A NUMERICAL STUDY OF A SPECTRAL PROBLEM IN SOLID-FLUID TYPE STRUCTURES

by

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ABSTRACT

This article presents a numerical study of a spectral problem which models the vibrations of a solid-fluid structure. It is a quadratic eigenvalue problem involving incompressible Stokes equations. In its numerical approximation we use Lagrange finite elements. To approximate the velocity, degree 2 polynomials on triangles were used, and for the pressure, degree 1 polynomials. The numerical results obtained confirm the theory, as they show in particular that the known theoretical bound for the maximum number of non-real eigenvalues admitted by such a system is optimal. The results also take account of the dependence of vibration frequencies with respect to determined physical parameters which have a bearing on the model.

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0. INTRODUCTION

This article aims to solve numerically a mathematical model which describes the vibrations of a solid-fluid type system. The problem of determining the vibratory eigenfrequencies and eigenmotions of this type of system has considerable importance in engineering, as this occurs naturally in the design and simulation of various sorts of industrial equipment. In recent decades much effort has been devoted to experimental and theoretical research into this type of problem, examples of which include the papers of R. Blevins [1977], [1984], H.J. Connors [1950], S.S. Chen [1975a], [1975b], D.J. Gorman [1977], M.J. Pettigrew [1981], M. Paidoussis [1982], and J. Planchard [1980], [1982], [1985], [1987]. In particular, the model which we study describes the vibrations of a bundle of K metallic tubes immersed in an incompressible viscous fluid. Among the publications which study this problem using similar mathematical techniques and models, one might cite the joint works of F. Aguirre & C. Conca [1988], C. Conca, J. Planchard & M. Vanninathan [1988], [1989] and [1990].

From the mathematical point of view, the model in question is a non-standard differential eigenvalue problem which involves the stationary Stokes equations. The variational formulation of this model is a quadratic eigenvalue problem whose coefficients are selfadjoint linear operators, acting on an infinite dimensional Hilbert space. In C. Conca, M. Durán & J. Planchard [1992a], a result is proved of existence of eigenvalues and eigenvectors of the model and it is shown that the spectrum of the problem consists of a countable infinite quantity of complex eigenvalues which converge in modulus to infinity. Furthermore, it is shown that the number of eigenvalues with a non-zero imaginary part is finite. An estimate of the number of non-real eigenvalues is provided in [C.,D.,P. 1992b], where it is shown that the spectrum of the solid-fluid system allows at most 4K imaginary eigenvalues. The mathematical techniques used to prove the above consist, in the first case, of identifying the unknown eigenvalues with the characteristic values of a non-selfadjoint compact operator, and using classical spectral theorems of functional analysis. In the second case, to obtain the estimate, we use abstract results concerning the variation of the spectrum of a bounded linear operator perturbed by a non-negative operator, of finite range. More details can be found in the above cited works and the references contained therein.

In the first section we present a summary of the above. The problem is then solved numerically. To this end a mixed variational formulation of the differential problem is introduced, in which the condition of incompressibility of the fluid, or of zero divergence, is dealt with implicitly. Then, the problem is approximated by the discretization of the Hilbert spaces involved. We approximate the velocity and pressure through continuous functions, which on each element of the triangulation of the domain are degree 1 and degree 2 polynomials, respectively. The bases chosen for the finite dimensional spaces are those which are usually used in the finite elements method. The family of triangulations of the domain is constructed in such a way as to satisfy a pair of conditions of non-degeneracy and regularity.

The discrete problem becomes, equivalently, a quadratic eigenvalue matrix problem. Using classical techniques, we show that this can be reduced to a generalized eigenvalue problem of the Sylvester type, which can be dealt with numerically using standard computer software packages for numerical calculations.

The numerical experiments were carried out in a test geometry with a single tube, using three domain triangulations. The numerical evidence collected confirms the previous theoretical results, in the sense that a countable spectrum was obtained which converges to infinity and which admits non-real eigenvalues, and which can in fact be found in the region of the complex plane indicated theoretically. Furthermore, this numerical experience provides valuable information about the spectral variation of the problem. In particular, it clearly shows the dependence which exists between the imbedded rigidity of the bundle of tubes (which we represent by k) and the non-real part of the spectrum of the problem. In this context, the numerical experiments take account of the bifurcation to the complex plane of a set of real eigenvalues which collapse together for certain threshold values of k. The insensitivity with respect to k of the higher eigenvalues should also be noted. The above can be seen from simple inspection of the result tables. Finally, the graphs are presented of velocity fields and isobaric lines of some of the structure's vibratory eigenmotions.

§1. MATHEMATICAL FORMULATION OF THE PROBLEM

In this section we present a summary of the deduction and theoretical study of the mathematical model which describes the physical problem in question. Firstly, the physical hypotheses assumed for the system are established, and the mathematical model which this represents is deduced. Then, the differential problem is rigorously formulated, and finally, the most important theorems of existence and location are summarised.

1.1 The physical problem. The physical problem which interests us is the study of the vibrations of a solid-fluid type system. To be more exact, the problem consists of determining the vibratory eigenfrequencies and eigenmotions of a bundle of metallic tubes immersed in an incompressible viscous fluid.

The fluid is assumed to be contained in a three dimensional cavity with rigid walls. It is assumed that they are parallel to each other, that they are perfectly rigid (they do not allow deformations) and that they are elastically mounted in such a way that they can only vibrate on a transverse plane, perpendicular to the bundle. Furthermore, axial effects are not considered and it is assumed that the tubes are of infinite length. The problem is then studied in two dimensions, restricting it to any of the sections of the cavity which are perpendicular to the tubes.

With respect to dynamics, it is assumed that the solid-fluid system undergoes small vibrations around a state of equilibrium.

1.2 Formulation of the eigenvalue problem. Let Ω_0 be an open bounded subset of $\mathbb{R}^2$, with a locally Lipschitz continuous boundary Γ_0 (see J. NEČAS [1967] Chapter I) and let $\{\Theta_i\}_{i=1,K}$ be a family of K open subsets of Ω_0, which has the following properties:

(1.1a) $\quad \forall i = 1, ..., K, \quad \Theta_i$ is a non-empty connected open subset of Ω_0.

(1.1b) $\quad \forall i = 1, ..., K, \quad \bar{\Theta}_i \subset \Omega_0$.

(1.1c) $\quad \forall i \neq j, \quad \bar{\Theta}_i \cap \bar{\Theta}_j = \phi$.

(1.1d) $\quad$ Each Θ_i has a locally Lipschitz boundary Γ_i.

Using the above notation we can define Ω as follows:

$$\Omega = \Omega_0 \setminus \bigcup_{i=1}^{K} \bar{\Theta}_i.$$

It should be observed that the boundary of Ω has $(K+1)$ connected components, which are $\Gamma_0, \Gamma_1, ..., \Gamma_K$.

The mathematical model which describes the solid-fluid interaction is a differential eigenvalue problem with non-local boundary conditions on the velocity. In the model, the tube sections are represented by the perforations $\{\Theta_i\}_{i=1,K}$, and the domain Ω represents the area occupied by the fluid. If the velocity of the fluid is denoted by $\mathbf{u}_0 = \mathbf{u}_0(x,t)$ and the pressure by $p_0 = p_0(x,t)$, then $(\mathbf{u}_0, p_0)$ satisfies:

$$\frac{\partial \mathbf{u}_0}{\partial t} - 2\nu \text{ div } e(\mathbf{u}_0) + \nabla p_0 = \mathbf{0} \quad \text{in } \Omega \tag{1.2a}$$

$$\text{div } \mathbf{u}_0 = 0 \quad \text{in } \Omega \tag{1.2b}$$

$$\mathbf{u}_0 = \mathbf{0} \quad \text{on } \Gamma_0 \tag{1.2c}$$

$$\mathbf{u}_0 = \frac{d\mathbf{s}_i}{dt} \quad \text{on } \Gamma_i, \quad \forall i = 1, ..., K, \ \forall t \in \mathbb{R}, \tag{1.2d}$$

where, in (1.2a), ν represents the kinematic viscosity of the fluid (ν is a strictly positive given constant) and $e(\mathbf{u}_0)$ is the linear strain tensor, defined by

$$2e(\mathbf{u}_0) = \nabla \mathbf{u}_0 + (\nabla \mathbf{u}_0)^t \ .$$

In equation (1.2d), $\mathbf{s}_i$ is the transverse displacement vector of the ith tube, which, due to the physical assumptions made, only depends on t. If it is furthermore assumed that there is no interaction between the tubes, and given that small oscillations are being considered around a state of equilibrium, the movement of the ith tube obeys a simple harmonic oscillation with a forced term, implied by its interaction with the fluid. Thus $\mathbf{s}_i$ satisfies the equation:

$$(1.2e) \qquad m_i \frac{d^2\mathbf{s}_i}{dt^2} + k_i\mathbf{s}_i = -\int_{\Gamma_i} \sigma(\mathbf{u}_0, p_0)\mathbf{n}ds,$$

where m_i is the mass per length unit of tube i and k_i is a strictly positive real constant, which represents the stiffness constant of the spring system supporting the ith tube (see J. PLANCHARD [1980]). The term $\sigma(\mathbf{u}_0, p_0)$ represents the stress tensor of the system. This satisfies Stokes's law:

$$(1.3) \qquad \sigma(\mathbf{u}_0, p_0) = -p_0 I + 2\nu\ e(\mathbf{u}_0),$$

where I is the identity matrix. Finally, in (1.2e), $\mathbf{n}$ represents the outward unit normal on the boundary of Ω. The equations (1.2c,d,e) describe the interactions between the fluid and the tubes. In particular, they model the fact that the fluid, being viscous, adheres to the rigid walls.

