^+(u_j^n) + F^-(u_{j+1}^n) - F^+(u_{j-1}^n) - F^-(u_j^n)\big) \le 0. \tag{3.7}$$

(2) *If $\{u^h\}$ is uniformly bounded in the L^∞ norm and converges almost everywhere to an element $u \in L^\infty$, then u satisfies* (1.1), (1.2), *and* (1.4) *in the sense of distributions.*

(3) *If the system* (1.1) *admits convex and bounded invariant regions and the splitting under consideration is a diagonalizable flux-splitting, the scheme preserves the invariant regions and therefore is stable in the L^∞ norm.*

(4) *Consider a strictly hyperbolic system of two conservation laws with genuinely nonlinear characteristic fields and assume that the splitting is an entropy flux-splitting for every convex entropy of* (1.1). *Also assume that the Riemann solutions for (2.2) exist for arbitrary large initial data and satisfy the entropy inequality (1.4). Assume finally that either the splitting is diagonalizable or* (1.1) *admits a convex and bounded invariant region. Then the scheme converges in the strong L^p norm ($p < \infty$) to an entropy solution of* (1.1) – (1.4).

Proof. By assumption, U is an entropy for both systems in (2.2) with the corresponding entropy flux functions $F^{\pm}$. The Riemann solution $R^{\pm}(u_l, u_r)$, by construction, satisfies the shock entropy inequality (1.4). According to Lax [37], it also satisfies the distributional entropy inequality:

$$\partial_t U(u^{\pm}) + \partial_x F^{\pm}(u^{\pm}) \leq 0. \tag{3.8}$$

Thanks to the assumption (3.4), the entropy inequality holds even if strict hyperbolicity fails at some point. The inequality

$$U(u^{\pm}_{j+1/2}) - U(u^n_j) + 2\lambda\big(F^{\pm}(u^n_{j+1}) - F^{\pm}(u^n_j)\big) \leq 0, \tag{3.9}$$

follows by writing (3.8) for the functions $u^h_{\pm}$, next integrating on the rectangle: $\big\{(t,x) : x_j < x < x_{j+1/2}, t_n < t < t_{n+1}\big\}$, and finally using Young's inequality for the convex function U. Then (3.8) is a consequence of (3.7) and Young's inequality. The passage to the limit in (3.7) is a classical matter by the Lax-Wendroff theorem since (3.7) has the usual conservative form. The principle relative to the invariant regions is due to Chueh-Conley-Smoller [12]. It has been established in Section 2 that (2.2) and (1.1) have the same Riemann invariants and invariant regions when the flux-splitting is a diagonalizable one.

After DiPerna [26], it suffices to observe, for any C^2 entropy-entropy flux pair (U, F), the dissipation measures satisfy:

$$\partial_t U(u^h) + \partial_x F(u^h) \in \text{ compact set of } \quad H^{-1}_{loc}, \tag{3.10}$$

$$\partial_t U(u^h) + \partial_x F(u^h) \leq E^h \to 0 \quad \text{in the sense of distributions as} \, h \to 0. \tag{3.11}$$

Indeed the conditions (3.10) and (3.11) follow from the entropy inequalities.

Remark 3.1.

(1) Partial results of L^∞ stability can be obtained for nondiagonalizable splittings, provided there exists at least one (not necessarily bounded in all directions) invariant region. For instance, the mass density in gas dynamics usually keeps its (non-negative) sign.
(2) Theorem 3.1 applies to not only the system of gas dynamics in Lagrangian coordinates studied in Section 2, but also the system of gas dynamics in Eulerian coordinates provided that attention is restricted to solutions without the vacuum points. Arbitrary solutions including possibly the vacuum are treated in Section 4.

We now extend the result concerning the entropy consistency in Theorem 3.1 to arbitrary flux-splitting schemes. In that case, the entropy inequality has a nonconservative form and is derived by using the notion of nonconservative product in Dal Maso-LeFloch-Murat [23] and LeFloch-Liu [41]. We first introduce some notations. Let $u_l = m^{\pm}_0, m^{\pm}_1, \cdots, m^{\pm}_N = u_r$ be the constant states in the Riemann solution $R^{\pm}(u_l, u_r)$. For each real- or vector- valued function a, the integral of a along the path naturally associated with $R^{\pm}(u_l, u_r)$ is defined by

$$\int_{R^{\pm}(u_l,u_r)} a = \sum_{k-\text{waves}} \int_0^1 a(m^{\pm}_k(s))ds, \tag{3.12}$$

where $m^{\pm}_k(s) = (1-s)m^{\pm}_{k-1} + s m^{\pm}_k$ if the wave connecting $m^{\pm}_{k-1}$ to $m^{\pm}_k$ is a shock or a contact discontinuity. If the wave is a rarefaction wave, $m^{\pm}_k$ satisfies the following equation for some $C > 0$:

$$\frac{dm^{\pm}_k(s)}{ds} = C r_k(m^{\pm}_k(s)), \qquad m^{\pm}_k(0) = m^{\pm}_{k-1}, \quad m^{\pm}_k(1) = m^{\pm}_k. \tag{3.13}$$

Given a pair (u_l, u_r), we denote the corresponding path by $\phi(u_l, u_r)$. It is not hard to see that the family of paths ϕ satisfies the assumption made in [23]. Henceforth, a nonconservative product $\alpha(u)\frac{du}{dx}$ of a function of bounded variation u by the measure $\frac{du}{dx}$ is well defined as a locally bounded Borel measure. Following the notation in [23], we denote this measure by $\left[\alpha(u)\frac{du}{dx}\right]_\phi$. Observe that the product depends upon the choice of the family of paths. Using the paths, described above, based on the Riemann solution is essential in what follows.

Theorem 3.2. *Let* (1.1) *be a system of N conservation laws and let* (3.1)–(3.3) *be a scheme based on an arbitrary flux-splitting for* (1.1). *Suppose that the set $\mathcal{O}$ is small enough. Then the following local discrete entropy inequalities hold:*

$$U(u^\pm_{j+1/2}) - U(u^n_j) + 2\lambda \int_{R^\pm(u^n_j, u^n_{j+1})} \nabla U \nabla f^\pm \leq 0, \tag{3.14}$$

$$U(u^{n+1}_j) - U(u^n_j) + \lambda \int_{R^+(u^n_{j-1}, u^n_j)} \nabla U \nabla f^+ + \lambda \int_{R^-(u^n_j, u^n_{j+1})} \nabla U \nabla f^- \leq 0. \tag{3.15}$$

Theorem 3.2 provides a weak form of the entropy inequality (3.7). Note that (3.15) has a nonconservative form and generally it is unclear whether (1.4) can be deduced from (3.15) even under the assumption of strong convergence of the scheme.

Proof of Theorem 3.2. The proof is similar to the one of (3.7) in Theorem 3.1. We have to prove that the solution $u^\pm$ to the Riemann problem $R^\pm(u_l, u_r)$ satisfies the following nonconservative version of the entropy inequality (3.4):

$$\partial_t U(u^\pm) + \left[\nabla U \nabla f^\pm \partial_x u\right]_\phi \leq 0. \tag{3.16}$$

Inequalities (3.14) and (3.15) follow by integration from (3.16).

