Convergence Rate of Some Domain Decomposition Methods for Overlapping and Nonoverlapping Subdomains

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Abstract

Three iterative domain decomposition methods are considered: simultaneous updates on all subdomains (Additive Schwarz Method), flow directed sweeps and double sweeps. By using some techniques of formal language theory we obtain a unique criterion of convergence for the three methods. The convergence rate is a function of the criterion and depends on the algorithm.

1 Introduction

We study the convergence of some domain decomposition methods. The equation we consider is a convection-diffusion operator. The domain is decomposed into N vertical strips with possible overlaps. Absorbing boundary conditions are used as interface conditions. Three iterative domain decomposition methods are considered (see § 2 for a precise statement). In these methods, the solution in each subdomain is updated in different ways. The first algorithm is the additive Schwarz method (ASM) which consists in updating the solution at the same time in each subdomain. In the second algorithm denoted by DSA, the update is

made by double sweeps over the domain. In the third method called FDA, the solution is updated by flow directed sweeps over the domain (flow directed Gauss-Seidel method, see [6]). The constant coefficient operator is considered here and the study is based on Fourier analysis. When there are two subdomains ($N = 2$), the convergence is proved by computing explicitly the convergence rate, see [9]. For an arbitrary number of subdomains, it seems difficult even in our simplified context to compute explicitly the effective convergence rate. Indeed, it amounts to computing the eigenvalues of a $2N-2 \times 2N-2$ matrix. In order to estimate the convergence rate, other methods have to be used. For instance in [7], [2] and [9], the convergence of the ASM algorithm is proved by using energy estimates. In [6] and [10], convergence is proved for the flow directed Gauss-Seidel method. In this paper, we prove convergence for the three algorithms (ASM, double-sweeps and flow oriented Gauss-Seidel algorithms) with or without overlaps. The originality of the proof presented here lies in the use of definitions coming from formal language theory, see [8]. We thus obtain a unique criterion of convergence for the three methods. Of course, the convergence rate which is a function of this criterion depends on the method.

The paper is organized as follows: in section 2, we define the algorithms and state the main result (theorem 2.4). In § 3, we prove the well-posedness of these algorithms. The three subsequent sections are devoted to the proof of theorem 2.4. In section 4, we write the three previous algorithms in the form of substructuring methods i.e. the unknowns are defined on the boundaries of the subdomains. Then, we take the Fourier transform of the resulting algorithms. In § 5, we prove some results in algebra by using some tools of formal language theory [8]. In section 6, we estimate the convergence rate in the Fourier space and we prove the theorem.

2 Statement of the problem

Notation. For a function $f(x,y) : \mathbf{R}^2 \longrightarrow \mathbf{R}$, $\hat{f}(k_1,k_2)$ denotes its Fourier transform w.r.t. to (x,y):

$$\hat{f}(k_1,k_2) = (\mathcal{F}f)(k_1,k_2) = \int e^{-i(k_1x+k_2y)} f(x,y)\,dx\,dy$$
$$\text{and} \quad f(x,y) = (\mathcal{F}^{-1}\hat{f})(x,y) = \int e^{i(k_1x+k_2y)} \hat{f}(k_1,k_2)\,\frac{dk_1\,dk_2}{(2\pi)^2}.$$

The partial Fourier transform of $f(x,y)$ w.r.t. y is denoted by $\hat{f}(x,k)$ or by $\mathcal{F}_y(f)(x,k)$.

Let $\mathcal{L}$ be a convection-diffusion operator

$$\mathcal{L} = c + a\partial_x + b\partial_y - \nu\Delta$$

where c, a, b and ν are real numbers and $a > 0$, $\nu > 0$, $c \geq 0$. We want to solve the following problem set on $\mathbf{R}^2$

$$\mathcal{L}(u) = f \text{ in } \mathbf{R}^2 \tag{1}$$

where f is a given function. We have obviously the

Lemma 2.1 *Let $m \in \mathbf{N}$ and $f \in H^m(\mathbf{R}^2)$ be such that $\frac{\hat{f}(k_1,k_2)(1+k_1^2+k_2^2)}{c+iak_1+ibk_2+\nu(k_1^2+k_2^2)} \in L^2(\mathbf{R}^2)$. Then, there exists a unique $u \in H^{2+m}(\mathbf{R}^2)$ solution to (1).*

In the sequel, the data f is supposed to satisfy at least the assumption of lemma 2.1 with $m = 0$.

Remark 2.2 *If $c > 0$, assumption (i) is satisfied for $f \in L^2(\mathbf{R}^2)$.*

2.1 Definition of the algorithms

The space $\mathbf{R}^2$ is decomposed into N vertical strips in the following manner: Let $\Omega_i =]l_i, L_i[\times\mathbf{R}$, $1 \leq i \leq N$ with $-\infty = l_1 < l_2 \leq L_1 < \ldots < L_{i-2} < l_i \leq L_{i-1} < \ldots < L_N = +\infty$, we have $\mathbf{R}^2 = \cup_{i=1}^N \bar{\Omega}_i$. As interface conditions we use the absorbing boundary conditions introduced in [3], [4] and [5]. For the left (resp. right) boundary of a subdomain, we take as interface conditions the operator $\partial_x - \Lambda_j^+$ (resp. $\partial_x - \Lambda_j^-$) (j =0,1 or 2) where:

$$\Lambda_0^+ = \frac{a + \sqrt{a^2 + 4\nu c}}{2\nu},\ \Lambda_0^- = \frac{a - \sqrt{a^2 + 4\nu c}}{2\nu}, \tag{2}$$

$$\begin{aligned} \Lambda_1^+ &= \tfrac{a+\sqrt{a^2+4\nu c}}{2\nu} + \tfrac{b}{\sqrt{a^2+4\nu c}}\partial_y \\ \Lambda_1^- &= \tfrac{a-\sqrt{a^2+4\nu c}}{2\nu} - \tfrac{b}{\sqrt{a^2+4\nu c}}\partial_y \end{aligned} \tag{3}$$

and

$$\begin{aligned} \Lambda_2^+ &= \tfrac{a+\sqrt{a^2+4\nu c}}{2\nu} + \tfrac{b}{\sqrt{a^2+4\nu c}}\partial_y - \tfrac{\nu}{\sqrt{a^2+4\nu c}}(1 + \tfrac{b^2}{a^2+4\nu c})\partial_{yy} \\ \Lambda_2^- &= \tfrac{a-\sqrt{a^2+4\nu c}}{2\nu} - \tfrac{b}{\sqrt{a^2+4\nu c}}\partial_y + \tfrac{\nu}{\sqrt{a^2+4\nu c}}(1 + \tfrac{b^2}{a^2+4\nu c})\partial_{yy}. \end{aligned} \tag{4}$$

The symbol of the operators $\Lambda_j^\pm$ are

$$\lambda_0^\pm(k) = \frac{a \pm \sqrt{a^2 + 4\nu c}}{2\nu} \tag{5}$$

$$\lambda_1^\pm(k) = \frac{a \pm \sqrt{a^2 + 4\nu c}}{2\nu} \mp ik\frac{b}{\sqrt{a^2 + 4\nu c}} \tag{6}$$

$$\lambda_2^\pm(k) = \frac{a \pm \sqrt{a^2 + 4\nu c}}{2\nu} \mp ik\frac{b}{\sqrt{a^2 + 4\nu c}} \pm \frac{\nu}{\sqrt{a^2 + 4\nu c}}\left(1 + \frac{b^2}{a^2 + 4\nu c}\right)k^2 \tag{7}$$

The infinite domain Ω_N requires a special treatment and we need the

Definition 2.3 *Let $m \in \mathbf{N}$, we introduce $H^{2+m}_{st}(\Omega_N) = \{v / \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y} \in H^{1+m}(\Omega_N)$ and $\forall L > l_N,\ v \in L^2(]l_N, L[\times\mathbf{R})\}$ endowed with the norm $||v||_{H^2_{st}(\Omega_N)} = ||\frac{\partial v}{\partial x}||_{H^{1+m}(\Omega_N)} + ||\frac{\partial v}{\partial y}||_{H^{1+m}(\Omega_N)} + ||v||_{L^2(]l_N, l_N+1[\times\mathbf{R})}$. Otherwise we set $H^2_{st}(\Omega_i) = H^2(\Omega_i)$ for $i = 1, \ldots, N-1$.*

Three domain decomposition methods are considered. They are defined recursively by starting from the initial estimates $u_i^0 \in H^2_{st}(\Omega_i)$ of the solution u of (1) in the domain Ω_i, $i = 1, \ldots, N$.

