Macroscopic Properties of a Stationary Non-Equilibrium Distribution for a Non-Gradient Interacting Particle System.

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Abstract

We consider a one dimensional generalized symmetric simple exclusion process where are permitted at most two particles per site. The system is open and at the boundary a stochastic dynamics is chosen to model two infinite reservoirs of particles with different densities. This simple model is non gradient. We prove that in the stationary state the particles empirical density field converges to the deterministic solution of a non linear elliptic equation as the microscopic size of the system goes to infinity. Fick's law of transport for the expected value of the current in the stationary state is also proven.

Introduction

Suppose one has to compute the density profile in a pipe connecting two infinite reservoirs containing a fluid with two different densities in a stationary regime. At a macroscopic level, i.e. when the distance between molecules is infinitely small compared to the size of the pipe, it is natural to set the equation for the density ρ

$$0 \;=\; \partial_t \rho \;=\; \partial_x J$$

and then to invoke Fick's Law in order to write the second equation for the current J:

$$J \;=\; \hat{a}(\rho)\partial_x \rho$$

All of this remains at a macroscopic level and ignores any description of the fluid in terms of a large number of interacting molecules. On the other hand if one is given the interaction potential between the molecules one should in principle be able to compute the coefficient $\hat{a}(\rho)$ and also to verify that Fick's law holds at least by taking appropriate limits.

In the past decade considerable efforts have been made to bridge the gap between the molecular description and the macroscopic one (usually known as *hydrodynamic limits* in

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the jargon of infinite particles systems). Due to the technical difficulties to this date, most of the rigorous results have been proven for stochastic caricatures as the one presented below, while Newtonian dynamics seem still out of reach.

Pioneering work on this subject was done in [GKMP] on the so called symmetric simple exclusion process, although it was not yet realized at that time that one was dealing with hydrodynamics so that one uses the term Fourier's Law rather than Fick's Law. However when properly translated this system gives rise to a constant coefficient $\hat{a}$ independent of ρ. Anyway the proof makes use of various special properties of the dynamics and cannot be extended to general situations.

In a recent paper Eyink, Lebowitz and Spohn (cf. [ELS]), using the robust techniques of [GPV], study the macroscopic properties of the stationary distribution for the class of all one-dimensional *gradient* lattice gas dynamics (see below for precise definitions) in contact with two infinite reservoirs of particles at different chemical potential. Notice that for such lattice models the stationary distribution is not explicitely computable. It is known that they are not local Gibbs states, and can be approximated by local Gibbs States only at a first order in the gradient of the density between the two reservoirs. In [ELS] it is shown for such models that the empirical density field of the particles converge to the deterministic solution of the stationary transport equation.

The results contained in [ELS] are limited to *gradient* lattice dynamics, i.e. to systems where the instantaneous current between two sites is given by the difference between a local function and its space translation, i.e. a spatial discrete gradient [1]. That means that the conservation law is already microscopically expressed as a second order difference equation. But the gradient condition imposes very restrictive assumptions on the dynamics.

The motivation of the present paper is the extension of the results of [ELS] to non-gradient dynamics of particles on a lattice. The simplest non-gradient lattice model[2] we could think of is a generalized exclusion model where particles performs symmetric random walks but no more than *two* particles per site are allowed (instead of one for the usual symmetric exclusion process), giving origin to a hard core exclusion if a particle attempts to jump to a site already occupied by two particles. For this model we have already studied (cf. [KLO]) the nonstationary macroscopic evolution when the system is closed (i.e. when it is isolated from exterior reservoirs, and the total number of particles is conserved).

We try now to give a heuristic idea of the increasing complexity of the various models. For the usual symmetric simple exclusion the situation is simple because the instantaneous current between two sites (say 0 and 1) is given by $\eta(1)-\eta(0)$. This is why the macroscopic equation is linear.

In [ELS], as we said above, the dynamics considered is such that the instantaneous current has the form of a gradient plus a 'time derivative' i.e.

$$W_{0,1} \;=\; \tau_1 h(\eta) - h(\eta) + LF(\eta)$$

[1] [ELS] results actually extend to systems with currents that are given by a spatial gradient plus a time derivative of local functions

[2] i.e. not in the class considered by [ELS] (see previous footnote)

were h and F are two local functions, L is the generator of the dynamics in the infinite volume (no boundary conditions), and τ is the space translation. The diffusion coefficent will depend only on h and not on F. The reason why the term LF is irrelevant is that its time fluctuations are orthogonal to the one of the gradient part $\tau_1 h(\eta) - h(\eta)$.

In our non-gradient situation there do not exist two such fixed functions h and F. Following Varadhan's ideas (cf. [V]), we show that the time fluctuations of $W_{0,1}$ with respect to the the dynamics generated by L in equilibrium, can be approximated by those of

$$\hat{a}(\rho)\,(\eta(1) - \eta(0)) - LF(\rho, \eta)$$

for some constant $\hat{a}(\rho)$ and function $F(\rho, \eta)$ that must depend on the parameter ρ that is the density of the equilibrium measure at which the these fluctuations are computed. So the gradient part of the current now will depend on the local density of particles. This picture is not complete: such a function $F(\rho, \eta)$ does not really exist. Formally it is given by some infinite sum and an approximation argument must be used.

The central problem in applying Varadhan's method here is to prove that locally the system is close to equilibrium. For this purpose we need a bound on the Dirichlet form of the stationary measure. To obtain this bound we have to modify an argument in [ELS] (cf. proposition 2, were the gradient condition was explicitely used).

As a corollary of the main result in this paper, we prove Fick's law which gives an explicit expression for the first order correction to the expectation of the current under the stationary measure.

1. Notation and Results.

For an integer N, let $T_N = \{0, \frac{1}{N}, \dots, 1\}$ and denote $\{0,1,2\}^{T_N}$ by X_N. We consider on X_N the Markov process, informally described in the introduction, which generator is given by

$$(L_N f)(\eta) = \sum_{k=0}^{N-1} (L_{k,k+1} f)(\eta) \;+\; (L_g f)(\eta) \;+\; (L_d f)(\eta)\,. \tag{1.1}$$

The elementary generators $L_{k,k+1}$, L_g and L_d are defined as follows.

$$(L_{k,k+1} f)(\eta) \;=\; d_{k,k+1}(\eta)\,[f(\eta^{k,k+1}) - f(\eta)] \;+\; g_{k,k+1}(\eta)\,[f(\eta^{k+1,k}) - f(\eta)] \tag{1.2}$$

and the rate functions $d_{k,k+1}$ and $g_{k,k+1}$ are given by

$$d_{k,k+1}(\eta) = \begin{cases} 1 & \text{if } \eta_k \geq 1,\ \eta_{k+1} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g_{k,k+1}(\eta) = \begin{cases} 1 & \text{if } \eta_k \leq 1,\ \eta_{k+1} \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

In the above formula $\eta^{k,i}$ stands for

$$\eta^{k,i}_j = \begin{cases} \eta_j & \text{if } j \neq k, i \\ \eta_k - 1 & \text{if } j = k \\ \eta_i + 1 & \text{if } j = i\,. \end{cases}$$

The "border" generators L_g, L_d are creation–destruction operators defined by

$$(L_g f)(\eta) \;=\; c_g^+(\eta_0)\left[f(\eta^{0,+}) - f(\eta)\right] \;+\; c_g^-(\eta_0)\left[f(\eta^{0,-}) - f(\eta)\right]$$

$$(L_d f)(\eta) \;=\; c_d^+(\eta_N)\left[f(\eta^{N,+}) - f(\eta)\right] \;+\; c_d^-(\eta_N)\left[f(\eta^{N,-}) - f(\eta)\right] \;,$$

where

$$\eta_j^{k,\pm} \;=\begin{cases} \eta_j & \text{if } j \neq k \\ \eta_k \pm 1 & \text{if } j = k \end{cases}$$

and the intensities $c_g^{\pm}$ and $c_d^{\pm}$ will be defined later. The exclusion operators $L_{k,k+1}$ correspond to jumps between sites k and $k+1$.

For an integer k, we denote by $W_{k,k+1}$ the current between sites k and $k+1$:

$$W_{k,k+1}(\eta) \;=\; d_{k,k+1}(\eta) \;-\; g_{k,k+1}(\eta)\,. \tag{1.3}$$

We will often omit the dependence of $W_{k,k+1}$ on η. Notice that with these notations, for an integer $1 \leq k \leq N-1$, $\quad L_N \eta_k$ writes

$$L_N \eta_k \;=\; W_{k-1,k} \;-\; W_{k,k+1}\;.$$

We consider the following one parameter family of product measures such that the probability to find r particles at a given site is proportional to λ^r. This formula makes sense for every nonnegative λ.

Notice that the mean occupation number of particles is given by

$$\frac{\lambda + 2\lambda^2}{1 + \lambda + \lambda^2}$$

which is obviously a strictly increasing function onto the interval $[0, 2)$.