In view of the definition of the paths in (3.12), the inequality (3.16) clearly holds in the rarefaction fans since a rarefaction fan is a smooth solution to (1.1). We only need to check (3.16) across a shock discontinuity. By computing its Taylor expansion, we now prove that

$$\Omega(u_r) = -\sigma(u_l, u_r)(U(u_r) - U(u_r)) - \int_0^1 \nabla U(u_s) \nabla f^\pm(u_s)(u_r - u_l)\, ds \leq 0, \tag{3.17}$$

provided that $|u_r - u_l|$ is small enough with fixed u_l and the left and right states satisfy the Rankine-Hugoniot relation as well as the entropy inequality (1.4).

One has

$$\Omega(u_r) = \int_0^1 \nabla U(u(s))\big(- \sigma(u_l, u_r) I - \nabla f^\pm(u(s))\big)(u_r - u_l)\, ds$$

since $u'(s) = u_r - u_l$, where I is the unit matrix. The Rankine-Hugoniot condition yields the relation

$$\int_0^1 \big(- \sigma(u_l, u_r) I - \nabla f^\pm(u(s))\big)(u_r - u_l)\, ds,$$

so that

$$\Omega(u_r) = \int_0^1 \big(\nabla U(u(s)) - \nabla U(u_l)\big)\big(- \sigma(u_l, u_r) I - \nabla f^\pm(u(s))\big)(u_r - u_l)\, ds.$$

At this stage we use the *linearity* of the path $u(s)$. Setting $u(s,\tau)=(1-\tau)u_l+\tau u(s)$, we have

$$\Omega(u_r)=\int_0^1\int_0^1 (u_r-u_l)^\top \nabla^2 U(u(s,\tau))\big(-\sigma(u_l,u_r)I-\nabla f^\pm(u(s))\big)(u_r-u_l)\tau s\,dsd\tau,$$

by using $u(s,\tau)-u_l=\tau s(u_r-u_l)$. Finally, after decomposition of u_r-u_l on the basis of eigenvectors for $\nabla f^\pm$: $u_r-u_l=\sum_{1\le k\le N}\alpha_k(s)r_k^\pm(u(s))$, we get the following formula

$$\Omega(u_r)=\sum_{1\le k\le N}\int_0^1\int_0^1 r_k^\pm(u(s))^\top\nabla^2U(u(s,\tau))\big(-\sigma(u_l,u_r)-\lambda_k^\pm(u(s))\big)r_k^\pm(u(s))\tau s\,dsd\tau. \tag{3.18}$$

Consider a genuinely nonlinear i–characteristic field. Then it is known from Lax [36] that

$$u_r=u_l+\epsilon_i r_i^\pm(u_l)+O(\epsilon_i^2),\quad \sigma(u_l,u_r)=\lambda_i^\pm(u_l)+\epsilon_i\nabla\lambda_i^\pm(u_l)r_i^\pm(u_l)+O(\epsilon_i^2),$$

where $\epsilon_i\le 0$. Therefore, (3.18) gives

$$\Omega(u_r)=(\epsilon_i)^3\big(\nabla\lambda_i^\pm(u_l)r_i^\pm(u_l)\big)r_i^\pm(u(s))^\top\nabla^2U(u(s,\tau))r_k^\pm(u(s)),$$

which is clearly non-positive.

The case of a linearly degenerate characteristic field is trivial, since then $\Omega(u_r)$ vanishes identically. When two or more eigenvalues coincide, part of the integral in (3.18) vanishes while the other is nonpositive. This completes the proof of (3.17).

An inequality similar to (3.15) was proved in [49] for the so-called Osher-Solomon scheme. Although this scheme is not quite a flux-splitting, it nevertheless shares many of their properties. Observe that the inequality in [49] has a conservative form, however.

Furthermore some of our results in this section, especially Theorem 3.2, remain valid for the class of schemes of the form [19]:

$$u_j^{n+1}=u_j^n-\lambda\int_{u_{j-1}^n}^{u_j^n}A^+(\bar u)d\bar u-\lambda\int_{u_j^n}^{u_{j+1}^n}A^-(\bar u)d\bar u,$$

where the matrix-valued functions $A^\pm$ satisfy

$$A^+(u)+A^-(u)=\nabla f(u),\quad \text{for all } u \quad \text{in } \mathcal{O},$$

and the above integrals are performed along a given family of paths in $\mathbf{R}^N$, say $\phi(s;u_j^n,u_{j+1}^n)$, $s\in[0,1]$, for all (u_j^n,u_{j+1}^n): for instance,

$$\int_{u_j^n}^{u_{j+1}^n}A^+(\bar u)d\bar u=\int_0^1 A^+(\phi(s;u_j^n,u_{j+1}^n))\partial_s\phi(s;u_j^n,u_{j+1}^n)ds.$$

Note that in this case the underlying systems have the following nonconservative form:

$$\partial_t v+A^+(v)\partial_x v=0 \qquad \text{and} \qquad \partial_t w+A^-(w)\partial_x w=0.$$

This idea of paths is due to [23], in which the hyperbolic systems in nonconservative form are studied. This approach was proposed and investigated recently in [19] for designing new schemes to treat systems with linearly degenerate fields, which is the general case for most physical systems.

4. A Family of Entropy Flux-Splittings for the Euler Equations

In this section we prove that the system of isentropic Euler equations admits a family of genuine entropy flux-splittings for the mechanical energy. The example in Theorem 4.1 can be regarded as an "entropy modification" of the Van Leer's splitting. Our formula is very similar to the one in [38]: a polynomial form with a limited number of terms. Next we prove in Theorem 4.2 that there exists a unique genuine entropy flux-splitting that satisfies *all* of the entropy inequalities. This splitting is diagonalizable and is uniquely determined by two nontrivial independent entropies. When $\gamma = 1 + \frac{1}{2m+1}$, $m \geq 2$ integer, it is actually a polynomial function of the Mach number. Finally we check in Theorem 4.4 that the latter splitting can be also derived by using the kinetic formulation in [44].

We restrict to the case of a polytropic perfect gas with adiabatic exponent $1 < \gamma \leq 3$. We refer to [9] for the treatment of the real gas case and the system of non-isentropic gas dynamics. Our results here, as well as in [9] on the real gas, are expected to extend to the situations where no kinetic formulation is available. Furthermore, we can argue that our variant to the Van Leer splitting is fully explicit. Kinetic splittings are often given by integral formulae that can be integrated out for special values of the adiabatic exponent γ only.

