Ergodicity in Infinite Hamiltonian Systems with Conservative Noise

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Abstract

We study the stationary measures of an infinite Hamiltonian system of interacting particles in $I\!R^3$ subject to a stochastic local perturbation conserving energy and momentum. We prove that the translation invariant measures that are stationary for the deterministic Hamiltonian dynamics, reversible for the stochastic dynamics, and that satisfy some regularity conditions are convex combination of "Gibbs" states. This result implies hydrodynamical behavior for the systems under consideration.

0. INTRODUCTION

The ergodicity problem in Hamiltonian dynamical systems is at the base of equilibrium statistical mechanics. While, beginning with the celebrated Sinai's paper [S], some result are known for finite system (see [LW] for a general approach to ergodicity in Hamiltonian systems), very little is known concerning infinite systems (some results are known for special systems with an arbitrary, but finite, number of particles [BLPS]). By ergodicity of an infinite system we mean that the only invariant measures, within a reasonably "regular" class, consist of convex combinations of Gibbs measures.

Furthermore, recent developments in non-equilibrium hydrodynamics (cf. [OVY]) show that the ergodicity of an infinite systems is a main ingredient in the rigorous derivation of Euler equations as a macroscopic description of the conservation laws for the density, the momentum and the energy (at least in the smooth regime of these equations).

Since no results are present for deterministic systems, it is natural to ask if a stochastic perturbation may help in proving ergodicity. The stochastic perturbation should conserve the energy, the momentum and the number of particles of the system, while destroying

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locally the other possible invariants of the motion.

A stochastic perturbation of this type is introduced in [OVY]: any two particles exchange randomly momentum in such a way as to preserve only the total momentum and energy of the two particles. The rate of exchange is assumed to decrease when the distance between the two particles increases, but the range is infinite. Accordingly, any particle is interacting stochastically with any other and the corresponding diffusion on the momenta space of any finite number of particles is of an elliptic type. This permits to characterize the distribution of the momenta of any finite number of particles conditioned to the positions: it must be a uniform measure on the corresponding invariant manifold in the momenta space. The equivalence of ensembles implies that the distribution of the momenta conditioned on the position is a convex combination of "Maxwellians". In addition, one can localize the invariance, under the Hamiltonian dynamics, of the distribution, and prove that the distribution of the positions satisfies the DLR equations with respect to the corresponding interaction.

The purpose of the present paper is to extend the foregoing argument to *finite* range stochastic interactions. Two difficulties arise immediately: one of local and the other of global type.

Locally, restricting oneself to a finite "chain" (or cluster) of particles interacting stochastically, the diffusion on the space of momenta is no longer elliptic; it becomes then necessary to prove that it is, at least, hypoelliptic. This is done quite easily with an inductive argument and in grand generality: only the convexity of the kinetic energy is needed.

The global obstacle is of a more serious nature. The diffusion on the momenta is hypoelliptic only when restricted to chains of interacting particles. But, several clusters of particles, too far apart to interact stochastically, may be present; hence, they could be at "different temperatures". We need the help of the deterministic Hamiltonian dynamics to "connect" distant clusters.

Taking commutators between the vector fields generating the stochastic dynamics and the Hamiltonian generator one obtains a Lie algebra of vector fields large enough to generate all the tangent space to the energy-momentum manifold on the phase space (i.e., position and momentum). This means that our system is invariant for these dynamics (that turn out to be local) so we can freely deform any cluster and "glue" it with the others. Proceeding in such a way we can obtain, in each sufficiently large finite box, a unique cluster and consequently prove that the momenta are uniformly distributed.

A further difficulty arises when the kinetic energy is quadratic (i.e., the usual "Gaussian case"). In fact, in this case all the above mentioned dynamics preserve also the center of mass of any finite cluster of particles. It is then necessary to perform cluster deformations that conserve the center of mass.

As in [OVY] we consider only stationary measures having finite entropy density with respect to a grancanonical Gibbs measure. This condition seems to characterize a nice class of regular measures. To complete our argument, various extra assumptions are necessary:

(a) The range of the stochastic interaction is finite but must be strictly larger than the one of the deterministic potential.
(b) The invariant measures considered must have sufficiently high particles density. More

precisely, we need to be sure that, for almost any configuration, any sufficiently large box contains at least two particles interacting stochastically. The bound on the density we ask here is very rough, and we believe it can be substantially improved by using a more refined argument.

(c) We assume that the measures considered are separately invariant for the deterministic and the stochastic dynamics. Furthermore, they must be reversible for the stochastic dynamics alone. The reversibility with respect to the global stochastic dynamics is a more general condition than the ones needed to derive the hydrodynamic limit: the invariance for each local stochastic dynamics (cf. proposition 2.2) would suffice (cf. [OVY]).

(d) In order to apply our results to obtain hydrodynamic limits following [OVY] we consider also kinetic energy that are not quadratic, since [OVY] does not apply to the quadratic case. Nonetheless, we must assume a mild restriction on the kinetic energy function: the local dynamics cannot have undesired invariants (like the center of mass in the Gaussian case, see lemma 2.5). We provide examples of kinetic energy functions that satisfy both our condition and the ones assumed in [OVY] (cf. Appendix 1).

As a consequence of our result, theorem 2.1 of [OVY] is valid for Hamiltonian dynamics with stochastic perturbation of the non-Gaussian type considered in the present paper.

For lattice systems the problem of ergodicity is solved in [FFL] in a more satisfactory way. In fact, there it is not needed condition (c), i.e., only the invariance with respect to the total dynamics (deterministic + stochastic) is required.

Concerning condition (c), notice that we could have asked the invariance for the finite stochastic dynamic in each finite box. We prove indeed that this is equivalent to the global reversibility (cf. proposition 2.2). Proposition 2.2 has an interest in itself: it says that if a stochastic dynamics on a lattice in finite dimension is hypoelliptic then for the corresponding infinite dynamics all the reversible measures are given by Gibbs measures. This generalize a result of M. Zhu (cf.[Z]) to the "hypoelliptic" situation.

The next section contains a more precise description of the results outlined here, together with the plan of the paper.

1. NOTATIONS AND RESULTS

Sample space

A point of $I\!R^3 \times I\!R^3$ will be denoted by (q,p) and the sample space Ω will consist of points $\omega = \{(q_\alpha, p_\alpha)\}$. Any bounded region B in $I\!R^3$ will contain only a finite number of particles, with positions q_α, in addition one can think of p_α as tags, and consider the corresponding finite configuration in $B \times I\!R^3$.

Interaction

We consider a repulsive finite range smooth pair potential $V(q_\alpha - q_\beta)$ such that:

i) $V(x) = 0 \quad |x| > R_0$ (finite range)

ii) $\langle x,\, V'(x)\rangle \leq 0 \quad \forall x$ (repelling interaction)

The repelling condition (ii) is of a technical nature and it should be possible to remove it by a more accurate analysis.

Kinetic Energy

It is given by a radial strictly convex function $\phi(p) \in C^3(\mathbb{R}^3)$. We consider two cases:

(G) $\phi(p)$ is a quadratic function of p, which is the classical *Gaussian* case.

(NG) $\phi(p) = \sum_{i=1}^{3} \varphi(p^i)$ with φ a strictly convex function on $\mathbb{R}$ ($\varphi'' > 0$) with

$$\frac{1}{2}\frac{d^2}{dx^2}(\varphi''(x))^2 = \varphi'''(x)^2 + \varphi^{iv}(x)\varphi''(x) \neq 0$$

apart from, at most, finitely many points. We will refer to this case as the *non-Gaussian* case.

Notice that, if $\frac{d^2}{dx^2}(\varphi''(x))^2 = 0$ for each x and if we require the natural condition $\varphi(x) = \varphi(-x)$, then $\varphi''(x)$ =constant, i.e., we have the Gaussian case. This shows that, morally, our conditions cover all the possible cases; yet, it could be interesting to carry out a more detailed investigation.

As already mentioned, our main motivation to treat the case (NG) is to apply the present results to the derivation of the hydrodynamic limit. To do so, the kinetic energy function must satisfy the conditions:

$$\left|\frac{\partial \phi}{\partial p^j}\right| \leq C', \quad \left|\frac{\partial^2 \phi}{\partial p^j \partial p^i}\right| \leq C'' \qquad \forall p \in \mathbb{R}^3$$

which are clearly not satisfied by the classical case (G).

Hamiltonian Dynamics

The Hamiltonian is defined by the formal expression:

$$\mathcal{H}(\omega) = \sum_{\alpha} \phi(p_\alpha) + \frac{1}{2}\sum_{\alpha}\sum_{\beta\neq\alpha} V(q_\alpha - q_\beta)$$

and the Liouville operator by

$$L = \sum_{\alpha}\sum_{i=1}^{3} \left[\partial_{p^i_\alpha}\mathcal{H}\ \partial_{q^i_\alpha} - \partial_{q^i_\alpha}\mathcal{H}\ \partial_{p^i_\alpha}\right]$$

In this paper, we are not concerned with the existence of the dynamics generated by L or its stochastic perturbations. Our aim is simply to characterize the probability measures on Ω that are 'formally' invariant (see Th. 1.1). For a more detailed description of the above objects see [*AGGLM*, §2].

Stochastic perturbation of the dynamics

We will use the notation $v^i_\alpha \equiv \phi_i(p_\alpha) \equiv \partial_{p^i_\alpha}\phi$.

For each smooth function $\eta_{\alpha\beta} : \mathbb{R}^6 \to \mathbb{R}^3$ we define the vector field

$$X(\eta_{\alpha\beta}) = \langle \eta_{\alpha\beta},\, D_{\alpha\beta}\rangle = \sum_{i=1}^{3} \eta^i_{\alpha\beta} D^i_{\alpha\beta}$$

where $D_{\alpha\beta} = \partial_{p_\alpha} - \partial_{p_\beta}$.

We are interested in vector fields with null divergence, i.e.,

$$\operatorname{div}(X(\eta_{\alpha\beta})) = \langle\, D_{\alpha\beta}, \eta_{\alpha\beta}\rangle = \sum_{i=1}^{3} D^i_{\alpha\beta}\eta^i_{\alpha\beta} = 0.$$

Furthermore, we ask that $X(\eta_{\alpha\beta})$ is tangent to the surfaces, $I\!R^3 \times I\!R^3$,

$$\Pi_c \equiv \begin{cases} p^i_\alpha + p^i_\beta = c^i & i = 1, 2, 3 \\ \phi(p_\alpha) + \phi(p_\beta) = c^0, \end{cases}$$

that is, the orthogonality relation $\langle \eta_{\alpha\beta},\, D_{\alpha\beta}(\phi(p_\alpha) + \phi(p_\beta))\rangle = 0$ (equivalently, $\langle \eta_{\alpha\beta},\, v_\alpha\rangle = \langle \eta_{\alpha\beta},\, v_\beta\rangle$).

Let $X(\eta_{\alpha\beta})^*$ be the adjoint of $X(\eta_{\alpha\beta})$ with respect to the measures

$$e^{\lambda_4(-\phi(p_\alpha)-\phi(p_\beta))+\lambda\cdot(p_\alpha+p_\beta)}\ dp_\alpha\ dp_\beta$$

for any $\lambda_4 > 0$ and $\lambda = (\lambda_1,\, \lambda_2,\, \lambda_3)$, with the restriction that

$$\int \exp(-\lambda_4\phi(p) + \lambda \cdot p)dp < +\infty\ .$$

We have, because the null divergence and the orthogonality property, that $X(\eta_{\alpha\beta})^* = -X(\eta_{\alpha\beta})$.