The first one is the additive Schwarz method (abbreviated to ASM) and writes:

$$\left\{ \begin{array}{l} u_i^{n+1} \in H^2_{st}(\Omega_i),\ 1 \leq i \leq N \\ \mathcal{L}(u_i^{n+1}) = f \text{ in } \Omega_i \\ (\partial_x - \Lambda_j^+)(u_i^{n+1}) = (\partial_x - \Lambda_j^+)(u_{i-1}^n) \text{ at } x = l_i,\ 2 \leq i \leq N \\ (\partial_x - \Lambda_j^-)(u_i^{n+1}) = (\partial_x - \Lambda_j^-)(u_{i+1}^n) \text{ at } x = L_i,\ 1 \leq i \leq N-1 \end{array} \right. \tag{8}$$

In the ASM, u_i^n is updated at the same time in each subdomain. In the second algorithm, the value of u_i^n is updated in one subdomain at a time. We proceed by double sweeps over the subdomains:

<u>left to right sweep</u>

$$\left\{ \begin{array}{l} u_i^{n+1/2} \in H^2_{st}(\Omega_i),\ 1 \leq i \leq N \\ \mathcal{L}(u_i^{n+1/2}) = f \text{ in } \Omega_i \\ (\partial_x - \Lambda_j^+)(u_i^{n+1/2}) = (\partial_x - \Lambda_j^+)(u_{i-1}^{n+1/2}) \text{ at } x = l_i,\ 2 \leq i \leq N \\ (\partial_x - \Lambda_j^-)(u_i^{n+1/2}) = (\partial_x - \Lambda_j^-)(u_{i+1}^n) \text{ at } x = L_i,\ 1 \leq i \leq N-1 \end{array} \right. \tag{9}$$

<u>right to left sweep</u>

$$\left\{ \begin{array}{l} u_i^{n+1} \in H^2_{st}(\Omega_i),\ 1 \leq i \leq N \\ \mathcal{L}(u_i^{n+1}) = f \text{ in } \Omega_i \\ (\partial_x - \Lambda_j^+)(u_i^{n+1}) = (\partial_x - \Lambda_j^+)(u_{i-1}^{n+1/2}) \text{ at } x = l_i,\ 2 \leq i \leq N \\ (\partial_x - \Lambda_j^-)(u_i^{n+1}) = (\partial_x - \Lambda_j^-)(u_{i+1}^{n+1}) \text{ at } x = L_i,\ 1 \leq i \leq N-1. \end{array} \right. \tag{10}$$

Let us remark that $u_N^{n+1/2} = u_N^{n+1}$. This algorithm is called DSA (for Double Sweep Algorithm).

The third algorithm denoted shortly by FDA is a slight modification of the second one. It consists only in flow directed sweeps:

<u>left to right sweep</u>

$$\left\{ \begin{array}{l} u_i^{n+1} \in H^2_{st}(\Omega_i),\ 1 \leq i \leq N \\ \mathcal{L}(u_i^{n+1}) = f \text{ in } \Omega_i \\ (\partial_x - \Lambda_j^+)(u_i^{n+1}) = (\partial_x - \Lambda_j^+)(u_{i-1}^{n+1}) \text{ at } x = l_i,\ 2 \leq i \leq N \\ (\partial_x - \Lambda_j^-)(u_i^{n+1}) = (\partial_x - \Lambda_j^-)(u_{i+1}^n) \text{ at } x = L_i,\ 1 \leq i \leq N-1 \end{array} \right. \tag{11}$$

Our aim is to prove the

Theorem 2.4 *We assume $u_i^0 \in H^2_{st}(\Omega_i)$, $1 \leq i \leq N$ and that $f \in L^2(\mathbf{R}^2)$ satisfies the condition of Lemma 2.1. There exists a constant $\xi_c > 0$, depending on $\frac{b}{a}$ and on $\frac{\nu c}{a^2}$ so that the following convergence results hold as soon as $\inf_{1\leq i\leq N} \frac{a}{\nu}(L_i - l_i) > \xi_c$.*
a) In the non-overlapping case: $\inf_{1\leq i\leq N-1}(L_i - l_{i+1}) = 0$.
The three algorithms achieve convergence in the sense that $\lim_{n\to\infty} ||u_i^n - u||_{H^2_{st}(\Omega_i)} = 0$, $1 \leq i \leq N$, if we add the conditions $f \in H^1(\mathbf{R}^2)$ and $u_i^0 \in H^3_{st}(\Omega_i)$ for the ASM and the FDA.
b) In the overlapping case: $\frac{1}{2}(L_i - l_i) > L_j - l_{j+1} > \delta > 0$, $\forall i, j \in \{1, \ldots, N-1\}$.
The three algorithms achieve geometric convergence with the estimates:
$||u_i^n - u||_{H^2_{st}(\Omega_i)} \leq C\rho^{[\frac{n}{2(N-1)}]} \sup_j ||u_j^0 - u||_{H^2_{st}(\Omega_j)}$, $n \geq 2N+1$, for the ASM,
$||u_i^n - u||_{H^2_{st}(\Omega_i)} \leq C\rho^n \sup_j ||u_j^0 - u||_{H^2_{st}(\Omega_j)}$ $n \geq 3$ for the DSA,
$||u_i^n - u||_{H^2_{st}(\Omega_i)} \leq C\rho^{[\frac{n}{N-1}]} \sup_j ||u_j^0 - u||_{H^2_{st}(\Omega_j)}$ $n \geq 2(N-1)+1$ for the FDA,
(writing [.] for the integer part) where C and ρ are constants, $C > 0$, $0 < \rho < 1$, C and ρ are independent of N.

Remark 2.5 *In [2], [9] and [10], convergence is proved in the non overlapping case for algorithms ASM and FDA under weaker assumptions. The proof is based on energy estimates and we only need $f \in L^2(\mathbf{R}^2)$. Nevertheless, in these works, the DSA was not considered.*

3 Well-posedness of the algorithms

Notation. We set $\sigma(j) = 0$ for $j = 0$ or 1 and $\sigma(j) = 1$ for $j = 2$.

We study the well-posedness of the following BVP

$$\begin{cases} \mathcal{L}(v) = f \\ (\partial_x - \Lambda_j^-)(v) = g^- \text{ at } x = L_i \\ (\partial_x - \Lambda_j^+)(v) = g^+ \text{ at } x = l_i \end{cases} \tag{12}$$

Theorem 3.1 *Let $m \in \mathbf{N}$, $f \in H^m$ and $j = 0$, 1 or 2, and $g^\pm \in H^{1/2+m-\sigma(j)}(\mathbf{R})$.*
a) For $1 \leq i \leq N-1$, there exists a unique solution u_i to (12) in $H^{2+m}(\Omega_i)$ and the operator $(g^+, g^-) \longrightarrow u_i$ (forget g^+ for $i = 1$) from $H^{1/2+m-\sigma(j)}(\mathbf{R}) \times H^{1/2+m-\sigma(j)}(\mathbf{R})$ to $H^{2+m}(\Omega_i)$ is continuous.
b) For $i = N$, there exists a unique solution u_N to (12) in $H^{2+m}_{st}(\Omega_N)$ and the operator $g^+ \longrightarrow u_N$ from $H^{1/2+m-\sigma(j)}(\mathbf{R})$ to $H^{2+m}_{st}(\Omega_i)$ is continuous. Moreover, if $c > 0$, $u_N \in H^{2+m}(\Omega_N)$.

Proof We consider the case $m = 0$. For $m \neq 0$, the proof is very similar. We distinguish according to the value of i.

Case $1 < i < N-1$ In fact, we just consider $2 \leq i \leq N-1$. The case $i = 1$ is carried out similarly and is even simpler: it suffices to take $\alpha(k) = 0$ in (13).