We consider the system of isentropic gas dynamics [21]:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0, \end{cases} \tag{4.1}$$

where $\rho \geq 0$ and v are the density and the velocity of the gas, respectively. For polytropic gas, $p(\rho) = \frac{\rho^\gamma}{\gamma}$ with $\gamma \geq 1$. System (4.1) has the form (1.1) with $u = (\rho, \rho v)^\top$ and $f(u) = (\rho v, \rho v^2 + p(\rho))^\top$. The following notations will be useful: $m = \rho v$ (moment), $c = \sqrt{p'(\rho)} = \rho^\theta$ (sound speed), $M = v/c$ (Mach number), and $\theta = \frac{\gamma-1}{2}$. Eigenvalues and a set of eigenvectors for the system (4.1) are given by $\lambda_1 = v - c$, $\lambda_2 = v + c$, $r_1 = (1, v-c)^\top$, and $r_2 = (1, v+c)^\top$. The flux can be written in the form:

$$f(u) = \begin{pmatrix} \rho c(\rho) a(M) \\ \rho c(\rho)^2 b(M) \end{pmatrix}, \quad a(M) = M, \quad b(M) = M^2 + \frac{1}{2\theta + 1}. \tag{4.2}$$

System (4.1) is strictly hyperbolic everywhere except at the points where $p'(\rho) = 0$, which occurs only at the vacuum state $\rho = 0$.

Our first objective is to derive a large family of entropy flux-splittings for (4.1). For simplicity, we are mostly interested in the case that the flux functions $f^\pm$ have the form:

$$f^\pm(u) = \begin{pmatrix} \rho c(\rho) a^\pm(M) \\ \rho c(\rho)^2 b^\pm(M) \end{pmatrix}. \tag{4.3}$$

The actual restriction is that the functions a and b are independent of ρ. This choice is consistent with (4.2). For a real gas, the functions a, $a^\pm$, etc, also depend upon ρ (cf. [9]). For a γ-law gas, one can construct indeed more general splittings than (4.3). Since

$$a^+(M) + a^-(M) = M, \quad b^+(M) + b^-(M) = M^2 + \frac{1}{2\theta + 1}, \tag{4.4}$$

it is enough to determine one of the two sets of functions $(a^\pm, b^\pm)$.

The following theorem describes all the possible entropy splittings. For definiteness, we consider those for the physical entropy $U_*(\rho, m) = m^2/(2\rho) + \rho^\gamma/(\gamma(\gamma-1))$. The corresponding entropy flux is $F_*(\rho, m) = m^3/(2\rho^2) + m\rho^{\gamma-1}/(\gamma - 1)$. The result below can be easily extended to any entropy functions.

Theorem 4.1. *Consider the Euler system* (4.1) *of isentropic gas dynamics for a γ-law gas. The entropy flux-splittings $f = f^+ + f^-$ for the mechanical energy U_* of the form (4.3) are given by:*

$$\begin{cases} a^+(M) = (3\theta+1)k(M) - \theta M k'(M), \\ b^+(M) = (3\theta+1)Mk(M) + (1-\theta M^2)k'(M), \end{cases} \tag{4.5}$$

where $k : \mathbf{R} \to \mathbf{R}$ is any function of class $W^{2,\infty}$ such that (2.1b) *holds. The splitting is genuine if, moreover,*

$$k(M) = \begin{cases} 0, & \text{for } M \le -M_*, \\ \frac{M}{2\theta+1}, & \text{for } M \ge M_* \end{cases} \tag{4.6}$$

for some $M_* > 0$. If $k(M)$ is of class C^{m+1}, the splitting is of class C^m. In particular the following explicit formula defines such a genuine entropy flux-splitting of class C^1:

$$\begin{cases} a^+(M) = \alpha\big((\theta+1)M^2 + 2(2\theta+1)AM + (3\theta+1)A^2\big), \\ b^+(M) = \alpha\big((\theta+1)M^3 + 2(2\theta+1)AM^2 + (3\theta+1)A^2)M + 2M + 2A\big), \end{cases} \tag{4.7}$$

where $1/\alpha = 4(2\theta+1)A^2$ and $A > 0$ is any number such that $A \ge 1/\theta$.

It can be checked that the classical Steger-Warming splitting and the Van Leer splitting are not entropy splittings even for another entropy rather than U_*. One can check that $k(M)$ in Theorem 4.1 can not be taken as a linear function. However, $k(M)$ can be taken as a quadratic function to recover a variant of the Van Leer splitting.

Our next result concerns the existence and uniqueness of an entropy flux splitting that satisfies *all* the entropy inequalities for the Euler equations.

Theorem 4.2.

(1) *The system (4.1) for a γ-law gas admits a unique genuine entropy flux-splitting $f(u) = f^+(u) + f^-(u)$ with the form* (4.3) *satisfying all the entropy inequalities.*

(2) *This splitting admits a closed integral form (given after the statement of the theorem) and satisfies*

$$\begin{cases} f^+(u) = f(u), \quad f^-(u) = 0, & \text{if} \quad M = v/c \ge 1/\theta, \\ f^+(u) = 0, \quad f^-(u) = f(u), & \text{if} \quad M = v/c \le -1/\theta. \end{cases} \tag{4.8}$$

When $1 < \gamma \le 3$, the functions $(a^\pm, b^\pm)$ are of class $C^{\frac{\gamma+1}{2(\gamma-1)}}(\mathbf{R})$ and the systems with flux functions f^+ and f^- are strictly hyperbolic in the regions $\{M > -1/\theta\}$ and $\{M < 1/\theta\}$, respectively. When $\gamma = 1 + \frac{2}{2m+1}$, $m \ge 1$ integer, the functions $a^\pm$ and $b^\pm$ in the interval $(-1/\theta, 1/\theta)$ are polynomial functions of degrees $2m+2$ and $2m+3$, respectively.

(3) *This splitting can be characterized as the only splitting that satisfies at least the entropy inequalities associated with the two entropies: $U_*(\rho, m)$ and $U_{**}(\rho, m) = m/\rho$. In other words, two nontrivial entropies are sufficient to select the unique splitting.*

(4) *This splitting is also the only splitting that is diagonalizable such that the matrices $\nabla f^\pm$ have the same eigenvectors as ∇f. In other words, the latter is the only splitting satisfying all invariant regions of the system* (4.1).

We will see in the proof of Theorem 4.2 that the functions $a^+(M)$ and $b^+(M)$ are given by

$$a^+(M) = \begin{cases} 0, & \theta M \le -1, \\ c_\theta M \int_{-1}^{\theta M} \big(1 - \frac{y}{\theta M}\big)(1-y^2)^{\frac{1-\theta}{2\theta}} dy, & |\theta M| \le 1, \\ M, & \theta M \ge 1, \end{cases} \tag{4.9}$$

and

$$b^{+}(M)=\begin{cases} 0, \qquad \theta M\le -1,\\ c_\theta\int_{-1}^{\theta M}\big(M^2+1-\frac{\theta+1}{\theta^2}y^2\big)(1-y^2)^{\frac{1-\theta}{2\theta}}\,dy, \qquad |\theta M|\le 1,\\ M^2+\frac{1}{2\theta+1}, \qquad \theta M\ge 1,\end{cases} \tag{4.10}$$

with $c_\theta=\int_{-1}^{1}(1-y^2)^{\frac{1-\theta}{2\theta}}\,dy$. The expressions for the functions $a^-(M)$ and $b^-(M)$ are similar and can be deduced from (4.4).