We use the previous vector fields to define an operator of the second order that will be the generator of the stochastic perturbation. Consider a finite number $K \geq 3$ of vectors $\{\eta^\theta_{\alpha\beta}\}$ with the properties above. We define the operator

$$\hat{L}_{\alpha\beta} = -\frac{1}{2}\sum_\theta\ X(\eta^\theta_{\alpha\beta})^* X(\eta^\theta_{\alpha\beta}).$$

Moreover, we require that, at each point, the linear combination of $\{\eta^\theta_{\alpha\beta}\}$ spans a two dimensional subspace of $I\!R^3$, eventually apart from a set $\widetilde{\Sigma}^s_{\alpha\beta}$ consisting of the countable union of codimension–two manifolds. Therefore, $\hat{L}_{\alpha\beta}$ is selfadjoint, and elliptic outside $\widetilde{\Sigma}^s_{\alpha\beta}$. For later purposes, we define $\widetilde{\Sigma}_{\alpha\beta} = \widetilde{\Sigma}^s_{\alpha\beta} \cup \{(p_\alpha,\, p_\beta) \mid v_\alpha = v_\beta\}$; by convexity follows that $\widetilde{\Sigma}_{\alpha\beta}$ is the finite union of smooth manifold with codimension two as well.

Let $\sigma(q)$ be a radial smooth function on $I\!R^3$, such that $\sigma(q) > 0$ for each $\|q\| < R_1$, and $\sigma(q) = 0$ for each $\|q\| \geq R_1 > R_0$. Then we consider the operator

$$\hat{L} = \sum_{\alpha,\beta} \sigma(q_\alpha - q_\beta)\hat{L}_{\alpha\beta}.$$

In the following considerations it will be important that σ is strictly positive for a radius R_1 strictly greater than R_0, i.e. that the range of the stocastic interaction is larger than the one of the 'deterministic' interaction.

Gibbs Measures

Let $\Lambda \subset \mathbb{R}^3$. Each configuration $\omega \in \Omega$ can be written as $\omega = \{\omega_\Lambda, \omega_{\Lambda^c}\}$ where $\omega_\Lambda = \{(q_\alpha, p_\alpha) \in \omega \mid q_\alpha \in \Lambda\}$.

Let P be a probability measure on Ω. If the P-conditional distribution of ω_Λ given the configuration outside ω_{Λ^c} is proportional to

$$\frac{1}{n!} \exp\left[\lambda_0 n + \sum_{\alpha=1}^{n}\sum_{i=1}^{3} \lambda_i p_\alpha^i - \lambda_4 \mathcal{H}_{\Lambda,n}(\omega_\Lambda, \omega_{\Lambda^c})\right]$$

then P is called Gibbs Measure (or grandcanonical Gibbs measure). In the above expression n is the number of particles in Λ (that we will denote by $|\omega_\Lambda|$) and the local Hamiltonian is defined by

$$\mathcal{H}_{\Lambda,n}(\omega_\Lambda, \omega_{\Lambda^c}) = \sum_{q_\alpha \in \Lambda}\left[\phi(p_\alpha) + \sum_{q_\beta \in \omega} V(q_\alpha - q_\beta)\right].$$

Statement of the result

Define

$$\begin{aligned}
\rho(\omega) &= \lim_{|\Lambda|\to\infty} |\Lambda|^{-1}|\omega_\Lambda| \\
\Pi(\omega) &= \lim_{|\Lambda|\to\infty} |\Lambda|^{-1}\sum_{q_\alpha\in\Lambda} p_a \\
E(\omega) &= \lim_{|\Lambda|\to\infty} |\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}\left[\phi(p_\alpha) + \sum_{q_\beta\in\omega} V(q_\alpha - q_\beta)\right]
\end{aligned}$$

Let $\rho_* = \frac{3}{4R_1^3\pi}$ and, for any square region Λ and any $n > \rho_*|\Lambda|$, consider $\Xi_\Lambda(n,\, \Pi,\, E,\, \omega_c) \subset \mathbb{R}^{6n}$ the surface of configurations for which the n particles contained in Λ have density of momentum Π, and density of energy E (with boundary conditions ω_c). The aim of this paper is to prove the following:

Theorem 1.1. *Let Q be a translation invariant probability measure on Ω, such that*

(i) For each box Λ the relative entropy of Q with respect to a fixed grancanonical Gibbs measure P in Λ is bounded by $C|\Lambda|$;

(ii) $Q(\{\omega \in \Omega \mid \rho(\omega) > \rho_*\}) \;=\; 1$;

(iii) Q is invariant w.r.t. the dynamics generated by L, in the sense that, for any smooth local function $F_\Lambda(\omega_\Lambda)$,

$$\mathbb{E}^Q(LF_\Lambda) \;=\; 0\,;$$

(iv) Q is reversible with respect to $\hat{L}$, i.e., for any two smooth local functions φ and ψ holds

$$\mathbb{E}^Q(\psi\hat{L}\varphi) \;=\; \mathbb{E}^Q(\varphi\hat{L}\psi)\,;$$

then Q is a convex combination of (gran canonical) Gibbs Measures.

The following lemma shows that the condition of finite entropy (i) implies some regularity on the averages considered:

Lemma 1.2. *If Q satisfies the hypothesis (i) of Theorem 1.1, then, for each $\Lambda \subset I\!R^3$,*

$$(i) \qquad Q\left(|\Lambda|^{-2}|\omega_\Lambda|^2\right) \le C_1 < \infty$$

$$(ii) \qquad Q\left(|\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}\|p_a\|\right) \le C_2 < \infty$$

$$(iii) \qquad Q\left(|\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}\left[\phi(p_\alpha)+\sum_{q_\beta\in\omega}V((q_\alpha-q_\beta)\right]\right) \le C_3 < \infty$$

where C_1, C_2, C_3 are constants independent by Λ.

Proof.

The inequalities (ii) and (iii) are consequences of the following entropy inequality:

$$Q(F) \le \frac{1}{\beta}\log P\left(\exp(\beta F)\right) \;+\; \frac{1}{\beta}H(Q|P)$$

which is valid for any function F and any constant $\beta > 0$, where $H(Q|P)$ is the relative entropy of Q with respect to P. By condition (i) on the relative entropy, for sufficiently small β, we have:

$$Q\left(|\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}\|p_a\|\right) \le \frac{1}{\beta|\Lambda|}\log P\left(\exp\left(\beta\sum_{q_\alpha\in\Lambda}\|p_a\|\right)\right) \;+\; \frac{1}{\beta}C \le \frac{1}{\beta}C'$$

In a similar way one can prove (iii). While (i) follows by the same argument and the superstability inequality:

there exists $B > 0$ and $A > 0$ such that for any finite box Λ and any configuration we have:

$$\sum_{q_a\in\Lambda}\sum_{\beta}V(q_\alpha-q_\beta) \;\ge \frac{A}{|\Lambda|}|\omega_\Lambda|^2 - B|\omega_\Lambda|$$

This inequality is satisfied by our repulsive interaction V (see [R]). □

The above Lemma, and the translation invariance of Q, insures that the limits $\rho(\omega)$, $E(\omega)$, $\Pi(\omega)$ exist Q-almost everywhere.

The proof of Theorem 1.1 will be carried out in three parts. In the next section we will construct a multitude of local dynamics that leave the finite dimensional restriction of the measure Q invariant. In sections three and four we show that these dynamics are sufficient in order to give a local characterization weaker than the one given by DLR equations, but sufficient to claim that the global distribution of the momenta, conditioned to the positions, is given by a convex combination of "Maxwellian" (corresponding to the proper ϕ). We conclude the argument in section five, along the line of [OVY], by proving that in the infinite limit the kinetic energy is "invariant" for the deterministic dynamic generated by L. Thus, each component of the convex combination is invariant for L. A classic argument (cf.[GV] and [OV]) shows that invariant distribution for L that have distribution of the momenta conditioned to the position given by a maxwellian are canonical Gibbs measures.

2. CLUSTERS AND LOCAL DYNAMICS

Given a configuration ω, we call "connected" two particles that are sufficiently close to interact stochastically (α and β are connected if $\sigma(q_\alpha - q_\beta) > 0$). We call "cluster" a set of particles such that any two can be joined by a chain of connected ones. We call "isolated cluster" a cluster such that no other particle is connected to any particle in the cluster.

Any configuration ω can be grouped in many isolated clusters $\{\Gamma_i(\omega_q)\}$. These clusters may be finite or infinite. Also, the restriction of any configuration to a finite region ω_Λ is grouped into finite clusters $\{\Gamma_i^\Lambda(\omega_q)\}$, where we have overlooked the connections with the particles outside Λ.

To simplify notations, we denote by $\widetilde{\mathbb{E}}$ the expectation of Q conditioned to a configuration of positions ω_q.

Consider a cluster Γ_i^Λ, and let n be the number of particles in it. Then $\hat{L}_{\Gamma_i^\Lambda} = \sum_{\alpha\beta\in\Gamma_i^\Lambda} \sigma(q_\alpha - q_\beta)\hat{L}_{\alpha\beta}$ is an operator on $\mathbb{R}^{3n}$ and it conserves the quantities

$$\sum_{q_\alpha\in\Gamma_i^\Lambda} p_\alpha^1, \qquad \sum_{q_\alpha\in\Gamma_i^\Lambda} p_\alpha^2, \qquad \sum_{q_\alpha\in\Gamma_i^\Lambda} p_\alpha^3, \qquad \sum_{q_\alpha\in\Gamma_i^\Lambda} \phi(p_\alpha)$$

Let us consider the corresponding connected hypersurfaces of dimension $3n-4$:

$$\Sigma_c \equiv \left\{ (p_1,\ldots,p_n) \;\middle|\; \sum_\alpha p_\alpha^1 = c^1, \sum_\alpha p_\alpha^2 = c^2, \sum_\alpha p_\alpha^3 = c^3, T = \sum_\alpha \phi(p_\alpha) = c^4 \right\}$$

And the sets

$$\widetilde{\Sigma}_c \equiv \left\{ (p_1,\ldots,p_n) \;\middle|\; (p_i,\, p_j) \in \widetilde{\Sigma}_{(ij)} \text{ for some } i,\, j \in \{1,\ldots,n\} \right\}$$

Note that $\widetilde{\Sigma}_c$ is the finite union of smooth submanifolds in Σ_c of codimension, at least, two; hence $\Sigma_c \setminus \widetilde{\Sigma}_c$ is connected.

Lemma 2.1. *$L_{\Gamma_i^\Lambda}$ is an hypoelliptic operator on $\Sigma_c \setminus \widetilde{\Sigma}$, i.e., the Lie algebra generated by*

$$\{X_{\alpha\beta}^\theta \mid \alpha, \beta \in \Gamma_i\}$$

spans the tangent space of Σ_c at each point in $\Sigma_c \setminus \widetilde{\Sigma}_c$.

Proof.

Let us fix our attention on an arbitrary point in $\Sigma_c \setminus \widetilde{\Sigma}_c$.

We can represent a cluster of n particles by a graph $\widetilde{\mathcal{G}}_n$ with n vertices. Each vertex of the graph corresponds to a particle; two vertices are joined by a bond (edge) if and only if they correspond to particles close enough to interact stochastically. In a cluster of n particles there are at least $n-1$ bonds:

$$\sharp\{(\alpha,\beta) \mid \sigma(q_\alpha - q_\beta) < R_1\} \geq \; n-1.$$

We consider then a minimal acyclical (i.e., the edges do not form any loop) connected sub-graph $\mathcal{G}_n$ of $\widetilde{\mathcal{G}}_n$. This amounts to choose $n-1$ bonds. We choose a set of $2n-2$ linearly independent vector fields $X^{\theta_i}_{\alpha\beta}$, $i \in \{1,2\}$, $(\alpha\beta)$ being a bond in $\mathcal{G}_n$.