As is usual in vibration problems, we additionally assume a periodic time dependence. We therefore want to find $(\mathbf{u}_0, p_0)$ by:

$$(1.4a) \qquad \mathbf{u}_0(x,t) = \mathbf{u}(x)\ e^{wt}$$

$$(1.4b) \qquad p_0(x,t) = p(x)\ e^{wt},$$

where w is the unknown vibratory frequency of the system.

Replacing (1.4) on the right hand side of (1.2e) one can explicitly calculate the unique solution of (1.2e). We have:

$$(1.5) \qquad \mathbf{s}_i(t) = \frac{-1}{k_i + m_i w^2}\Big(\int_{\Gamma_i} \sigma(\mathbf{u}, p)\mathbf{n}ds\Big)\ e^{wt}\ .$$

Now, combining equations (1.4), (1.5) with (1.2a,—,d), it follows that the triplet $(w, \mathbf{u}, p)$ ought to be the solution of the following spectral problem on Ω :

$$(1.6a) \qquad -2\nu \text{ div } e(\mathbf{u}) + \nabla p + w\mathbf{u} = \mathbf{0} \quad \text{in } \Omega$$

$$(1.6b) \qquad \text{div } \mathbf{u} = 0 \quad \text{in } \Omega$$

$$(1.6c) \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_0$$

$$(1.6d) \qquad \mathbf{u} = \frac{-w}{k_i + m_i w^2}\Big(\int_{\Gamma_i} \sigma(\mathbf{u}, p)\mathbf{n}ds\Big) \quad \text{on } \Gamma_i,\ \forall i = 1, ..., K.$$

To obtain the variational formulation of problem (1.6), the following Sobolev space is introduced:

$$H = \{\mathbf{v} \in H^1(\Omega)^2 \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0 \text{ and}$$
$$\mathbf{v} \text{ is a constant vector on } \Gamma_i,\ \forall i = 1, ..., K\}.$$

Clearly H is a closed vector subspace of $H^1(\Omega)^2$, and therefore a Hilbert space with the induced norm. If one now considers the semi-norm:

$$|e(\mathbf{v})|^2_{0,\Omega} = \int_\Omega e(\mathbf{v}) : e(\bar{\mathbf{v}}) dx\ ,$$

by virtue of Korn's inequality (see P.-A. Raviart & J.-M. Thomas [1983] Chapter 2), it is the case that $|e(\cdot)|_{0,\Omega}$ is a norm in H, equivalent to the standard norm induced by $H^1(\Omega)^2$. From now, we will consider H to be equipped with this norm.

Multiplying (1.6a) by $\bar{\mathbf{v}}$ in H, integrating by parts in Ω, and using (1.6c,d) and (1.3), it follows that if the triplet $(w, \mathbf{u}, p)$ is a solution to (1.6), then the pair $(w, \mathbf{u})$ is a solution of the following variational eigenvalue problem:

(1.7a) $\quad$ Find $w \in \mathbb{C}$

(1.7b) $\quad$ $\mathbf{u} \in H, \mathbf{u} \neq \mathbf{0}$ such that $\quad \forall \mathbf{v} \in H$

(1.7c)

$$2\nu \int_\Omega e(\mathbf{u}) : e(\bar{\mathbf{v}}) dx + w \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}} dx + \sum_{i=1}^{K} (\frac{k_i + m_i w^2}{w}) \gamma_i(\mathbf{u}) \cdot \gamma_i(\bar{\mathbf{v}}) = 0,$$

where, in (1.7c), $\gamma_i(\mathbf{u})$ denotes the trace of $\mathbf{u}$ on Γ_i. Conversely, a standard application of the De Rham Theorem (see J.-L. Lions' book [1969] Chapter 1 or R. Temam [1977] Propositions 1.1 and 1.2) shows that if the pair $(w, \mathbf{u})$ is a solution of (1.7), then there is a function $p \in L^2(\Omega)$ such that $(w, \mathbf{u}, p)$ is a solution of (1.6). Below we deal with problem (1.7) with a view to solving it theoretically and numerically.

1.3 Theorem of existence and location of eigenvalues. In this section we expound the existence and location theorem of eigenfrequencies of problem (1.7), which is proved in [C.,D.,P. 1992a]. The technique used to prove existence consists of showing that the spectrum of (1.7) coincides with the characteristic values of a quadratic eigenvalue problem involving three linear operators T_1, T_2 and Q, which are defined as follows:

(1.8a) $\quad T_1 : H \to H$

(1.8b) $\quad T_1\mathbf{u} = \varphi_1 \qquad \forall \mathbf{u} \in H,$

(1.9a) $\quad T_2 : \mathbb{C}^{2K} \to H,$

(1.9b) $\quad T_2\mathbf{s} = \varphi_2 \qquad \forall \mathbf{s} \in \mathbb{C}^{2K},$

and

$$(1.10a) \qquad Q : H \to \mathbb{C}^{2K}$$
$$(1.10b) \qquad Q\mathbf{u} = (\gamma_1(\mathbf{u}), \cdots, \gamma_K(\mathbf{u})) \quad \forall \mathbf{u} \in H,$$

where functions φ_1, φ_2 are the unique solutions of the following variational problems:

$$(1.11a) \qquad \text{Find } \varphi_1 \in H \text{ such that}$$
$$(1.11b) \qquad 2\nu \int_\Omega e(\varphi_1) : e(\bar{\mathbf{v}})dx + \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}} dx = 0 \quad \forall \mathbf{v} \in H$$

and

$$(1.12a) \qquad \text{Find } \varphi_2 \in H \text{ such that}$$
$$(1.12b) \qquad 2\nu \int_\Omega e(\varphi_2) : e(\bar{\mathbf{v}})dx + \sum_{i=1}^{K} \mathbf{s}_i \cdot \gamma_i(\bar{\mathbf{v}}) = 0 \quad \forall \mathbf{v} \in H.$$

Given that $\omega = 0$ is not a solution of (1.7), then

$$(1.13) \qquad \lambda = \frac{1}{\omega},$$

is correctly defined. As is shown in [C.,D.,P. 1992a], problem (1.7) is equivalent to the following eigenvalue problem:

$$(1.14a) \qquad \text{Find } \lambda \in \mathbb{C},\ \mathbf{u} \in H, \mathbf{u} \neq \mathbf{0}$$
$$(1.14b) \qquad (T_1 + T_2 ddiag(m_i)Q)\mathbf{u} - \lambda\mathbf{u} + \lambda^2 T_2 ddiag(k_i)Q\mathbf{u} = \mathbf{0},$$

where $ddiag(m_i)$ and $ddiag(k_i)$ are diagonal matrices of dimension $2K \times 2K$, whose diagonals entries are $\{m_1, m_1, ..., m_K, m_K\}$ and $\{k_1, k_1, ..., k_K, k_K\}$, respectively.

Problem (1.14) is a quadratic eigenvalue problem which brings into play selfadjoint compact linear operators. The result expounded below summarises the principal characteristics of the spectrum of (1.14), more details are available in the work referred to above.

Theorem 1.1

The spectrum of (1.14) *consists of a countable infinite quantity of complex numbers which converge to zero:*

$$\lambda_1, ..., \lambda_\ell, ... \to 0. \tag{1.15}$$

Moreover, the eigenvalues have the following properties:

(i) $Re(\lambda_i) < 0 \qquad \forall i \geq 1.$

(ii) *If* $Im(\lambda_i) \neq 0$, *then*

$$\frac{1}{|\lambda_i|^2} < k/m, \tag{1.16}$$

where k/m *denotes the quantity* $\max\limits_{1\leq j\leq K}\{k_j/m_j\}$.

Proof.- Let us define

$$\mathbf{v} = \mathbf{u} - \frac{1}{\lambda}(T_1 + T_2 ddiag(m_i)Q)\mathbf{u}. \tag{1.17}$$

It is easy to show that if $(\lambda, \mathbf{u})$ is a solution of (1.14), then $\mathbf{v}$ belongs to the finite dimensional space F, defined by

$$F = Im(T_2) = T_2(\mathbb{C}^{2K}) \subset H,$$

and $(\lambda, (\mathbf{u}, \mathbf{v}))$ is a solution of the following generalized eigenvalue problem

$$\begin{bmatrix} T_1 + T_2 ddiag(m_i)Q & 0 \\ 0 & I_F \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \lambda \begin{bmatrix} I_H & -i_F \\ T_2 ddiag(k_i)Q & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}, \tag{1.18}$$

where I_H and I_F are the identity operators of spaces H and F, respectively. On the right hand side of (1.18), i_F denotes the canonic imbedding of F into H.

Conversely, it is straightforward to see that if $(\lambda, (\mathbf{u}, \mathbf{v})) \in \mathbb{C} \times (H \times F)$ is a solution of (1.18), then the couple $(\lambda, \mathbf{u})$ is a solution of (1.14) and $\mathbf{v}$ verifies (1.17). Thus (1.14) and (1.18) are equivalent.

It is appropriate to note that the right hand side of (1.18) is a non-selfadjoint linear operator in $H \times F$. Thus the spectrum of (1.18), as well as the spectrum of (1.14), may admit non-real eigenvalues.