For every ρ in $[0, 2]$, we denote by ν_ρ the above product measure with $\lambda(\rho)$ such that the density of particles on each site is ρ with the convention:

$$\nu_\rho\{\eta_0\} = \rho.$$

We will use indifferently the same notation ν_ρ for the product measure on X_N or on the infinite product $X = \{0, 1, 2\}^{\mathbf{Z}}$. As a shorthand for the expectation with respect to ν_ρ, we will use the notation

$$\int f(\eta)\nu_\rho(d\eta) = \langle f \rangle_\rho.$$

Observe that the operators $L_{k,k+1}$ are symmetric with respect to the measures ν_ρ for any $\rho \in \;[0, 2]$.

To define the creation and destruction rates at the border, we fix two densities ρ_g and ρ_d in $[0, 2]$. We choose $c_g^{\pm}(\cdot)$ and $c_d^{\pm}(\cdot)$ so as to make the creation and destruction process at the left (right) border reversible with respect to ν_{ρ_g} (ν_{ρ_d}). This corresponds to imposing to the rates to satisfy the equations:

$$\begin{aligned} c_g^+(i) &= \lambda(\rho_g)\; c_g^-(i+1) \quad i=0,1 \\ c_d^+(i) &= \lambda(\rho_d)\; c_d^-(i+1) \quad i=0,1\;. \end{aligned} \tag{1.4}$$

Notice that for each border we obtain a two parameter family of rates.

Given two densities ρ_g, ρ_d (the densities of particles we want to have respectively to the left and to the right of our system), the Markov process generated by L_N has a unique invariant measure that we will denote with μ_{ss}^N, and f_{ss} will denote its density with respect to a fixed product measure ν_ρ, taken as a reference measure. Our goal is to establish a law of large numbers for the density field under μ_{ss}^N and show that it converges, as N increases to ∞, to the solution of a non linear elliptic equation. In order to define the diffusion coefficient $\hat{a}(\rho)$ of the elliptic equation, we need to establish some notation and to consider the generalized exclusion process in the infinite space $X = \{0,1,2\}^{\mathbf{Z}}$.

For an integer k, denote by $L_{k,k+1}$ the generator of the generalized exclusion process restricted to sites k, $k+1$ defined in (1.2). For a density ρ, denote by $\mathcal{D}_\rho$ the Dirichlet form associated to the jumps between sites 0 and 1 in the infinite space X. Thus for $f: X \to \mathbf{R}$,

$$\mathcal{D}_\rho(f) = \left\langle d_{0,1}(\eta)[\sqrt{f(\eta^{0,1})} - \sqrt{f(\eta)}]^2 + g_{0,1}(\eta)[\sqrt{f(\eta^{1,0})} - \sqrt{f(\eta)}]^2 \right\rangle_\rho .$$

We will omit the index ρ when no confusion arises.

For an integer k define the operators $\nabla_{k,k+1}^g$ and $\nabla_{k,k+1}^d$ as

$$\begin{aligned} (\nabla_{k,k+1}^g f)(\eta) &= g_{k,k+1}(\eta)[f(\eta^{k+1,k}) - f(\eta)] \\ (\nabla_{k,k+1}^d f)(\eta) &= d_{k,k+1}(\eta)[f(\eta^{k,k+1}) - f(\eta)]. \end{aligned}$$

To keep notation simple set

$$(\nabla^g f)(\eta) = (\nabla_{0,1}^g f)(\eta) \quad \text{and} \quad (\nabla^d f)(\eta) = (\nabla_{0,1}^d f)(\eta).$$

Using the internal product on $\mathbf{R}^2$, the Dirichlet form $\mathcal{D}$ can be rewritten as

$$\mathcal{D}(f) = \left\langle \left\| \begin{pmatrix} \nabla^d \sqrt{f} \\ \nabla^g \sqrt{f} \end{pmatrix} \right\|^2 \right\rangle_\rho .$$

In order to define the diffusion coefficient, for every cylinder function $F: X \to \mathbf{R}$, consider the formal sum

$$\Gamma_F(\eta) = \sum_k \tau_k F$$

which does not make sense but for which

$$\nabla \Gamma_F = \begin{pmatrix} \nabla^d \Gamma_F \\ \nabla^g \Gamma_F \end{pmatrix}$$

are well defined since they involve only a finite number of non zero differences.

We are now in position to define the diffusion coefficient. For each ρ, define

$$\hat{a}(\rho) = \frac{1}{2\chi(\rho)} \inf_F \Big\langle \Big\| \begin{pmatrix} d_{0,1} \\ -g_{0,1} \end{pmatrix} - \begin{pmatrix} \nabla^d \Gamma_F \\ \nabla^g \Gamma_F \end{pmatrix} \Big\|^2 \Big\rangle_\rho . \tag{1.5}$$

In this formula χ stands for the usual static compressibility which in our case is equal to

$$\chi(\rho) = \Big\langle \eta_0^2 \Big\rangle_\rho - \Big\langle \eta_0 \Big\rangle_\rho^2 .$$

In [KLO] we proved that the diffusion coefficient $\hat{a}(\rho)$ is continuous strictly positive and not constant.

Before stating the main theorems of this article, we have to clarify what we mean by weak solutions. Denote by $C_0^2([0,1])$ the space of real, twice continuously differentiable functions vanishing at the border as well as its first and second derivatives.

A bounded function $\rho(x)\colon [0,1] \to \mathbf{R}$ is said to be a weak solution in H_{-1} of the nonlinear elliptic equation with boundary conditions

$$\begin{cases} \partial_x \left(\hat{a}(\rho) \partial_x \rho \right) = 0 \\ \rho(0) = \rho_g, \ \rho(1) = \rho_d \ . \end{cases} \tag{1.6}$$

if

(i) ρ has a derivative in $L^2([0,1])$:

$$\int_0^1 (\partial_x \rho)^2(x)\, dx \ < \ \infty \ ;$$

(ii) for every function $\mathcal{F}$ in $C_0^2([0,1])$,

$$\int_0^1 \mathcal{F}''(x) \hat{A}(\rho(x))\, dx \ = \ 0 \ .$$

In this last formula $\hat{A}$ represents the integral of $\hat{a}$:

$$\hat{A}(\rho) \ = \ \int_0^\rho \hat{a}(z)\, dz \qquad \text{for} \qquad \rho \in [0,2]. \tag{1.7}$$

Observe that $\hat{A}$ is a strictly increasing function.

It is easy to prove an uniqueness result of weak solution in H_{-1} of (1.6).

The first main result of this article is the following law of large numbers for the empirical measure under the stationary regime of the process.

Theorem 2.1. *For any $J \in C([0,1])$ and for any $\delta > 0$*

$$\lim_{N\to\infty} \mu_{ss}^N \left(\left| \frac{1}{N} \sum_{k=0}^{N} J\left(k/N\right) \eta_k \; - \; \int J(x)\rho(x)dx \right| > \delta \right) = 0$$

where $\rho(x)$ is the weak solution in H_{-1} of the nonlinear elliptic equation (1.6).

Theorem 2.2. *(Fick's law.) For every $x \in [0,1]$,*

$$\lim_{N\to\infty} N \; \mu_{ss}^N \Big\{ W_{[xN],[xN]+1} \Big\} \; = \; -\hat{a}(\rho(x))\partial_x \rho(x) \; ,$$

where ρ is the unique weak solution in H_{-1} of (1.6).

Observe that the last expression does not depend on x. In fact, we have:

$$\lim_{N\to\infty} N \; \mu_{ss}^N \Big\{ W_{[xN],[xN]+1} \Big\} \; = \; \hat{A}(\rho_g) - \hat{A}(\rho_d) \; .$$

3. Entropy Production.

In this section we prove that the Dirichlet form of the density of the stationary measure μ_{ss}^N with respect to a reference measure ν_ρ is bounded by C_0/N for some constant C_0. This is one of the main ingredient needed in the proof of the so called 1 and 2 blocks estimates.

For a first reading, the reader may wish to skip the proof of this result and go directly to the next section where Theorems 2.1 and 2.2 are proved.

For a positive integer ℓ and an integer k, we denote by η_k^ℓ the density of particles in a box of length $2\ell + 1$ centered at k:

$$\eta_k^\ell = \frac{1}{2\ell+1} \sum_{|j-k|\le \ell} \eta_j \; .$$

We will need often the following integration by parts formula. For an integer $0 \le k \le N-1$ and a cylinder function f,

$$\langle W_{k,k+1} f \rangle \; = \; -\frac{1}{2} \left\langle \begin{pmatrix} d_{k,k+1} \\ -g_{k,k+1} \end{pmatrix} \cdot \nabla_{k,k+1} f \right\rangle \; . \tag{3.1}$$

Proposition 3.1. *For every positive density ρ,*

$$\sum_{i=0}^{N-1} \left\langle \| \nabla_{i,i+1} \sqrt{f_{ss}} \|^2 \right\rangle \le \; 4 \left[\log \frac{\lambda(\rho_g)\lambda(\rho_d)}{\lambda(\rho)^2} \right]^2 N^{-1} \; .$$

Proof. By the definition of stationary state we have:

$$\begin{aligned} 0 &= \langle f_{ss} L_N \log f_{ss} \rangle \\ &= \left\langle f_{ss} \sum_k L_{k,k+1} \log f_{ss} \right\rangle + \langle f_{ss} L_g \log f_{ss} \rangle + \langle f_{ss} L_d \log f_{ss} \rangle \cdot \end{aligned}$$