Two bonds are said to be contiguous if they have one vertex in common. There are at least $n-2$ couples of contiguous bonds. If we compute the commutator between two contiguous bonds $\{\alpha\beta\}$ and $\{\gamma\beta\}$ we obtain:

$$[X^{\theta_i}_{\alpha\beta}, X^{\theta_j}_{\gamma\beta}] \; = \langle \eta^{\theta_i}_{\alpha\beta},\, A_{\theta_j} D_{\gamma\beta}\rangle - \langle \eta^{\theta_j}_{\gamma\beta},\, B_{\theta_i} D_{\alpha\beta}\rangle$$

where A_{θ_j} is the matrix whose element kl is given by $D^k_{\alpha\beta}(\eta^{\theta_j}_{\gamma\beta})_l$ (that is, the derivative, with respect to the k component of $p_\alpha - p_\beta$, of the l component of $\eta^{\theta_j}_{\gamma\beta}$) and B_{θ_i} is the matrix $D^k_{\gamma\beta}(\eta^{\theta_i}_{\alpha\beta})_l$.

Let $\mathcal{G}_k$, $k < n$ be a sequence of connected sub-graph of $\mathcal{G}_n$ containing k vertices, $\mathcal{G}_k \subset \mathcal{G}_{k+1}$, and $\mathcal{A}_k$ the Lie algebra generated by the vector fields $X^{\theta_i}_{\alpha\beta}$, $\alpha,\, \beta \in \mathcal{G}_k$. We will show by induction that $\mathcal{A}_n$ contains $3n-4$ linearly independent vector fields.

The fact is clear for $\mathcal{A}_2$. Let us suppose that it is true for $\mathcal{A}_k$ and let us show that $\mathcal{A}_{k+1}$ contains three extra vector fields linearly independent from all the previous ones.

We start by noticing that $\mathcal{G}_{k+1}$ is obtained by adding a particle (vertex), say α, to $\mathcal{G}_k$ and that such particle is connected with the rest of the graph by only one bond. Let us call β the particle to which α is connected and γ a particle, in $\mathcal{G}_k$, connected to β. $\mathcal{A}_{k+1}$ is larger (or equal) than the set of vector fields generated by all the vectors in $\mathcal{A}_k$ and $X^{\theta_i}_{\alpha\beta}$, $[X^{\theta_i}_{\alpha\beta},\, X^{\theta_j}_{\gamma\beta}]$. If our inductive hypothesis is false, then there exist $\lambda_i \in I\!R$, $\mu_{ij} \in I\!R$ and $Y_{ij} \in \mathcal{A}_k$ such that

$$0 = \lambda_1 X^{\theta_1}_{\alpha\beta} + \lambda_2 X^{\theta_2}_{\alpha\beta} + \mu_{ij}[X^{\theta_i}_{\alpha\beta},\, X^{\theta_j}_{\gamma\beta}] + Y_{ij}$$

for each $i,j \in \{1,2\}$. To see that this is impossible we apply the above tangent vector to p_α

$$0 = \lambda_1 \eta^{\theta_1}_{\alpha\beta} + \lambda_2 \eta^{\theta_2}_{\alpha\beta} - \mu_{ij}\,(B_{\theta_i})^T\, \eta^{\theta_j}_{\gamma\beta}$$

multiplying such relation by $D_{\alpha\beta}T$ (here we use the fact that, for points not in $\widetilde{\Sigma}$, $D_{\alpha\beta}T \neq 0$) we get

$$0 = \mu_{ij}\langle B_{\theta_i} D_{\alpha\beta}T,\, \eta^{\theta_j}_{\gamma\beta}\rangle$$

To simplify the previous expression we recall that, by definition, $\langle \eta^{\theta}_{\alpha\beta},\, D_{\alpha\beta}T\rangle = 0$ and applying the operator $D_{\gamma\beta}$ to such an equality we obtain

$$0 = B_{\theta_i} D_{\alpha\beta} T + H_\beta \eta^{\theta_i}_{\alpha\beta}$$

where H_β is the matrix $\partial_{p_\beta}\partial_{p_\beta}T$. It is essential to notice that the Hessian H_β is positive definite, given the convexity of ϕ.

By collecting the previous relations follows

$$0 = \mu_{ij}\langle H_\beta \eta^{\theta_i}_{\alpha\beta},\, \eta^{\theta_j}_{\gamma\beta}\rangle$$

To conclude the proof it is enough to show that at least one of the scalar products is different from zero. Notice that $\{\eta^{\theta_i}_{\alpha\beta}\}$ and $\{\eta^{\theta_j}_{\gamma\beta}\}$ span two two-dimensional planes in $\mathbb{R}^3$. Such planes must intersect, at least, at a line. Let $\zeta \neq 0$ be a vector belonging to such line. Clearly there exists τ_i τ'_i such that $\sum_l \tau_l \eta^{\theta_l}_{\alpha\beta} = \zeta$ and $\sum_l \tau'_l \eta^{\theta_l}_{\alpha\beta} = \zeta$. This yields to the contradiction $0 = \langle H_\beta \zeta,\, \zeta\rangle$. That is, the vectors $X^{\theta_1}_{\alpha\beta}$, $X^{\theta_2}_{\alpha\beta}$, $\sum_{ij}\tau_i\tau'_j[X^{\theta_i}_{\alpha\beta},\, X^{\theta_j}_{\gamma\beta}]$ are linearly independent with respect to themselves and with respect to the ones in the algebra $\mathcal{A}_k$. □

Proposition 2.2. *The condition (iv) of reversibility of the measure Q implies that, for any bond corresponding to connected particles $b = \{\alpha, \beta\}$ in the configuration ω_q and any smooth local function ϕ, we have*

$$\mathbb{E}^{Q}(X^{\theta}_{\alpha\beta}\phi \mid \omega_q) \equiv \widetilde{\mathbb{E}}\left(X^{\theta}_{\alpha\beta}\phi\right) = 0$$

provided that $supp(\phi) \cap \widetilde{\Sigma} = \emptyset$.

Proof.

Let φ and ψ be any smooth local functions with support in a finite region Λ', and Λ a region containing Λ' and so large that particles inside Λ' cannot interact with particles outside Λ. Since the size of the support of a test function can be assumed arbitrarily small, without loss of generality, we will carry out a local argument; namely we will assume that the support of ψ is contained in a conveniently small neighborhood of an arbitrary configuration not belonging to $\widetilde{\Sigma}$ (i.e., the support must be so small as not to intersect $\widetilde{\Sigma}$). If the configuration ω_q does not contain any cluster in Λ, then the Proposition is obviously true. Next, we will assume that only one isolated cluster[1] Γ_Λ is present (the case in which several isolated clusters are present can be treated in the same way, as we will remark at the end of the proof). The graph associated to Γ_Λ is finite and it will contain M bonds. The reversibility condition (iv) implies:

$$-\widetilde{\mathbb{E}}\left(\psi\hat{L}\varphi\right) = \sum_{b\in\Gamma_\Lambda}\sum_{\theta}\sigma_b\widetilde{\mathbb{E}}\left(X^\theta_b\psi X^\theta_b\varphi\right)$$

where $b = \{\alpha, \beta\}$ is a generic bond in Γ_Λ, i.e., $\sigma_b = \sigma(q_\alpha - q_\beta) > 0$. Note that the operators $X_{\alpha\beta}$ with q_α or q_β not in Λ do not appear in the right hand side of the above equation, although it is possible that $\sigma(q_\alpha - q_\beta) \neq 0$; this is due to the fact that, since the test

[1] Remember that an "isolated cluster" in Λ can still have interactions with particles outside Λ.

functions depend only on the particles in Λ', if $q_\alpha \notin \Lambda$ and $\sigma(q_\alpha - q_\beta) \neq 0$, then $q_\beta \notin \Lambda'$ which implies $X_{\alpha\beta}\psi = 0 = X_{\alpha\beta}\varphi$.

A technical obstacle to our proof is that, in general, the vector fields $\{X^\theta_{b_1}, \ldots, X^\theta_{b_M}\}$ are neither linearly independent nor their linear combinations span all the Lie algebra that they generate. Typically, only $L \leq KM$ [2] such vector fields will be linearly independent, while the Lie algebra will be $N \geq L$ dimensional. To overcome such problem we choose, among $\{X^\theta_{b_1}, \ldots, X^\theta_{b_M}\}$ and their commutators, a subset of linearly independent vector fields $\{X_1, \ldots, X_N\}$ that form a base for the Lie algebra.[3] In addition, we require

$$\{X_1, \ldots, X_L\} \subset \{X^\theta_{b_1}, \ldots, X^\theta_{b_M}\}.$$

Thus, the original KM vector fields can be expressed as linear combinations of the indipendent vector fields $\{X_1, \ldots, X_L\}$:

$$X^\theta_{b_j} = \sum_{i=1}^{L} \nu^\theta_{ji} X_i \qquad\qquad j = 1, \ldots, KM.$$

Since,

$$[X^\theta_{b_j}, X^{\theta'}_{b_k}] = \sum_{l,p} [\nu^\theta_{jl} X_l, \nu^{\theta'}_{kp} X_p] = \sum_{l,p} \left\{ \nu^\theta_{jl}(X_l \nu^{\theta'}_{kp}) X_p - \nu^{\theta'}_{kp}(X_p \nu^\theta_{jl}) X_l + \nu^\theta_{jl}\nu^{\theta'}_{kp}[X_l, X_p] \right\}$$

it is clear that $\{X_1, \ldots, X_L\}$ generates the complete Lie algebra under consideration.

Let A be the $L \times L$ matrix with elements defined by

$$a_{i,k} = \sum_{j=1}^{M} \sum_{\theta} \sigma_{b_j} \nu^\theta_{ji} \nu^\theta_{jk}$$

then

$$\sum_{b \in \Gamma_\Lambda} \sum_{\theta} \sigma_b \widetilde{\mathbb{E}} \left(X^\theta_b \psi X^\theta_b \varphi \right) = \sum_{i,k=1}^{L} \widetilde{\mathbb{E}} \left(a_{ik} X_i \psi X_k \varphi \right).$$

It is easy to check that the matrix A is positive defined, and therefore invertible.

According to Lemma 2.1 $\{X_1, \ldots, X_N\}$ span the tangent space of Σ_c (the surfaces associated to the cluster Γ_Λ). Since such surfaces foliate the phase space of the particles contained in Γ_Λ, we can choose coordinates (c, y) such that the vector fields $\{Y_i\}_1^N$, associated to the coordinates $\{y_i\}_1^N$, generates the tangent space of Σ_c (i.e., for each c, $\{y\}$ is a system of coordinates for Σ_c). This implies, $\forall i, j$,

$$[Y_i, Y_j] = 0 \quad ; \qquad Y_i^* = -Y_i \quad ; \qquad Y_i y_j = \delta_{ij} \quad .$$

[2] KM is the cardinality of $\{X^\theta_{b_1}, \ldots, X^\theta_{b_M}\}$.

[3] This is possible provided the support of ψ is sufficiently small.