Let us next define A and B by the following rules:

$$A : H \times F \to H \times F, \tag{1.19a}$$

$$A(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} T_1 + T_2 ddiag(m_i)Q & 0 \\ 0 & I_F \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}, \tag{1.19b}$$

and

$$B : H \times F \to H \times F, \tag{1.20a}$$

$$B(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} I_H & -i_F \\ T_2 ddiag(k_i)Q & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}. \tag{1.20b}$$

Therefore (1.18) (and also (1.14)) is nothing but the generalized eigenvalue problem associated with the operators A and B, i.e., find $\lambda \in \mathbb{C}$ and $(\mathbf{u}, \mathbf{v}) \in H \times F$, $(\mathbf{u}, \mathbf{v}) \neq (\mathbf{0}, \mathbf{0})$ such that

$$A(\mathbf{u}, \mathbf{v}) = \lambda B(\mathbf{u}, \mathbf{v}). \tag{1.21}$$

We can easily prove, using classical theorems of Functional Analysis, that the operator B has a linear continuous inverse. Then, it follows that the problems (1.14) and (1.21) are in fact equivalent to the following eigenvalue problem: find $\lambda \in \mathbb{C}$ and $(\mathbf{u}, \mathbf{v}) \in H \times F$, $(\mathbf{u}, \mathbf{v}) \neq (\mathbf{0}, \mathbf{0})$ such that

$$T(\mathbf{u}, \mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}), \tag{1.22}$$

where the operator T is defined by

$$T = B^{-1}A. \tag{1.23}$$

Since T_1 and $T_2 ddiag(m_i)Q$ are compact operators, the operator A is compact. Thus T is also compact. Applying the spectral theory of compact operators (see T. Kato [1976] Chapter III), it follows that the spectrum of T consists of a countable sequence of complex numbers whose only possible accumulation point is zero. We denote the eigenvalues of T by $\lambda_1, ..., \lambda_\ell, ...$, where they are always assumed to be numbered so that the same eigenvalue is repeated according to its geometric multiplicity.

Now, we prove the inequalities. Let $(\lambda, \mathbf{u})$ be a solution of (1.14). Taking the scalar product of equation (1.14b) by $\mathbf{u}$, we obtain

$$[(T_1 + T_2 ddiag(m_i)Q)\mathbf{u}, \mathbf{u}] - \lambda |e(\mathbf{u})|_{0,\Omega}^2 + \lambda^2 [T_2 ddiag(k_i)Q\mathbf{u}, \mathbf{u}] = 0.$$

Taking $\mathbf{v}$=$\mathbf{u}$ in (1.7) and using (1.12), this identity can be rewritten as follows

$$|\mathbf{u}|^2_{0,\Omega} + \sum_{i=1}^{K} m_i|\gamma_i(\mathbf{u})|^2 + 2\nu\lambda|e(\mathbf{u})|^2_{0,\Omega} + \lambda^2 \sum_{i=1}^{K} k_i|\gamma_i(\mathbf{u})|^2 = 0,$$

which implies

$$|\mathbf{u}|^2_{0,\Omega} + \sum_{i=1}^{K} m_i|\gamma_i(\mathbf{u})|^2 + 2\nu Re(\lambda)|e(\mathbf{u})|^2_{0,\Omega}$$

$$\text{(1.24}a\text{)} \qquad + (Re^2(\lambda) - Im^2(\lambda)) \sum_{i=1}^{K} k_i|\gamma_i(\mathbf{u})|^2 = 0,$$

$$\text{(1.24}b\text{)} \qquad 2Im(\lambda)\big(\nu|e(\mathbf{u})|^2_{0,\Omega} + Re(\lambda)\sum_{i=1}^{K} k_i|\gamma_i(\mathbf{u})|^2\big) = 0.$$

If $Im(\lambda) = 0$, (1.24a) clearly implies that $Re(\lambda) \leq 0$. On the other hand, if $Im(\lambda) \neq 0$, (1.24b) reduces to

$$\nu|e(\mathbf{u})|^2_{0,\Omega} + Re(\lambda)\sum_{i=1}^{K} k_i|\gamma_i(\mathbf{u})|^2 = 0,$$

which clearly implies that $Re(\lambda) < 0$. Combining these results, and using the fact that $\lambda = 0$ is not an eigenvalue of (1.14), we conclude that $Re(\lambda) < 0$.

Let us next prove (ii). To this end, we assume that $Im(\lambda) \neq 0$ and we introduce the vector functions ϕ_1, ϕ_2, which are defined by

$$\text{(1.25}a\text{)} \qquad \phi_1 = T_1\mathbf{u},$$

$$\text{(1.25}b\text{)} \qquad \phi_2 = T_2 ddiag(\lambda^2 k_i + m_i)Q\mathbf{u}.$$

In terms of ϕ_1, ϕ_2, (1.14b) can be rewritten as follows:

$$\text{(1.26)} \qquad \phi_1 + \phi_2 = \lambda\mathbf{u}.$$

Using the definitions of the operators T_1, T_2 (cf.(1.8), (1.11) and (1.9), (1.12), respectively), it follows that ϕ_1, ϕ_2 verify

$$\text{(1.27}a\text{)} \qquad 2\nu[\phi_1, \mathbf{v}] = -\int_\Omega \mathbf{u}\cdot\bar{\mathbf{v}}\,dx \quad \forall\mathbf{v} \in H,$$

$$\text{(1.27}b\text{)} \qquad 2\nu[\phi_2, \mathbf{v}] = -\sum_{i=1}^{K}(\lambda^2 k_i + m_i)\gamma_i(\mathbf{u})\cdot\gamma_i(\bar{\mathbf{v}}) \quad \forall\mathbf{v} \in H.$$

Multiplying (1.26) by ϕ_1, we obtain

(1.28) $$|e(\phi_1)|^2_{0,\Omega} + [\phi_2, \phi_1] = \lambda[\mathbf{u}, \phi_1].$$

Taking $\mathbf{v} = \mathbf{u}$ as test function in (1.27a), we have

$$[\mathbf{u}, \phi_1] = -\frac{1}{2\nu}|\mathbf{u}|^2_{0,\Omega},$$

and replacing this identity in (1.28), we can write equivalently

(1.29) $$|e(\phi_1)|^2_{0,\Omega} + \frac{\lambda}{2\nu}|\mathbf{u}|^2_{0,\Omega} + [\phi_2, \phi_1] = 0.$$

On the other hand, multiplying (1.26) by ϕ_2, we obtain

(1.30) $$[\phi_1, \phi_2] + |e(\phi_2)|^2_{0,\Omega} = \lambda[\mathbf{u}, \phi_2].$$

But, taking $\mathbf{v} = \mathbf{u}$ in (1.27b), we have

$$[\mathbf{u}, \phi_2] = -\frac{1}{2\nu}\sum_{i=1}^{K}(\bar{\lambda}^2 k_i + m_i)|\gamma_i(\mathbf{u})|^2,$$

and so, replacing $[\mathbf{u}, \phi_2]$ in (1.30), we obtain

(1.31) $$[\phi_1, \phi_2] = -|e(\phi_2)|^2_{0,\Omega} - \frac{\lambda}{2\nu}\sum_{i=1}^{K}(\bar{\lambda}^2 k_i + m_i)|\gamma_i(\mathbf{u})|^2.$$

Combining (1.29) with (1.31), we deduce

$$|e(\phi_1)|^2_{0,\Omega} - |e(\phi_2)|^2_{0,\Omega} + \frac{\lambda}{2\nu}|\mathbf{u}|^2_{0,\Omega} - \frac{\bar{\lambda}}{2\nu}\sum_{i=1}^{K}(\lambda^2 k_i + m_i)|\gamma_i(\mathbf{u})|^2 = 0,$$

hence we infer that

$$|\mathbf{u}|^2_{0,\Omega} + \sum_{i=1}^{K} m_i|\gamma_i(\mathbf{u})|^2 - |\lambda|^2 \sum_{i=1}^{K} k_i|\gamma_i(\mathbf{u})|^2 = 0,$$

which implies the following inequality

$$\sum_{i=1}^{K}(\frac{k_i}{m_i} - \frac{1}{|\lambda|^2})m_i|\gamma_i(\mathbf{u})|^2 > 0,$$

and so, we conclude that

$$\frac{k}{m} - \frac{1}{|\lambda|^2} > 0.$$

This shows (1.16) and completes the proof. ∎

It is interesting to observe in (ii) that the spectrum of (1.14) can only contain a finite quantity of eigenvalues whose imaginary part is different from zero. It should be noted that Theorem 1.1 does not provide an estimate of the number of imaginary (non-real) eigenvalues; this is provided by the following result, the proof of which can be referred to in [C.,D.,P. 1992b].

Theorem 1.2

The spectrum of (1.14) *admits a maximum of* $4K$ *imaginary eigenvalues. Moreover, if we denote by*

$$0 < \cdots \leq \mu_m \leq \cdots \leq \mu_1$$

the additive inverses of the real eigenvalues of (1.14), *and by*

$$0 < \cdots \leq \xi_m \leq \cdots \leq \xi_1$$

the eigenvalues of the operator $-(T_1 + T_2 ddiag(m_i)Q)$, *then the following interlacing inequalities hold:*

$$\xi_j \leq \mu_j \leq \xi_{j-2K} \quad \forall j \geq 2K+1 \tag{1.32a}$$

$$\xi_j \leq \mu_j \qquad \forall j = 1, ..., 2K. \quad \blacksquare \tag{1.32b}$$

§2. NUMERICAL ANALYSIS OF THE PROBLEM

In this section a numerical solution of problem (1.6) is undertaken, by means of Lagrange type finite elements on triangles. To this end, certain simplifications are made in the geometry of the problem, which allow a more appropriate numerical approach.