We first obtain an upper bound for the second and third expressions of the right hand side of the last equality. For the second one, for instance, define R as $\lambda(\rho)/\lambda(\rho_g)$.

$$\begin{aligned} &\langle f_{ss} L_g \log f_{ss} \rangle \\ &\quad = \left\langle f_{ss}(\eta) c_g^+(\eta) [\log f_{ss}(\eta^{0,+}) - \log f_{ss}(\eta)] \right\rangle + \left\langle f_{ss}(\eta) c_g^-(\eta) [\log f_{ss}(\eta^{0,-}) - \log f_{ss}(\eta)] \right\rangle \\ &\quad = \left\langle f_{ss}(\eta) c_g^+(\eta) \log \frac{R f_{ss}(\eta^{0,+})}{f_{ss}(\eta)} \right\rangle + \left\langle f_{ss}(\eta) c_g^-(\eta) \log \frac{f_{ss}(\eta^{0,-})}{R f_{ss}(\eta)} \right\rangle \\ &\qquad\qquad - (\log R) \left\langle f_{ss}(\eta)(c_g^+ - c_g^-)(\eta) \right\rangle . \end{aligned}$$

Now by a change of variables and from property (1.4) of $c_g^\pm$ this last expression is equal to

$$\begin{aligned} &\left\langle c_g^-(\eta) \Big[R^{-1} f_{ss}(\eta^{0,-}) - f_{ss}(\eta) \Big] \log \frac{f_{ss}(\eta)}{R^{-1} f_{ss}(\eta^{0,-})} \right\rangle - (\log R) \left\langle f_{ss}(\eta)(c_g^+ - c_g^-)(\eta) \right\rangle \\ &\quad \le - \log \frac{\lambda(\rho)}{\lambda(\rho_g)} \left\langle f_{ss}(\eta)(c_g^+ - c_g^-)(\eta) \right\rangle . \end{aligned}$$

We denote by $W_{-1,0}$ the current at the left boundary:

$$W_{-1,0}(\eta) = c_g^+(\eta) - c_g^-(\eta) .$$

We proved therefore that

$$\langle f_{ss} L_g \log f_{ss} \rangle \le - \log \frac{\lambda(\rho)}{\lambda(\rho_g)} \langle f_{ss} W_{-1,0} \rangle .$$

Identical computation gives a similar inequality for the third expression of the last formula in this proof:

$$\langle f_{ss} L_d \log f_{ss} \rangle \le - \log \frac{\lambda(\rho)}{\lambda(\rho_d)} \langle f_{ss} W_{N,N+1} \rangle ;$$

where $W_{N,N+1}$ is the current at the right boundary:

$$W_{N,N+1}(\eta) = c_+^d(\eta) - c_-^d(\eta) .$$

Putting these two inequalities together we have

$$- \left\langle f_{ss} \sum_{k=0}^{N-1} L_{k,k+1} \log f_{ss} \right\rangle \le \log \left(\frac{\lambda(\rho_g)}{\lambda(\rho)} \right) \langle W_{-1,0} \rangle_{ss} + \log \left(\frac{\lambda(\rho_d)}{\lambda(\rho)} \right) \langle W_{N,N+1} \rangle_{ss} .$$

The second step of the proof consists in obtaining an upper bound for the right hand side of this last expression by means of the stationarity of μ_{ss} and the Schwarz inequality.

By the stationarity of f_{ss} and the conservation law of the $L_{i,i+1}$ we have

$$\langle f_{ss} W_{-1,0} \rangle \;=\; \langle f_{ss} W_{N,N+1} \rangle \;=\; \langle f_{ss} W_{i,i+1} \rangle \;=\; \frac{1}{N} \sum_{i=0}^{N-1} \langle f_{ss} W_{i,i+1} \rangle$$

From formula (3.1) and the previous equality, we have that

$$\langle f_{ss} W_{-1,0} \rangle \;=\; \langle f_{ss} W_{N,N+1} \rangle \;=\; -\frac{1}{2N} \sum_{i=0}^{N-1} \left\langle \begin{pmatrix} d_{i,i+1} \\ -g_{i,i+1} \end{pmatrix} \cdot \nabla_{i,i+1} f_{ss} \right\rangle \,.$$

By Schwarz inequality the right hand side of the last expression is bounded above by

$$\left(\frac{4}{N} \sum_{i=0}^{N-1} \left\langle \| \nabla_{i,i+1} \sqrt{f_{ss}} \|^2 \right\rangle \right)^{\frac{1}{2}}$$

Therefore,

$$- \left\langle f_{ss} \sum_{i=0}^{N-1} L_{i,i+1} \log f_{ss} \right\rangle \;\leq\; 2 \left[\log \frac{\lambda(\rho_g)\lambda(\rho_d)}{\lambda(\rho)^2} \right] \left(\frac{1}{N} \sum_{i=0}^{N-1} \left\langle \| \nabla_{i,i+1} \sqrt{f_{ss}} \|^2 \right\rangle \right)^{\frac{1}{2}} \,.$$

We conclude the proof of this proposition bounding the left side of this inequality by the Dirichlet form by means of an elementary inequality.

Since $a(\log b - \log a) \leq 2\sqrt{a}(\sqrt{b} - \sqrt{a})$,

$$\left\langle \| \nabla_{i,i+1} \sqrt{f_{ss}} \|^2 \right\rangle \;\leq\; - \; \langle f_{ss} L_{i,i+1} \log f_{ss} \rangle$$

for every $0 \leq i \leq N-1$. The last two inequalities give the bound on the Dirichlet form:

$$\sum_{i=0}^{N-1} \left\langle \| \nabla_{i,i+1} \sqrt{f_{ss}} \|^2 \right\rangle \leq \; 4 \left[\log \frac{\lambda(\rho_g)\lambda(\rho_d)}{\lambda(\rho)^2} \right]^2 N^{-1} \,. \quad \square$$

Recalling that the product measure ν_{ρ_g} is reversible for the generator L_g, we obtain the following bounds on the Dirichlet forms associated to L_g and L_d.

Corollary 3.2.

$$- \left\langle \sqrt{f_{ss}} L_g \sqrt{f_{ss}} \right\rangle_{\rho_g} \leq \; 4 \left[\log \frac{\lambda(\rho_d)}{\lambda(\rho_g)} \right]^2 N^{-1} \;;$$

$$- \left\langle \sqrt{f_{ss}} L_d \sqrt{f_{ss}} \right\rangle_{\rho_d} \leq \; 4 \left[\log \frac{\lambda(\rho_g)}{\lambda(\rho_d)} \right]^2 N^{-1} \,.$$

Proof: Again by stationarity we have:

$$
\begin{aligned}
- \left\langle \sqrt{f_{ss}} L_g \sqrt{f_{ss}} \right\rangle_{\rho_g} &\le - \langle f_{ss} L_g \log f_{ss} \rangle_{\rho_g} \\
= \left\langle f_{ss} \sum_{k=0}^{N-1} L_{k,k+1} \log f_{ss} \right\rangle_{\rho_g} &+ \langle f_{ss} L_d \log f_{ss} \rangle_{\rho_g} \\
&\le \langle f_{ss} L_d \log f_{ss} \rangle_{\rho_g}
\end{aligned}
$$

and we have already an upper bound on the last term. □

4. Hydrodynamical limit.

In this section we prove Theorems 2.1 and 2.2. Let $\mathcal{M}([0,1]^2)$ be the space of all positive measures on the square $[0,1]^2$ with total mass bounded by 4. Let π^N be the positive measure obtained from a configuration η by the relation

$$
\pi^N(\eta) \;=\; \frac{1}{N^2} \sum_{k,j=0}^{N} \eta_k \eta_j \delta_{(k/N, j/N)} \ ,
$$

where $\delta_{(u,v)}$ denotes the Dirac measure on (u,v). We will often omit the dependence on η in the notation of π^N.

Define Q^N as the measure on $\mathcal{M}([0,1]^2)$ obtained as the image of the stationary measure μ_{ss}^N by the application π^N:

$$
Q^N[A] \;=\; \mu_{ss}^N \left\{ \eta;\ \pi^N(\eta) \in A \right\} \ .
$$

In section 6 we prove that the sequence $(Q^N)_{N \ge 1}$ is tight and that every limit point Q^* is concentrated on measures absolutely continuous with respect to the Lebesgue measure and whose density is equal to the product of the marginal densities. More precisely, every limit point Q^* is concentrated on measures $\pi(dx,dy)$ such that

$$
\begin{aligned}
\pi(dx,dy) \;&=\; \rho(x,y)\, dx dy \\
\rho(x,y) \;&=\; \rho(x)\rho(y) dx dy \ .
\end{aligned}
$$

Moreover, we show that the marginal density has a derivative in $L^2(dx)$:

$$
Q^* \left[\int_0^1 (\partial_x \rho)^2(x)\, dx \right] \;<\; \infty \ .
$$

The strategy of the proof of Theorem 2.1 is simple. Recall the definition of the strictly increasing function $\hat{A}$ defined in (1.6) and denote by $\bar{\rho}$ the weak solution in H_{-1} of (1.6). We have to show that Q^* is concentrated on the path $\bar{\rho}$. One way of doing it is to prove that the positive function $I\colon [0,1] \to \mathbf{R}_+$ defined by

$$
I(x) \;=\; Q^* \left[\{\rho(x) - \bar{\rho}(x)\} \left\{ \hat{A}(\rho(x)) - \hat{A}(\bar{\rho}(x)) \right\} \right]
$$

is identically equal to 0 or to prove that the function $I: [0,1]^2 \to \mathbf{R}$ defined by

$$I(x,y) \;=\; Q^* \left[\{\rho(x) - \bar\rho(x)\} \left\{ \hat A(\rho(y)) - \hat A(\bar\rho(y)) \right\} \right]$$

vanishes on the diagonal.