Uniqueness Since problem (12) is linear, it suffices to prove that $v = 0$ is the only solution to the homogeneous problem. Let v be a solution to (12) with $f = 0$ and $g = 0$. We have

$$0 = \int\int \mathcal{L}(v) \times v = \int\int cv^2 + \nu|\nabla v|^2 + \tfrac{a}{2}\int_{\mathbf{R}} v^2(L_i, y)\,dy - \tfrac{a}{2}\int_{\mathbf{R}} v^2(l_i, y)\,dy \\ -\nu\int_{\mathbf{R}} v\tfrac{\partial v}{\partial x}(L_i, y) + \nu\int_{\mathbf{R}} v\tfrac{\partial v}{\partial x}(l_i, y)$$

From the boundary conditions we have,

$$0 = \int\int cv^2 + \nu|\nabla v|^2 + \nu\int_{\mathbf{R}} v(L_i, y)(\frac{a}{2\nu}v - \Lambda_j^-(v))\,dy - \nu\int_{\mathbf{R}} v(l_i, y)(\frac{a}{2\nu}v - \Lambda_j^+(v))\,dy$$

Since $Re(\frac{a}{2\nu} - \lambda_j^-) > 0$ and $Re(\lambda_j^+ - \frac{a}{2\nu}) > 0$, the integrals over $\mathbf{R}$ are positive and we have thus $\nabla v = 0$. Thus, v is constant. Since $v \in H^2(\Omega_i)$, $v = 0$.

Existence We want to find $v \in H^2(\Omega_i)$ solution to (12). We seek $v = w + u_{|\Omega_i}$ with w in the form

$$\hat{w} = \alpha(k)e^{\lambda^-(k)(x-l_i)} + \beta(k)e^{\lambda^+(k)(x-L_i)} \tag{13}$$

where

$$\lambda^+(k) = \frac{a + \sqrt{a^2 + 4\nu c - 4i\nu bk + 4k^2\nu^2}}{2\nu},\ Re(\lambda^+) > 0 \\ \lambda^-(k) = \frac{a - \sqrt{a^2 + 4\nu c - 4i\nu bk + 4k^2\nu^2}}{2\nu},\ Re(\lambda^-) \leq 0 \tag{14}$$

so that $\mathcal{L}(\mathcal{F}^{-1}(e^{\lambda\pm(k)(x-l_i)})) = 0$ and u is the solution to (1). From the boundary conditions at $x = L_i$ and at $x = l_i$, we have:

$$\alpha(\lambda^- - \lambda_j^-)e^{\lambda^-(k)(L_i - l_i)} + \beta(k)(\lambda^+ - \lambda_j^-) = \hat{g}^- - \mathcal{F}_y((\partial_x - \Lambda_j^-)(u)) \equiv h^-$$

and

$$\alpha(\lambda^- - \frac{a}{\nu} + \lambda_j^-) + \beta(k)(\lambda^+ - \frac{a}{\nu} + \lambda_j^-)e^{\lambda^+(k)(l_i - L_i)} = \hat{g}^+ - \mathcal{F}((\partial_x - \Lambda_j^+)(u)) \equiv h^+,$$

$\hat{h}^\pm \in H^{1/2-\sigma(j)}(\mathbf{R})$. The linear system in α, β is uniquely solvable iff (we have used $\lambda^+ = \frac{a}{\nu} - \lambda^-$)

$$\triangle = -(\lambda^- - \lambda_j^-)^2 e^{(\lambda^- - \lambda^+)(L_i - l_i)} + (\lambda^- - \frac{a}{\nu} + \lambda_j^-)^2 \neq 0$$

iff

$$\left(\frac{\lambda^- - \lambda_j^-}{\lambda^- - \frac{a}{\nu} + \lambda_j^-}\right)^2 \neq e^{(\lambda^+ - \lambda^-)(L_i - l_i)}$$

One one hand, we have $Re(\lambda^+ - \lambda^-) \geq \frac{1}{\nu}\sqrt{a^2 + 4\nu c}$ so that $|e^{-(\lambda^- - \lambda^+)(L_i - l_i)}| > 1$. On the other hand we prove below in Lemma 3.2 that $|\frac{\lambda^- - \lambda_j^-}{\lambda^- - \frac{a}{\nu} + \lambda_j^-}| < 1$. It is then clear that $\triangle \neq 0$.

Lemma 3.2 *a) The quantity* $\rho_j(k) = \dfrac{\lambda^- - \lambda_j^-}{\lambda^- - \frac{a}{\nu} + \lambda_j^-}$ *satisfies*

$$|\rho_j(k)| < 1, \quad \forall k \in \mathbf{R}.$$

b) Moreover there exist two positive constants $\eta_c > 0$ *and* $d > 0$ *only depending on* $\frac{\nu c}{a^2}$ *and* $\frac{b}{a}$ *so that*

$$|\rho_j(k)| < 1 - \frac{d}{\frac{\nu|k|}{a}}, \quad \forall k, \ \frac{\nu|k|}{a} \geq \eta_c.$$

Proof We first write

$$\rho_j(k) = \frac{1 - \frac{\lambda_j^- - \frac{a}{2\nu}}{\lambda^- - \frac{a}{2\nu}}}{1 - \frac{\lambda_j^- + \frac{a}{2\nu}}{\lambda^- - \frac{a}{2\nu}}}$$

and we notice the relationship

$$1 - \left|\frac{1+z}{1-z}\right|^2 = \frac{4Re(z)}{|1+z|^2} = \frac{4Re(1/z)}{|1+1/z|^2}, \quad \forall z \in \mathbf{C}.$$

a) One easily checks in every case ($j = 0, 1, 2$) $Re(\lambda^- - \frac{a}{2\nu}) < 0$, $Re(\lambda_j^- - \frac{a}{2\nu}) < 0$ and $sgn(Im(\lambda^- - \frac{a}{2\nu}) = sgn(Im(\lambda_j^- - \frac{a}{2\nu}) = sgn(bk)$, which implies $Re(\frac{\lambda_j^- - \frac{a}{2\nu}}{\lambda^- - \frac{a}{2\nu}}) > 0$.
b) The quantities $\frac{2\nu}{a}(\lambda^- - \frac{a}{2\nu})$ and $\frac{2\nu}{a}(\lambda_j^- - \frac{a}{2\nu})$ can be expressed in terms of $\beta = \frac{b}{a}$, $\gamma = \frac{\nu c}{a^2}$ and $\eta = \frac{\nu k}{a}$. Straightforward calculations give asymptotics as $|\eta| \to \infty$:

$$\frac{\lambda_0^- - \frac{a}{2\nu}}{\lambda^- - \frac{a}{2\nu}} = \frac{\sqrt{1+4\gamma}}{2|\eta|} + O_{\beta\gamma}(\frac{1}{\eta^2}),$$

$$\frac{\lambda_1^- - \frac{a}{2\nu}}{\lambda^- - \frac{a}{2\nu}} = \left(\frac{\sqrt{1+4\gamma}}{2|\eta|} + \frac{\beta^2}{2\sqrt{1+4\gamma}\|\eta\|}\right) - i\frac{\beta sgn(\eta)}{\sqrt{1+4\gamma}} + O_{\beta\gamma}(\frac{1}{\eta^2}),$$

and

$$\left(\frac{\lambda_2^- - \frac{a}{2\nu}}{\lambda^- - \frac{a}{2\nu}}\right)^{-1} = \frac{\sqrt{1+4\gamma}}{\left(1+\frac{\beta^2}{1+4\gamma}\right)|\eta|} + O_{\beta\gamma}(\frac{1}{\eta^2}),$$

which yields the result.□

Thus, we get the following expressions for α and β:

$$\begin{aligned} \alpha &= \frac{(\lambda^- - \frac{a}{\nu} + \lambda_j^-)\hat{h}^+ + (\lambda_j^- - \lambda^-)e^{\lambda^+(l_i - L_i)}\hat{h}^-}{-(\lambda^- - \lambda_j^-)^2 e^{(\lambda^- - \lambda^+)(L_i - l_i)} + (\lambda^- - \frac{a}{\nu} + \lambda_j^-)^2} \\ \beta &= \frac{(\lambda^- - \frac{a}{\nu} + \lambda_j^-)\hat{h}^- + (\lambda_j^- - \lambda^-)e^{\lambda^-(L_i - l_i)}\hat{h}^+}{(\lambda^- - \lambda_j^-)^2 e^{(\lambda^- - \lambda^+)(L_i - l_i)} - (\lambda^- - \frac{a}{\nu} + \lambda_j^-)^2} \end{aligned} \tag{15}$$