Firstly, the discretization of the problem is presented, and then its matrix formulation, which turns out to be a generalized eigenvalue problem of the Sylvester type. Finally, the numerical results which were obtained are presented.

2.1 Discretization of the problem. The first geometrical hypothesis consists in assuming that $\Omega \subseteq I\!R^2$ is a convex bounded polygon with which we associate a regular family of triangulations $\{\tau_h\}_{h>0}$ (in the sense of Ph. Ciarlet [1979], Chapter 2, Section 2.1), such that

$$\bar{\Omega} = \bigcup_{T \in \tau_h} T \qquad \forall h > 0,$$

where h is defined by $h \equiv \max_{T \in \tau_h} h_T$; h_T being the diameter of triangle T.

Another simplification adopted is to assume that all the tubes are identical and that they are supported by the same system of springs. Due to these considerations the constants k_i and m_i $\forall i = 1, .., K$, have a single value, say k and m, respectively. That is,

$$k \equiv k_1 = ... = k_K \qquad m \equiv m_1 = ... = m_K \tag{2.1}$$

To carry out the numerical analysis of (1.6), we shall begin by reformulating the problem variationally. While the theoretical study of (1.6) only requires the use of a primal variational formulation (see (1.7)), its numerical study needs a mixed variational formulation, in which the incompressibility condition is dealt with implicitly. To this end, we introduce the Sobolev space V, defined by:

$$V = \{\mathbf{v} \in H^1(\Omega)^2 \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0,\ \mathbf{v} \text{ is a constant vector on } \Gamma_i,\ \forall i = 1, ..., K\}. \tag{2.2}$$

Clearly, V is a closed vector subspace of $H^1(\Omega)^2$ (thus, a Hilbert space by itself with the standard induced norm).

Multiplying (1.6a) by $\mathbf{v} \in V$ and (1.6b) by $q \in L^2(\Omega)$, it is easy to prove that a mixed variational formulation of (1.6) is:

$$\text{Find } (w, \mathbf{u}, p) \in \mathbb{C} \times V \times L^2(\Omega), \text{ such that} \tag{2.3a}$$

$$a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) + w\ d(\mathbf{u}, \mathbf{v}) + \left(\frac{k + m\ w^2}{w}\right) c(\mathbf{u}, \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in V \tag{2.3b}$$

$$b(\mathbf{u}, q) = 0 \qquad \forall q \in L^2(\Omega), \tag{2.3c}$$

where, in (2.3b,c), the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$, $d(\cdot,\cdot)$ are defined by:

$$a(\mathbf{u}, \mathbf{v}) = 2\nu \int_\Omega e(\mathbf{u}) : e(\bar{\mathbf{v}}) dx$$

$$b(\mathbf{v}, p) = \int_\Omega p \text{ div } \bar{\mathbf{v}} dx$$

$$c(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{K} \gamma_i(\mathbf{u}) \cdot \gamma_i(\bar{\mathbf{v}})$$

$$d(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}} dx\ .$$

To approximate V and $L^2(\Omega)$, we introduce the finite dimensional spaces:

$$V_h \equiv \{\mathbf{v}_h \in C^\circ(\bar{\Omega})^2 \mid \mathbf{v}_h|_T \in P_2(T)^2 \ \ \forall T \in \tau_h,\ \mathbf{v}_h = \mathbf{0} \text{ on } \Gamma_0 \text{ and } \mathbf{v}_h \text{ is a constant vector on } \Gamma_i, \forall i = 1, ..., K\},$$

$$L_h^2 \equiv \{\ q_h \in C^\circ(\bar{\Omega}) \mid q_h|_T \in P_1(T) \ \ \forall T \in \tau_h\},$$

where, in the above definitions, $P_1(T)$ ($P_k(T), k \geq 0$ in the general case) denotes the space of polynomials of degree less than or equal to 1 ($\leq k$, respectively) and $C^\circ(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$.

The positive integers $2N_h$ and M_h are defined as the dimensions of spaces V_h and L_h^2, respectively. Unless it would lead to confusion, we shall omit the subindex h from now on, with the aim of simplifying notation. Let $\{\mathbf{v}_1, ..., \mathbf{v}_{2N}\}$ and $\{\phi_1, ..., \phi_M\}$ be bases of V_h and L_h^2, respectively. In what follows we shall assume that the family of triangulations

$\{\tau_h\}_{h>0}$ satisfies the following hypothesis of non-degeneracy, concerning the degrees of freedom associated with the velocity and pressure:

$$2N - M > 0. \tag{2.4}$$

In these conditions, we can now approximate problem (2.3) by:

$$\text{Find } (w_h, \mathbf{u}_h, p_h) \in \mathbb{C}\times V_h \times L_h^2, \text{ such that } \forall \mathbf{v}_h \in V_h \tag{2.5a}$$

$$w_h^2\{d(\mathbf{u}_h, \mathbf{v}_h) + mc(\mathbf{u}_h, \mathbf{v}_h)\} + w_h a(\mathbf{u}_h, \mathbf{v}_h) + kc(\mathbf{u}_h, \mathbf{v}_h) = w_h b(\mathbf{v}_h, p_h) \tag{2.5b}$$

$$b(\mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in L_h^2. \tag{2.5c}$$

The rate of convergence estimates can be derived for the approximation of eigenvalues and eigenvectors of this mixed method. We refer to C. Conca, M. Durán & J. Rappaz [1994]. Briefly, we prove that the problem (2.3) (discretized problem (2.5), respectively) can be put in the standard form of a Brezzi-Babuška saddlepoint problem satisfying some adequate hypotheses, which allow the use of several closely related abstract results on eigenvalue approximation. These abstract results can be found in the works of J.H. Bramble & J.E. Osborn [1973], I. Babuška & A. Aziz [1973], J.E. Osborn [1975], J. Descloux, N. Nassif & J. Rappaz [1977], [1978], W.G. Kolata [1978] and B. Mercier, J.E. Osborn, J. Rappaz & P.-A. Raviart [1981].

2.2 Matrix formulation of (2.5). In this section, problem (2.5) is reformulated from a matrix point of view. This will allow the approximated solutions $(w_h, \mathbf{u}_h, p_h)$ to be calculated explicitly. We begin by finding $(\mathbf{u}_h, p_h)$ of the form:

$$\mathbf{u}_h = \sum_{i=1}^{2N} \alpha_i \mathbf{v}_i \tag{2.6a}$$

$$p_h = \sum_{i=1}^{M} \beta_i \phi_i \tag{2.6b}$$

where $\alpha_i, \beta_i \in \mathbb{C}$. By introducing (2.6a) and (2.6b) into (2.5b), we obtain:

$$w_h^2 \sum_{i=1}^{2N} \alpha_i\{d(\mathbf{v}_i, \mathbf{v}_h) + mc(\mathbf{v}_i, \mathbf{v}_h)\} + w_h \sum_{i=1}^{2N} \alpha_i a(\mathbf{v}_i, \mathbf{v}_h) + k \sum_{i=1}^{2N} \alpha_i c(\mathbf{v}_i, \mathbf{v}_h) = w_h \sum_{i=1}^{M} \beta_i b(\mathbf{v}_h, \phi_i).$$

This expression is valid for all $\mathbf{v}_h \in V_h$, in particular for every $\mathbf{v}_j$, $j = 1, .., 2N$. It follows that

$$\left\{ \begin{aligned} & w_h^2 \sum_{i=1}^{2N} \alpha_i\{d(\mathbf{v}_i, \mathbf{v}_j) + mc(\mathbf{v}_i, \mathbf{v}_j)\} + w_h \sum_{i=1}^{2N} \alpha_i a(\mathbf{v}_i, \mathbf{v}_j) \\ & \qquad + k \sum_{i=1}^{2N} \alpha_i c(\mathbf{v}_i, \mathbf{v}_j) = w_h \sum_{i=1}^{M} \beta_i b(\mathbf{v}_j, \phi_i) \qquad \forall j = 1, .., 2N. \end{aligned} \right. \tag{2.7}$$

Thus, defining the matrices A, B, C, D by:

$$A = [a_{ij}] = a(\mathbf{v}_i, \mathbf{v}_j) \qquad i, j = 1, ..., 2N \tag{2.8a}$$

$$B = [b_{ij}] = b(\mathbf{v}_i, \phi_j) \qquad i = 1, ..., 2N, j = 1, ..., M \tag{2.8b}$$

$$C = [c_{ij}] = c(\mathbf{v}_i, \mathbf{v}_j) \qquad i, j = 1, ..., 2N \tag{2.8c}$$

$$D = [d_{ij}] = d(\mathbf{v}_i, \mathbf{v}_j) \qquad i, j = 1, ..., 2N \tag{2.8d}$$

and the vectors α, β by:

$$\alpha = (\alpha_1, ..., \alpha_{2N})^t \tag{2.9a}$$

$$\beta = (\beta_1, ..., \beta_M)^t, \tag{2.9b}$$

it follows that (2.7), expressed in matrix terms, is equivalent to

$$w_h^2 \ \{D + mC\}\alpha + w_h \ A\alpha + kC\alpha = w_h B\beta \tag{2.10}$$

In the same way, introducing (2.6a) in (2.5c), we obtain

$$\sum_{i=1}^{2N} \bar{\alpha}_i b(\mathbf{v}_i, \phi_j) = 0 \qquad \forall j = 1, .., M. \tag{2.11}$$

Using the above notation, and conjugating (2.11), it follows that

$$B^* \alpha = \mathbf{0}, \tag{2.12}$$

where B^* is the transposed conjugate matrix of B.