This strategy is acomplished in three steps. In Proposition 4.1 and Corollary 4.3, we show that for every pair of functions $\mathcal{F}$ and $\mathcal{G}$ in $C_0^2([0,1])$,

$$Q^* \left[\int_0^1 dx \int_0^1 dy \left\{ \mathcal{G}''(x) \hat A(\rho(x)) \mathcal{F}(y) \rho(y) \;+\; \mathcal{G}(x) \rho(x) \mathcal{F}''(y) \hat A(\rho(y)) \right\} \right] \;=\; 0 \; .$$

Moreover, in Proposition 4.4, we show that for every function $\mathcal{F}$ in $C_0^2([0,1])$,

$$Q^* \left[\int_0^1 dx \mathcal{F}''(x) \hat A(\rho(x)) \right] \;=\; 0 \; .$$

From this two results and since $\bar\rho$ is the weak solution in H_{-1} of (2.5), we obtain that

$$\begin{aligned} Q^* \Bigg[\int_0^1 dx \int_0^1 dy \Big\{ & \mathcal{G}''(x) \mathcal{F}(y) \left[\hat A(\rho(x)) - \hat A(\bar\rho(x)) \right] [\rho(y) - \bar\rho(y)] \\ & + \mathcal{G}(x) \mathcal{F}''(y) [\rho(x) - \bar\rho(x)] \left[\hat A(\rho(y)) - \hat A(\bar\rho(y)) \right] \Big\} \Bigg] \;=\; 0 \end{aligned}$$

for every pair of functions in $C_0^2([0,1])$. This means that, in weak sense, the function $I(x,y)$ defined above has a second partial derivative with respect to the first argument skew symmetric:

$$(\partial_x^2 I)(x,y) \;+\; (\partial_x^2 I)(y,x) \;=\; 0$$

in the interior of $[0,1]^2$.

This first step concerns only the behavior of I inside the square $[0,1]^2$. The second step consists in proving that I is identically 0 at the border. This is done in Proposition 4.5 by proving a law of large numbers for the density at the border under the stationary measure.

Finally, we show that all functions I with the two previous properties vanish on the diagonal.

We now turn to the proofs.

Proposition 4.1. *Let $\mathcal{F}$, $\mathcal{G}: [0,1] \to \mathbf{R}$ be functions of class C^2 with compact support on $(0,1)$. For every limit point Q^* of the sequence Q^N,*

$$Q^* \left[\int_0^1 dx \int_0^1 dy \left\{ \mathcal{G}''(x) \hat A(\rho(x)) \mathcal{F}(y) \rho(y) \;+\; \mathcal{G}(x) \rho(x) \mathcal{F}''(y) \hat A(\rho(y)) \right\} \right] \;=\; 0 \; .$$

Proof. Assume without loss of generality that the sequence Q^N converges to Q^*.

Let $\iota: \mathbf{R} \to \mathbf{R}_+$ be a smooth approximation of the identity with support contained in $[-1,1]$:

$$\iota \geq 0 \; ; \qquad \int \iota(x)\, dx \;=\; 1 \; ; \qquad \iota(x) \;=\; 0 \quad \text{if} \quad |x| \geq 1 \; .$$

For a positive δ, denote by ι_δ the approximation of the identity ι scaled by δ:

$$\iota_\delta(\cdot) \;=\; \delta^{-1} \iota(\cdot/\delta)$$

and by $\mathcal{H}_\delta$ 1 plus the integral of the difference of two translations of ι_δ:

$$\mathcal{H}_\delta(x) \;=\; 1 \;+\; \int_{-\infty}^{x} \Big[\iota_\delta(y - 3\delta) - \iota_\delta(y + 3\delta) \Big]\, dy \; .$$

Notice that $\mathcal{H}_\delta$ is identically 0 in a neighborhood of the origin and identically 1 outside a larger neighborhood of the origin.

Since μ_{ss}^N is a stationary measure,

$$0 \;=\; N^{-2} \sum_{k,j} \mathcal{F}(k/N) \mathcal{H}_\delta((k-j)/N) \mathcal{G}(j/N) \Big\langle N^2 L_N[\eta_k \eta_j] \Big\rangle_{ss} \; .$$

Since $\mathcal{H}_\delta$ is identically 0 in a neighborhood of the origin, in the above formula we have only to compute $L_N \eta_k \eta_j$ for k and j far apart. In this case the generator acts separetely on each term. Recalling the definition (1.3) of the current, we obtain that

$$L_N \eta_k \eta_j \;=\; \eta_k \left[W_{j-1,j} - W_{j,j+1} \right] \;+\; \eta_j \left[W_{k-1,k} - W_{k,k+1} \right] .$$

Therefore, by a discrete integration by part, since $\mathcal{F}$ and $\mathcal{G}$ have compact support in $(0,1)$, we obtain that the last expression is equal to

$$\begin{aligned} & N^{-1} \sum_{k,j} \mathcal{G}(k/N) \partial^N \left(\mathcal{F}(j/N) \mathcal{H}_\delta((k-j)/N) \right) < \eta_k W_{j,j+1} >_{ss} \\ & \quad + \; N^{-1} \sum_{k,j} \mathcal{F}(j/N) \partial^N \left(\mathcal{G}(k/N) \mathcal{H}_\delta((k-j)/N) \right) < \eta_j W_{k,k+1} >_{ss} \quad . \end{aligned} \tag{4.1}$$

In the above formula, ∂^N stands for the discrete derivative:

$$\partial^N \mathcal{F}(k/N) \;=\; N[\mathcal{F}((k+1)/N) - \mathcal{F}(k/N)] \; .$$

We consider the two expression separately. Define $\Phi_N: [0,1] \times [0,1] \to \mathbf{R}$ as the smooth functions $\mathcal{G} \partial^N (\mathcal{F}\mathcal{H})$:

$$\Phi_N(k/N, j/N) \;=\; \mathcal{G}(k/N) \partial^N \left(\mathcal{F}(j/N) \mathcal{H}_\delta((k-j)/N) \right) \; .$$

Notice that for sufficiently large N Φ_N is a smooth function with compact support on E, where

$$E \;=\; \{(x,y) \in (0,1)^2 ; \; x \neq y\} \; . \tag{4.2}$$

For every positive integer ℓ, let

$$X_{N,\ell}(\eta):\ = N^{-1} \sum_{k,j} \Phi_N(k/N, j/N) \eta_k W_{j,j+1} \\ + N^{-1} \sum_{k,j} \Phi_N(k/N, j/N) \eta_k \, \hat{a}\left(\eta_j^\ell\right) (2\ell+1)^{-1} [\eta_{j+\ell} - \eta_{j-\ell}] \, .$$

In the next section we will prove the following proposition which is the crucial part of the proof.

Proposition 4.2. *For every functions $\mathcal{F}$, $\mathcal{G}$ in $C_K^2((0,1))$ and positive δ,*

$$\lim_{\epsilon\to 0} \lim_{N\to\infty} \Big\langle X_{N,N\epsilon} \Big\rangle_{ss} \ = \ 0 \, .$$

With this result it is easy to conclude the proof of Theorem 4.1. Indeed, from Proposition 4.2 and (4.1), we have that

$$0 \ = \ \lim_{\epsilon\to 0} \lim_{N\to\infty} \Bigg\{ N^{-1} \sum_{k,j} \Phi_N(k/N, j/N) \Big\langle \eta_k \, \hat{a}\left(\eta_j^{N\epsilon}\right) (2N\epsilon + 1)^{-1} [\eta_{j+\epsilon N} - \eta_{j-\epsilon N}] \Big\rangle_{ss} \\ + \ N^{-1} \sum_{k,j} \Phi'_N(k/N, j/N) \Big\langle \eta_k \, \hat{a}\left(\eta_j^{N\epsilon}\right) (2N\epsilon + 1)^{-1} [\eta_{j+\epsilon N} - \eta_{j-\epsilon N}] \Big\rangle_{ss} \Bigg\} \, ,$$

where Φ'_N is defined as Φ_N with the roles of $\mathcal{F}$ and $\mathcal{G}$ interchanged.