In addition, there exists an invertible $N \times N$ matrix λ, such that

$$X_i = \sum_{i=1}^{N} \lambda_{ij} Y_j \ .$$

Let us choose as function φ a coordinate function y_j multiplied by a smooth function with value one on the support of ψ, which, consequently, can be ignored. Applying $\hat{L}$ we have

$$\begin{aligned} \hat{L}\ y_j &= \sum_{k,i} \ X_k\ a_{ki}\ X_i\ y_j \\ &= \sum_{k,i} \ X_k\ a_{ki}\ \lambda_{ij} \end{aligned}$$

where we have used

$$X_i\ y_j \ = \ \sum_{l} \ \lambda_{il}\ Y_l\ y_j \ = \ \lambda_{ij}.$$

The reversibility relation then gives us:

$$-\sum_{k,i} \widetilde{I\!E}\ (\psi X_k\ \ (a_{ki}\ \lambda_{ij})) \ = \ \sum_{k,i} \widetilde{I\!E}\ (a_{ki}\ \lambda_{ij}\ X_k\ \psi)$$

which is equivalent to

$$\sum_{k,i} \widetilde{I\!E}\ (X_k\, (\psi\ a_{ki}\ \lambda_{ij})) \ = \ 0$$

Let $V = \lambda^{-1} I\!R^L$, then $A\lambda\ :\ V \to I\!R^L$ is one to one and onto. Which means that for each $e^k \in I\!R^L$, $e^k = (0, \dots, 1, \dots, 0)$, there exists $\alpha^k \in V$ such that $A\lambda\alpha^k = e^k$. Moreover, in some small neighborhood of any configuration, α^k will vary smoothly.

We can make the following L^2 different choices of ψ

$$\psi_{jh} \ = \ \alpha_j^h \phi$$

where ϕ is a function with sufficiently small support around the configuration we are considering.

Summing over j we obtain

$$0 = \sum_{i,j,k} \widetilde{I\!E}\ \left(\ X_k(\alpha_j^h\ \lambda_{ij} a_{ki}\ \phi)\right) = \sum_{k} \widetilde{I\!E}\ \left(X_k e_k^h \phi\right),$$

that is to say

$$\widetilde{I\!E}\ (X_h\ \phi) \ = \ 0 \qquad \forall\, h \in \{0, \dots, L\}$$

which implies our thesis.

The generalization to the situation where many clusters appear in the region Λ is straightforward since, in the above argument, the coordinate functions y_j are localized on the particular cluster we are considering. Hence, the argument simply factors over the different clusters. □

Up to now we have seen that the measure is invariant with respect to vector fields that generate the tangent space to the surfaces of the momenta of the clusters Γ_i^Λ with constant kinetic energy and momentum. This was done only by using the reversibility of the stochastic dynamics. Now we will use the Hamiltonian dynamics and we will study the Lie algebra generated by $\{X^\theta_{\alpha\beta};\ [X^\theta_{\alpha\beta},\, L]\}_{q_\alpha, q_\beta \in \Lambda}$.

Lemma 2.3. *For each region* Λ*, each local smooth function* φ *localized in* Λ*, calling* $\mathcal{A}_\Lambda$ *the Lie algebra generated by the operators, we have* $\{X^\theta_{\alpha\beta};\ [X^\theta_{\alpha\beta},\, L]\}_{q_\alpha, q_\beta \in \Lambda}$

$$\mathbb{E}\,\left(X\varphi \mid |\omega_\Lambda| = n;\, \omega_{\Lambda^c}\right) = 0$$

for each $X \in \mathcal{A}_\Lambda$.

Proof.

Let $\chi_\varepsilon(q)$ be a smooth function equal to one if $q \in \Lambda$, and equal to 0 if the distance between q and Λ is larger than ε. We can then define $N_\varepsilon \equiv \sum_\alpha \chi_\varepsilon(q_\alpha)$ to be an approximation of the number of particles in Λ. Clearly, when ε goes to zero, N_ε tends to the number of particles contained in the closure of Λ, which, by absolute continuity, equals almost everywhere the number of particles contained in the interior.

Let h be a smooth function on $\mathbb{R}^+$ with compact support and φ a smooth local function with support contained in the interior of Λ; in addition, we consider arbitrary smooth functions $\psi_{\alpha\beta} : \mathbb{R}^6 \to \mathbb{R}$ with support in $\Lambda \times \Lambda$ and we use them to define the local operators $X(\psi) \equiv \sum\limits_{\alpha\beta} \psi_{\alpha\beta}(q_\alpha,\, q_\beta)\sigma(q_\alpha - q_\beta)X_{\alpha\beta}$ (clearly all these operators are part of the Lie algebra $\mathcal{A}_\Lambda$). Using the previous definitions, we have

$$0 = \mathbb{E}\,([L,\, X(\psi)]\varphi h(N_\varepsilon)) = \mathbb{E}\,(h[L,\, X(\psi)]\varphi) - \mathbb{E}\,(X(\psi)\varphi Lh) - \mathbb{E}\,(\varphi X(\psi) Lh).$$

Since $Lh(N_\varepsilon) = h'(N_\varepsilon) \sum\limits_{\gamma\in\Lambda^c} \langle p_\gamma,\, \nabla\chi_\varepsilon(q_\gamma)\rangle$ which, as far as $X(\psi)$ is concerned, is a constant, it follows

$$0 = \mathbb{E}\,(h(N_\varepsilon)[L,\, X(\psi)]\varphi).$$

The above equation shows that it is possible to condition with respect to the number of particles in Λ; a similar computation shows that it is possible to condition with respect to the configuration outside Λ as well. □

Let Λ_0 be a box larger than Λ; such that a particle outside Λ_0 cannot interact with particles contained in Λ. Let

$$P_{\Lambda_0} = \sum_{\gamma\in\Lambda_0} p_\gamma; \quad Q_{\Lambda_0} = \sum_{\gamma\in\Lambda_0} q_\gamma; \quad E_{\Lambda_0} = \sum_{\gamma\in\Lambda_0} \phi(p_\gamma) + \sum_{\delta\neq\gamma} V(q_\gamma - q_\delta).$$

Lemma 2.4. *The Lie Algebra $\mathcal{A}_\Lambda$ is tangent to the surface P_{Λ_0}=constant, E_{Λ_0}=constant, and acts only on observables depending on the coordinates of the particles inside Λ.*

Proof.

Given two particles $\alpha,\ \beta \in \Gamma$ we have

$$
\begin{aligned}
&X^\theta_{\alpha\beta} P_{\Lambda_0} = 0\\
&X^\theta_{\alpha\beta} E_{\Lambda_0} = 0\\
&[X^\theta_{\alpha\beta},\, L] P_{\Lambda_0} = X^\theta_{\alpha\beta} \sum_\gamma \frac{\partial T}{\partial q_\gamma} = 0\\
&[X^\theta_{\alpha\beta},\, L] E_{\Lambda_0} = X^\theta_{\alpha\beta} \left[\sum_\gamma \langle \frac{\partial \mathcal{H}}{\partial p_\gamma},\, \frac{\partial E_{\Lambda_0}}{\partial q_\gamma} \rangle - \langle \frac{\partial \mathcal{H}}{\partial q_\gamma},\, \frac{\partial E_{\Lambda_0}}{\partial p_\gamma} \rangle \right]
\end{aligned}
$$

Letting $\Delta = \mathcal{H} - E_{\Lambda_0} = \sum_{\gamma \notin \Lambda_0} \phi(p_\gamma) + \frac{1}{2} \sum_{\gamma \notin \Lambda_0} \sum_{\delta \neq \gamma} V(q_\gamma - q_\delta)$, and $H_\gamma = \left(\frac{\partial^2 \phi(p_\gamma)}{\partial p^i_\gamma \partial p^j_\gamma} \right)$, we can rewrite the last equation as

$$
\begin{aligned}
[X^\theta_{\alpha\beta},\, L] E_{\Lambda_0} &= X^\theta_{\alpha\beta} \left[\sum_\gamma \langle \frac{\partial \Delta}{\partial p_\gamma},\, \frac{\partial E_{\Lambda_0}}{\partial q_\gamma} \rangle - \langle \frac{\partial \Delta}{\partial q_\gamma},\, \frac{\partial E_{\Lambda_0}}{\partial p_\gamma} \rangle \right]\\
&= -\langle \frac{\partial \Delta}{\partial q_\alpha},\, H_\alpha \eta^\theta_{\alpha\beta} \rangle + \langle \frac{\partial \Delta}{\partial q_\beta},\, H_\beta \eta^\theta_{\alpha\beta} \rangle = 0
\end{aligned}
$$

since $\frac{\partial \Delta}{\partial q_\alpha} = -\frac{1}{2} \sum_{\gamma \notin \Lambda_0} V'(q_\gamma - q_\alpha) = 0 = \frac{\partial \Delta}{\partial q_\beta}$ (remember that the deterministic range of interaction is shorter than the stochastic one).

Similarly, a direct computation shows that, if $q_\gamma \notin \Lambda$, then

$$
\begin{aligned}
&[X^\theta_{\alpha\beta},\, L] q_\gamma = 0\\
&[X^\theta_{\alpha\beta},\, L] p_\gamma = 0\,.
\end{aligned}
$$

□

At this point we have to distinguish between the Gaussian and the non-Gaussian case. The difference is emphasized by the following lemma:

Lemma 2.5. *If ϕ is quadratic, the Lie Algebra $\mathcal{A}_\Lambda$ is tangent to the surface P_{Λ_0}=constant, E_{Λ_0}=constant, and Q_{Λ_0}=constant.*

Proof.

All we need to compute is

$$
\begin{aligned}
&X^\theta_{\alpha\beta} Q_{\Lambda_0} = 0\\
&[X^\theta_{\alpha\beta},\, L] Q_{\Lambda_0} = X^\theta_{\alpha\beta} \sum_{\gamma \in \Lambda_0} \frac{\partial \phi}{\partial p_\gamma} = (H_\alpha - H_\beta) \eta^\theta_{\alpha\beta} = 0
\end{aligned}
$$

since, in the present case, $H_\alpha = H_\beta$=constant. $\square$

This means that, in the Gaussian case, the vector fields we are considering conserve the center of mass, even if this is not conserved by L; accordingly, the Lie Algebra generated by $\{X^\theta_{\alpha\beta},\, [X^\theta_{\alpha\beta},\, L]\}$, for some $\alpha, \beta \in \Gamma$ (Γ being some cluster in Λ), can be at most five dimensional. We prove that the algebra has the largest possible dimension.

Lemma 2.6. *If ϕ is quadratic, and $\alpha, \beta \in \Gamma$ are connected, then the Lie Algebra generated by $\{X^\theta_{\alpha\beta};\, [X^\theta_{\alpha\beta},\, L]\}$ is five dimensional.*

Proof.