The equations (2.10), (2.12) reduce problem (2.5) to:

(2.13a) Find $(w_h, \alpha, \beta) \in \mathbb{C} \times \mathbb{C}^{2N} \times \mathbb{C}^M, \alpha \neq \mathbf{0}, \beta \neq \mathbf{0}$ such that

$$w_h^2 \ \{D + mC\}\alpha + w_h \ A\alpha + kC\alpha = w_h B\beta \tag{2.13b}$$

$$B^* \alpha = \mathbf{0} \ , \tag{2.13c}$$

which is a quadratic eigenvalue problem of finite dimension.

Problem (2.13) can still be transformed if the family of triangulations $\{\tau_h\}_{h>0}$ satisfies certain conditions of non-degeneracy and regularity. Let us thus observe that if α satisfies (2.13c), then

$$\alpha \in Ker(B^*)$$

and α can be written as a linear combination of elements of a basis of $Ker(B^*)$. The following result provides information about this subspace.

Theorem 2.1

If the family of triangulations $\{\tau_h\}_{h>0}$ *satisfies* (2.4) *and if*

$$(2.14) \qquad 2\ |\tau_h| + \ K \leq \ 2N \qquad \forall h > 0,$$

where $|\tau_h|$ *denotes the cardinality of* τ_h*, then*

$$dim\ Ker(B^*) = 2N - M + 1,$$

where $2N$ *and* M *are the dimensions of* V_h *and* L_h^2*, respectively.*

Proof.- Let $\mathbf{x} \in Ker(B)$ be any element. By definition, the vector $\mathbf{x} = (x_1, .., x_M)^t$ satisfies:

$$\sum_{j=1}^{M} x_j b_{ij} = 0 \qquad \forall i = 1, .., 2N.$$

By introducing (2.8b) into the above expression, we obtain:

$$(2.15a) \qquad \sum_{j=1}^{M} x_j \int_{\Omega} \phi_j \ \text{div}\ \bar{\mathbf{v}}_i dx = 0 \qquad \forall i = 1, .., 2N.$$

Thus, if we define $\varphi = \sum_{j=1}^{M} x_j \phi_j$, then (2.15a) is equivalent to:

$$(2.15b) \qquad \int_{\Omega} \varphi \ \text{div}\ \bar{\mathbf{v}} dx = 0 \qquad \forall \mathbf{v} \in V_h.$$

Integrating (2.15b) by parts, we obtain:

$$(2.15c) \qquad -\int_{\Omega} \nabla\varphi \cdot \bar{\mathbf{v}} dx + \sum_{\ell=1}^{K} \int_{\Gamma_\ell} \varphi\ \gamma_\ell(\bar{\mathbf{v}}) \cdot \mathbf{n} ds = 0 \qquad \forall \mathbf{v} \in V_h.$$

Now, as $\varphi|_T \in P_1(T)\ \ \forall T \in \tau_h$, then $\nabla\varphi$ is a constant vector of $\mathbb{C}^2$, in every triangle T of τ_h. Let us denote

$$\mathbf{c}_T = \nabla\varphi|_T \qquad \forall T \in \tau_h.$$

Using this notation, and the fact that $\gamma_\ell(\mathbf{v})$ is constant on $\Gamma_\ell \quad \forall \ell = 1,..,K$, we have:

$$(2.15d) \qquad -\sum_{T\in\tau_h} \mathbf{c}_T \cdot \int_T \bar{\mathbf{v}} dx + \sum_{\ell=1}^{K} \int_{\Gamma_\ell} \varphi \ \mathbf{n} ds \cdot \gamma_\ell(\bar{\mathbf{v}}) = 0 \qquad \forall \mathbf{v} \in V_h.$$

Furthermore, (2.15d) can be rewritten as follows:

$$(2.15e) \qquad -\sum_{T\in\tau_h} \mathbf{c}_T \cdot \int_T \bar{\mathbf{v}} dx + \sum_{\ell=1}^{K} (\sum_{T\in A_\ell} \int_{L_T} \varphi \ \mathbf{n} ds) \cdot \gamma_\ell(\bar{\mathbf{v}}) = 0 \qquad \forall \mathbf{v} \in V_h,$$

where $A_\ell \equiv \{T \in \tau_h \mid T$ having at least one side on $\Gamma_\ell\}$. For every $T \in A_\ell$, we define $L_T \equiv \Gamma_\ell \cap \bar{T}$.

If it can be proved that a $\mathbf{v}^* \in V_h$ exists which fulfils the following properties:

$$(2.16a) \qquad \int_T \mathbf{v}^* dx = -\mathbf{c}_T \qquad \forall T \in \tau_h$$

$$(2.16b) \qquad \gamma_\ell(\bar{\mathbf{v}}^*) \ \cdot \sum_{T\in A_\ell} \int_{L_T} \varphi \ \mathbf{n} ds = 0 \qquad \forall \ell = 1,..,K,$$

we can then conclude the proof. Effectively, taking $\mathbf{v} = \mathbf{v}^*$ in (2.15e), it follows that

$$\mathbf{c}_T = \nabla\varphi|_T = \mathbf{0} \qquad \forall T \in \tau_h,$$

which implies that φ is a constant function on Ω, which in turn implies that

$$Ker(B) =< (1,..,1)^t > .$$

Thus, range $(B) = M - 1$. However,

$$dim \ Ker(B^*) = dim \ Im(B)^\perp,$$

and it can then be concluded that

$$dim \ Ker(B^*) = 2N - M + 1.$$

Let us now prove the existence of such $\mathbf{v}^* \in V_h$. To do this, let us begin by introducing a basis $\{\mathbf{z}_1, .., \mathbf{z}_{2N}\}$ of V_h, with the following properties:

$$\gamma_\ell(\mathbf{z}_i) = \mathbf{0} \quad \forall \ell = 1, .., K, \ \forall i = 2K+1, .., 2N, \tag{2.17a}$$

$$\gamma_\ell(\mathbf{z}_{2i-1}) = (1, 0)\ \delta_{\ell i} \quad \forall \ell = 1, .., K, \ \forall i = 1, .., K, \tag{2.17b}$$

$$\gamma_\ell(\mathbf{z}_{2i}) = (0, 1)\ \delta_{\ell i} \quad \forall \ell = 1, .., K, \ \forall i = 1, .., K. \tag{2.17c}$$

The existence of a basis which fulfils these conditions is evident, if (2.4) is satisfied. We look for $\mathbf{v}^*$ in the form

$$\mathbf{v}^* = \sum_{i=1}^{2N} \rho_i \ \mathbf{z}_i \ .$$

The equations (2.16a) impose $2|\tau_h|$ linearly independent restrictions on the coefficients $\{\rho_i\}_{i=1}^{2N}$, which are:

$$\sum_{i=1}^{2N} \rho_i \ (\int_T \mathbf{z}_i dx) = -\mathbf{c}_T \qquad \forall T \in \tau_h.$$

On the other hand, since the functions $\{\mathbf{z}_i\}_{i=1}^{2N}$ satisfy (2.17), $\mathbf{v}^*$ will satisfy (2.16b) if the coefficients $\{\rho_i\}_{i=1}^{2N}$ verify the following equations:

$$\rho_{2\ell-1} \sum_{T \in A_\ell} \int_{L_T} \varphi n_1 ds + \ \rho_{2\ell} \sum_{T \in A_\ell} \int_{L_T} \varphi n_2 ds = 0 \qquad \forall \ell = 1, .., K.$$

Thus, on the 2N coefficients ρ_i, we impose $2|\tau_h| + K$ linearly independent restrictions. As triangulation τ_h fulfils (2.14), one can conclude the existence of at least one $\mathbf{v}^* \in V_h$ which satisfies (2.16). This completes the proof of Theorem 2.1. ∎

Let us denote $n = dim\ Ker(B^*)$, and let $\{\varphi_1, ..., \varphi_n\}$ be a basis of $Ker(B^*)$. Let us decompose α in this basis, as follows:

$$\alpha = \sum_{j=1}^{n} \eta_j \varphi_j = \Phi \eta,$$

where $\eta_j \in \mathbb{C}$, $\forall j = 1, .., n$, and where Φ is the $2N$ by n matrix whose columns are the vectors φ_j.

Multiplying (2.13b) by Φ^*, the following homogeneous equation is obtained:

$$w_h^2\ \Phi^*(D + mC)\Phi\eta + w_h\ \Phi^* A \Phi \eta + k \Phi^* C \Phi \eta = 0. \tag{2.18a}$$

Let us now fix the following notation:

$$
\begin{aligned}
E &= \Phi^*(D + mC)\Phi \\
F &= \Phi^* A\Phi \\
G &= k\Phi^* C\Phi
\end{aligned}
$$

and the matrix equation (2.18a) can then be rewritten as follows:

$$
w_h^2\ E\eta + w_h\ F\eta + G\eta = 0 \tag{2.18b}
$$

which is clearly a quadratic eigenvalue problem.