It is easy to check that $w \in H^2(\Omega_i)$ by using Parseval-Plancherel equality and the expression of the Fourier transform of w. For instance, we have

$$\begin{aligned} &\int_{l_i}^{L_i} \int_{\mathbf{R}} (1+k^2)^2 \, |\hat{w}(x,k)|^2 dx \, dk \\ &\leq 2\int_{\mathbf{R}} |\alpha|^2(k)(1+k^2)^2 \frac{|e^{2Re(\lambda^-)(L_i - l_i)} - 1|}{-2\lambda^-} dk + 2\int_{\mathbf{R}} |\beta|^2(k)(1+k^2)^2 \frac{|-e^{2Re(\lambda^+)(l_i - L_i)} + 1|}{2\lambda^+} dk \\ &\leq 4\int_{\mathbf{R}} \left(|\hat{h}^+|^2 \frac{|\lambda^+ - \frac{a}{\nu} + \lambda_j^-|^2}{|\triangle|^2} + |\hat{h}^-|^2 \frac{|\lambda^+ - \lambda_j^-|^2 e^{2Re(\lambda^+)(l_i - L_i)}}{|\triangle|^2} \right) (1+k^2)^2 \frac{|e^{2Re(\lambda^-)(L_i - l_i)} - 1|}{-2\lambda^-} dk \\ &+4\int_{\mathbf{R}} \left(|\hat{h}^-|^2 \frac{|\lambda^+ - \frac{a}{\nu} + \lambda_j^-|^2}{|\triangle|^2} + |\hat{h}^+|^2 \frac{|\lambda^+ - \lambda_j^-|^2 e^{2Re(\lambda^-)(L_i - l_i)}}{|\triangle|^2} \right) (1+k^2)^2 \frac{|-e^{2Re(\lambda^+)(l_i - L_i)} + 1|}{2\lambda^+} dk \end{aligned}$$

If $j = 0$ or 1, we know that $|\hat{h}^\pm|^2(1+k^2)^{1/2} \in L^1(\mathbf{R})$. The boundedness of the integrals is thus proved if the following functions from $\mathbf{R}$ to $\mathbf{R}$ are bounded:

$$\begin{aligned} \phi_1 &: k \to \frac{|\lambda^+ - \frac{a}{\nu} + \lambda_j^-|^2}{|\triangle|^2}(1+k^2)^{3/2} \frac{|e^{2Re(\lambda^-)(L_i - l_i)} - 1|}{-2\lambda^-} \\ \phi_2 &: k \to \frac{|\lambda^+ - \lambda_j^-|^2 e^{2Re(\lambda^+)(l_i - L_i)}}{|\triangle|^2}(1+k^2)^{3/2} \frac{|e^{2Re(\lambda^-)(L_i - l_i)} - 1|}{-2\lambda^-} \\ \phi_3 &: k \to \frac{|\lambda^+ - \frac{a}{\nu} + \lambda_j^-|^2}{|\triangle|^2}(1+k^2)^{3/2} \frac{|-e^{2Re(\lambda^+)(l_i - L_i)} + 1|}{2\lambda^+} \\ \phi_4 &: k \to \frac{|\lambda^+ - \lambda_j^-|^2 e^{2Re(\lambda^-)(L_i - l_i)}}{|\triangle|^2}(1+k^2)^{3/2} \frac{|-e^{2Re(\lambda^+)(l_i - L_i)} + 1|}{2\lambda^+} \end{aligned}$$

We study the behavior of these functions as k tends to infinity. For k large, $|\lambda^\pm|$ behaves as $|k|$, $|\lambda^+ - \frac{a}{\nu} + \lambda_j^-|^2$ and $|\lambda^+ - \lambda_j^-|^2$ as k^2 and $|\triangle|^2$ as k^4. Thus the functions $\phi_l(k)$ go to constants as k tends to infinity. Since they are continuous, they are bounded. This proves that w, w_y and w_{yy} are in $L^2(\Omega_i)$.

If $j = 2$, we know that $|\hat{h}^{\pm}|^2(1+k^2)^{-1/2} \in L^1(\mathbf{R})$. The boundedness of the integrals is thus proved if the functions $\psi_l(k)$, $1 \leq l \leq 4$, formally equal to $(1+k^2)\phi_l(k)$ with $\lambda_j = \lambda_2$, are bounded. We study the behavior of these functions as k approaches infinity. For k large, $|\lambda^+ - \frac{a}{\nu} + \lambda_2^-|^2$ and $|\lambda^+ - \lambda_2^-|^2$ behave as k^4 and $|\triangle|^2$ as k^8. Thus the functions $\psi_l(k)$ converge to constants as k tends to infinity. Since they are continuous, they are bounded. This proves that w, w_y and w_{yy} are in $L^2(\Omega_i)$.

With similar arguments, one checks that w_x and w_{xx} belong to $L^2(\Omega_i)$, $1 \leq i \leq N-1$ for any $j = 0, 1, 2$. Now, let $v = u_{|\Omega_i} + w$ with w defined by (13)-(15), it is clear that $v \in H^2(\Omega_i)$ is solution to (12) and that v depends continuously on $g^{\pm}$.

Case $i = N$: Again due to the positive orientation of the drift term, one cannot hope here estimates in $H^{2+m}(\Omega_N)$. That is the reason why the spaces $H_{st}^{2+m}(\Omega_N)$, which include only a local control of the L^2-norm, were introduced.

Uniqueness The proof is based on the

Lemma 3.3 *Let v be a solution to (12) with $f = 0$ and $g^+ = 0$ and let $w = v\, e^{-\frac{a}{2\nu}(x-l_N)}$. Then, w has the following properties*
i) $w \in H^2(\Omega_N)$
ii) w satisfies $(\frac{a^2}{4\nu} + c)\, w + b\, w_y - \nu \Delta w = 0$ in Ω_N.
iii) $w_x - (\Lambda_j^+ - \frac{a}{2\nu})w = 0$ at $x = l_N$.

Proof *i)* Let $L > l_N$, we have

$$\begin{aligned}\int_{l_N}^{L}\int_{\mathbf{R}} w^2 \;&= \int_{l_N}^{L}\int_{\mathbf{R}} v^2 e^{-\frac{a(x-l_N)}{\nu}} = \int_{l_N}^{L}\int_{\mathbf{R}} -\frac{\nu}{a} v^2 \partial_x (e^{-\frac{a(x-l_N)}{\nu}}) \\ &= \int_{l_N}^{L}\int_{\mathbf{R}} \frac{2\nu}{a} v v_x e^{-\frac{a(x-l_N)}{\nu}} - \frac{\nu}{a}\int \left[v^2 e^{-\frac{a(x-l_N)}{\nu}}\right]_{l_N}^{L}\end{aligned}$$

By Cauchy-Schwarz we have

$$\int_{l_N}^{L}\int_{\mathbf{R}} w^2 \leq \frac{2\nu}{a}\left(\int_{l_N}^{L}\int_{\mathbf{R}} w^2\right)^{1/2}\left(\int_{l_N}^{L}\int_{\mathbf{R}} (v_x)^2 e^{-\frac{a(x-l_N)}{\nu}}\right)^{1/2} + \frac{\nu}{a}\int v^2(l_N, y)\, dy$$

so that

$$\int_{l_N}^{L}\int_{\mathbf{R}} w^2 \leq \frac{2\nu}{a}\left(\int_{l_N}^{L}\int_{\mathbf{R}} w^2\right)^{1/2} \|v_x\|_{L^2(\Omega_N)} + \frac{\nu}{a}\int v^2(l_N, y)\, dy.$$

The integral of w^2 over $]l_N, L[\times\mathbf{R}$ is thus bounded independently of L. This proves that $w \in L^2(\Omega_N)$ and, since $w \in H_{st}^2(\Omega_N)$ like v, that $w \in H^2(\Omega_N)$.
Properties *ii)* and *iii)* are obtained by substituting $w\, e^{\frac{a(x-l_N)}{2\nu}}$ for v in (12).□

Uniqueness is now easy to prove. We multiply by w the equation it satisfies and we integrate by parts over Ω_N to obtain

$$\begin{array}{l} 0 = \int\int (\frac{a^2}{4\nu} + c)w^2 + \nu|\nabla w|^2 + \nu \int_{x=l_N} w(l_N, y)\frac{\partial w}{\partial x}(l_N, y) \\ = \int\int (\frac{a^2}{4\nu} + c)w^2 + \nu|\nabla w|^2 + \nu \int_{x=l_N} w(\Lambda^+ - \frac{a}{2\nu})(w) \end{array}$$

so that $w = 0$ and thus $v = 0$.