Proof of Theorem 4.2. The Jacobian matrix of $f^\pm(u)$ is computed by differentiation of the formula (4.3):

$$Df^\pm=\begin{pmatrix} c(\theta+1)\big(a^\pm-M\frac{da^\pm}{dM}\big) & \frac{da^\pm}{dM}\\ c^2\big((2\theta+1)b^\pm-(\theta+1)M\frac{db^\pm}{dM}\big) & c\frac{db^\pm}{dM}\end{pmatrix}. \tag{4.11}$$

Let U be an arbitrary entropy for system (4.1). The matrix $\nabla^2U\nabla f^\pm$ is symmetric if and only if the coefficients a and b (we drop the indices from now on) satisfy the following first order equation:

$$\begin{aligned}\partial^2_{\rho\rho}U\,\frac{da}{dM}+c\,\partial^2_{\rho m}U\,\big((\theta+1)(M\frac{da}{dM}-a)+\frac{db}{dM}\big)\\ +c^2\,\partial^2_{mm}U\big((\theta+1)M\frac{db}{dM}-(2\theta+1)b\big)=0.\end{aligned} \tag{4.12}$$

On the other hand, any entropy U satisfies the second order hyperbolic equation:

$$\partial^2_{\rho\rho}U+2\,c\,M\,\partial^2_{\rho m}U+c^2\,(M^2-1)\,\partial^2_{mm}U=0. \tag{4.13}$$

Combining (4.12) and (4.13), we obtain

$$\begin{aligned}c\,\partial^2_{\rho m}U\,\big((\theta-1)M\frac{da}{dM}-(\theta+1)a+\frac{db}{dM}\big)\\ +c^2\,\partial^2_{mm}U\big((1-M^2)\frac{da}{dM}+(\theta+1)M\frac{db}{dM}-(2\theta+1)b\big)=0.\end{aligned} \tag{4.14}$$

Plugging the Hessian matrix ∇^2U_* of the mechanical energy in (4.17), we have

$$(1-\theta M^2)\frac{da}{dM}+(\theta+1)Ma+\theta M\frac{db}{dM}-(2\theta+1)b=0. \tag{4.15}$$

Note in passing that, plugging (4.5) in (4.15), we see that (4.15) indeed holds for any choice of function $k(M)$. This provides us with an alternative way to derive (4.5).

Plugging now the entropy U_{**} in (4.14), we have

$$(1-M^2)\frac{da^\pm}{dM}+(1+\theta)M\frac{db^\pm}{dM}-(2\theta+1)b^\pm=0. \tag{4.16}$$

Therefore, in view of (4.15) and (4.16), the functions a and b satisfy the following two equations:

$$\begin{cases}(1-\theta)M\frac{da^\pm}{dM}+(1+\theta)a^\pm-\frac{db^\pm}{dM}=0,\\ (1-M^2)\frac{da^\pm}{dM}+(1+\theta)M\frac{db^\pm}{dM}-(2\theta+1)b^\pm=0.\end{cases} \tag{4.17}$$

We now derive the conditions corresponding to the diagonalizable splittings. The flux-splitting has the form:

$$\nabla f^{\pm} = P\Lambda^{\pm}P^{-1} \equiv \begin{pmatrix} 1 & 1 \\ v-c & v+c \end{pmatrix} \begin{pmatrix} \lambda_1^{\pm} & 0 \\ 0 & \lambda_2^{\pm} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ v-c & v+c \end{pmatrix}^{-1},$$

that is,

$$\nabla f^{\pm} = \begin{pmatrix} (M+1)\lambda_1^{\pm} + (-M+1)\lambda_2^{\pm} & (-\lambda_1^{\pm} + \lambda_2^{\pm})/c \\ (M^2-1)c\lambda_1^{\pm} + (-M^2+1)c\lambda_2^{\pm} & (-M+1)\lambda_1^{\pm} + (M+1)\lambda_2^{\pm} \end{pmatrix}. \tag{4.18}$$

From (4.3) and (4.18), we obtain a set of differential equations for the coefficients $\lambda_i^{\pm}$, $a^{\pm}$, and $b^{\pm}$. Using the new unknowns $\tilde{\lambda}_i^{\pm} \equiv \frac{1}{c}\lambda_i^{\pm}$, one has

$$\begin{aligned}
(k+1)a^{\pm} - (k+1)M\frac{da^{\pm}}{dM} &= (M+1)\tilde{\lambda}_1^{\pm} + (-M+1)\tilde{\lambda}_2^{\pm}, \\
\frac{da^{\pm}}{dM} &= -\tilde{\lambda}_1^{\pm} + \tilde{\lambda}_2^{\pm}, \\
(2k+1)b^{\pm} - (k+1)M\frac{db^{\pm}}{dM} &= (M^2-1)\tilde{\lambda}_1^{\pm} + (-M^2+1)\tilde{\lambda}_2^{\pm}, \\
\frac{db^{\pm}}{dM} &= (-M+1)\tilde{\lambda}_1^{\pm} + (M+1)\tilde{\lambda}_2^{\pm}.
\end{aligned}$$

By elimination of the $\tilde{\lambda}_i^{\pm}$'s and some additional computation, we conclude that $a^{\pm}(M)$ and $b^{\pm}(M)$ must necessarily satisfy the same set of equations as before (cf. (4.17))! The proof of the part (iii) of the theorem is completed.

We now solve the system of partial differential equations (4.17). The functions $a^{\pm}$ can be determined first, and the functions $b^{\pm}$ be deduced from the $a^{\pm}$'s. Namely, $a^{\pm}$ solves the following second order equations:

$$(1 - M^2\theta^2)\frac{d^2a^+}{dM^2}(M) + \theta(\theta+1)M\frac{da^+}{dM}(M) - \theta(\theta+1)a^+ = 0, \tag{4.19}$$

and $b^+(M)$ satisfies first order equations

$$\begin{cases} \frac{db^{\pm}}{dM} = (1-\theta)M\frac{da^{\pm}}{dM} + (1+\theta)a^{\pm}, \\ (2\theta+1)b^{\pm} = (1-\theta^2)\frac{da^{\pm}}{dM} + (1+\theta)^2Ma^{\pm}. \end{cases} \tag{4.20}$$

The first equation is automatically satisfied and b^+ is an explicit function of a^+:

$$b^+(M) = \frac{1}{2\theta+1}\big\{(1-\theta^2M^2)\frac{da^+}{dM}(M) + (\theta+1)^2Ma^+(M)\big\}. \tag{4.21}$$

We now give the construction of the function $a^+(M)$; the case of $a^-(M)$ is similar. We consider the domain $\Delta = \{M \mid |M| \le 1/\theta\}$ and its complement Δ^c. We first solve the differential equation in the domain Δ^c with the following condition at infinity:

$$a^+(M) \to \begin{cases} M, & M \to \infty, \\ 0, & M \to -\infty. \end{cases} \tag{4.22}$$