The above procedure consists in first solving equation (2.18b), that is, calculating the velocity, and then obtaining the coefficients of the pressure from (2.13b). That is, first, the following problem is solved:

$$
\text{Find } (w_h, \eta) \in \mathbb{C} \times \mathbb{C}^n, \eta \neq 0 \text{ such that} \tag{2.19a}
$$

$$
w_h^2\ E\eta + w_h\ F\eta + G\eta = 0, \tag{2.19b}
$$

and next the coefficients of the pressure are computed by solving the linear system:

$$
B\beta = w_h\ (D + mC)\Phi\eta + A\Phi\eta + \frac{k}{w_h} C\Phi\eta. \tag{2.20}
$$

Problem (2.19) is solved by reducing it to a generalized eigenvalue problem. To be more exact, by introducing ξ, defined by:

$$
\xi = (F + \frac{1}{w_h} G)\ \eta. \tag{2.21a}
$$

From (2.19b), it follows that ξ satisfies the equation:

$$
E\eta = -\frac{1}{w_h}\xi. \tag{2.21b}
$$

By writing (2.21) as a matrix system, we obtain:

$$
\begin{bmatrix} -E & 0 \\ -F & I \end{bmatrix} \begin{bmatrix} \eta \\ \xi \end{bmatrix} = \frac{1}{w_h} \begin{bmatrix} 0 & I \\ G & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \xi \end{bmatrix}. \tag{2.22}
$$

Thus, if (w_h, η) is a solution to (2.19), then (w_h, η, ξ) is a solution of (2.22). Conversely, every solution (w_h, η, ξ) is such that ξ fulfils (2.21a), and accordingly (w_h, η) is a solution of (2.19). If we use P and Q, respectively, to denote the left-hand and right-hand matrices of (2.22), we find that (2.19) is equivalent to:

$$
\text{Find } (w_h, \eta, \xi) \in \mathbb{C} \times \mathbb{C}^{2n}, (\eta, \xi) \neq \mathbf{0} \text{ such that} \tag{2.23a}
$$

$$
P(\eta, \xi) = \frac{1}{w_h}\ Q(\eta, \xi). \tag{2.23b}
$$

The following theorem summarises the above and describes the spectrum of (2.5).

Theorem 2.2

The following statements hold true:

(i) *Problem* (2.23) *admits exactly* $(n + range(G))$ *different solutions.*

(ii) *Problem* (2.23) *and system* (2.20) *provide all the solutions of the discretized problem* (2.5).

Proof.- Matrix E is invertible, therefore P is also invertible. Thus, defining

$$R = P^{-1}Q,$$

it is sufficent to note that the degree of the polynomial which provides the non-zero eigenvalues of R is (n+ range(G)), which proves (i). Claim (ii) follows from what was done to deduce (2.23). ∎

It is interesting to note that the spectrum of the discretized problem (2.5) might contain non-real eigenvalues, as seen in problem (2.23), in which matrices P and Q are not Hermitian.

2.3 Numerical Results. The numerical experiments were carried out in a test geometry which consists of a square cavity (transverse section) with a single perforation, also square (K=1), located in the center of the cavity. This geometry corresponds to the case of having only one tube immersed in the fluid.

For the effective numerical solution of (2.5), the selected bases of spaces V_h and L^2_h are those usually used in the finite elements method. To be more precise, we used finite Lagrange elements on triangles, with the characteristic that the velocity is approximated by continuous functions, which when restricted to each triangle are degree 2 polynomials, and that the pressure is approximated by continuous functions which, on each triangle, are degree 1 polynomials. Thus, associated with each triangle not bearing boundary conditions, there are six degrees of freedom for the velocity (vertices and middle points of the triangles of τ_h), and with every triangle of τ_h, three degrees of freedom for the pressure (all vertices of τ_h) are associated.

The boundary condition on Γ_1 (see (1.6d)) was handled explicitly, taking the whole Γ_1 as a single (independent) node of the triangulation. The generalized eigenvalue problem of the Sylvester type (see (2.22)) was solved by using standard computer software package for numerical calculations.

Three triangulations were considered, τ_{h_1}, τ_{h_2} and τ_{h_3} of the domain Ω. Table 2.1 shows a summary of the principal characteristics of each of them. It might be observed that all these triangulations satisfy the hypothesis (2.4) of non-degeneracy, and hypothesis (2.14) of Theorem 2.1.

Tables 2.2—2.4 provide the computed values of the eigenfrequencies for each triangulations of Ω. The viscosity of the fluid was fixed on the value $\nu = 1$. In all three tables the calculations were made with five different values of the constant k/m: 0.01, 0.1, 1.0, 10.0 and 100.0. The graphs of some eigenfunctions and their pressure fields (isobaric lines) are included (see Figures 2.5 and 2.6, where we illustrate by means of some examples the different types of symmetries that the solutions can present).

As can be seen in Tables 2.2—2.4, the numerical experiments confirm the existing abstract results for this mathematical model. To be more precise, it can be observed that the countable spectrum of the problem converges in modulus to infinity and it admits non-real eigenvalues, which can be found in the region of the complex plane indicated theoretically (see Theorem 1.1). It is interesting to note that the quantity of non-real eigenvalues are no more than $4K$ (see Theorem 1.2). In addition, the numerical results provide valuable information on the behaviour of the spectrum when k varies (considering that $m = 1.0$, fixed). In particular, it can be observed that for small values of the parameter k, the spectrum is only made up of real numbers, and that as its value increases, imaginary eigenvalues appear, due to the bifurcation on the complex plane of pairs of real eigenvalues which collapse together. It is also interesting to note that the eigenvalues of simple multiplicity and large magnitude are insensitive to k's variation, as well as, all the graphs of velocity and pressure fields.

REFERENCES

AGUIRRE F. & CONCA C. [1988] Eigenfrequencies of a tube bundle immersed in a fluid, *Appl. Math. Optim.* **18**, 1–38.

BABUŠKA I. & AZIZ A. [1973] Survey lectures on the mathematical foundations of the finite element method, in *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, A.K. AZIZ ed., Academic Press, New-York.

BLEVINS R. [1977] *Flow-Induced Vibrations*, Van Nostrand Reinhold, New-York.

BLEVINS R. [1984] Sound induced by vortex shedding from cylinders, *J. Sound Vibration* **92**, 455–470.

BRAMBLE J. & OSBORN J.E. [1973] Rate of convergence estimates for non-selfadjoint eigenvalue approximations, *Math. Comp.* **27**, 525–549.

CHEN S.S. [1975a] Vibrations of nuclear fuel-bundles, *Nuclear Engineering Design* **35**, 399–422.

CHEN S.S. [1975b] Vibration of a row of circular cylinders in a liquid, *J. Engrg. Industry Trans. A.S.M.E. Ser. B* **97**, 1212–1218.

CIARLET PH. [1979] *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam.

Conca C., Durán M. & Planchard J. [1992a] A quadratic eigenvalue problem involving Stokes equations, *Comput. Methods Appl. Mech. Engrg.* **100**, 295–313.

Conca C., Durán M. & Planchard J. [1992b] A bound for the number of nonreal solutions of a quadratic eigenvalue problem, *Adv. Math. Sci. Appl.* **1**, 229–249.

Conca C., Durán M. & Rappaz J. [1994] Rate of convergence estimates for the spectral approximation of a problem in solid-fluid type structures, Rapport Interne, Centre de Mathématiques Appliquées, Ecole Polytechnique de Paris, Palaiseau, France (to appear).

Conca C., Planchard J. & Vanninathan M. [1989] Existence and location of eigenvalues for fluid-solid structures, *Comput. Methods Appl. Mech. Engrg.* **77**, 253–291.

Conca C., Planchard J. & Vanninathan M. [1990] Limits of the resonance spectrum of tube arrays immersed in a fluid, *J. Fluids Structures* **4**, 541–558.

Conca C. & Vanninathan M. [1988] A spectral problem arising in fluid-solid structures, *Comput. Methods Appl. Mech. Engrg.* **69**, 215–242 .

Connors H.J. [1950] Fluid elastic vibrations of tube arrays excited by cross-flow, in *Flow Induced Vibration in Heat Exchangers*, R.D. Reiff ed., A.S.M.E., New-York.

Descloux J., Nassif N. & Rappaz J. [1977] Various results on spectral approximation, Rapport Interne, Departement de Mathématiques, Ecole Polytechnique Fédérale de Lausanne, Lausanne, Suisse.

Descloux J., Nassif N. & Rappaz J. [1978] On spectral approximation, Part 2: Error estimates for the Galerkin method, *RAIRO Modél. Math. Anal. Numér.* **7**, 113–119.

Gorman D.J. [1977] Experimental study of peripheral problems to liquid flow-induced vibration in heat exchangers and steam generators, Paper F4/6, Trans. SMIRT, San Francisco.

Kato T. [1976] *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin.

Kolata W.G. [1978] Approximation of variationally posed eigenvalue problems, *Numer. Math.* **29**, 159–171.

Lions J.-L. [1969] *Quelques Méthodes de Résolution des Problèmes aux Limites Non-Linéaires*, Dunod, Paris.

Mercier B., Osborn J.E., Rappaz J. & Raviart P.-A. [1981] Eigenvalue approximation by mixed and hybrid methods, *Math. Comp.* **36**, 427–453.

Nečas J. [1967] *Les Méthodes Directes en Théorie des Equations Elliptiques*, Masson, Paris.

Osborn J.E. [1975] Spectral approximation for compact operators, *Math. Comp.* **29**, 712–725.

Paidoussis M. [1982] A review of flow-induced vibrations in reactors and reactor components, *Nuclear Engineering Design* **74**, 31–60.

PETTIGREW M.J. [1981] Flow induced vibration phenomena in nuclear power station components, *Power Industry Research* **1**, 97–133.

PLANCHARD J. [1980] Computation of the acoustic eigenfrequencies of cavities containing a tube bundle, *Comput. Methods Appl. Mechanics Engrg.* **24**, 125–135.

PLANCHARD J. [1982] Eigenfrequencies of a tube-bundle placed in a confined fluid, *Comput. Methods Appl. Mech. Engrg.* **30**, 75–93.

PLANCHARD J. [1985] Vibrations of nuclear fuel assemblies: a simplified model, *Nuclear Engineering and Design* **86**, 383–391 (1985).