Existence We want to find $v \in H^2(\Omega_N)$ solution to (12). We seek $v = w + u_{|\Omega_N}$ with w in the form $\hat{w} = \alpha(k)e^{\lambda^-(k)(x-l_i)} + \beta(k)e^{\lambda^-(k)(x-L_i)}$. Since Ω_N is not bounded in the x direction, the only possibility to have w_x in $L^2(\Omega_N)$ is to take $\beta = 0$. The boundary condition at $x = l_N$ gives:

$$\alpha = \frac{\hat{h}^+}{\lambda^+ - \frac{a}{\nu} + \lambda_j^-}$$

It can be checked easily that $w \in H^2_{st}(\Omega_N)$. The need for introducing $H^2_{st}(\Omega_N)$ comes from the fact that for $c = 0$, w is not necessarily in $L^2(\Omega_N)$ since $\lambda^-(0) = 0$.□

4 Reformulation of the algorithm

Let $e_i^n = u_i^n - u$, since the equations and the algorithms are linear e_i^n satisfies the same algorithms as u_i^n with $f = 0$. We have to prove the convergence to zero of the sequence e_i^n. In order to introduce a zero order term in the equation, we define $v_i^n(x, y) = e^{\frac{-a}{2\nu}x}e_i^n(x, y)$. The derivative ∂_x is changed into $\partial_x + \frac{a}{2\nu}$, the operator $c + a\partial_x + b\partial_y - \nu\Delta$ into

$$\mathcal{L}_a = (c + \frac{a^2}{4\nu}) + b\partial_y - \nu\Delta$$

and the boundary conditions by subtracting $\frac{a}{2\nu}$ to $\Lambda_j^{\pm}$. We note that $\Lambda_j^+ - \frac{a}{2\nu} = -(\Lambda_j^- - \frac{a}{2\nu})$ and we set $\Lambda_j = \Lambda_j^+ - \frac{a}{2\nu} = -(\Lambda_j^- - \frac{a}{2\nu})$. As an example the ASM now writes

$$\left\{ \begin{array}{l} \mathcal{L}_a(v_i^{n+1}) = 0,\ v_i^{n+1} \in H^2(\Omega_i)\ 1 \le i \le N \\ (\partial_x - \Lambda_j)(v_i^{n+1}) = (\partial_x - \Lambda_j)(v_{i-1}^n) \text{ at } x = l_i,\ 2 \le i \le N \\ (\partial_x + \Lambda_j)(v_i^{n+1}) = (\partial_x + \Lambda_j)(v_{i+1}^n) \text{ at } x = L_i,\ 1 \le i \le N-1 \end{array} \right. \tag{16}$$

We introduce some notations. Let $\Gamma_{i,l} = \{l_i\} \times \mathbf{R}$ and $\Gamma_{i,r} = \{L_i\} \times \mathbf{R}$ be the boundaries of subdomain Ω_i, $\vec{n}_{i,r}$ (resp. $\vec{n}_{i,l}$) is the outward normal from Ω_i on $\Gamma_{i,r}$ (resp. $\Gamma_{i,l}$). We set

$$h_{i,r}^n = (\frac{\partial}{\partial \vec{n}_{i,r}} + \Lambda_j)(v_i^n) \text{ on } \Gamma_{i,r} \text{ for } 1 \le i \le N-1$$

and

$$h^n_{i,l} = (\frac{\partial}{\partial \vec{n}_{i,l}} + \Lambda_j)(v^n_i) \text{ on } \Gamma_{i,l} \text{ for } 2 \leq i \leq N$$

The relationship between the norms of $h^n_{i,r \text{ or } l}$ and the traces of e^n_i come at once from $v^n_i = e^{-\frac{a}{2\nu}x} e^n_i$.

Lemma 4.1 *With the above definitions, we have*

$$||h^n_{i,r}||_{H^{1/2-\sigma(j)}(\mathbf{R})} = e^{-\frac{a\,L_i}{2\nu}} ||(\partial_x - \Lambda^-_j)(e^n_i)(L_i, .)||_{H^{1/2-\sigma(j)}(\mathbf{R})} \text{ for } 1 \leq i \leq N-1$$

and

$$||h^n_{i,l}||_{H^{1/2-\sigma(j)}(\mathbf{R})} = e^{-\frac{a\,l_i}{2\nu}} ||(\partial_x - \Lambda^+_j)(e^n_i)(l_i, .)||_{H^{1/2-\sigma(j)}(\mathbf{R})}. \text{ for } 2 \leq i \leq N$$

We consider the operator S_i, $1 \leq i \leq N$ defined by $S_i(h_l, h_r) = v$ (forget h_l for $i = 1$ and h_r for $i = N$) with

$$\begin{aligned} \mathcal{L}_a(v) &= 0 \text{ in } \Omega_i \\ (\frac{\partial}{\partial \vec{n}_{i,l}} + \Lambda_j)(v) &= h_l \text{ on } \Gamma_{i,l} \ (2 \leq i \leq N) \\ (\frac{\partial}{\partial \vec{n}_{i,r}} + \Lambda_j)(v) &= h_r \text{ on } \Gamma_{i,r} \ (1 \leq i \leq N-1) \ . \end{aligned} \tag{17}$$

Similarly to what we did for Theorem 3.1 one easily checks the

Lemma 4.2 *The operators S_i, $1 \leq i \leq N$ are continuous from $H^{1/2-\sigma(j)}(\Gamma_{i,l}) \times H^{1/2-\sigma(j)}(\Gamma_{i,r})$ into $H^2(\Omega_i)$ (forget $H^{1/2-\sigma(j)}(\Gamma_{1,l})$ for $i = 1$ and $H^{1/2-\sigma(j)}(\Gamma_{N,r})$ for $i = N$).*

Now, the ASM described in (8) writes

$$\begin{aligned} h^{n+1}_{2,l} &= (\tfrac{\partial}{\partial \vec{n}_{2,l}} + \Lambda_j) S_1(h^n_{1,r}) \\ h^{n+1}_{3,l} &= (\tfrac{\partial}{\partial \vec{n}_{3,l}} + \Lambda_j)(S_2(h^n_{2,l}, 0) + S_2(0, h^n_{2,r})) \\ &\vdots \\ h^{n+1}_{N,l} &= (\tfrac{\partial}{\vec{n}_{N,l}} + \Lambda_j)(S_{N-1}(h^n_{N-1,l}, 0) + S_{N-1}(0, h^n_{N-1,r})) \\ h^{n+1}_{N-1,r} &= (\tfrac{\partial}{\partial \vec{n}_{N-1,r}} + \Lambda_j) S_N(h^n_{N,l}) \\ h^{n+1}_{N-2,r} &= (\tfrac{\partial}{\partial \vec{n}_{N-2,r}} + \Lambda_j)(S_{N-1}(h^n_{N-1,l}) + S_{N-1}(0, h^n_{N-1,r})) \\ &\vdots \\ h^{n+1}_{1,r} &= (\tfrac{\partial}{\vec{n}_{1,r}} + \Lambda_j)(S_2(h^n_{2,l}) + S_2(0, h^n_{2,r})). \end{aligned} \tag{18}$$

We introduce some other notations in order to write the algorithm (8) (18) in a compact form. We set $H^{1/2-\sigma(j)}(\Gamma) = H^{1/2-\sigma(j)}(\Gamma_{2,l}) \times \ldots \times H^{1/2-\sigma(j)}(\Gamma_{N,l}) \times H^{1/2-\sigma(j)}(\Gamma_{1,r}) \ldots \times H^{1/2-\sigma(j)}(\Gamma_{N-1,r})$ so that the $2(N-1)$-uplets $(h_{2,l}, \ldots, h_{N,l}, h_{1,r}, \ldots, h_{N-1,r})$ belongs to $H^{1/2-\sigma(j)}(\Gamma)$ and we define the continuous operator $\mathcal{T} : H^{1/2-\sigma(j)}(\Gamma) \to H^{1/2-\sigma(j)}(\Gamma)$ by

$$\mathcal{T}(h_{2,l}, \ldots, h_{N,l}, h_{1,r}, \ldots, h_{N-1,r}) = \begin{bmatrix} (\frac{\partial}{\partial \vec{n}_{2,l}} + \Lambda_j) S_1(h_{1,r}) \\ (\frac{\partial}{\partial \vec{n}_{3,l}} + \Lambda_j)(S_2(h_{2,l}, 0) + S_2(0, h_{2,r})) \\ \vdots \\ (\frac{\partial}{\partial \vec{n}_{N,l}} + \Lambda_j)(S_{N-1}(h_{N-1,l}, 0) + S_2(0, h_{N-1,r})) \\ (\frac{\partial}{\partial \vec{n}_{1,r}} + \Lambda_j)(S_2(0, h_{2,r}) + S_2(h_{2,l}, 0)) \\ \vdots \\ (\frac{\partial}{\partial \vec{n}_{N-2,r}} + \Lambda_j)(S_{N-1}(0, h_{N-1,r}) + S_{N-1}(h_{N-1,l}, 0)) \\ (\frac{\partial}{\partial \vec{n}_{N-1,r}} + \Lambda_j) S_N(h_{N,l}, 0) \end{bmatrix}$$