Applying the vector fields to q_α we have

$$\begin{aligned}
[X^\theta_{\alpha\beta},\, L]q_\alpha &= H_\alpha \eta^\theta_{\alpha\beta} \\
[X^\theta_{\alpha\beta},\, [X^\theta_{\alpha\beta},\, L]]q_\alpha &= H_\alpha D_{\alpha\beta}(\eta^\theta_{\alpha\beta})\eta^\theta_{\alpha\beta}
\end{aligned}$$

This vectors span a three dimensional vector space and are linearly independent with respect with the vectors $X^\theta_{\alpha\beta}$. To see this, it is sufficient to consider a generic linear combination, equal it to 0, and multiply it by $H_\alpha^{-1} D_{\alpha\beta} E$, then

$$0 = \sum_i \mu_i \langle D_{\alpha\beta}E,\, \eta^{\theta_i}_{\alpha\beta}\rangle + \nu \langle D_{\alpha\beta}E,\, D_{\alpha\beta}(\eta^{\theta_1}_{\alpha\beta})\eta^{\theta_1}_{\alpha\beta}\rangle$$

Next, remember that $\langle D_{\alpha\beta}E,\, \eta^\theta_{\alpha\beta}\rangle = 0$, differentiating such an expression by $D_{\alpha\beta}$ one gets

$$(H_\alpha + H_\beta)\eta^\theta_{\alpha\beta} + D_{\alpha\beta}(\eta^\theta_{\alpha\beta})^T D_{\alpha\beta}E = 0$$

and, multiplying by $\eta^\theta_{\alpha\beta}$,

$$\langle \eta^\theta_{\alpha\beta},\, (H_\alpha + H_\beta)\eta^\theta_{\alpha\beta}\rangle = -\langle D_{\alpha\beta}E,\, D_{\alpha\beta}(\eta^\theta_{\alpha\beta})\eta^\theta_{\alpha\beta}\rangle.$$

Using the above equalities we obtain

$$\nu \langle \eta^{\theta_1}_{\alpha\beta},\, (H_\alpha + H_\beta)\eta^{\theta_1}_{\alpha\beta}\rangle = 0$$

that is $\nu = 0$. From this follows $\mu_i = 0$. $\square$

In the non-Gaussian case the center of mass is not conserved by the vector fields we are considering, and we have no other obvious conserved quantity. We impose a conditions on the noise to make sure that there are no conserved quantities, beside those considered in Lemma 2.4. More precisely we require the following:

Condition on the Noise. *For each two particles α, β, interacting stochastically, we require that the Lie algebra generate by the vectors $X^\theta_{\alpha\beta}$ and $[X^\theta_{\alpha\beta},\, L]$ is eighth dimensional at each point of every surface with fixed total energy and total momentum except, at most, for the finite union of smooth manifolds of codimension two $\widetilde{\Sigma}_{\alpha\beta}$.*

In Appendix 1 we will show that if ϕ satisfy (NG) then the above condition is satisfied.

We introduce two family of surfaces in $\mathbb{R}^{6n}$,

$$\Xi(n,\,\Pi,\,E,\,\omega_c)=\left\{(q,\,p)\in\mathbb{R}^{6n}\;\middle|\;\sum_i p_i=\Pi;\;\sum_i\phi(p_i)+\sum_i\sum_j V(q_i-q_j)\right.$$
$$\left.+\sum_i\sum_{q\in\omega_c}V(q_i-q)=E\right\}$$
$$\Xi(n,\,\Theta,\,\Pi,\,E,\,\omega_c)=\left\{(q,\,p)\in\Xi(n,\,\Pi,\,E,\,\omega_c)\;\middle|\;\sum_i q_i=\Theta\right\}$$

and let $\widetilde{\Xi}$ be the union of the sets for which $(p_1,\,p_j)\in\widetilde{\Sigma}_{(i,j)}$, for some $i\neq j$. By hypotheses $\widetilde{\Xi}$ has at least codimension two in Ξ, in additions it has zero Lebesgue measure.

Lemma 2.7. *For almost all Θ, Π, E, for each $X\in\mathcal{A}_\Lambda$, and any local function φ with support contained in Λ and disjoint from $\widetilde{\Xi}$*

$$\mathbb{E}\left(X\varphi\mid\omega_\Lambda\in\Xi,\,\omega_{\Lambda^c}\right)=0.$$

In addition, if Λ contains a unique cluster, then the Lie algebra $\mathcal{A}_\Lambda$ contains all the tangent space of Ξ at each point of $\Xi\backslash\widetilde{\Xi}$.

Proof.

The first condition follows from lemma 2.3 and lemma 2.4 (or lemma 2.5 for the Gaussian case). To address the second part of the lemma we start an induction argument similar to the one used in lemma 2.1. We want to generate a 6n-7 dimensional Lie algebra in the Gaussian case and a 6n-4 dimensional Lie algebra in the non-Gaussian case. In both cases we need at each step of the induction argument, i.e., for every particle α that we add to a cluster, six new independent vector fields.

From the proof of Lemma 2.1 we have already three independent vector fields generated by $\{X^{\theta^i}_{\alpha\beta},X^{\theta^j}_{\beta\gamma}\}$. All these are acting only in the direction of the momenta, so all we need is to look at the action of the new vector fields on the positions direction to establish their linear independence.

Define

$$\widetilde{L}^{\theta_k}_{\alpha\beta}=[X^{\theta_k}_{\alpha\beta},\,L]$$

$$\overline{L}^{\theta_k\theta_l}_{\alpha\beta\gamma}=[X^{\theta_l}_{\beta\gamma},\,\widetilde{L}^{\theta_k}_{\alpha\beta}].$$

Applying these vector fields to q_α we have:

$$\widetilde{L}^{\theta_k}_{\alpha\beta}q_\alpha=X^{\theta_k}_{\alpha\beta}Lq_\alpha=H_\alpha\eta^{\theta_k}_{\alpha\beta}$$
$$\overline{L}^{\theta_k\theta_l}_{\alpha\beta\gamma}q_\alpha=X^{\theta_l}_{\beta\gamma}H_\alpha\eta^{\theta_k}_{\alpha\beta}=H_\alpha\left(D_\beta\eta^{\theta'}_{\alpha\beta}\right)^T\eta^{\theta}_{\beta\gamma}$$

It is then enough to prove that the vectors $X^{\theta_i}_{\alpha\beta}$, $[X^{\theta_i}_{\alpha\beta},\, X^{\theta_j}_{\gamma\beta}]$, $\widetilde{L}^{\theta_i}_{\alpha\beta}$, $\sum_{ij} \xi_{ij} \overline{L}^{\theta_i\theta_j}_{\alpha\beta\gamma}$, for some choice of ξ_{ij}, and Y (where Y belongs to the lie algebra generated by the vectors already considered during the induction procedure) are linearly independent. Again we assume that it is not so, i.e.,

$$0 = \sum_{k=1}^{2} \lambda_k X^{\theta_k}_{\alpha\beta} + \mu_{ij}[X^{\theta_i}_{\alpha\beta},\, X^{\theta_j}_{\gamma\beta}] + \sum_{k=1}^{2} \nu_k \widetilde{L}^{\theta_k}_{\alpha\beta} + \tau \sum_{kl} \xi_{kl} \overline{L}^{\theta_k\theta_l}_{\alpha\beta\gamma} + Y$$

for some λ_i, μ_{ij}, ν_i, τ, Y. We apply the previous expression to, q_α and obtain

$$0 = \sum_{k=1}^{2} \nu_k H_\alpha \eta^{\theta_k}_{\alpha\beta} + \tau \sum_{kl} \xi_{kl} H_\alpha \left(D_\beta \eta^{\theta_l}_{\alpha\beta} \right)^T \eta^{\theta_k}_{\gamma\beta}$$

If we multiply by $H_\alpha^{-1} D_{\alpha\beta} E$, recalling the properties of η, we obtain

$$0 = \tau \sum_{kl} \xi_{kl} \langle H_\alpha \eta^{\theta_k}_{\alpha\beta},\, \eta^{\theta_l}_{\gamma\beta} \rangle$$

Which shows that, out of $\widetilde{\Sigma}$, it is always possible to choose ξ_{kl} such that the sum is different from zero. This implies $\tau = 0$ and allows us to conclude the proof in complete analogy with lemma 2.1. □

3. CLUSTERING (THE NON-GAUSSIAN CASE).

In the previous section, starting from our dynamics, we have constructed local Lie algebras $\mathcal{A}_\Lambda$ of vector fields such that

$$I\!E^{\,Q} \left(X F_\Lambda \;\middle|\; |\omega_\Lambda| = n,\, \Pi(\omega_\Lambda) = \Pi,\, E(\omega_\Lambda) = E,\, \omega_{\Lambda^c} \right) = 0$$

for any $X \in \mathcal{A}_\Lambda$, and any local function F_Λ with support disjoint from the singular codimension two set. Using the properties of these vector fields we will make a step forward in the proof of Theorem 1. For the already mentioned reasons, it is necessary to separate the argument in the Gaussian and the non-Gaussian case. In this section we will discuss the non-Gaussian case, being the Gaussian case more complex. In the next section we will say how to modify the argument in the Gaussian case.

We need here the properties of high density of the measure Q, in order to find at least a cluster of two particles in any sufficiently large box. Let us set $\rho_* = \frac{3}{4R_1^3\pi}$. We consider an increasing sequence of cubes Λ_k and a region Λ such that

$$\Lambda \subset \Lambda_0, \qquad \Lambda_k \subset \Lambda_{k+1}, \qquad \bigcup_{k=0}^{\infty} \Lambda_k = I\!R^3$$

Let $\rho_k(\omega)$, $\Pi_k(\omega)$, $E_k(\omega)$ be respectively the density, the momentum and the energy of the particles in the region Λ_k for the configuration ω. Then we define $\Omega_k = \{\omega \mid \rho_k(\omega) \geq \rho_*\}$

and χ_k, the characteristic function of the set Ω_k. By the assumption (ii) of theorem 1.1, for any bounded F holds

$$\lim_{k\to\infty} I\!E^{\,Q}(\chi_k F) = I\!E^{\,Q}(F) \tag{3.1}$$

Let us fix some $k > 0$ large enough. We call n the number of particles, Π the density of momentum, and E the density of energy in the region Λ_k; ω_c the configuration of the particles outside Λ_k; $(q,\, p) \in I\!R^{6n}$ the coordinates of a particle in the region and $\Xi(n,\, \Pi,\, E,\, \omega_c) \subset I\!R^{6n}$ the surface on which the n particles have density of momentum Π, density of energy E, (note that the density of energy inside Λ_k depends on ω_c).

What we have proven in the previous section is that our Lie algebra generates the tangent space of $\Xi(n,\, \Pi,\, E,\, \omega_c)$ only at those points corresponding to a unique cluster.

Notice that such surfaces $\Xi(n,\, \Pi,\, E,\, \omega_c)$ need not to be connected. Let us consider then a connected component $\Xi_j(n,\, \Pi,\, E,\, \omega_c)$ and let us call $d\mu_{n,\,\Pi,\,E,\,j,\,\omega_c}(q,\, p)$ the measure Q conditioned on the surface $\Xi_j(n,\, \Pi,\, E,\, \omega_c)$. Since all the quantities we have conditioned on are conserved by the vector fields of the Lie subalgebra generated only by the particles in Λ_k, this conditioned measure $d\mu$ is invariant for such a subalgebra (lemma 2.7); moreover, the subalgebra is composed by null divergence vector fields. This implies that, in the neighborhood of points corresponding to configurations with a unique cluster, the measure $d\mu_n$ is proportional to the Lebesgue measure. More precisely let us choose a δ small enough and a configuration $(q_*,\, p_*) \in \Xi$ such that any configuration with $\|q - q_*\| + \|p - p_*\| < \delta$ has the same unique cluster and $v_i \neq v_j$ for every $i,\, j$. Thus, give a function ϕ with support contained in a neighborhood of $(q_*,\, p_*)$ of radius δ follows

$$\begin{aligned}
&\int_{\Xi_j(n,\Pi,\,E,\,\omega_c)} \phi(q,\, p) F_\Lambda(q,\, p) d\mu_{n,\,\Pi,\,E,\,j,\,\omega_c}(q,\, p) \\
&= Z(n,\, \Pi,\, E,\, j,\, \omega_c) \int_{\Xi_j(n,\Pi,\,E,\,\omega_c)} \phi(q,\, p) F_\Lambda(q,\, p) dS(q,\, p)
\end{aligned}$$

where dS is the Lebesgue measure on Ξ_j and Z is a normalization constant.