PLANCHARD J. [1987] Global behaviour of large elastic tube bundles immersed in a fluid, *Computational Mechanics* **2**, 105–118.

PLANCHARD J. [1988] Comportement des faisceaux de tubes immergés, Lecture at the Summer School on Numerical Analysis EDF-CEA-INRIA, in J. DONEA & al., *Aspects Théoriques et Numériques en Dynamique des Structures*, Collection de la Direction des Etudes et Recherches d'Electricité de France **70**, 165–242, Eyrolles, Paris.

RAVIART P.-A. & THOMAS J.-M. [1983] *Introduction à l'Analyse Numérique des Equations aux Dérivées Partielles*, Masson, Paris.

TEMAM R. [1977] *Navier-Stokes Equations*, North-Holland, Amsterdam.

Table 2.1 : Main features of the triangulations.

	$\mathcal{T}_{h_1}$	$\mathcal{T}_{h_2}$	$\mathcal{T}_{h_3}$
Number of triangles	32	64	128
Total number of vertices and middle points	80	160	288
Number of degrees of freedom for the velocity ($2N$)	98	194	450
Number of degrees of freedom for the pressure (M)	24	48	80
Number of eigenvalues of (1.14) ($2N - M + 1 + range(G)$)[1]	77	149	373

[1]See Theorem 2.2.

Table 2.2 : Characteristic values of problem (1.14)
(Triangulation τ_{h_1})

i	$k/m = 0.01$ $\omega_i = 1/\lambda_i$	$k/m = 0.1$ $\omega_i = 1/\lambda_i$	$k/m = 1.0$ $\omega_i = 1/\lambda_i$	$k/m = 10.0$ $\omega_i = 1/\lambda_i$	$k/m = 100.0$ $\omega_i = 1/\lambda_i$
1	-1.967×10^{-4}	-1.967×10^{-3}	-1.975×10^{-2}	-0.206	-2.170 + 1.798 i
2	-1.967×10^{-4}	-1.967×10^{-3}	-1.975×10^{-2}	-0.206	-2.170 − 1.798 i
3	-2.686	-2.686	-2.686	-2.686	-2.170 + 1.798 i
4	-3.942	-3.941	-3.926	-3.777	-2.170 − 1.798 i
5	-3.942	-3.941	-3.926	-3.777	-2.689
6	-7.132	-7.132	-7.132	-7.132	-7.132
7	-8.937	-8.937	-8.937	-8.937	-8.937
8	-9.164	-9.164	-9.163	-9.153	-9.060
9	-9.164	-9.164	-9.163	-9.153	-9.060
10	-11.841	-11.841	-11.841	-11.841	-11.841
11	-12.173	-12.173	-12.173	-12.172	-12.155
12	-12.173	-12.173	-12.173	-12.172	-12.156
13	-12.596	-12.596	-12.596	-12.596	-12.596
14	-14.285	-14.285	-14.285	-14.285	-14.285
15	-14.637	-14.637	-14.637	-14.637	-14.637
16	-16.110	-16.110	-16.110	-16.110	-16.106
17	-16.110	-16.110	-16.110	-16.110	-16.106
18	-16.643	-16.643	-16.643	-16.643	-16.643
19	-16.669	-16.669	-16.669	-16.669	-16.669
20	-17.073	-17.073	-17.073	-17.073	-17.073
21	-17.073	-17.073	-17.073	-17.073	-17.073
22	-17.912	-17.912	-17.912	-17.912	-17.912
23	-18.750	-18.750	-18.750	-18.750	-18.750
24	-20.004	-20.004	-20.004	-20.004	-20.004
⋮	⋮	⋮	⋮	⋮	⋮
etc.	*etc.*	*etc.*	*etc.*	*etc.*	*etc.*
76	-88.292	-88.292	-88.292	-88.291	-88.283
77	-88.292	-88.292	-88.292	-88.291	-88.283

Table 2.3 : Characteristic values of problem (1.14)
(Triangulation τ_{h_2})

i	$k/m = 0.01$ $\omega_i = 1/\lambda_i$	$k/m = 0.1$ $\omega_i = 1/\lambda_i$	$k/m = 1.0$ $\omega_i = 1/\lambda_i$	$k/m = 10.0$ $\omega_i = 1/\lambda_i$	$k/m = 100.0$ $\omega_i = 1/\lambda_i$
1	-1.861×10^{-4}	-1.862×10^{-3}	-1.869×10^{-2}	-0.195	-2.085 + 1.743 i
2	-1.861×10^{-4}	-1.862×10^{-3}	-1.869×10^{-2}	-0.195	-2.085 − 1.743 i
3	-2.650	-2.650	-2.650	-2.650	-2.085 + 1.743 i
4	-3.907	-3.905	-3.891	-3.739	-2.085 − 1.743 i
5	-3.907	-3.905	-3.891	-3.739	-2.650
6	-8.511	-8.511	-8.511	-8.511	-8.511
7	-8.705	-8.705	-8.705	-8.705	-8.705
8	-8.901	-8.901	-8.900	-8.898	-8.879
9	-8.901	-8.901	-8.900	-8.898	-8.879
10	-9.442	-9.442	-9.442	-9.442	-9.442
11	-10.633	-10.633	-10.632	-10.627	-10.572
12	-10.633	-10.633	-10.632	-10.627	-10.572
13	-10.839	-10.839	-10.839	-10.839	-10.839
14	-10.921	-10.921	-10.921	-10.921	-10.921
15	-12.877	-12.877	-12.877	-12.874	-12.845
16	-12.877	-12.877	-12.877	-12.874	-12.845
17	-14.466	-14.466	-14.466	-14.466	-14.466
18	-14.491	-14.491	-14.491	-14.491	-14.491
19	-16.281	-16.281	-16.281	-16.281	-16.281
20	-16.704	-16.704	-16.703	-16.701	-16.675
21	-16.704	-16.704	-16.703	-16.701	-16.675
22	-19.408	-19.408	-19.408	-19.408	-19.408
23	-20.378	-20.378	-20.377	-20.376	-20.361
24	-20.378	-20.378	-20.377	-20.376	-20.361
⋮	⋮	⋮	⋮	⋮	⋮
etc.	*etc.*	*etc.*	*etc.*	*etc.*	*etc.*
148	-198.154	-198.154	-198.154	-198.154	-198.154
149	-198.776	-198.776	-198.776	-198.776	-198.776

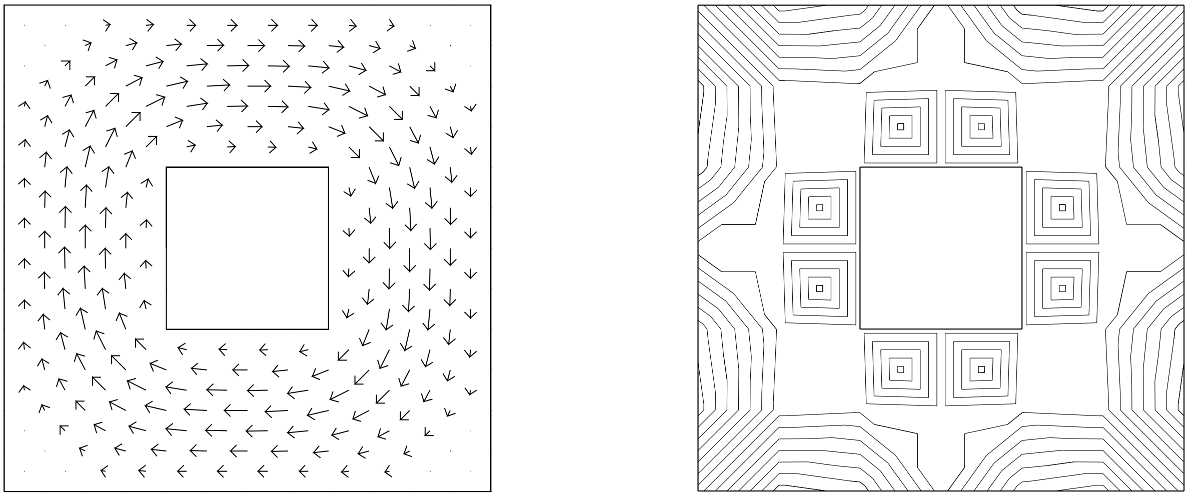
Table 2.4 : Characteristic values of problem (1.14) (Triangulation τ_{h_3})

i	$k/m = 0.01$ $\omega_i = 1/\lambda_i$	$k/m = 0.1$ $\omega_i = 1/\lambda_i$	$k/m = 1.0$ $\omega_i = 1/\lambda_i$	$k/m = 10.0$ $\omega_i = 1/\lambda_i$	$k/m = 100.0$ $\omega_i = 1/\lambda_i$
1	-1.868×10^{-4}	-1.869×10^{-3}	-1.876×10^{-2}	-0.196	-2.067 + 1.752 i
2	-1.868×10^{-4}	-1.869×10^{-3}	-1.876×10^{-2}	-0.196	-2.067 − 1.752 i
3	-2.625	-2.625	-2.625	-2.625	-2.067 + 1.752 i
4	-3.863	-3.862	-3.847	-3.696	-2.067 − 1.752 i
5	-3.863	-3.862	-3.847	-3.696	-2.625
6	-7.909	-7.909	-7.909	-7.909	-7.909
7	-7.934	-7.934	-7.934	-7.934	-7.934
8	-8.167	-8.167	-8.167	-8.164	-8.138
9	-8.167	-8.167	-8.167	-8.164	-8.138
10	-9.223	-9.223	-9.223	-9.223	-9.223
11	-9.554	-9.554	-9.554	-9.554	-9.554
12	-9.844	-9.844	-9.844	-9.841	-9.818
13	-9.844	-9.844	-9.844	-9.841	-9.818
14	-10.318	-10.318	-10.318	-10.318	-10.318
15	-11.574	-11.574	-11.573	-11.567	-11.504
16	-11.574	-11.574	-11.573	-11.567	-11.504
17	-11.924	-11.924	-11.924	-11.924	-11.924
18	-12.173	-12.173	-12.173	-12.173	-12.173
19	-13.724	-13.724	-13.724	-13.722	-13.701
20	-13.724	-13.724	-13.724	-13.722	-13.701
21	-14.096	-14.096	-14.096	-14.096	-14.096
22	-17.256	-17.256	-17.256	-17.256	-17.256
23	-17.758	-17.758	-17.758	-17.757	-17.748
24	-17.758	-17.758	-17.758	-17.757	-17.748
⋮	⋮	⋮	⋮	⋮	⋮
etc.	*etc.*	*etc.*	*etc.*	*etc.*	*etc.*
372	-431.518	-431.518	-431.518	-431.518	-431.518
373	-431.518	-431.518	-431.518	-431.518	-431.518