The operator $\mathcal{T}$ can be written in the form of an operator valued matrix

$$\mathcal{T}(H) = \begin{bmatrix} 0 & & & 0 & \times & & & 0 \\ \times & \ddots & & & & \ddots & & \\ & \ddots & \ddots & & & & \ddots & \\ 0 & & \times & 0 & 0 & & & \times \\ \times & & & 0 & 0 & \times & & 0 \\ & \ddots & & & & \ddots & \ddots & \\ & & \ddots & & & & \ddots & \times \\ 0 & & & \times & 0 & & & 0 \end{bmatrix} \begin{bmatrix} h_{2,l} \\ \vdots \\ \vdots \\ h_{N,l} \\ h_{1,r} \\ \vdots \\ \vdots \\ h_{N-1,r} \end{bmatrix}$$

where the crosses correspond to non zero operators.

From (8), we see that the additive Schwarz method corresponds to a Jacobi algorithm:

$$H^{n+1} = \mathcal{T}(H^n) \tag{19}$$

Consider now the DSA (9)-(10) and the FDA (11). In order to write them in a compact form, we introduce four $(2N-2) \times (2N-2)$ operator valued matrices:

$$\mathcal{M}l = (\mathcal{M}l)_{1 \le m,\, n \le 2N-2} \text{ with } \mathcal{M}l_{mn} = \begin{cases} \mathcal{T}_{mn} & 1 \le m,\, n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{A}l = (\mathcal{A}l)_{1\leq m,\, n\leq 2N-2} \text{ with } \mathcal{A}l_{mn} = \begin{cases} \mathcal{T}_{mn} & 1 \leq m \leq N-1,\ N \leq n \leq 2N-2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{A}r = (\mathcal{A}r)_{1\leq m,\, n\leq 2N-2} \text{ with } \mathcal{A}r_{mn} = \begin{cases} \mathcal{T}_{mn} & N \leq m \leq 2N-2,\ 1 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{M}r = (\mathcal{M}r)_{1\leq m,\, n\leq 2N-2} \text{ with } \mathcal{M}r_{mn} = \begin{cases} \mathcal{T}_{mn} & N \leq m,\ n \leq 2N-2 \\ 0 & \text{otherwise} \end{cases}$$

so that we have $\mathcal{T} = \mathcal{M}l + \mathcal{A}l + \mathcal{M}r + \mathcal{A}r$. With these notations, we see the DSA (9)-(10) corresponds to the algorithm:

$$\begin{aligned} H^{n+1/2} &= (\mathcal{M}l + \mathcal{A}l)(H^{n+1/2}) + (\mathcal{A}r + \mathcal{M}r)(H^n) \\ H^{n+1} &= (\mathcal{M}l + \mathcal{A}l)(H^{n+1/2}) + (\mathcal{A}r + \mathcal{M}r)(H^{n+1}), \end{aligned}$$

that is

$$\begin{aligned} H^{n+1} &= (Id - \mathcal{A}r - \mathcal{M}r)^{-1}(\mathcal{M}l + \mathcal{A}l)(Id - \mathcal{M}l - \mathcal{A}l)^{-1}(\mathcal{A}r + \mathcal{M}r)\, H^n \\ &= (Id - \mathcal{A}r - \mathcal{M}r)^{-1}(Id - \mathcal{M}l - \mathcal{A}l)^{-1}(\mathcal{M}l + \mathcal{A}l)(\mathcal{A}r + \mathcal{M}r) = \mathcal{T}_{ds}\, H^n, \end{aligned}$$

while the FDA (11) corresponds to a Gauss-Seidel algorithm (see for instance [11], [1]):

$$H^{n+1} = (\mathcal{M}l + \mathcal{A}l)(H^{n+1}) + (\mathcal{A}r + \mathcal{M}r)(H^n),$$

that is

$$H^{n+1} = (Id - \mathcal{M}l - \mathcal{A}l)^{-1}(\mathcal{A}r + \mathcal{M}r)\, H^n = \mathcal{T}_{fl}\, H^n$$

The convergence of the algorithms is studied by making a Fourier analysis. The Fourier transform of the operators $\mathcal{T}, \mathcal{A}r, \mathcal{A}l, \mathcal{M}r, \mathcal{M}l, \mathcal{T}_{ds}$ and $\mathcal{T}_{fl}$ are matrices of symbols denoted by $T, Ar, Al, Mr, Ml, T_{ds}$ and T_{fl}. For sake of simplicity we suppose the overlap to be constant, $L_i - l_{i+1} = \delta,\ \forall i = 1, \ldots, N-1$. Indeed, such an assumption does not affect the validity of the following arguments. A straightforward computation derived from (13), (14) and (15) shows that the nonzero coefficients equal:

$$\begin{aligned} (Ar)_{n+N-1,n}(k) &= \frac{\rho_j(e^{-\lambda\delta} + e^{-\lambda(L_{n+1}-l_{n+1})})}{1 - {\rho_j}^2 e^{-2\lambda(L_{n+1}-l_{n+1})}}, \quad 1 \leq n \leq N-1 \\ (Al)_{n,n+N-1}(k) &= \frac{\rho_j(e^{-\lambda\delta} + e^{-\lambda(L_n-l_n)})}{1 - {\rho_j}^2 e^{-2\lambda(L_n-l_n)}}, \quad 1 \leq n \leq N-1 \\ (Ml)_{n+1,n}(k) &= \frac{e^{-\lambda(L_{n+1}-l_{n+1}-\delta)}(1 - {\rho_j}^2)}{1 - {\rho_j}^2 e^{-2\lambda(L_{n+1}-l_{n+1})}}, \quad 1 \leq n \leq N-2 \\ (Mr)_{n,n+1}(k) &= \frac{e^{-\lambda(L_{n-N+2}-l_{n-N+2}-\delta)}(1 - {\rho_j}^2)}{1 - {\rho_j}^2 e^{-2\lambda(L_{n-N+2}-l_{n-N+2})}}, \quad N \leq n \leq 2N-3 \end{aligned} \tag{20}$$

where $\lambda = \lambda(k) = \lambda^+(k) - \frac{a}{2\nu} = -(\lambda^-(k) - \frac{a}{2\nu})$ and $\rho_j = \rho_j(k)$ is defined in Lemma 3.2.

The purpose of the next section is to estimate the norms $||T^n||$, $||{T_{ds}}^n||$ and $||{T_{fl}}^n||$ in terms of the norms of Ar, Al, Mr and Ml by using the following important properties:

$$\begin{array}{lll} Mr^{N-1} = Ml^{N-1} = 0; & Ml\,Mr = Mr\,Ml = 0; & Al^2 = Ar^2 = 0 \\ Al\,Ml = Ar\,Mr = 0; & Ml\,Ar = Mr\,Al = 0 & \end{array} \tag{21}$$

It is worth noticing that these relations come from the structure of the matrices and do not depend on the value of the coefficients.