It is a simple matter to check that $a \equiv M$ is then the trivial solution of (4.19) in the domain Δ. The condition at infinity (4.22) is thus sufficient to show that the splitting is genuine. We next solve the system in Δ using suitable boundary conditions deduced from (4.22). Specifically, we have to find a function a^+, a solution to (4.19), defined in the domain Δ satisfying the positivity conditions

$$\lambda_i^- \leq 0 \quad \text{and} \quad \lambda_i^+ \geq 0, \tag{4.23}$$

and the two boundary conditions at the boundary $\partial\Delta$:

$$a^+(-1/\theta) = 0, \quad a^+(1/\theta) = 1/\theta. \tag{4.24}$$

The ordinary differential equation (4.19) is singular: the coefficient of the term $\frac{d^2a^+}{dM^2}$ vanishes at the end points $M = \pm 1/\theta$. Existence of a solution to this two-point boundary problem is not a standard matter. However, it is not hard to show by a direct calculation that there is a unique solution to the boundary value problem (4.19) and (4.24) given by the following formula:

$$a^+(M) = -\frac{2M \int_{-\frac{1}{\theta}}^{M} \frac{(1-\theta^2y^2)^{\frac{\theta+1}{2\theta}}}{y^2} dy}{\theta(\theta+1)\int_{-\frac{1}{\theta}}^{\frac{1}{\theta}} (1-\theta^2y^2)^{\frac{1-\theta}{2\theta}} dy} \tag{4.25}$$

for all $M \neq 0$ in the interval $(-1/\theta, 1/\theta)$. After integration by parts in the above formula, we obtain

$$a^+(M) = \begin{cases} 0, \quad M \leq -\frac{1}{\theta}, \\ \dfrac{\left\{(1-\theta^2M^2)^{\frac{\theta+1}{2\theta}} + \theta(\theta+1)M \int_{-\frac{1}{\theta}}^{M} (1-\theta^2y^2)^{\frac{1-\theta}{2\theta}} dy\right\}}{\theta(\theta+1)\int_{-\frac{1}{\theta}}^{\frac{1}{\theta}} (1-\theta^2y^2)^{\frac{1-\theta}{2\theta}} dy}, \quad -\frac{1}{\theta} \leq M \leq \frac{1}{\theta}, \\ M, \quad M \geq \frac{1}{\theta}. \end{cases} \tag{4.26}$$

The corresponding function b^+ is

$$b^+(M) = \begin{cases} 0, \quad M \leq -\frac{1}{\theta}, \\ \dfrac{\left\{(\theta+1)^2M(1-\theta^2M^2)^{\frac{\theta+1}{2\theta}} + \theta(\theta+1)(1+(2\theta+1)M^2)\int_{-\frac{1}{\theta}}^{M} (1-\theta^2y^2)^{\frac{1-\theta}{2\theta}} dy\right\}}{\theta(\theta+1)(2\theta+1)\int_{-\frac{1}{\theta}}^{\frac{1}{\theta}} (1-\theta^2y^2)^{\frac{1-\theta}{2\theta}} dy}, \quad -\frac{1}{\theta} \leq M \leq \frac{1}{\theta}, \\ M^2 + \frac{1}{2\theta+1}, \quad M \geq \frac{1}{\theta}. \end{cases} \tag{4.27}$$

We now show that the Jacobian matrix Df^+ is non-negative. It suffices to consider the range of values $-1/\theta < M < 1/\theta$. We want to show that

$$Df^+(\rho, m) = \frac{D}{D(\rho, m)} \begin{pmatrix} a^+(M) \\ b^+(M) \end{pmatrix} > 0.$$

The eigenvalues of $Df^+(\rho, m)$ are given by:

$$\lambda_1^+ = c(\theta A + a^+ - \frac{da^+}{dM}), \quad \text{and} \quad \lambda_2^+ = c(\theta A + a^+ + \frac{da^+}{dM}),$$

with

$$A \equiv A(M) = -M\frac{da^+}{dM}(M) + a^+(M) = \frac{(1-\theta^2M^2)^{\frac{\theta+1}{2\theta}}}{\theta(\theta+1)\int_{-\frac{1}{\theta}}^{\frac{1}{\theta}} (1-\theta^2y^2)^{\frac{1-\theta}{2\theta}} dy}.$$

We observe that

$$\theta A(M) + a^+(M) \pm \frac{da^+}{dM}(M) = \frac{g(M)}{\int_{-\frac{1}{\theta}}^{\frac{1}{\theta}} (1-\theta^2 y^2)^{\frac{1-\theta}{2\theta}}\, dy},$$

where

$$g(M) = (1-\theta^2 M^2)^{\frac{1+\theta}{2\theta}} + \theta(M \pm 1) \int_{-\frac{1}{\theta}}^{M} (1-\theta^2 y^2)^{\frac{1-\theta}{2\theta}}\, dy$$

satisfies the conditions

$$\begin{cases} g(-\frac{1}{\theta}) = g'(-\frac{1}{\theta}) = 0, \\ g'(M) = \theta(1-\theta)(1+\theta^2)(1-\theta^2 M^2)^{\frac{1-3\theta}{2\theta}} \geq 0. \end{cases}$$

Therefore $\lambda_1^+ > 0$, $\lambda_2^+ > 0$, when $-\frac{1}{\theta} < M < \frac{1}{\theta}$, which proves the desired inequality for $-1/\theta < M < 1/\theta$.

Furthermore, we have

$$\lambda_2^+ - \lambda_1^+ = c\frac{da^+}{dM}(M) = \frac{\int_{-\frac{1}{\theta}}^{M} (1-\theta^2 y^2)^{\frac{1-\theta}{2\theta}}\, dy}{\int_{-\frac{1}{\theta}}^{\frac{1}{\theta}} (1-\theta^2 y^2)^{\frac{1-\theta}{2\theta}}\, dy} > 0, \quad -\frac{1}{\theta} < M < \frac{1}{\theta}.$$

It follows that the system with the flux f^+ is strictly hyperbolic for all $M > -1/\theta$. We can also check that the system with the flux f^- is strictly hyperbolic for all $M > -1/\theta$. This completes the proof of Theorem 4.2.

Finally we show that the splitting in Theorem 4.2 can be also derived by the so-called kinetic approximation. We follows here the approach in [51] for the systems of non-isentropic gas dynamics and extend the idea to our case, the isentropic Euler system.

Theorem 4.3. *The kinetic flux-splitting, from the kinetic formulation due to Lions-Perthame-Tadmor* [44]*, coincides with the entropy flux-splitting for all the entropy functions derived in Theorem 4.3.*

Proof. In view of the formulation in [44] and using the notation in this section, we obtain

$$f_1^+(u) = \int_{\{\xi \geq 0\}} (\theta\xi + (1-\theta)v) \left(\frac{c(\rho)^2}{\theta^2} - (\xi - v)^2\right)_+^{\frac{1-\theta}{2\theta}} d\xi,$$

where $z_+ = \max(0, z)$ by definition. The formula (4.31) can be written as

$$f_1^+(u) = \rho^{\theta+1}\theta^{\frac{\theta-1}{\theta}} \int_M^\infty (M + \theta y) \left(\frac{1}{\theta^2} - y^2\right)_+^{\frac{1-\theta}{2\theta}} dy \equiv \rho c(\rho) a^+(M).$$

This yields the same formula (4.9). Checking b is similar.