Since we assumed that the sequence Q^N converges to Q^*, the first line of the last expression is equal to

$$\lim_{\epsilon\to 0} Q^* \Bigg[\int dx \int dy \, \rho(x) \mathcal{G}(x) \partial_y \left[\mathcal{F}(y) \mathcal{H}_\delta(y) \right] \hat{a} \left((2\epsilon)^{-1} \int_{y-\epsilon}^{y+\epsilon} \rho(z) \, dz \right) \\ \times (2\epsilon)^{-1} [\rho(y+\epsilon) - \rho(y-\epsilon)] \Bigg] \, .$$

In Appendix 1 we prove that all limit points of the sequence Q^N are concentrated on measure whose densities with respect to the Lebesgue measure are in $L^2(dx)$. Therefore the last expression is equal to

$$Q^* \left[\int dx \int dy \, \rho(x) \mathcal{G}(x) \partial_y \left[\mathcal{F}(y) \mathcal{H}_\delta(y) \right] \hat{a} \left(\rho(y) \right) \partial_y \rho(y) \right] \, .$$

From the definition of $\mathcal{H}_\delta$, its derivative is equal to the difference of an approximation of the identity and its translation by 6δ. Since, under Q^*, $\partial_y \rho$ belongs to $L^2(dx)$, the limit of the last expression, as δ decreases to 0, is equal to

$$Q^* \left[\int dx \int dy \, \rho(x) \mathcal{G}(x) (\partial_y \mathcal{F})(y) \hat{a} \left(\rho(y) \right) \partial_y \rho(y) \right] \, .$$

An integration by parts conclude the proof of Theorem 4.1. $\square$

Corollary 4.3. *The statement of Proposition 4.1 holds for every function $\mathcal{F}$ and $\mathcal{G}$ in $C_0^2([0,1])$.*

Proof. Since every limit point of the sequence Q^N is concentrated on measures $\pi(dx,dy)$ whose density is bounded, Corollary 4.3 follows by approximating functions in $C_0^2([0,1])$ by functions in $C_K^2([0,1])$. $\square$

Proposition 4.4. *Let $\mathcal{F}\colon[0,1]\to\mathbf{R}$ be functions in $C_0^2([0,1])$. For every limit point Q^* of the sequence Q^N,*

$$Q^*\left[\int_0^1 dx\mathcal{F}''(x)\hat{A}(\rho(x))\right] \;=\; 0\,.$$

The proof is omitted since it is similar to the one of Proposition 4.1 and Corollary 4.3.

Let $\bar\rho$ be the unique weak solution in H_{-1} of (1.6). From Propositions 4.1 and 4.4, it follows that for every function $\mathcal{F}$ and $\mathcal{G}$ in $C_0^2([0,1])$, we have

$$\begin{aligned} Q^*\Bigg[\int_0^1 dx\int_0^1 dy\Big\{&\mathcal{G}''(x)\mathcal{F}(y)\left[\hat{A}(\rho(x))-\hat{A}(\bar\rho(x))\right][\rho(y)-\bar\rho(y)] \\ &+\mathcal{G}(x)\mathcal{F}''(y)[\rho(x)-\bar\rho(x)]\left[\hat{A}(\rho(y))-\hat{A}(\bar\rho(y))\right]\Big\}\Bigg] \;=\; 0 \end{aligned} \tag{4.3}$$

for every limit point Q^* of the sequence Q^N.

Up to this point we have obtained properties of the limit points Q^* on the interior of the interval $[0,1]$. We now turn to the proof of a law of large numbers for the density on the borders.

Proposition 4.5. *For any $\delta>0$:*

$$\lim_{\epsilon\to0}\lim_{N\to\infty}\mu_{ss}^N\left(\Big|\frac{1}{\epsilon N}\sum_{j=0}^{\epsilon N}\eta_j \;-\; \rho_g\Big|\geq\delta\right) \;=\; 0\,;$$

$$\lim_{\epsilon\to0}\lim_{N\to\infty}\mu_{ss}^N\left(\Big|\frac{1}{\epsilon N}\sum_{j=0}^{\epsilon N}\eta_{N-j} \;-\; \rho_d\Big|\geq\delta\right) \;=\; 0\,.$$

Proof. This follows from Corollary 3.2. In fact

$$\mu_{ss}^N\left(\Big|\frac{1}{\epsilon N}\sum_{j=0}^{\epsilon N}\eta_j \;-\; \rho_g\Big|\right)\leq\mu_{ss}^N\left(\Big|\frac{1}{\epsilon N}\sum_{j=0}^{\epsilon N}\eta_j \;-\; \frac{1}{l}\sum_{j=0}^{l}\eta_j\Big|\right)+\mu_{ss}^N\left(\Big|\frac{1}{l}\sum_{j=0}^{l}\eta_j \;-\; \rho_g\Big|\right)\,.$$

The first term on the right is then controlled by a two block argument (cf. Lemma 4.2 in [KLO]) and goes to zero as $N \to \infty$ and then $\epsilon \to 0$. The second term goes to zero as $N \to \infty$ and then $l \to \infty$ by the law of large numbers, since by Corollary 3.2. the Dirichlet form of μ_{ss}^N restricted to $\{0,1,2\}^{\{0,\dots,l\}}$ converges to 0 as $N \uparrow \infty$. $\square$

Corollary 4.6. *Every limit point Q^* of the sequence Q^N is such that*

$$Q^*[\rho(0) \;=\; \rho_g] \;=\; 1 \qquad \textit{and} \qquad Q^*[\rho(1) \;=\; \rho_d] \;=\; 1\;.$$

Proof. Let Q^* be a limit point. From Proposition 4.5,

$$\lim_{\epsilon \to 0} Q^* \left(\left| \epsilon^{-1} \int_0^\epsilon \rho(z)\, dz \;-\; \rho_g \right| \right) \;=\; 0\;.$$

Since, by Proposition A.1.1, Q^* is concentrated on profiles with continous densities, the integral in the last formula converges to $\rho(0)$ Q^* a.s. . $\square$

Proof of Theorem 2.1. Let $\bar\rho(x)$ be the solution of the equation (1.6). Fix a limit point Q^* of the sequence Q^N. Define the function $I\colon [0,1]^2 \to \mathbf{R}$ as

$$I(x,y) = Q^* \left(\left[\hat A\left(\rho(x)\right) - \hat A\left(\bar\rho(x)\right) \right] \left(\rho(y) - \bar\rho(y)\right) \right)\;.$$

Notice that I is positive on the diagonal.

By equation (4.3) we have that in a weak sense

$$\partial_x^2 I(x,y) + \partial_y^2 I(y,x) = 0$$

in the interior of $[0,1]^2$. This says that there exists a function $g(x,y)$ skew–symmetric such that

$$\partial_x^2 I(x.y) \;=\; g(x,y)\;.$$

From Proposition 4.5, we know the behavior of the function I at the boundary. Indeed, we have that

$$\lim_{\epsilon \to 0} \epsilon^{-1} \int_0^\epsilon I(x,y)\, dx \;=\; 0$$

uniformly in y and similar limits for the three other boundary of $[0,1]^2$.

Let $G(x,x')$ be the corresponding Green function. We have

$$\int_0^1 I(x,x) dx = \int_0^1 \int_0^1 G(x,y) g(x,y) dx dy\;.$$

The Green function G is symmetric and g is skew–symmetric. Therefore the right hand side of the last expression is equal to 0.

Since $I(x,x) \ge 0$ this implies that $I(x,x) = 0$ a. e.. $\hat{A}$ is a strictly increasing function, so we have that

$$Q^*(\rho(x) = \bar{\rho}(x) \text{ a. e. }) = 1 \ .$$

This conclude the proof of Proposition 2.1. $\square$

We now turn to the proof of Fick's law. It is a simple consequence of the bound on the Dirichlet form obtained in Proposition 3.1 and of Proposition 4.2.

Proof of Theorem 2.2. Since μ_{ss}^N is stationary, $\left\langle W_{i,i+1} \right\rangle_{ss}$ does not depend on i. For this reason, if for a positive δ, $\mathcal{F}_\delta$ denotes a smooth function with compact support on $(0,1)$ such that

$$0 \le \mathcal{F}_\delta(x) \le 1 \qquad \text{and} \qquad \mathcal{F}_\delta(x) = 1 \quad \text{for} \quad \delta \le x \le 1-\delta \ ,$$

we have that

$$\begin{aligned} N\left\langle W_{[xN],[xN]+1} \right\rangle_{ss} &= \sum_{k=0}^{N-1} \left\langle W_{k,k+1} \right\rangle_{ss} \\ &= \sum_{k=0}^{N-1} \mathcal{F}_\delta(k/N) \left\langle W_{k,k+1} \right\rangle_{ss} + \sum_{k=0}^{N-1} (1-\mathcal{F}_\delta)(k/N) \left\langle W_{k,k+1} \right\rangle_{ss} \ . \end{aligned}$$

The second sum of the last line is of order $O(\delta)$. Indeed, since μ_{ss}^N is stationary, this expression may be rewritten as

$$\left(\sum_{k=0}^{N-1} (1-\mathcal{F}_\delta)(k/N) \right) \left(\frac{1}{N} \sum_{j=0}^{N-1} \left\langle W_{j,j+1} \right\rangle_{ss} \right) \ .$$

By the integration by part formula (3.1) and Proposition 3.1 the absolute value of this expression is bounded above by,

$$2\delta N \left\{ N^{-1} \sum_{k=0}^{N-1} \left\langle \| \nabla_{i,i+1} \sqrt{f_{ss}} \|^2 \right\rangle \right\}^{1/2} \ \le \ C_0 \delta \ .$$