5 An algebraic trick

In this section, we prove the

Theorem 5.1 *Let $||.||$ be an algebra norm on $\mathcal{M}_{2N-2}(\mathbf{C})$, $||AB|| \leq ||A||\,||B||$, so that $||Id|| = 1$. For a $k \in \mathbf{R}$ we suppose*

$$\rho(k) = ||Ar(k)||\,||Al(k)||\left(\sum_{n=0}^{N-2}||Ml(k)||^n\right)\left(\sum_{n=0}^{N-2}||Mr(k)||^n\right) < 1.$$

Then the estimates

$$\begin{array}{ll} ||T^n(k)|| \leq C(k)\,\frac{1}{1-\rho(k)}\,\rho(k)^{\left[\frac{n}{2N-2}-\frac{3}{2}\right]}, & \forall n \geq 2N \\ ||{T_{ds}}^n(k)|| \leq C(k)\,(1+\rho(k))\rho(k)^{n-1}, & \forall n \geq 2 \\ ||{T_{fl}}^n(k)|| \leq C(k)\,\frac{1}{1-\rho(k)}\,\rho(k)^{\left[\frac{n}{N-1}\right]-2}, & \forall n \geq 2(N-1) \end{array}$$

hold with $C(k) = \left(1 + \frac{\rho(k)}{||Al(k)||}\right)\left(1 + \frac{\rho(k)}{||Ar(k)||} + \frac{\rho(k)}{||Ar(k)||||Al(k)||}\right)$

Remark 5.2 *Note that the expressions $\frac{\rho(k)}{||Al(k)||}$, $\frac{\rho(k)}{||Ar(k)||}$ and $\frac{\rho(k)}{||Al(k)||||Ar(k)||}$ make sense even if $||Ar(k)|| = 0$ or $||Al(k)|| = 0$. Actually such a case is interesting. It corresponds to the situation $\lambda_j = \lambda$ where the algorithms are implemented with the exact absorbing boundary conditions. Then the above result can be improved: It comes from the form of T, T_{ds} and T_{fl} that $T^{N-1} = 0$, $T_{ds} = 0$ and $T_{fl}^{N-1} = 0$.*

Remark 5.3 *Notice that the convergence rate equals $\rho(k)^{\frac{1}{2(N-1)}}$ for ASM and $\left(\rho(k)^{\frac{1}{2(N-1)}}\right)^2$ for FDA. This relationship often occurs in the comparison of Jacobi and Gauss-Seidel algorithms (see [1]).*

As we said, this theorem is a consequence of the nilpotency relationships (21). It is a purely combinatorial result. Its proof requires some techniques of formal language theory.

5.1 An incursion in formal language theory

Part of the notions introduce here are standard. We refer the interested reader to [8] for further details and applications.
First we recall that a monoid (or semigroup) is a set M endowed with an associative binary law: $(m, m') \in M \times M \to m.m' \in M$ and a neutral element 1_M. A morphism $\phi : (M, .) \to (N, .)$ is an application satisfying $\phi(m\, m') = \phi(m)\phi(m')$ and $\phi(1_M) = 1_N$. A submonoid of $(M, .)$ is a subset P, $1_M \in P \subset M$, so that $(P, ._{|P\times P})$ is a monoid. The binary operation . can be extended to the subsets of M by setting

$$XY = \{xy / x \in X,\, y \in Y\}.$$

It is clear that a subset $N \subset M$ is a submonoid iff $1_M \in N$ and $NN \subset N$. If X is a subset of M, there exists a unique submonoid N of M, minimal for set inclusion: Indeed $N = \cap_{X\subset P, P \text{ submonoid}} P$. We say that X is a generator of N or that N is generated by X. With any set A one can canonically attach a monoid A^* of which A is a generator. It is defined as the set of finite sequences $w = a_1\, a_2 \ldots a_n$, $a_i \in A$, endowed with the concatenating operation

$$(a_1\, a_2 \ldots a_n)(b_1\, b_2 \ldots b_m) = a_1\, a_2 \ldots a_n\, b_1\, b_2 \ldots b_m$$

and completed with a neutral element 1, $1.w = w.1 = w$. Conventionally, the elements of A are called letter, the elements of A^* words, 1 the empty word, A^* the monoid of words on A and A an alphabet. For any submonoid P of A^* there exists a unique generating set X_P minimal for set inclusion. Indeed one easily checks

$$X_P = (P - 1) - (P - 1)^2.$$

A monoid M is generated by a subset X iff there exists a morphism ϕ_X from X^* onto M so that $\phi_X(x) = x$, $\forall x \in X$ (this morphism is clearly uniquely defined). In view of this property, we introduce the following notion

Definition 5.4 *A monoid M is said to be free if there exists an alphabet B so that B^* and M are isomorphic.*

The next result comes at once from the definitions and the above remarks.

Proposition 5.5 *a) A monoid M is free iff there is a generating set X so that the morphism $\phi_X : X^* \to M$ is one to one. In such a case X is the unique generating set M minimal for set inclusion, denoted by X_M.*

b) Let P be a submonoid of A^* and X_P be its minimal generating set. Then P is free iff any equality

$$x_1 x_2 \dots x_n = y_1 y_2 \dots y_m, \qquad n, m \geq 0,\ x_i,\ y_j \in X$$

implies $n = m$ and $x_i = y_i,\ 1 \leq i \leq n$.

Note that not every submonoid of a free monoid is free. Consider as an example the submonoid of $\{x, y, z\}^*$ generated by $X = \{xy, yz, zx, xyz\}$: X is clearly the minimal generating set while we have $(xyz)(xyz) = (xy)(zx)(yz)$. The minimal generating set X_M of a free monoid of M is also called the basis of M. In the sequel, if a submonoid P of A^* is free, we always identify it with X_P^*.

Definition 5.6 *Let M be a free monoid M and let X be a subset of its basis X_M. We set for any $m \in M$,*

$$|m|_X = |x_1 \dots x_n|_X = \sum_{i=1}^{n} 1_X(x_i)$$

where $m = x_1 \dots x_n$, $x_i \in X_M$, and 1_X is the characteristic function of $X \subset X_M$.

This definition makes sense because the decomposition $m = x_1 \dots x_n,\ x_i \in X_M$ is unique. As an example when $X = X_M$, $|m|_{X_M}$ counts the number of factors in X_M the element m is made of. When $w \in A^*$, $|w|_A$ is nothing but the length of the word w (if the context is clear we simply write $|w|$) while $|w|_{\{a\}}$ with $a \in A$ counts the occurrences of the letter a in the word w.

Next we only work with monoids of words. Let $A = \{a_1, \dots, a_M\}$ be a finite alphabet and let B be a Banach algebra with unity endowed with a norm $||.||$ so that $||1||=1$. For a given finite sequence $(A_i)_{1 \leq i \leq M}$ in B, we consider the multiplicative morphism $mop : A^* \to B$ defined by $mop(a_i) = A_i,\ 1 \leq i \leq M$.

Definition 5.7 *For every subset P of A^*, we define its weight $||P|| \in [0, \infty]$ by*

$$||P|| = \sum_{m \in P} ||mop(m)||$$

Lemma 5.8 *a) Let P_1 and P_2 be two subsets of A^*, we have:*
$P_1 \subset P_2 \Longrightarrow ||P_1|| \leq ||P_2||$
and $||P_1.P_2|| \leq ||P_1||\,||P_2||$.
b) If P is the basis of a free submonoid $P^* \subset A^*$, then we have
$||\{m \in P^* / \, |m|_P = n\}|| \leq ||P||^n,\ \forall n \in \mathbf{N}$.

Proof The first assertion is obvious.For the second one, we write

$$\begin{aligned} ||P_1.P_2|| &\leq \sum_{m_1\in P_1,\, m_2\in P_2} ||mop(m_1.m_2)|| \\ &\leq \sum_{m_1\in P_1,\, m_2\in P_2} ||mop(m_1)||\,||mop(m_2)|| = ||P_1||\,||P_2||. \end{aligned}$$

b) For $n=0$, the inequality is satisfied since $||1||=1$. Hence the property is a consequence of a) and

$$\{m\in P^*/\ |m|_P = n+1\} = \{m\in P^*/\ |m|_P = n\}P.\square$$

5.2 Application

We consider an alphabet with four letters $A=\{a_r,\, a_l,\, m_r,\, m_l\}$ and the following subsets of A^*:

$W = \{m\in A^*; m_r^{N-1}, m_l^{N-1}, a_r^2, a_l^2, m_rm_l, m_lm_r,$
m_ra_l, m_la_r, a_rm_r and a_lm_l are not factors of m
and $|m|_{\{a_r,a_l\}}\geq 2\}$,

$W_1 = \{m_r^i a_r/\ 0\leq i,j\leq N-2\}\cup\{1\}$
$W_2 = \{m_l^i a_l m_r^j/\ 0\leq j\leq N-2\}\cup\{m_l^i/\ 1\leq i\leq N-2\}\cup\{1\}$

and $P=\{m_l^i a_l m_r^j a_r/\ 0\leq i,j\leq N-2\}$.
(We say that v is a factor of $m\in A^*$ if there exist $u,\, w\in A^*$ so that $m=uvw$.) It is clear from Proposition 5.5 that P is the basis of a free submonoid of A^*, identified with P^*.