5. Convergence of a Flux-Splitting Scheme for the Euler Equations

Now we extend to the Euler equations the convergence result of the flux-splitting schemes in Section 3. We shall prove the strong convergence of the scheme constructed from the flux-splitting derived in Theorem 4.2. Section 3 already guarantees the convergence of the scheme provided that the initial data takes its values in a small neighborhood of a given point away from the vacuum. The result in the present section holds for arbitrary large initial data possibly containing vacuum states. It is convenient to study first the Riemann problems for the flux functions f^+ and f^-.

Theorem 5.1. *The Riemann problem with arbitrary left and right states, for the system* (2.2*a*) *(resp.* (2.2*b*)*) with the flux-function f^+ (resp. the flux f^-) given by Theorem 4.2, admits a globally defined solution composed of two elementary waves that are either a shock wave or a rarefaction wave. Each wave has a non-negative (resp. a non-positive) speed and each shock wave with speed σ connecting the left state (ρ_l, m_l) with the right state (ρ_r, m_r) satisfies the entropy inequality:*

$$\sigma(U(\rho_r, m_r) - U(\rho_l, m_l)) - (F(\rho_r, m_r) - F(\rho_l, m_l)) \geq 0, \tag{5.1}$$

for any convex entropy-entropy flux pair (U, F) of the system (4.1)*. Furthermore, the system admits bounded and convex invariant regions for the Riemann problem.*

Proof. For definitness we treat the case of the function f^+ and, for the sake of simplicity, we shall write a for a^+, etc. Since the system with flux f^+ has the same right eigenvectors as the original system (4.1), these two systems have the same set of rarefaction curves. In particular, the rarefaction curves are globally defined and convex. In order to solve the Riemann problem, we only need prove that the shock curves exist globally and are convex and that the entropy condition is satisfied along the shock curves.

We consider the shock wave curves in the phase plane (ρ, m). Let (ρ_0, m_0) be the state on the left of a discontinuity traveling with speed σ. For definiteness, we only consider the case that (ρ_0, m_0) belongs to the region $\mathcal{D} = \big\{|M| < 1/\theta\big\}$, i.e., $1 \pm \theta M_0 > 0$. Other cases are either trivial or simpler.

The state (ρ, m) on the right of the discontinuity must satisfy the Rankine-Hugoniot relations

$$\left\{\begin{array}{ll} \sigma\,(\rho - \rho_0) & = \frac{1}{2}\left(\rho^{\theta+1} a^+(\frac{m}{\rho^{\theta+1}}) - \rho_0^{\theta+1} a^+(\frac{m_0}{\rho_0^{\theta+1}})\right), \\ \sigma\,(m - m_0) & = \frac{1}{2}\left(\rho^{2\theta+1} b^+(\frac{m}{\rho^{\theta+1}}) - \rho_0^{2\theta+1} b^+(\frac{m_0}{\rho_0^{\theta+1}})\right), \end{array}\right. \tag{5.2}$$

and the Lax entropy condition [37]: $\lambda_i^+(\rho, m) \leq \sigma_i^+ \leq \lambda_i^+(\rho_0, m_0)$, $i = 1, 2$. We will focus our attention on the part of the shock curve which lies inside the region $\mathcal{D}$ and use the abreviated notation $(a, b, \lambda, \sigma) = (a^+, b^+, \lambda^+, \sigma^+)$. Other regions and other cases for M_0 are also similar or much simpler.

The equation for the state on the right of the discontinuity is

$$\frac{m - m_0}{\rho - \rho_0} = \frac{\rho^{2\theta+1} b(M) - \rho_0^{2\theta+1} b(M_0)}{\rho^{\theta+1} a(M) - \rho_0^{\theta+1} a(M_0)},$$

or equivalently

$$\frac{t^{\theta+1} M - M_0}{t - 1} = \frac{t^{2\theta+1} b(M) - b(M_0)}{t^{\theta+1} a(M) - a(M_0)}, \tag{5.3}$$

where $t = \rho/\rho_0$.

Let us check first that the equation (5.3) has two roots. They will be denoted by $M^\pm = M^\pm(t, M_0)$, for each value $t > 0$, $t \neq 1$. Clearly, one has $M^\pm(1, M_0) = M_0$. It suffices to prove that, for any $t > 0, t \neq 1$, there exists a unique $\bar{M}(t, M_0)$ such that

$$(M - \bar{M}) F_M(M, t; M_0) > 0, \quad M \neq \bar{M}, \tag{5.4}$$

where the function F is defined in view of (5.5) by

$$F(M, t; M_0) = (t^{\theta+1} M - M_0)(t^{\theta+1} a(M) - a(M_0)) - (t-1)(t^{2\theta+1} b(M) - b(M_0)).$$

The first and second order derivatives of F with respect to the variable M are

$$\begin{aligned} F_M(M, t; M_0) &= t^{\theta+1}\big\{t^{\theta+1} a(M) - a(M_0) + a'(M)(t^{\theta+1} M - M_0) - (t-1) t^\theta b'(M)\big\}, \\ F_{MM}(M, t; M_0) &= t^{\theta+1}\big\{2 t^\theta a'(M) + (t^\theta(\theta t + 1 - \theta) M - M_0) a''(M)\big\} \equiv G(M, t; M_0). \end{aligned}$$

Using the equation satisfied by the function a, we explicitly compute the first order derivative of the function $G = F_{MM}$:

$$\begin{aligned} G_M(M, t; M_0) &\equiv C_0 (1 - \theta^2 M^2)^{\frac{1-3\theta}{2\theta}} H(M, t; M_0), \\ C_0 &= \frac{2\theta(\theta+1)}{\theta(\theta+1) \int_{-\frac{1}{\theta}}^{\frac{1}{\theta}} (1 - \theta^2 y^2)^{\frac{1-\theta}{2\theta}} dy}, \\ H(M, t; M_0) &= -\theta t^\theta(\theta t + 1 + \theta) M^2 + \theta(1-\theta) M_0 M + t^\theta(\theta t + 3 - \theta). \end{aligned}$$

In the case $M_0 < 0$, we have

$$G_M(M, t; M_0) = C_0 (1 - \theta^2 M^2)^{\frac{1-3\theta}{2\theta}} H(M, t; M_0) > 0, \quad \text{for all} \quad M \in [-\frac{1}{\theta}, 0],$$