On the other hand, by arguments similar to the ones used in the proof of Proposition 4.2, for each fixed δ, the first sum of the last line converges to

$$- \int_0^1 dx \mathcal{F}_\delta(x) \hat{a}(\rho(x)) \partial_x \rho(x) \ ,$$

where ρ is the solution of (1.6). Notice that for such a function $\hat{a}(\rho(x))\partial_x \rho(x)$ does not depend on x. In particular, the last expresssion converges to

$$-\hat{a}(\rho(x))\partial_x \rho(x)$$

when δ decreases to 0. $\square$

5. Proof of Proposition 4.2.

In order to prove Proposition 4.2, we need to introduce some notation. For a cylinder function F and $0 \le \rho \le 2$ define $A(F,\rho)$ as

$$A(F,\rho) = \frac{1}{2}\Big\langle \Big\| \begin{pmatrix} d_{0,1} \\ -g_{0,1} \end{pmatrix} - \nabla \Gamma_F \Big\|^2 \Big\rangle_\rho - \chi(\rho)\hat{a}(\rho).$$

From the definition (1.4) of $\hat{a}(\cdot)$, it is clear that $A(F,\rho)$ is positive and bounded. Moreover, for every $0 \le \rho \le 2$, the infimum of $A(F,\rho)$ over all cylinder functions satisfies

$$\inf_F A(F,\rho) = 0.$$

We need to consider functions F that depend also on the density. Let us denote by $\mathcal{U}$ the space of functions $F\colon [0,2] \times X \to \mathbf{R}$ such that

(i) For each $\rho \in [0,2]$, $F(\rho,\cdot)$ is a cylinder function with uniform support. That is there exist a finite set Λ such that for each ρ in $[0,2]$ the support of $F(\rho,\cdot)$ is contained in Λ.

(ii) For each configuration η, $F(\cdot,\eta)$ is a smooth function.

In what follows, for a function F in $\mathcal{U}$, we denote by $b = b(F)$ the common support of the cylinder functions $F(\rho,\cdot)$:

$$\mathrm{supp} F(\rho,\cdot) \subset \{0,1,2\}^{\{-b,\dots,b\}} \,. \tag{5.1}$$

The application A can be extended to act on functions of $\mathcal{U}$. Indeed, for a function F in $\mathcal{U}$, define $A(F,\rho)$ as

$$A(F,\rho) = A(F(\rho,\cdot),\rho).$$

Notice that

$$\inf_{F\in\mathcal{U}} \sup_{\rho\in[0,2]} A(F,\rho) \equiv 0$$

and that $A(F,\cdot)$ is continuous for every F in $\mathcal{U}$ since by Theorem A.3.2 in [KLO] $\hat{a}$ is continuous.

For a function F in $\mathcal{U}$ and a positive integer ℓ define a cylinder function F_ℓ by

$$F_\ell(\eta) = F(\eta_0^\ell, \eta) \,.$$

The proof of Proposition 3.2 follows from the following two lemmas. Recall from (4.2) the definition of the set E.

Lemma 5.1. *Let $\Phi\colon (0,1)^2 \to \mathbf{R}$ be a smooth function with compact support on E. For every positive integer ℓ and every function F in $\mathcal{U}$, let*

$$\begin{aligned} X_{N,\ell}^{F,\Phi}(\eta) :&= N^{-1} \sum_{k,j} \Phi(k/N, j/N) \eta_k W_{j,j+1} \\ &+ N^{-1} \sum_{k,j} \Phi(k/N,j/N)\eta_k\, \hat{a}\left(\eta_j^\ell\right) (2\ell)^{-1} [\eta_{j+\ell} - \eta_{j-\ell}] \\ &- N^{-1} \sum_{k,j} \Phi(k/N,j/N)\, \eta_k\, L_N(\tau_j F_\ell) \,. \end{aligned}$$

Then,

$$\inf_{F\in\mathcal{U}} \lim_{\epsilon\to 0} \lim_{N\to\infty} \left\langle X^F_{N,N\epsilon} \right\rangle_{ss} \;=\; 0\ .$$

Lemma 5.2. *Let $\Phi\colon(0,1)^2 \to \mathbf{R}$ be a smooth function with compact support on E. For every function F in $\mathcal{U}$,*

$$\lim_{N\to\infty} N^{-1} \sum_{k,j} \Phi(k/N, j/N)\Big\langle \eta_k\, L_N(\tau_j F_{N\epsilon})\Big\rangle_{ss} \;=\; 0\ .$$

We start with the proof of Lemma 5.2 which is simpler.

Proof of Lemma 5.2. Fix a function F in $\mathcal{U}$. By the stationarity of μ^N_{ss},

$$0 \;=N^{-1} \sum_{k,j} \Phi(k/N, j/N) \left\langle L_N \left[\eta_k\, (\tau_j F_{N\epsilon})\right] \right\rangle_{ss}\ .$$

Since Φ vanishes in a neigbourhood of the diagonal of $(0,1)^2$, for sufficiently large N and small ϵ, the generator L_N acts separetely on the product $\eta_k(\tau_j F_{N\epsilon})$. Therefore the last expression rewrites

$$\begin{aligned} 0 \;=N^{-1} &\sum_{k,j} \Phi(k/N, j/N)\Big\langle \eta_k\, L_N(\tau_j F_{N\epsilon})\Big\rangle_{ss} \\ &+\; N^{-1} \sum_{k,j} \Phi(k/N, j/N)\Big\langle [W_{k-1,k} - W_{k,k+1}]\tau_j F_{N\epsilon}\Big\rangle_{ss}\ . \end{aligned}$$

Therefore, to conclude the proof of the lemma it is enough to show that the second term of the last expression converges to 0. Since Φ has compact support on $(0,1)^2$, a discrete integration by part gives that it is equal to

$$N^{-2} \sum_{k,j} \partial^N_1 \Phi(k/N, j/N)\Big\langle W_{k,k+1}(\tau_j F_{N\epsilon}) f_{ss}\Big\rangle\ .$$

In this last formula, ∂^N_1 denotes the discrete partial derivative with respect to the first coordinate. By the integration by parts formula, since Φ vanishes in a neighborhood of the diagonal, for sufficiently large N and small ϵ, this last expression is equal to

$$\frac{-1}{2N^2} \sum_{k,j} \partial^N_1 \Phi(k/N, j/N)\Big\langle (\tau_j F_{N\epsilon}) \begin{pmatrix} d_{k,k+1} \\ -g_{k,k+1} \end{pmatrix} \cdot \nabla_{k,k+1} f_{ss} \Big\rangle\ ,$$

where the product $\cdot$ is the usual internal product. Since F is a bounded function, the absolute value of this last expression is bounded above by

$$\|F(\partial_x \Phi)\|_\infty \frac{1}{2N} \sum_k \left\langle \left| \begin{pmatrix} d_{k,k+1} \\ -g_{k,k+1} \end{pmatrix} \cdot \nabla_{k,k+1} f_{ss} \right| \right\rangle\ .$$

By Schwarz inequality and, by now, standard arguments, this last sum is bounded above by

$$\|F(\partial_x\Phi)\|_\infty \left\{ \frac{4}{N} \sum_k \left\langle \left\| \nabla_{k,k+1}\sqrt{f_{ss}} \right\|^2 \right\rangle \right\}^{1/2} \;\le\; \frac{C_1}{N}.$$

In the last step we used Proposition 3.1 to bound the Dirichlet form of the density f_{ss}. □

Proof of Lemma 5.1. The proof is divided in two steps. The first one consists in reducing the problem of a small macroscopic block to the same problem for a large microscopic block.

First of all, there is no spatial average on $W_{j,j+1}$ in $X_{N,N\epsilon}$. Such a spatial average is crucial and can easily be inserted since

$$N^{-1}\sum_{k,j} \Phi(k/N, j/N)\eta_k \left[(2\ell-1)^{-1} \sum_{|i-j|\le \ell-1} W_{i,i+1} \;-\; W_{j,j+1} \right]$$

is of order ℓ^2/N as one can see after using discrete integration by part and from the presence of a discrete laplacian.

The second step in reducing the problem of a small macroscopic block to the same problem for a large microscopic block is to replace the term involving the generator L_N by a simpler one. In order to do it we need to introduce some notation.

For a function F in $\mathcal{F}$ denote by $(L_N F)(\rho,\eta)$ the generator acting on the cylinder function $F(\rho,\cdot)$ and taken at the configuration η. Moreover, for a positive integer ℓ, $(L_N F)(\eta_0^\ell,\eta)$ denotes the smooth (in the first variable) function $(L_N F)(\cdot,\eta)$ taken at the density $\rho = \eta_0^\ell$:

$$(L_N F)(\eta_0^\ell,\eta) = (L_N F)(\cdot,\eta)\Big|_{\rho=\eta_0^\ell}.$$

The difference between $(L_N F)(\eta_0^{N\epsilon},\eta)$ and $L_N F_{N\epsilon}(\eta)$ which appears in the definition of $X_{N,N\epsilon}$ is that the latter takes into account the changes of the density of particles in the box $[-N\epsilon, N\epsilon]$ while the former does not. We claim that we may replace the latter by the former in the definition of $X_{N,N\epsilon}$. This is the content of the next lemma.