In the rest of the paragraph we will drop the index j and Ξ will indicate a connected component. Our strategy will be the following: we will divide Ξ in small sets, then we shall study the restriction of the measure μ on each set by comparing it with the measure in the neighborhood of $(q_*,\, p_*)$. This will be accomplished by constructing maps with the property of being measure preserving both for the measure $d\mu$ and the measure dS.

To be more concrete, calling $\widetilde{\Xi} = \{(q,\, p) \in \Xi \mid (p_i,\, p_j) \in \widetilde{\Sigma}_{ij} \text{ for some } i,\, j\}$, $B(\xi_*) \subset I\!R^{4n}$ the neighborhood of radius δ of $\xi_* = (q_*,\, p_*)$, we will prove the following:

Lemma 3.1. *For each $\xi = (q,\, p) \in \Xi\backslash\widetilde{\Xi}$ there exists a number $C(\xi)$, a neighborhood $B(\xi)$ of diameter $d(\xi)$, where $d(\xi) \leq \frac{1}{2}\min\{C(\xi)^{-1}\delta,\, dist(\xi,\, \widetilde{\Xi})\}$, and a map $\Upsilon_\xi : B(\xi) \to B(\xi_*)$ with the following properties*

$$\begin{aligned}
o)&\quad \Upsilon_\xi(\xi) = (q_*,\, p_*) \\
i)&\quad \|\Upsilon_\xi(\xi_1) - \Upsilon_\xi(\xi_2)\| \leq C(\xi)\|\xi_1 - \xi_2\| \qquad \forall \xi_1,\, \xi_2 \in B(\xi) \\
ii)&\quad \mu(\Upsilon_\xi(A)) = \mu(A) \quad \text{for all } A \subset B(\xi) \\
iii)&\quad S(\Upsilon_\xi(A)) = S(A) \quad \text{for all } A \subset B(\xi).
\end{aligned}$$

Observe that, by condition (i) on Q (finite entropy), μ is absolutely continuous with respect to the Lebesgue measure, and $\widetilde{\Xi}$ has μ-measure zero.

We will show how to use this lemma before proving it.

Since the manifold Ξ is compact, and $\widetilde{\Xi}$ is the finite union of smooth submanifolds, there exist countably many points ξ_j such that $\cup_j B(\xi_j) = \Xi \backslash \widetilde{\Xi}$ and each $B(\xi_j)$ intersects only finitely many other (the number of intersections can increase unboundedly near $\widetilde{\Xi}$). It is then possible to construct a smooth partition of unity $\{\chi_j\}$ for $\Xi \backslash \widetilde{\Xi}$ (i.e. $\chi_j \geq 0$ and $\sum_j \chi_j(\xi)$ is equal 0 for $\xi \in \widetilde{\Xi}$ and equal 1 otherwise) such that χ_j has support contained in some $B(\xi_{k_j})$. Accordingly, using the Lemma, for each F_Λ in the appropriate class, we have

$$
\begin{aligned}
&\int_{\Xi(n,\Pi,\,E,\,\omega_c)} F_\Lambda(q,\,p)d\mu_{n,\,\Pi,\,E,\,\omega_c}(q,\,p)\\
&= \sum_j \int_{\Xi(n,\Pi,\,E,\,\omega_c)} \chi_j(q,\,p)F_\Lambda(q,\,p)d\mu_{n,\,\Pi,\,E,\,\omega_c}(q,\,p)\\
&= \sum_j \int_{B(q_*,\,p_*)\cap\Xi(n,\Pi,\,E,\,\omega_c)} \chi_j \circ \Upsilon^{-1}_{\xi_{k_j}}(q,\,p)F_\Lambda \circ \Upsilon^{-1}_{\xi_{k_j}}(q,\,p)d\mu_{n,\,\Pi,\,E,\,\omega_c}(q,\,p)\\
&= \sum_j Z(n,\,\Pi,\,E,\,\omega_c) \int_{B(q_*,\,p_*)\cap\Xi(n,\Pi,\,E,\,\omega_c)} \chi_j \circ \Upsilon^{-1}_{\xi_{k_j}}(q,\,p)F_\Lambda \circ \Upsilon^{-1}_{\xi_{k_j}}(q,\,p)dS(q,\,p)\\
&= \sum_j Z(n,\,\Pi,\,E,\,\omega_c) \int_{\Xi(n,\Pi,\,E,\,\omega_c)} \chi_j(q,\,p)F_\Lambda(q,\,p)dS(q,\,p)\\
&= Z(n,\,\Pi,\,E,\,\omega_c) \int_{\Xi(n,\Pi,\,E,\,\omega_c)} F_\Lambda(q,\,p)dS(q,\,p)
\end{aligned}
$$

Since $S(\widetilde{\Xi}) = 0$, and μ_{n,Π,E,ω_c} is absolutely continuous with respect to S for almost all Π, E, ω_c, the equality holds true for each F_Λ. We have used that the support of $\chi_j \circ \Upsilon^{-1}_{\xi_{k_j}}$ is contained in $B(\xi_*)$ (by property (i) of Lemma 3.1) and the invariance of the measures with respect to the maps $\Upsilon_{\xi_{k_j}}$ (properties (ii), (iii) of Lemma 3.1).

Hence, we have proven the following:

Proposition 3.2 (non-Gaussian case). *For each $n \geq |\Lambda_k|\rho_*$ the measure $d\mu_{n,\,\Pi,\,E,\,j,\,\omega_c}(q,\,p)$ is proportional to the restriction of the Lebesgue measure on each $\Xi_j(n,\,\Pi,\,E,\,\omega_c)$, for almost all Π and E.*

Proof of Lemma 3.1.

Observe first that since $\widetilde{\Xi}$ has, at least, codimension 2 in Ξ, therefore $\Xi \backslash \widetilde{\Xi}$ is connected.

We will construct explicitly a map Υ that satisfies the wanted conditions. We will first describe a curve γ_j, connecting ξ_j to $\xi_* = (q_*,\,p_*)$. Then, we will define a one parameter family of vector fields; the wanted map will be obtained by integrating such family in a neighborhood of the curve γ_j. In the following we will look at a generic j, and we will suppress the j-subscript.

We will insure directly properties (o), (i) while property (ii) will follow from the fact that the vector field will be constructed with elements of the Lie algebra, which leaves the measure Q invariant, and property (iii) will follows because all these vector fields have zero divergence and therefore preserve the Lebesgue measure as well.

Let us call "strong cluster" a cluster that is connected by a series of bonds each of length less or equal than $R_1 - \delta$. From the definition follows that if we perturb by an amount δ the positions and velocities of the particles belonging to a strong cluster the particles still form a cluster.

Since $n > \rho_* \Lambda$ in the configuration ξ there must exist a two particle strong cluster inside Λ_k. We start by changing the velocities in order to keep a uniform distance from the manifold $\tilde{\Xi}$, then we move the particles in the cluster keeping fixed the positions and momenta of all the other particles. We wait for the first time one of the fixed particles get closer than $R_1 - \delta$ to one of the particles in the cluster we are moving. When this happens, we include the particle in the moving cluster and proceed to deform the larger cluster so obtained. A little thought will show that this procedure will eventually produce a unique cluster inside Λ_k. We can finally deform this last cluster until we obtain the configuration ξ_*. The sequence of deformations just described constitute our curve γ.

We parametrize γ by arc-length. Clearly, $\gamma \subset \Xi \backslash \tilde{\Xi}$. Let $L = \text{length}(\gamma)$ then $\gamma(0) = \xi$ and $\gamma(L) = \xi_*$. Moreover, for each $s \in [0, L]$, $\gamma'(s)$ can be written as $\sum_{i=1}^{m} a_i(s) X_i$ where $a_i(s)$ are smooth functions depending only on s and X_i are vector fields, in the Lie algebra associated to the moving cluster, that leave invariant Q, and m is some integer depending only on γ. In fact, we have already seen that the union of such vector fields contains the tangent space of the manifold with constant momentum and energy associated to the corresponding cluster. Then we define a time dependent vector field by

$$V(q, p, s) = \sum_{i=1}^{m} a_i(s) X_i(q, p)$$

and the corresponding flow Φ by

$$\begin{aligned} &\frac{d}{ds}\Phi(q, p, s) = V(\Phi(q, p, s), s) \\ &\Phi(q, p, 0) = (q, p). \end{aligned}$$

We can then define Υ:

$$\Upsilon(\xi) = \Phi(\xi, L).$$

Since V is always tangent to Ξ it follows that Υ is a well defined function from $\Xi \backslash \tilde{\Xi}$ to $\Xi \backslash \tilde{\Xi}$. We are left with the task of verifying properties $(o) - (iii)$.

Property (o) follows immediately from the construction ($\gamma(s)$ is a solution of the differential equation). The existence of the constant C in (i) follows from the continuous dependence from the initial conditions. More precisely, it follows from the fact that $\|DX\|_\infty < \infty$ for each $X \in \mathcal{A}_*$, $\sup\limits_{i,s} |a_i(s)| < \infty$, and that the length of the curves γ_j is bounded, by using Gronwal's inequality.

By the construction of γ, the particles belonging to the moving cluster at the time s form a strong cluster in the configuration $\gamma(s)$, and therefore they form a cluster in a δ-neighborhood of $\gamma(s)$. This, and property (i), implies that, for each $s \in [0, L]$, the vector field $\chi_{B(\xi)} \circ \Phi(\zeta, s)V(\Phi(\zeta, s), s)$, $\zeta \in \Xi$, generate a flow that leaves $d\mu$ invariant. Thus property (ii) holds.

Property (iii) follows from checking that the vector fields X_i have zero divergence and are tangent to the surfaces of constant momentum and energy in Λ_k. $\square$

4.CLUSTERING (THE GAUSSIAN CASE).

As in the previous section, we consider an increasing sequence of boxes Λ_k, to be more specific, let us assume that the Λ_k are concentrical cubic boxes of size $l_k = kl_{k-1}$. We define Ω_k as in the previous section and we will choose k large enough and consider only configurations ω belonging to $\Omega_k \cap \Omega_{k+1}$. Contrary to the previous section, we will use n, Π, E for the quantities in the box Λ_{k+2}, while ω_c will be the configuration outside Λ_{k+2}. In addition, we will call Θ the sum of the positions of all the particles in Λ_{k+2} (for convenience we set the origin of the coordinate system to coincide with the center of the boxes), and m the number of particles in Λ_k.

A surface $\Xi(n, \Pi, E, \omega_c) \in I\!\!R^{6n}$ can be decomposed in surfaces with fixed Θ and m: $\Xi(n, m, \Pi, E, \Theta, \omega_c)$. As before such surfaces do not need to be connected, and in general will not be. We are therefore forced to decompose them in their connected components $\Xi_j(n, m, \Pi, E, \Theta, \omega_c)$. Let us call $d\mu_{n,m,\Pi,E,\Theta,j,\omega_c}$ the measure Q conditioned to the surface $\Xi_j(n, m, \Pi, E, \Theta, \omega_c)$.

Lemma 4.1 (The Gaussian case). *For each $n \geq |\Lambda_{k+1}|\rho_*$, and $m \geq |\Lambda_k|\rho_*$ the measure $d\mu_{n,m,\Pi,E,\Theta,j,\omega_c}$ is proportional to the restriction of the Lebesgue measure on each $\Xi_j(n, m, \Pi, E, \Theta, \omega_c)$.*

Proof.