$G(-\frac{1}{\theta}, t; M_0) = 0$, and $G_M(M, t; M_0) > 0$ for all $M \geq 0$. Therefore, for $M_0 < 0$, we obtain

$$G(M, t; M_0) \geq 0. \tag{5.5}$$

In the case $M_0 > 0$, one easily checks that, for all $M > \frac{M_0}{t^\theta(\theta t + 3 - \theta)}$,

$$F_{MM}(M, t; M_0) = t^{\theta+1}\{2 t^\theta (a'(M) - M a''(M)) + (t^\theta(\theta t + 3 - \theta) M - M_0) a''(M)\} > 0,$$

and there exist two values M_1 and M_2 with $-\frac{1}{\theta} < M_1 < 0 < \frac{M_0}{t^\theta(\theta t + 3 - \theta)} < M_2 < \frac{1}{\theta}$ such that

$$G_M(M, t; M_0) > 0 \quad \text{for all} \quad M \in (M_1, M_2). \tag{5.6}$$

From (5.5), (5.6), and the relations $F_M(-\frac{1}{\theta}, t; M_0) = -t^{\theta+1} a(M_0) < 0$ and $F_M(M, t; M_0) \to \infty$ as $M \to \infty$, we deduce that there exists a point $\bar{M} \in (-\frac{1}{\theta}, \infty)$ such that

$$(M - \bar{M}) F_M(M, t; M_0) > 0. \tag{5.7}$$

This completes the proof of (5.4) and thus establishes the existence of two curves satisfying the Rankine-Hugoniot condition. In the following, we consider ρ_0 and M_0 as fixed and ρ (or equivalently $t = \rho/\rho_0$) as a parameter along the shock curve. The shock curves (actually half curves) are selected as follows:

(1) along the 1–shock curve: $t > 1$, $M > M_0$,
(2) along the 2–shock curve: $t > 1$, $M < M_0$.

Next we show that the shock curves are convex. It is convenient to show that the slope of the curve in (ρ, m) is a monotone function of t. By a lengthy but direct computation, we obtain the following expression for the derivative of the slope $(m-m_0)/(\rho-\rho_0)$:

$$\frac{d\left(\frac{m-m_0}{\rho-\rho_0}\right)}{d\rho} = \frac{t^{\theta+1}a'(M)}{\rho_0(t^{\theta+1}a(M)-a_0(M))^4 F_M}\left\{t^{2\theta}-\left(\frac{t^\theta M - M_0}{t-1}\right)^2\right\},$$

where $M^\pm = M^\pm(t, M_0)$. One can check by elementary calculations that

$$t^{2\theta}-\left(\frac{t^\theta M^\pm(t,M_0) - M_0}{t-1}\right)^2 \neq 1, \quad \text{for all} \quad t \neq 1,$$

which implies that

$$\frac{d}{d\rho}\left(\frac{m-m_0}{\rho-\rho_0}\right) \neq 0. \tag{5.8}$$

This implies that the two shock curves are convex in the $\rho - m$ plane.

Furthermore, after a tedious but direct computation, the derivative of the shock speed $\sigma(\rho)$ along the shock curve is

$$\begin{aligned} 2\sigma'(\rho) &= \frac{d}{d\rho}\left(\frac{\rho^{\theta+1}a(M)-\rho_0^{\theta+1}a(M_0)}{\rho-\rho_0}\right) \\ &= \frac{\rho_0^{\theta-1}t^{\theta+1}}{F_M}\left\{\left(\frac{t^{\theta+1}a(M)-a(M_0)}{t-1}+t^\theta(\theta M a'(M)-(\theta+1)a(M))\right)^2 - t^{2\theta}(a'(M))^2\right\}. \end{aligned}$$

It can be checked that the term in the left hand side never vanishes for $t \neq 1$. In view of the Lax entropy condition (5.3), we thus obtain

(1) along the 1-shock curve: $\rho > \rho_0, \quad \sigma'(\rho) \leq 0$,
(2) along 2-shock curve: $\rho < \rho_0, \quad \sigma'(\rho) \geq 0$.

Therefore the Riemann problem for the system with flux f_+ and arbitrary initial states admits a global solution satisfying the entropy condition (5.1). This completes the proof of Theorem 5.1.

Finally we prove the convergence of the flux-splitting scheme.

Theorem 5.2. *Consider the Cauchy problem for the system of gas dynamics in Eulerian coordinates* (4.1) *satisfying* $0 \leq \rho^h \leq M$ *and* $0 \leq |v_0(x)| \leq M$ *for some* M. *Let* (ρ^h, m^h) *be the approximate solutions given by the flux-splitting scheme* (3.1)-(3.3) *and Theorem 4.1. Then the sequence* (ρ^h, m^h) *converges strongly to an entropy weak solution to* (4.1).

Proof. We use the compactness framework developed in [6,25]. It is sufficient to prove that the approximate solutions $u^h \equiv (\rho^h, m^h)$ satisfy the following three conditions:

(A_1). $0 \leq \rho^h, |\frac{m^h}{\rho^h}| \leq M < \infty$,

(A_2). $\partial_t U(u^h) + \partial_x F(u^h) \in$ compact set of H^{-1}_{loc},

(A_3). $\partial_t U(\rho^h, m^h) + \partial_x F(\rho^h, m^h) \leq E^h \to 0$ in the sense of distributions as $h \to 0$, for any C^2 convex weak entropy pair (U, F).

The condition (A_1) is a consequence of the convex decomposition (3.6) and the property of the Riemann solutions. We recall that the Riemann problem admits bounded convex invariant regions.

In order to check the conditions (A_2) and (A_3), we study the dissipation measures $\partial_t U(u^h) + \partial_x F(u^h)$ for any C^2 convex weak entropy pair (U, F). For any $\phi \geq 0, \phi \in C_0^1(\Omega_T)$ with $\Omega_T = (-\infty, \infty) \times [0, T), T = mh$, we can use Green's formula and obtain

$$\begin{aligned} &\iint_{\Omega_T} \big(U(u^h)\partial_t\phi + F(u^h)\partial_x\phi\big)\,dxdt + \int_{\mathbf{R}} \phi(0,x)U(u^h(0,x))dx \\ &= S^h(\phi) + L_1^h(\phi) + L_2^h(\phi) + M^h(\phi), \end{aligned} \tag{5.9}$$

with

$$\begin{aligned} S^h(\phi) &= \int_0^T \sum \{\sigma[U] - [F]\}\phi(x(t),t)dt, \\ L_1^h(\phi) &= \sum_{j,n} \phi_j^n \int_{x_{j-1/2}}^{x_{j+1/2}} (U(u_-^{t_n}) - U(u_j^n))dx, \\ L_2^h(\phi) &= \sum_{j,n} \int_{x_{j-1/2}}^{x_{j+1/2}} (U(u_-^{t_n}) - U(u_j^n))(\phi - \phi_j^n)dx, \\ M^h(\phi) &= \int_{\mathbf{R}} \phi(x,T)U(u^h(x,T))dx - \int_{\mathbf{R}} \phi(0,x)U(u^h(0,x))dx, \end{aligned}$$

where $u_-^{t_n} = u^h(t_n - 0, x) = \frac{1}{2}\big(u_-^{+h}(t_n - 0, x) + u_-^{-h}(t_n - 0, x)\big)$, $\phi_j^n = \phi(t_n, x_j)$, and the summation in $S^h(\phi)$ is taken (for each $t > 0$) over all shock waves in $u^h(t)$ with location $x(t)$.