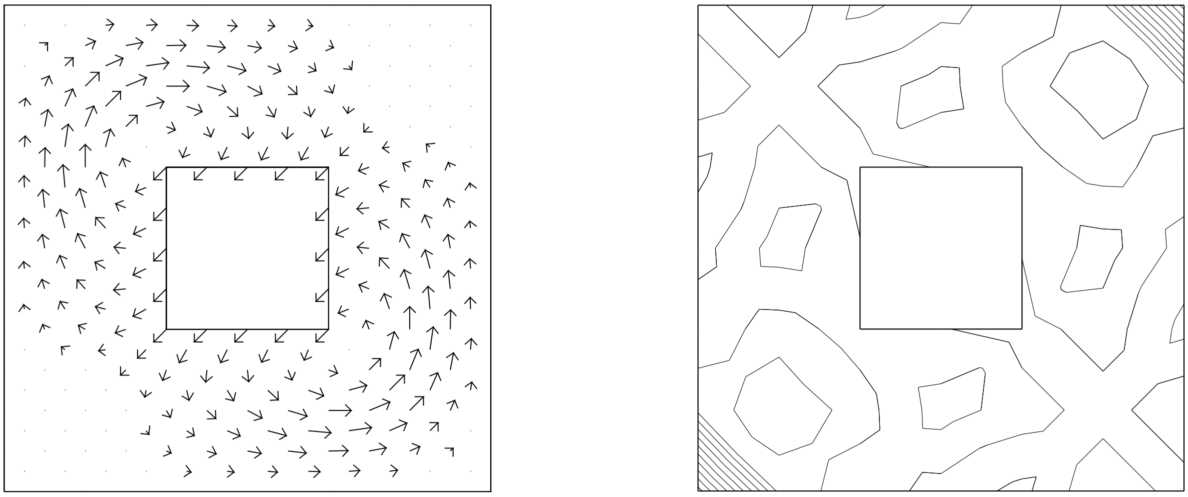
Graphical view of velocity field and isobaric lines
(Triangulation τ_{h_3}, case $k/m = 1.0$)

Figure 2.5

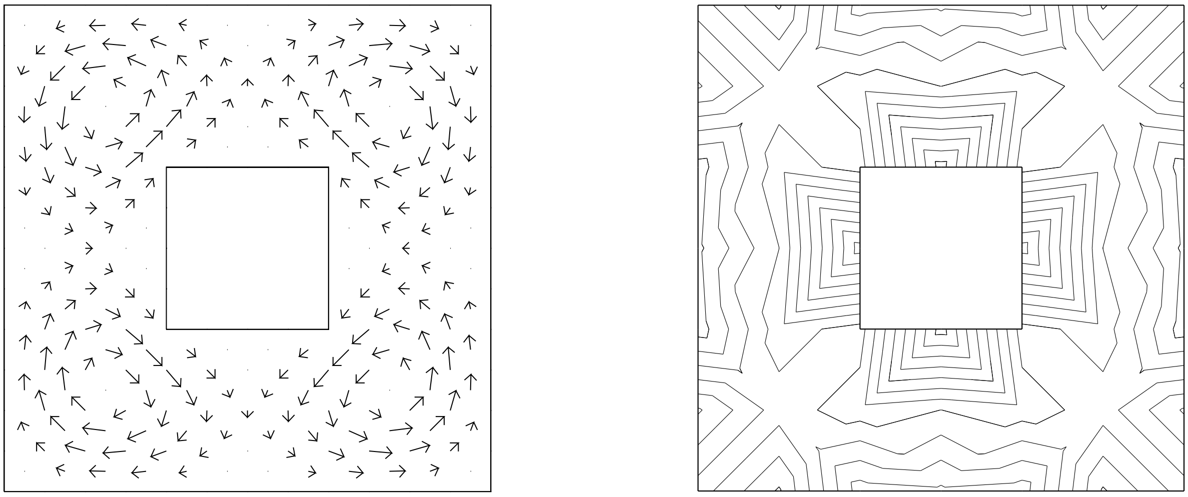
Eigenvalue $\omega_3 = -2.625$



Eigenvalue $\omega_4 = -3.847$



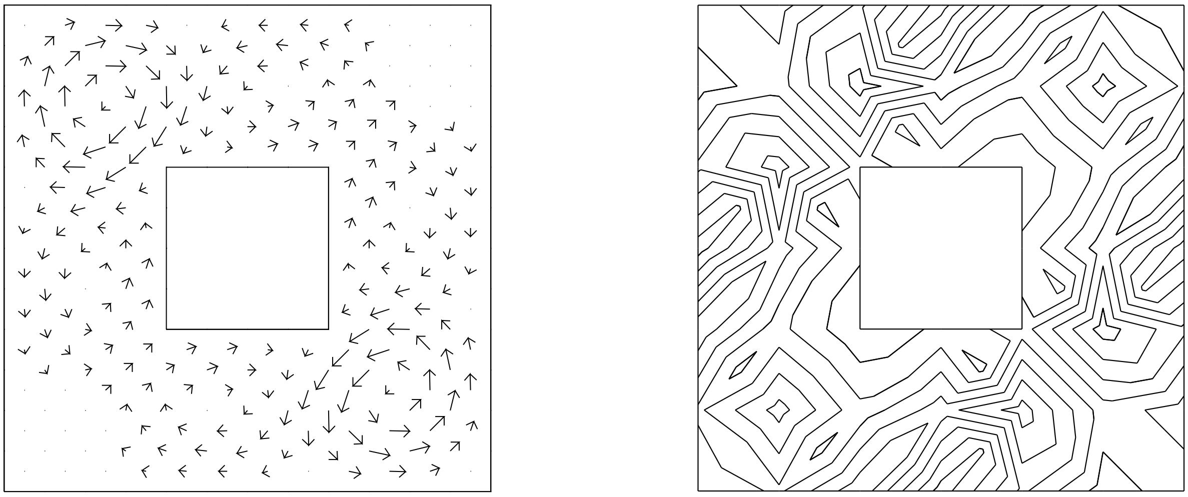
Eigenvalue $\omega_7 = -7.934$



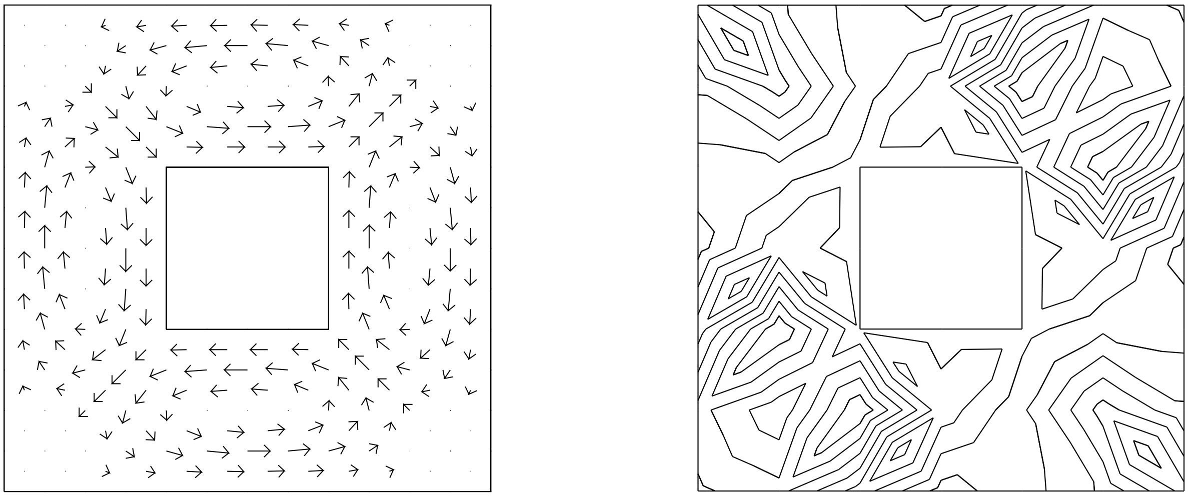
Graphical view of velocity field and isobaric lines
(Triangulation τ_{h_3}, case $k/m = 1.0$)

Figure 2.6

Eigenvalue $\omega_8 = -8.167$



Eigenvalue $\omega_{10} = -9.223$



Eigenvalue $\omega_{11} = -9.554$