Since the general ideas of the proof are very similar to the ones employed in §3, we will simply sketch the argument, but we will emphasize the changes necessary to deal with the present context. First, $\Xi_j(n, \Pi, E, \Theta, \omega_c) = \cup_{m \geq |\Lambda_k \rho_*|}\Xi_j(n, m, \Pi, E, \Theta, \omega_c)$ must contains at least a configuration that corresponds to a unique cluster,[4] let us call ω_* such a configuration. As in the previous section, given any configuration in $\Xi_j(n, m, \Pi, E, \Theta, \omega_c)\backslash\widetilde{\Xi}$ we will find a neighborhood that can be mapped into a fixed neighborhood of ω_* with a map that leave invariant both $d\mu$ and the Lebesgue measure. Since we know already that, in a neighborhood of ω_*, $d\mu$ is proportional to the Lebesgue measure, this will conclude the argument.

We have already seen that all is needed to construct such a map is to construct a curve connecting any initial configuration to ω_* and whose tangent vector are in the admissible

[4] Given two separate clusters in Λ_{k+2}, it is always possible to move, or deform, them closer without changing their common center of mass or having particles cross the boundaries of the region.

Lie algebra.

The curve may be constructed in the following way: since we have an high density of particles in Λ_k we must have at least a two particle cluster. We consider then all the clusters in Λ_k, we start to deform each cluster so that its particles are aligned and the distances among them are all larger than $R_1 - 2\delta$, but shorter than $R_1 - \delta$; clearly this can be done without changing the center of mass of the cluster. Moreover, we can rotate the clusters around their centers of mass. In doing so the particles belonging to a cluster may come closer than $R_1 - \delta$ to some other particle, each time that this happens we include the new particle in the cluster (or we fuse together two clusters, if the particle belongs itself to a different cluster), provided that, in doing so, we do not move the center of mass of the cluster outside Λ_{k+1}.

These deformations are certainly admissible and lead to larger and larger clusters. In fact, rotating a linear cluster it is possible to sweep a very large volume and if no particles are present in the volume potentially swept by all the cluster, then the density must be very low and this is forbidden by our bond on the density in Λ_k and Λ_{k+1}.

The above construction can be carried out until one of the clusters does not reach the boundaries of Λ_{k+2}. But, a cluster long enough to have its center of mass in Λ_{k+1} and touch the boundary of Λ_{k+2} contains enough particles to sweep all Λ_{k+1}. Thus, we are lead to a configuration that contains a unique cluster in Λ_{k+1}, and such a cluster must contains enough particles so that every other particle in Λ_{k+2} can be reached by deforming it.

Once we obtain a unique cluster we can deform it until we obtain the configuration ω_*, since the surfaces under consideration are connected. $\square$

5. PROOF OF THEOREM 1

Since the center of mass depends only on the positions of the particles, proposition 3.2 for the case NG and 4.1 for the case G lead to the same conclusion, illustrated by the following lemma:

Lemma 5.1. *For almost every configuration of the positions ω_q and any sufficiently large (depending on ω_q) box Λ, the conditional measure Q given ω_q and*

$$\sum_{q_j \in \Lambda_k} \phi(p_j) = const$$

$$\sum_{q_j \in \Lambda_k} p_j = const$$

is the Lebesgue measure on the corresponding surface.

Proof.

By the condition (ii) on the high density, we can consider only the configurations such that there exists a k_0 such that, in any box $\Lambda \supset \Lambda_{k_0}$ the density of the particles is larger or equal than ρ_*. We can then consider the distribution conditioned on the energy, momentum, density and external configuration for Λ. By the previous lemma this conditional distribution is the Lebesgue measure on each connected component $\Xi(n,\, \Pi,\, E,\, j,\, \omega_c)$. These connected components can be distinguished only from the positions of the particles in a R_1–larger box Λ'. Thus, for any $q \in \Lambda^{3n}$, the surface $\{p \in \mathbb{R}^{3n} \mid (q,\, p) \in \Xi(n,\, \Pi,\, E,\, j,\, \omega_c)\}$ is connected; this implies that, conditioning on the positions in Λ, we obtain the Lebesgue measure on the corresponding connected surface in momenta space. □

At this point we are in the same situation as in [OVY] (after lemma (4.5) there). In fact, as a consequence of the previous lemma, the distribution of the momentum conditioned on the positions is given by a convex combination of measures of the form

$$\pi(dp \,|\, \lambda) = \exp\Big[\sum_{i=1}^{3}\sum_{\alpha} \lambda^i p^i_\alpha - \lambda^4 \sum_{\alpha} \phi(p_\alpha)\Big] \,/\, \text{Normalization},$$

Lemma 5.2. *For any configuration $\omega = \{(x_\alpha, p_\alpha)\}$, let $\vec{z}(\omega)$ be the density, momenta and kinetic energy associated with the configuration defined by*

$$z^0(\omega) = \lim_{\delta\to 0} z^\mu_{\chi,\delta}(\omega) = \lim_{\delta\to 0} \delta^3 \sum_{\alpha=1}^{N} \chi(\delta x_\alpha)$$

$$z^\mu(\omega) = \lim_{\delta\to 0} z^\mu_{\chi,\delta}(\omega) = \lim_{\delta\to 0} \delta^3 \sum_{\alpha=1}^{N} \chi(\delta x_\alpha) p^\mu_\alpha(\omega)\ ,\ \ \mu = 1, 2, 3$$

$$z^4(\omega) = \lim_{\delta\to 0} z^\mu_{\chi,\delta}(\omega) = \lim_{\delta\to 0} \delta^3 \sum_{\alpha=1}^{N} \chi(\delta x_\alpha)\, \phi(p_\alpha)\ .$$

Here χ is a cutoff function of total integral one, $\vec{z}^\mu(\omega)$ exist almost everywhere and are independent of the cutoff χ. Furthermore, $\vec{z}(\omega)$ are constants of the motion for L in the sense that

$$\int h(\vec{z}(\omega))\, LF(\omega)\, dQ = 0,$$

for all local smooth functions F and all smooth functions h with compact support.

Proof.

This was proven in [OVY] for bounded ϕ'. For completeness, we present here the proof for unbounded ϕ'.

Because of the translation invariance of Q and the conditionals of p_α being independent, by the same argument used in lemma (1.2) these limits clearly exist and are independent of the cutoff χ.

By condition (iii) in theorem (1.1)

$$\begin{aligned} 0 &= \int L(F h(z^\mu_{\chi,\delta}(\omega)))\, dQ \\ &= \int (LF)\, h(z^\mu_{\chi,\delta}(\omega))\, dQ + \int F\, L\, h(z^\mu_{\chi,\delta}(\omega))\, dQ\ . \end{aligned}$$

The first term converges to $\int h(z^\mu_\chi(\omega))LF\,dQ$ as $\delta \to 0$. We only have to show that the second term converges to zero as $\delta \to 0$. Clearly, it suffices to show that as $\delta \to 0$

$$\int |Lz^\mu_{\chi,\delta}|\,dQ \to 0\ , \qquad \mu = 0, \ldots, 4\ . \tag{5.1}$$

This is easy to show for $\mu = 0, 1, 2, 3$ (as in [OVY] pag. 544).

For $\mu = 4$ we have

$$\begin{aligned}
\mathbb{E}^{\,Q}\left(|Lz^4_{\chi,\delta}|\right) =& \mathbb{E}^{\,Q}\left(|\delta\,\delta^3 \sum_{i,\alpha} \chi_i(\delta q_\alpha)\,\phi_i(p_\alpha)\,\phi(p_\alpha)|\right)\\
&+ \mathbb{E}^{\,Q}\left(|\delta^3 \sum_{\alpha\neq\beta}\sum_i \chi(\delta q_\alpha)\,\phi_i(p_\alpha)\,V_i(q_\alpha - q_\beta)|\right)
\end{aligned}$$

Only the second term of the right end side present difficulties. Let $w_i(\vec{z})$ and $\sigma_i(\vec{z})$ denote the expectation and variance of $\phi_i(p_\alpha)$ with respect to Q conditioned on $\vec{z}$. These can be computed explicitly by using the characterization of the conditional measure given ω_q and $\vec{z}$. We can bound the second term of the RHS of the above expression by

$$\begin{aligned}
&\mathbb{E}^{\,Q}\left(|\delta^3 \sum_i \sum_{\alpha\neq\beta} \chi(\delta q_\alpha)\,\phi_i(p_\alpha)\,V_i(q_\alpha - q_\beta)|\right)\\
=&\mathbb{E}^{\,Q}\left(|\delta^3 \sum_i\sum_\alpha \chi(\delta q_\alpha)\big[\phi_i(p_\alpha) - w_i\big] \sum_{\beta\neq\alpha} V_i(q_\alpha - q_\beta)|\right)\\
&+ \mathbb{E}^{\,Q}\left(|\delta^3 \sum_{\alpha\neq\beta}\sum_i \chi(\delta q_\alpha)\,V_i(q_\alpha - q_\beta)\,w_i|\right)\,.
\end{aligned}$$

The second term can be bounded as before. Using the Schwartz inequality the first term can be bounded by

$$\begin{aligned}
&\sum_i \mathbb{E}^{\,Q}\left(\mathbb{E}^{\,Q}\left(\left[\delta^3 \sum_\alpha \chi(\delta q_\alpha)\big[\phi_i(p_\alpha) - w_i\big]\sum_{\beta\neq\alpha} V_i(q_\alpha - q_\beta)\right]^2 \Big|\, \vec{z}\right)^{1/2}\right)\\
=&\sum_i \mathbb{E}^{\,Q}\left(\sqrt{\sigma_i(\vec{z})}\delta^3\,\mathbb{E}^{\,Q}\left(\sum_\alpha \chi(\delta q_\alpha)^2\,\big(\sum_\beta V_i(x_\alpha - x_\beta)\big)^2\Big|\,\vec{z}\right)^{1/2}\right)\\
\leq& \sum_i \mathbb{E}^{\,Q}(\sigma_i(\vec{z}))^{1/2}\delta^3\,\mathbb{E}^{\,Q}\left(\sum_\alpha \chi(\delta q_\alpha)^2\big(\sum_{\beta\neq\alpha} V_i(x_\alpha - x_\beta)\big)^2\right)^{1/2}\,.
\end{aligned}$$

By the condition on ϕ and the entropy argument we have that $I\!\!E^{Q}(\sigma_i(\vec{z}))$ is finite. To bound the second expectation, let us divide the set $\{x \,|\, |x| \leq 2\delta^{-1}\}$ into boxes of size $2R_0$ (R_0 is the range of V). Let σ index the boxes and let N_σ be the number of particles in the σ box.

$$\delta^3 \sum_i I\!\!E^{Q}\left(\sum_\alpha \chi(\delta q_\alpha)^2 \left[\sum_\beta V_i(x_\alpha - x_\beta)\right]^2\right)^{1/2} \leq \text{const. } \delta^3\, I\!\!E^{Q}\left(\sum_\sigma N_\sigma^3\right)^{1/2}$$

By convexity and the inequality $(\sum_\sigma N_\sigma^3)^{1/3} \leq (\sum_\sigma N_\sigma^2)^{1/2}$ we see that the above expression is bounded by

$$\text{const. } \delta^3 I\!\!E^{Q}\left(\left[\sum_\sigma N_\sigma^2\right]^{3/4}\right) \leq \text{ const. } \delta^3 \left[I\!\!E^{Q}\left(\sum_\sigma N_\sigma^2\right)\right]^{3/4}.$$

By lemma (1.2)(i) and the translation invariance, $I\!\!E^{Q}(\sum_\sigma N_\sigma^2)$ is bounded by δ^{-3}; hence, the quantity under consideration is bounded by const. $\delta^{3/4}$. This concludes the proof of the lemma 4.2. $\square$

By the previous lemma Q conditioned on $\vec{z}(\omega)$ is still invariant for L. Since we assume that Q is translation invariant, we can apply lemma 4.10 in [OVY] and obtain that these conditioned distribution are given by grancanonical Gibbs measures, concluding our proof.