Lemma 5.3. *Under the assumptions of Lemma 5.1,*

$$\lim_{N\to\infty} N^{-1}\sum_{k,j} \Phi(k/N,j/N)\Big\langle \eta_k\tau_j \left[L_N F_{N\epsilon}(\eta) - (L_N F)(\eta_0^{N\epsilon},\eta) \right] \Big\rangle_{ss} \;=\; 0.$$

The proof of this lemma is omitted since it is identical to the one of Lemma 4.1 in [KLO]. We have just to keep in mind two facts. First of all, Φ has compact support on E. For this reason the cylinder function η_k appearing in the statement of Lemma 5.3 will not prevent us of using the integration by parts (3.1). Secondly, in section 3, we proved that the Dirichlet form of the density f_{ss} is bounded by C/N. These are all ingredients needed in the proof of Lemma 4.2 of [KLO].

In Lemma 5.3 we replaced the expression $(L_N F_{N\epsilon})(\eta)$ appearing in the definition of $X_{N,N\epsilon}$ by a simpler one $(L_N F)(\eta_0^{N\epsilon}, \eta)$. The aim of this first step as explained above is to transform the original problem which involves the density of particles in small macroscopic boxes to the same problem with large microscopic boxes. Therefore we need to replace the density $\eta_0^{N\epsilon}$ appearing in the last expression by a density in a large microscopic block. This is the content of the next lemma.

Lemma 5.4. *Under the assumptions of Lemma 5.1,*

$$\lim_{\ell\to\infty} \lim_{\epsilon\to 0} \lim_{N\to\infty}$$

$$N^{-1} \sum_{k,j} \Phi(k/N, j/N) \Big\langle \eta_k \, \tau_j \left[(L_N F)(\eta_0^{N\epsilon}, \eta) - (L_N F)(\eta_0^{\ell}, \eta) \right] \Big\rangle_{ss} = 0.$$

The proof of this lemma is similar to the one of Lemma 4.2 in [KLO] and therefore is omitted.

The last step in the localization of the term involving the generator L_N is to introduce a space average on the configurations which will be crucial in what follows. In order to do it we introduce some notation.

For an integer j, $(\tau_j F)(\rho, \eta)$ denotes the cylinder function $F(\rho, \cdot)$ translated by j and taken at the configuration η. Moreover, $(\tau_j F)(\eta_0^\ell, \eta)$ denotes the function $(\tau_j F)(\rho, \eta)$ taken at the density $\rho = \eta_0^\ell$:

$$(\tau_j F)(\eta_0^\ell, \eta) = F(\rho, \tau_j \eta)\Big|_{\rho = \eta_0^\ell}.$$

Recall from (5.1) that we denote by b the common support of the functions $F(\rho, \cdot)$. Define ℓ_b by $\ell_b = \ell - b - 1$. The previous two lemmas show that in the definition of $X_{N,N\epsilon}$ we can replace $(L_N F_{N\epsilon})(\eta)$ by $(L_N F)(\eta_0^\ell, \eta)$. On the other hand, the expression

$$N^{-1} \sum_{k,j} \Phi(k/N, j/N) \eta_k \, \tau_j \left[(L_N F)(\eta_0^\ell, \eta) - \frac{1}{2\ell_b + 1} \sum_{i=-\ell_b}^{\ell_b} (\tau_i L_N F)(\eta_0^\ell, \eta) \right]$$

is of order ℓ^2/N. This can be seen using a discrete integration by part since it will appear a discrete laplacian. Notice that the average on $L_N F$ uses only those translations that remain inside the box $\{-\ell, \ldots, \ell\}$.

The final step in the first part of the proof consists in replacing

$$\hat{a}(\eta_j^{N\epsilon}) \frac{1}{2N\epsilon} \sum_{j-N\epsilon}^{j+N\epsilon-1} [\eta(i+1) - \eta(i)]$$

in the definition of $X_{N,N\epsilon}^{F,\Phi}$ by

$$\hat{a}(\eta_j^{\ell}) \frac{1}{2\ell} \sum_{j-\ell}^{j+\ell-1} [\eta(i+1) - \eta(i)] \, .$$

This is the content of the next lemma.

Lemma 5.5. *Under the assumptions of Lemma 5.1,*

$$\lim_{\ell\to\infty}\lim_{\epsilon\to 0}\lim_{N\to\infty} N^{-1}\sum_{k,j}\Phi(k/N,j/N)$$

$$\Big\langle \eta_k\Big[\hat a(\eta_j^{N\epsilon})\frac{1}{2N\epsilon}\sum_{j-N\epsilon}^{j+N\epsilon-1}[\eta_{i+1}-\eta_i]-\hat a(\eta_j^{\ell})\frac{1}{2\ell}\sum_{j-\ell}^{j+\ell-1}[\eta_{i+1}-\eta_i]\Big]\Big\rangle_{ss} \;=\; 0.$$

The proof of this lemma is omitted since it is similar to the one of Lemma 4.5 in [KLO].

Up to this point we reduced the proof of Lemma 5.1 to the proof of the following statement.

For a positive integer ℓ define $V_\ell^F(\eta)$ as

$$V_\ell^F(\eta)=\frac{1}{2\ell}\sum_{j=-\ell}^{\ell-1}W_{j,j+1}(\eta)+\frac{1}{2\ell}\hat a\left(\eta_0^\ell\right)[\eta_\ell-\eta_{-\ell}]-\frac{1}{2\ell_b+1}\sum_{j=-\ell_b}^{\ell_b}(\tau_j L_N F)(\eta_0^\ell,\eta)\,.$$

Lemma 5.6. *Under the assumptions of Lemma 5.1,*

$$\inf_{F\in\mathcal{U}}\lim_{\ell\to\infty}\lim_{N\to\infty}N^{-1}\sum_{k,j}\Phi(k/N,j/N)\Big\langle\eta_k(\tau_j V_\ell^F)\Big\rangle_{ss}\;=\;0.$$

Proof. Since by Proposition 3.1 the Dirichlet form of f_{ss} is bounded by C_0/N, for every positive β, the above expected value is bounded by

$$\sup_f\left\{N^{-1}\sum_{k,j}\Phi(k/N,j/N)\left\langle\eta_k(\tau_j V_\ell^F)f\right\rangle\;-\;N\beta D_N(f)\right\}\;+\;\beta C_0\,.$$

In the above formula, the supremum is taken over all densities with respect to the product measure ν_ρ. Therefore, to prove the theorem it is enough to show that for every positive β,

$$\inf_{F\in\mathcal{U}}\limsup_{\ell\to\infty}\limsup_{N\to\infty}\sup_f\left\{N^{-1}\sum_{k,j}\Phi(k/N,j/N)\Big\langle\eta_k(\tau_j V_\ell^F)f\Big\rangle\;-\;\beta N D_N(f)\right\}\;\le 0\,.$$

For a density f and for an integer $0\le k\le N$, define the density f_k as

$$f_k(\eta)\;=\;Z_k^{-1}(f)\eta_k f(\eta)\,,$$

where $Z_k(f)$ is a renormalizing constant:

$$Z_k(f) \;=\; \int \eta_k f(\eta) \nu_\rho(d\eta)\; .$$

Notice that $Z_k(f)$ is bounded by 2 since there are at most 2 particles per site and f is a density.

Since there are at most two particles per site and since for integers i at distance at least 2 from k, η_k behaves like a constant for jumps between sites i and $i+1$, the Dirichlet form restricted to the bond i, $i+1$ of f_k is bounded by the one of f:

$$I_{i,i+1}(f_k) \;\le\; 2 Z_k^{-1}(f) I_{i,i+1}(f)$$

for $|i-k|\ge 2$.

For a density f_k and for an integer $0\le j\le N$, denote by $f_{k,j}$ the conditional expectation of f_k with respect to the σ-algebra generated by $\eta_{j-\ell},\dots,\eta_{j+\ell}$. For integers j such that $|j-k|\ge \ell+1$, for the same reasons presented above and from the convexity of the Dirichlet form, a standard computation (cf. [GPV], [KLO]) shows that

$$I_{i,i+1}(f_{k,j}) \;\le\; I_{i,i+1}(f_k)\; .$$

Since the cylinder function V_ℓ^F depends on η only through $\eta_{-\ell},\dots,\eta_\ell$, from the above two remarks on the Dirichlet forms and since Φ vanishes in a neighborhood of the diagonal, for N sufficiently large the last supremum is bounded by

$$\sup_f \left\{ \sum_j \left[\frac{1}{N}\sum_k \Phi(k/N, j/N) Z_k(f) \Big\langle (\tau_j V_\ell^F) f_{k,j}\Big\rangle \;-\; \frac{Z_k(f)\beta N}{4\ell} \sum_{i=j-\ell}^{j+\ell-1} I_{i,i+1}(f_{k,j}) \right]\right\}\; .$$

For a fixed integer j, since $Z_k(f)$ is bounded by 2, the above supremum is bounded above by

$$\frac{\beta N^2}{2\ell} \sup_{|\lambda|\le 1} \sup_f \left\{ \frac{4\lambda \ell \|\Phi\|_\infty}{\beta N} \Big\langle V_\ell^F f\Big\rangle \;-\; \tilde D_\ell(f)\right\}\; . \tag{5.2}$$

In the last formula the second supremum is taken over all densities on $\{0,1,2\}^{\Lambda_\ell}$, where

$$\Lambda_\ell \;=\; \{-\ell,\dots,\ell\}$$

and $\tilde D_\ell$ is the Dirichlet form on $\{0,1,2\}^{\Lambda_\ell}$ defined by:

$$\tilde D_\ell(\cdot) \;=\; \sum_{i=-\ell}^{\ell-1} I_{i,i+1}(\cdot)\; .$$

Now we project on hyperplanes since because of the existence of conserved quantities ergodicity holds only on hyperplanes.