Due to the finiteness of the propagation speed of the approximate solutions, we can assume without loss of generality that the initial data $(\rho_0(x), m_0(x))$ have compact support. Say $u^h = (\rho^h, m^h)$ are compactly supported in the region Ω_T and $\int_{-\infty}^{\infty} U_*(u_0(x))dx \leq C < \infty$. For simplicity in the notation, we drop the index h in $u^h(t,x)$. Substitute $(U, F, \phi) = (U_*, F_*, 1)$ in the identity (5.9). We find

$$\begin{aligned} &\sum_{j,n} \int_{x_{-1/2}}^{x_{j+1/2}} (U_*(u_-^{t_n}) - U_*(u_j^n))dx + \int_0^T \sum \{\sigma[U_*] - [F_*]\}\phi(t, x(t))dt \\ &= \int_{\mathbf{R}} U_*(u^h(0,x))dx - \int_{\mathbf{R}} U_*(u^h(x,T))dx \leq \int_{\mathbf{R}} U_*(u_0(0,x))dx \leq C. \end{aligned}$$

Note that U_* is a convex entropy for both splitting systems and the original system. The Riemann solutions are constants in one of half cells $(x_{j-1}, x_{j-1/2}) \times (t_n, t_{n+1})$ and $(x_{j-1/2}, x_j) \times (t_n, t_{n+1})$, and satisfy the entropy condition : $\sigma[U_*] - [F_*] \geq 0$ across the shock waves. Moreover, using the averaging feature of the splitting scheme, we have

$$\begin{aligned} &\sum_{j,n} \phi_j^n \int_{x_{j-1/2}}^{x_{j+1/2}} (U_*(u_-^{t_n}) - U_*(u_j^n))dx \\ &= \sum_{j,n} \phi_j^n \int_{x_{j-1/2}}^{x_{j+1/2}} dx \int_0^1 (1-\theta)(u_-^{t_n} - u_j^n)^\top \nabla^2 U_*(u_j^n + \theta(u_-^{t_n} - u_j^n))(u_-^{t_n} - u_j^n)d\theta \geq 0. \end{aligned}$$

We obtain

$$\begin{aligned} &\int_0^T \sum \{\sigma[U_*] - [F_*]\}\phi(t, x(t))dt \leq C, \\ &\sum_{j,n} \phi_j^n \int_{x_{j-1/2}}^{x_{j+1/2}} dx \int_0^1 (1-\theta)(u_-^{t_n} - u_j^n)^\top \nabla^2 U_*(u_j^n + \theta(u_-^{t_n} - u_j^n))(u_-^{t_n} - u_j^n)d\theta \leq C. \end{aligned}$$

In particular, we have

$$\sum_{j,n}\int_{x_{j-1/2}}^{x_{j+1/2}}|u_-^{t_n}-u_j^n|^2dx\le C.$$

Consider a convex entropy U of the systems (4.1). The Riemann solutions take constant values in one of the half cells $(x_{j-1},x_{j-1/2})\times(t_n,t_{n+1})$ and $(x_{j-1/2},x_j)\times(t_n,t_{n+1})$ and satisfy the entropy condition. Thus, for any $\phi\ge 0,\phi\in C_0^1(\Omega_T)$, one has $S^h(\phi)\ge 0$, and

$$\begin{aligned}&\sum_{j,n}\phi_j^n\int_{x_{j-1/2}}^{x_{j+1/2}}(U(u_-^{t_n})-U(u_j^n))dx\\&=\sum_{j,n}\phi_j^n\int_{x_{j-1/2}}^{x_{j+1/2}}dx\int_0^1(1-\theta)(u_-^{t_n}-u_j^n)^\top\nabla^2U(u_j^n+\theta(u_-^{t_n}-u_j^n))(u_-^{t_n}-u_j^n)d\theta\ge 0.\end{aligned}$$

Moreover, for any $\phi\in C_0^\alpha(\Omega_T)$, $\frac{1}{2}<\alpha<1$, we have

$$\begin{aligned}|L_2^h(\phi)|&\le\sum_{j,n}\int_{x_{j-1/2}}^{x_{j+1/2}}|\phi(x,t_n)-\phi_j^n||U(u_-^{t_n})-U(u_j^n)|dx\\&\le h^{\alpha-1/2}\|\nabla U\|_{L^\infty}\|\phi\|_{C_0^\alpha}\left(\sum_{j,n}\int_{x_{j-1/2}}^{x_{j+1/2}}|u_-^{t_n}-u_j^n|^2dx\right)^{1/2}\\&\le Ch^{\alpha-1/2}\|\phi\|_{C_0^\alpha}\le Ch^{\alpha-1/2}\|\phi\|_{W_0^{1,p}},\quad p>\frac{n}{1-\alpha};\end{aligned}$$

that is,

$$\|L_2^h(\phi)\|_{W^{-1,q_0}}\le C\,h^{\alpha-1/2}\to 0,\ (h\to 0),\quad 1<q_0<\frac{n}{n-1+\alpha}<\frac{n}{n-1}.$$

We have $|M^h(\phi)|\le C\|\phi\|_{C_0}$, which implies

$$M^h\ \in\ \text{compact set of}\ W^{-1,q_0}.$$

Note that

$$E^h\equiv S^h+L_1^h+L_2^h+M^h\in\ \text{bounded set of}\ {}^{-1,q},\quad 1<p\le\infty,$$

so that one has

$$\begin{aligned}&E^h-L_2^h-M^h\in\ \text{bounded set of}\ W^{-1,q_0},\\&E^h-L_2^h-M^h\ge 0\quad\text{in}\ W^{-1,q_0}.\end{aligned}$$

Using Murat's Lemma [46] that the embedding of the positive cone of W^{-1,q_0} in W^{-1,q_1} is completely continuous for all $q_1<q_0$, we have

$$E^h-L_2^h-M^h\ \in\ \text{compact set of}\ W^{-1,q_1}.$$

Therefore we obtain

$$E^h\equiv S^h+L^h+M^h\ \in\ \text{compact set of}\ W^{-1,q_1}.$$

Using a lemma in [25], we have that the embedding of a compact set of $W^{-1,q}$ and a bounded set of $W^{-1,r}$ is compact in H^{-1} for $1<q\le 2<r<\infty$. From the above, we deduce that

$$S^h+L^h+M^h\in\ \text{compact set of}\ H_{loc}^{-1}.$$

This proves the condition (A_2). The condition (A_3) can be obtained similarly as in [6].

Acknowledgements. This work was initiated when the first author visited the Courant Institute and the second author was a Courant instructor from 1990 to 1992. The authors are grateful to Jonathan Goodman, Robert Kohn, Peter D. Lax, and Cathleen Morawetz for their interest in this work and motivating discussions. This work also benefited from stimulating discussions with Constantine M. Dafermos during a visit of the authors to Brown University. The authors also thank D. Serre for his observations. Gui-Qiang Chen's research was supported in part by NSF grant DMS-9207080, ONR grant N00014-91-J-1384, and by an Alfred P. Sloan Foundation Fellowship. Philippe LeFloch's research was supported in part by NSF grant DMS-92-09326 and ONR grant N00014-91-J-1384 when he visited the University of Chicago, and by NSF grant DMS-88-06731 when he was at Courant Institute.

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