APPENDIX 1.

To show that our condition on the noise and the kinetic energy in the non-Gaussian case are far from empty, we give here an example of stochastic perturbation that satisfies such condition. This is the only point where we use our requirement on the form of the kinetic energy ϕ.

Lemma A.1. *If $\{\eta^\theta_{\alpha\beta}\} = \{e_1 \wedge D_{\alpha\beta}E,\ e_2 \wedge D_{\alpha\beta}E,\ e_3 \wedge D_{\alpha\beta}E\}$ and $\phi(p_a) = \sum_{i=1}^3 \varphi(p^i_\alpha)$, with $(\varphi''')^2 + \varphi^{iv}\varphi'' = 0$ at most at finitely many points, then condition (*) on the noise is satisfied.*

Proof.

A simple computation shows

$$\begin{aligned}
[X^i_{\alpha\beta},\, L]q_\alpha =& H_\alpha \eta^i_{\alpha\beta} \\
[X^i_{\alpha\beta},\, L]q_\beta =& -\, H_\beta \eta^i_{\alpha\beta} \\
[X^j_{\alpha\beta},\, [X^i_{\alpha\beta},\, L]]q^l_\alpha =& (\eta^j_{\alpha\beta})_l(\eta^i_{\alpha\beta})_l H'_{\alpha ll} + H_{\alpha ll}\left(e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta}\right)_l \\
[X^i_{\alpha\beta},\, [X^j_{\alpha\beta},\, L]]q^l_\beta =& (\eta^j_{\alpha\beta})_l(\eta^i_{\alpha\beta})_l H'_{\beta ll} - H_{\beta ll}\left(e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta}\right)_l
\end{aligned}$$

where $H_{\alpha ll}$ stand for the element ll of the diagonal matrix H_α. The matrix H'_α is the derivative of the matrix H_α; $(\cdot)_l$ stands for the l–th component of the corresponding vectors. Now, let us take the six vectors obtained by letting i, j vary only in $\{1, 2\}$. We define the vectors w^{ij} by

$$w^{ij}_l = (\eta^i_{\alpha\beta})_l(\eta^j_{\alpha\beta})_l.$$

Let us consider

$$\sum_{i=1}^{2} \mu_i [X^i_{\alpha\beta}, L] + \sum_{i,j=1}^{2} \nu_{ij} [X^j_{\alpha\beta}, [X^i_{\alpha\beta}, L]] = 0$$

Applying the above vector fields to q_α, q_β, we have

$$0 = \sum_{i=1}^{2} \mu_i H_\alpha \eta^i_{\alpha\beta} + \sum_{i,j=1}^{2} \nu_{ij} \left[H'_\alpha w^{ij} + H_\alpha \left(e^i \wedge (H_\alpha + H_\beta) \eta^j_{\alpha\beta} \right) \right]$$

$$0 = -\sum_{i=1}^{2} \mu_i H_\beta \eta^i_{\alpha\beta} + \sum_{i,j=1}^{2} \nu_{ij} \left[H'_\beta w^{ij} - H_\beta \left(e^i \wedge (H_\alpha + H_\beta) \eta^j_{\alpha\beta} \right) \right]$$

If we multiply the first by $(H_\alpha)^{-1}$, the second by $(H_\beta)^{-1}$, and add one to the other, then we get

$$0 = \sum_{i,j=1}^{2} \nu_{ij} A w^{ij}$$

where $A = 1/2\{H'_\alpha H_\alpha^{-1} + H'_\beta H_\beta^{-1}\}$. Notice that A is invertible out of a set of codimension 1 (see later for more details), consequently

$$0 = \sum_{i,j=1}^{2} \nu_{ij} w^{ij}$$

$$0 = \sum_{i=1}^{2} \mu_i \eta^i_{\alpha\beta} + \sum_{i,j=1}^{2} \nu_{ij} e^i \wedge (H_\alpha + H_\beta) \eta^j_{\alpha\beta}$$

To conclude we need an explicit representations of the vectors involved in the previous equations. Let $D^i_{\alpha\beta} E = \zeta_i$, $h_i = H_{\alpha ii} + H_{\beta ii}$, then a direct computation yields

$$\begin{aligned}
\eta^1 &= (0,\, -\zeta_3,\, \zeta_2) \\
\eta^2 &= (\zeta_3,\, 0,\, -\zeta_1) \\
w^{11} &= (0,\, \zeta_3^2,\, \zeta_2^2) \\
w^{12} &= w^{21} = (0,\, 0,\, -\zeta_1\zeta_2) \\
w^{22} &= (\zeta_3^2,\, 0,\, \zeta_1^2) \\
e^1 \wedge (H_\alpha + H_\beta)\eta^1_{\alpha\beta} &= (0,\, -\zeta_2 h_3,\, -\zeta_3 h_2) \\
e^1 \wedge (H_\alpha + H_\beta)\eta^2_{\alpha\beta} &= (0,\, \zeta_1 h_3,\, 0) \\
e^2 \wedge (H_\alpha + H_\beta)\eta^1_{\alpha\beta} &= (\zeta_2 h_3,\, 0,\, 0) \\
e^2 \wedge (H_\alpha + H_\beta)\eta^2_{\alpha\beta} &= (-\zeta_1 h_3,\, 0,\, -\zeta_3 h_1).
\end{aligned}$$

Immediately follows $\nu_{22} = \nu_{11} = 0$ and $\nu_{12} = -\nu_{21}$, which, substituted in the remaining equations, yields

$$\Omega \begin{pmatrix} \mu_1 \\ \mu_2 \\ \nu_{12} \end{pmatrix} = 0$$

For some matrix Ω with $\det(\Omega) = \zeta_1\zeta_2\zeta_3(h_2 + h_3)$. Since the determinant is equal zero on a set of codimension one, we have that the vector are linearly independent, out of a set of codimension one.

This set of codimension one consists of $\cup_i\{p \mid \varphi'''(p^i_\alpha)\varphi''(p^i_\alpha) = -\varphi'''(p^i_\beta)\varphi''(p^i_\beta)\}$, where the matrix A is not invertible[5], and $\cup_i\{p^i_\alpha = p^i_\beta\}$, where the matrix Ω is not invertible. To get codimension two we have to analyze all the different cases one by one, since they are treated all in the same way we will consider only the points on the set $\{p^1_\alpha = p^1_\beta\}$, and we will leave the rest to the skeptical reader. We can clearly ignore points of the above set that also belong to some other singular set: they belong to a set of codimension two. For points in the set under consider we will have $\zeta_1 = 0$, while all the other components will be different from zero. This implies that $w^{12} = w^{21} = e^1 \wedge (H_\alpha + H_\beta)\eta^2_{\alpha\beta} = 0$, we need then to produce more vectors, i.e., compute more commutators. It turns out to be sufficient to compute

$$\begin{aligned}
[X^2, [X^1_{\alpha\beta}, [X^2_{\alpha\beta}, L]]]q_\alpha &= H_\alpha v_1 + H'_\alpha v_2 \\
[X^2, [X^1_{\alpha\beta}, [X^2_{\alpha\beta}, L]]]q_\beta &= -H_\beta v_1 + H'_\beta v_2 \\
[X^2, [X^2_{\alpha\beta}, [X^1_{\alpha\beta}, L]]]q_\alpha &= H'_\alpha v_2 \\
[X^2, [X^2_{\alpha\beta}, [X^1_{\alpha\beta}, L]]]q_\beta &= H'_\beta v_2
\end{aligned}$$

where $v_1 = (0, \zeta_3 h_1 h_3, 0)$, and $v_2 = (0, 0, -\zeta_2\zeta_1 h_1)$. We have then to study the linear combination

$$\sum_{i=1}^{2} \mu_i [X^i_{\alpha\beta}, L] + \sum_{i,j=1}^{2} \nu_{ij}[X^j_{\alpha\beta}, [X^i_{\alpha\beta}, L]] + \varepsilon_1 [X^2, [X^1_{\alpha\beta}, [X^2_{\alpha\beta}, L]]] + \varepsilon_2 [X^2, [X^2_{\alpha\beta}, [X^1_{\alpha\beta}, L]]] = 0$$

where ν_{12}, ν_{21} are taken to be zero since the corresponding commutators, when restricted to the q_α, q_β space, would not contribute anything of interest. As before, we apply the vectors to the coordinates q_α, q_β, we multiply by H_α^{-1} and H_β^{-1} and add the corresponding equations, in so doing we obtain

$$\sum_{i,j=1}^{2} \nu_{ij} w^{ij} + (\varepsilon_1 + \varepsilon_2) v_2 = 0$$

[5] The condition of the hypothesis insure that such set is a smooth codimension one manifold unless $\varphi'''(p^i_\alpha)^2 + \varphi^{iv}(p^i_\alpha)\varphi''(p^i_\alpha) = \varphi'''(p^i_\beta)^2 + \varphi^{iv}(p^i_\beta)\varphi''(p^i_\beta) = 0$, which can happen only on a set of codimension two.

from this follows immediately $\nu_{11} = \nu_{22} = 0$, $\varepsilon_1 = -\varepsilon_2$. Substituting in the original equation we get

$$0 = \sum_{i=1}^{2} \mu_i H_\alpha \eta^i_{\alpha\beta} + \varepsilon_1 H_\alpha v_1$$

which implies $\mu_i = \varepsilon_i = 0$ on a set of codimension two. $\square$

REFERENCES.

[AGGLM] M. Aizemann, S. Goldstein,C. Gruber, J.L. Lebowitz, P. Martin : On the Equivalence between KMS-States and Equilibrium States for Classical Systems, Comm. Math. Phys. **53** (1977), 209–220.

[AGL] M. Aizemann, S. Goldstein, J.L. Lebowitz: Conditional Equilibrium and Equivalence of Microcanonical and Grancanonical Ensembles in the Thermodynamic Limit, Comm. Math. Phys. **62** (1978), 279–302.

[BLSP] L.Bunimovich, C.Liverani, S.Pellegrinotti, Y.Suhov: Ergodic systems of n balls in a billiard table, Communications in Mathematical Physics, **146** (1992) 357–396.

[ES] G.Eyink, H. Spohn: Space-time Invariant Measures of the Ideal Gas with Finite Number, Energy and Entropy Densities, preprint (1991).

[FFL] J. Fritz, T. Funaki, J.L. Lebowitz: Stationary States of Random Hamiltonian Systems, preprint 1992.

[GV] G. Gallavotti, E. Verboven: On the Classical KMS Condition, Nuovo Cimento (1976), 274–286.

[LW] C.Liverani, M.P.Wojtkowski: Ergodicity in Hamiltonian systems, to appear in Dynamics Reported.

[OVY] S. Olla, S.R.S. Varadhan, H. T. Yau : Hydrodynamic Limit for a Hamiltonian System with Weak Noise, Comm. Math. Phys. **155** (1993), 523–560.

[OV] S. Olla, S.R.S. Varadhan : Scaling limit for interacting Ornstein-Uhlenbeck processes, Comm. Math. Phys. **135** (1991) 355–378.

[R] D.Ruelle: Superstable interactions in classical statistical mechanics, Comm. Math. Phys. **18** (1970) 127-159.

[Si] Ya.G.Sinai: Dynamical systems with elastic reflections, Russ.Math.Surveys **25** (1970), 137–189.

[Sp] H.Spohn: Large Scale Dynamics of Interacting Particles, Springer, Berlin, 1991.

[Z] M.Zhu: The Reversible Measures of a Conservative System with Finite Range Interactions, preprint 1992.