For $0 \leq K \leq 2(2\ell+1)$ denote by $\mu_{\ell,K}$ the measure ν_ρ conditioned on the hyperplane $\{\xi;\ \sum_{k=-\ell}^{\ell} \xi(k) = K\}$:

$$\mu_{\ell,K}(\cdot) = \nu_\rho(\cdot \mid \sum_{k=-\ell}^{\ell} \xi(k) = K).$$

Notice that the r.h.s. of the last expression does not depend on ρ.

For $0 \leq K \leq 2(2\ell+1)$ and for a density f on $\{0,1,2\}^{\{-\ell,\dots,\ell\}}$ denote by $f_{\ell,K}$ the projection of f on the hyperplane $\{\xi;\ \sum_{k=-\ell}^{\ell} \xi(k) = K\}$:

$$f_{\ell,K}(\xi) = \frac{f(\xi)}{\int f(\zeta)\mu_{\ell,K}(d\zeta)} \quad \text{for} \quad \xi;\ \sum_{k=-\ell}^{\ell} \xi(k) = K.$$

With this notation just introduced, for a density f on $\{0,1,2\}^{\{-\ell,\dots,\ell\}}$

$$\Big\langle V_\ell^F(\eta) f(\eta) \Big\rangle$$

may be written as

$$\sum_{K=0}^{2(2\ell+1)} c(f,K) \Big\langle\ V_\ell^F(\eta) f_{\ell,K}(\eta) \Big\rangle_{\ell,K}.$$

In this formula $c(f,K)$ stands for

$$c(f,K) = \int \mathbf{1}\{\xi;\ \sum_{k=-\ell}^{\ell} \xi(k) = K\} f(\xi)\nu_\rho(d\xi)$$

and $< \cdot >_{\ell,K}$ denotes the expectation with respect to the measure $\mu_{\ell,K}$.

Finally, since the total number of particles is conserved by the dynamics, the Dirichlet form of a density f may be written as

$$\tilde{D}_\ell(f) = \sum_{K=0}^{2(2\ell+1)} c(f,K)\tilde{D}_{\ell,K}(f_{\ell,K})$$

where $c(f,K)$ is defined just above and $\tilde{D}_{\ell,K}$ is the Dirichlet form

$$\tilde{D}_{\ell,K}(f) = \sum_{k=-\ell}^{\ell-1} \Big\langle d_{k,k+1}(\eta)\left(\nabla^d_{k,k+1}\sqrt{f}\right)^2 + g_{k,k+1}(\eta)\left(\nabla^g_{k,k+1}\sqrt{f}\right)^2 \Big\rangle_{\ell,K}.$$

Therefore, decomposing the integral and the Dirichlet form of (5.2) as a sum over hyperplanes with fixed total number of particles, then taking the supremum inside the sum we get that it is bounded above by

$$\frac{\beta N^2}{2\ell} \sup_{0\leq K\leq 2(2\ell+1)} \sup_{|\lambda|\leq 1} \sup_f \left\{ \frac{4\lambda\ell\|\Phi\|_\infty}{\beta N} \Big\langle V_\ell^F(\eta) f(\eta) \Big\rangle_{\ell,K} - \tilde{D}_{\ell,K}(f) \right\} .$$

From the bound on the largest eigenvalue of a small perturbation of a generator, stated as Theorem A.1.1 of [BKL], we obtain that the limit when N increases to ∞ of this expression is less than or equal to

$$\frac{4\|\Phi\|_\infty^2}{\beta} \sup_K \left\{ 2\ell \Big\langle V(-L_\ell)^{-1} V \Big\rangle_{\ell,K} \right\} .$$

From section 5 of [KLO], the limit of the expression inside the supremum, when ℓ increases to ∞ and K/ℓ converges to ρ, is equal to $A(F,\rho)$ defined at the begining of this section, uniformly in ρ. From the definition of A,

$$\inf_{F\in\mathcal{U}} \sup_{0\le\rho\le 2} A(F,\rho) \;=\; 0 .$$

This concludes the proof of the theorem. $\square$

Appendix 1. Tightness.

In this section we prove that the sequence Q^N introduced in section 4 is tight and that all limit points are concentrated on measures with certain properties.

Proposition A.1.1. *The sequence Q^N is tight. Moreover, every limit point Q^* of the sequence Q^N is such that*

$$\begin{aligned} E^{Q^*}\left[\pi(dx,dy) \;=\; \rho(x)\rho(y)dxdy\right] \;&=\; 1 ,\\ Q^*\left[0 \le \rho(x) \le 2\right] \;&=\; 1\\ E^{Q^*}\left(\int_0^1 (\partial_x \rho(x))^2\, dx \right) \;&\le\; C . \end{aligned}$$

Proof. The sequence Q^N is tight since it is a family of probabilities over a compact space (the space of positive measures with total mass bounded by 4 on the compact space $[0,1]^2$).

Each limit point is concentrated on absolutely continuous measures whose density are positive and bounded by 4 because in the model considered there are at most two particles per site. Their densities are product by construction.

We now prove the last property of the limit points. Since μ_{ss}^N is stationary and by the entropy inequality, for any smooth J in $C_K^2((0,1))$:

$$\begin{aligned} &\mu_{ss}^N \left(\sum_{i=1}^N J(\frac{i}{N})(\,\eta(i) - \eta(i+1)\,) \right)\\ &\quad = E^{\mu_{ss}^N} \left(\int_0^1 dt \sum_{i=1}^N J(\frac{i}{N})(\,\eta_t(i) - \eta_t(i+1)\,) \right)\\ &\quad\quad \le \frac{1}{N} \log E^{\nu_\rho} \left(e^{N\int_0^1 dt\, \sum_{i=1}^N J(\frac{i}{N})(\,\eta_t(i)-\eta_t(i+1)\,)} \right) \;+\; C_0 ; \end{aligned}$$

where C_0 is the bound on the relative entropy of μ_{ss}^N with respect to the product measure ν_ρ.

By Feynman-Kac formula this last expression is bounded by

$$\frac{1}{N}\lambda_N \ + C_0,$$

where the eigenvalue λ_N is given by

$$\lambda_N \ = \ \sup_f \left\{ N \sum J(i/N) < (\eta(i) - \eta(i+1))f > + N^2 \Big\langle 2^{-1}(L_N^* + L_N)\sqrt{f}, \sqrt{f} \Big\rangle \right\} \ .$$

In this last formula, L_N^* denotes the adjoint of L_N with respect to the product measure ν_ρ. The Dirichlet form $-\Big\langle 2^{-1}(L_N^* + L_N)\sqrt{f}, \sqrt{f} \Big\rangle$ which appears in the above formula is equal to the Dirichlet form

$$\sum_{i=0}^{N-1} \Big\langle \|\nabla_{i,i+1}\sqrt{f_{ss}}\|^2 \Big\rangle \ .$$

By the formula of integration by parts (3.1) we have

$$\begin{aligned} < (\eta(i) - \eta(i+1))f > &= \left\langle \Psi \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \nabla \tau_{-i} f \right\rangle \\ &= \left\langle \Psi \begin{pmatrix} \sqrt{\tau_{-i}f(\eta^{0,1})} + \sqrt{\tau_{-i}f(\eta)} \\ \sqrt{\tau_{-i}f(\eta^{1,0})} + \sqrt{\tau_{-i}f(\eta)}) \end{pmatrix} \cdot \nabla \sqrt{\tau_{-i}f} \right\rangle \end{aligned}$$

and by Schwarz inequality the last line is bounded by

$$B \left\langle \left\| \nabla \sqrt{\tau_{-i}f} \right\|^2 \right\rangle^{\frac{1}{2}}$$

for some constant B.

Then we have

$$\begin{aligned} \lambda_N \ &\le \sum_i \ \sup_f \left\{ N J(i/N) B \left\langle \left\| \nabla \sqrt{\tau_{-i}f} \right\|^2 \right\rangle^{\frac{1}{2}} - N^2 \left\langle \left\| \nabla \sqrt{\tau_{-i}f} \right\|^2 \right\rangle \right\} \\ &\le \frac{B^2}{4} \sum_i \ J^2(i/N). \end{aligned}$$

Taking the limits as $N \to \infty$ we have

$$E^{Q^*} \left(\int \ J'(x)\rho(x) \ dx \right) \ \le \ \frac{B^2}{4} \ \int J^2(x) \ dx \ + C_0.$$

By Lemma 7.4 in [KLO] we can take a supremum over the functions J inside the expectation and obtain:

$$E^Q \left(\sup_J \left\{ \int J'(x)\rho(x)\, dx - \frac{B^2}{4} \int J^2(x)\, dx \right\} \right) \le C_0 \ .$$

This concludes the proof. $\square$

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