Lemma 5.9 *With the above notations, we have*

$$W = \{m\in W_1.P^*.W_2,\ |m|_{\{a_l,a_r\}}\geq 2\}. \tag{22}$$

Proof A careful examination of the definition of W shows that a word $m\in W$ is nothing but a finite alternated sequence of a_l and a_r with at least two $a_{r\text{ or }l}$ and with factors of the form m_l^i (resp. m_r^i), $1\leq i\leq N-2$, inserted before an a_r (resp. a_l). For a word $m\in W$ we call $fa(m)$ the first element of $\{a_r,\, a_l\}$ occurring in m and $la(m)$ the last one. The result comes at once by discussing the four possibilities $fa(m)=a_{r\text{ or }l}$, $la(m)=a_{r\text{ or }l}$ ($|m|_{\{a_r,\,a_l\}}\geq 2$).

According to the preceding paragraph we consider the multiplicative morphism mop : $A^*\to\mathcal{M}_{2N-2}(\mathbf{C})$ with the norm of Theorem 5.1 defined by $mop(a_{r\text{ or }l}) = A_{r\text{ or }l}(k)$ and $mop(m_{r\text{ or }l}) = M_{r\text{ or }l}(k)$. The weights of the sets W_1, W_2 and P are estimated by

$$||W_1||\leq 1+\frac{\rho(k)}{||A_l(k)||},\ ||W_2||\leq 1+\frac{\rho(k)}{||A_r(k)||}+\frac{\rho(k)}{||A_r(k)||\,||A_l(k)||} \text{ and } ||P||=\rho(k)$$

while Lemma 5.8 yields

$$||\{m \in P^*,\ |m|_P = n\}|| \leq \rho(k)^n$$
$$\text{and } ||\{m \in P^*,\ |m|_P \geq n\}|| \leq \frac{1}{1-\rho(k)}\rho(k)^n,\ \forall n \in \mathbf{N}$$

We now close the
Proof of Theorem 5.1 a) We first consider

$$T^n = (Ar + Mr + Al + Ml)^n = \sum_{m \in A^*,\, |m|=n} mop(m)$$

with $n \geq 2N$. By eliminating the vanishing terms we get exactly ($mop(m) \neq 0$ and $|m| \geq 3(N-1)$ imply $|m|_{\{al,ar\}} \geq 2$)

$$T^n = \sum_{m \in W,\, |m|=n} mop(m)$$

which implies $||T^n|| \leq ||\{m \in W, |m| = n\}||$. Since $W \subset W_1 P^* W_2$ with $|w_1| \leq N-1$ for any $w_1 \in W_1$ and $|w_2| \leq 2N-3$ for any $w_2 \in W_2$, we have

$$\{m \in W,\ |m| = n\} \subset W_1\{m \in P^*,\ |m| \geq n - (3N-4)\}W_2.$$

By noticing that the maximal length of a word in P is $2N-2$, we obtain $|m|_P \geq \frac{|m|}{2N-2}$ for any $m \in P^*$ and

$$\{m \in W,\ |m| = n\} \subset W_1\{m \in P^*,\ |m|_P \geq \frac{n}{2N-2} - \frac{3}{2}\}W_2.$$

Hence the first estimate of Theorem 5.1 is an outcome of Lemma 5.8.
b) For T_{ds}, we first remark the identities,

$$(Id - Ar - Mr)^{-1} = \sum_{i=0}^{N-2} Mr^i(Id + Ar),\quad (Id - Al - Ml)^{-1} = \sum_{i=0}^{N-2} Ml^i(Id + Al)$$

and $(Ml + Al)(Mr + Ar) = Al(Mr + Ar)$ which lead to

$$T_{ds} = \sum_{i=0}^{N-2} Mr^i(Id + Ar) \sum_{i=0}^{N-2} Ml^i\, Al(Mr + Ar).$$

It can be checked easily that there exists $W_{ds} \subset W_1.P^*.W_2$ such that $T_{ds} = \sum_{m \in W_{ds}} mop(m)$ and that the submonoid of A^* generated by W_{ds} is free with W_{ds} (locate the letter a_l). We have

$$T_{ds}^n = \sum_{m \in W_{ds}^*, |m|_{W_{ds}} = n} mop(m).$$

By noticing that $|m|_{\{a_l\}} = 1$ when $m \in W_{ds}$, we obtain for $n \geq 2$

$$\{m \in W_{ds}^* / \; ||m|| \neq 0 \text{ and } |m|_{W_{ds}} = n\} \subset \{m \in W_{ds}^* \cap W / \; |m|_{W_{ds}} = n\} \subset \{m \in W / \; |m|_{\{a_l\}} = n\}$$

which implies

$$||{T_{ds}}^n|| \leq ||\{m \in W_1.P^*.W_2 / \; |m|_{\{a_l\}} = n\}||.$$

By noticing that $|m|_{\{a_l\}} = 1$ when $m \in P$, $|m|_{\{a_l\}} = 1$ when $m \in W_1$ and $|m|_{\{a_l\}} = 0$ when $m \in W_2$, we obtain for $n \geq 2$

$$||{T_{ds}}^n|| \leq ||W_1||\,||W_2||\,||\{m \in P^* / n - 1 \leq |m|_P \leq n\}||$$

and we conclude by using Lemma 5.8.

c) Let us first remark that

$$T_{fl} = \sum_{i=0}^{N-2} Ml^i\,(Id + Al)(Mr + Ar).$$

It can be checked easily that there exists $W_{fl} \subset W_1.P^*.W_2$ so that $T_{fl} = \sum_{m \in W_{fl}} mop(m)$ and that the submonoid of A^* generated by W_{fl} is free with basis W_{fl} (locate the element of $\{m_r,\, a_r\}$). We have

$$T_{fl}^n = \sum_{m \in W_{fl}^*, |m|_{W_{fl}} = n} mop(m).$$

By noticing that $|m|_{\{a_r, m_r\}} = 1$ when $m \in W_{fl}$, we obtain for $n \geq 2(N-1)$

$$\{m \in W_{ds}^* / \; ||m|| \neq 0 \text{ and } |m|_{W_{fl}} = n\} \subset \{m \in W / \; |m|_{W_{\{a_r, m_r\}}} = n\}$$

which implies

$$||{T_{fl}}^n|| \leq ||\{m \in W_1.P^*.W_2 / \; |m|_{\{a_r, m_r\}} = n\}||.$$

By noticing that $|m|_{\{a_r, m_r\}} \leq N - 1$ when $m \in P$, $|m|_{\{a_r, m_r\}} \leq N - 1$ when $m \in W_1$ and $|m|_{\{a_r, m_r\}} \leq N - 2$ when $m \in W_2$, we obtain for $n \geq 2(N-1)$

$$||{T_{fl}}^n|| \leq ||W_1||\,||W_2||\,||\{m \in P^* / n - 1 \leq |m|_P \geq \frac{n - (2N-3)}{N-1}\}||$$

and we conclude by using Lemma 5.8.

Remark 5.10 *Note that we never use in the proof the fact that $Ar(k)$, $Al(k)$, $Mr(k)$ and $Ml(k)$ are matrices. Indeed, Theorem 5.1 holds in any Banach algebra with unity if the nilpotency relationships (21) are satisfied for a finite N and $||1|| = 1$.*

6 Estimates of the convergence rate

On $\mathcal{M}_{2N-2}(\mathbf{C})$ we choose the norm $||A|| = \sup_{1\leq i\leq 2N-2} \sum_{1\leq j\leq 2N-2} |A_{ij}|$ which satisfies the assumptions of Theorem 5.1 and $||Au|| \leq ||A||\,||u||$ if the norm of the $(2N-2)$ vector u is defined by $||u|| = \sup_{1\leq i\leq 2N-2} |u_i|$. With such a norm, the quantities $||Al(k)||$, $||Ar(k)||$, $||Ml(k)||$ and $||Mr(k)||$ are easily derived from (20). We obtain

$$\rho(k) \leq |\rho_j|^2 (\frac{e^{-Re(\lambda)\delta} + e^{-Re(\lambda)L}}{1-|\rho_j|^2 e^{-2Re(\lambda)L}})^2 \left(\sum_{n=0}^{N-2} \left(\frac{|1-{\rho_j}^2| e^{-2Re(\lambda)(L-\delta)}}{1-|\rho_j|^2 e^{-2Re(\lambda)L}} \right)^n \right)^2$$

Lemma 6.1 *We set $L = \inf_i(L_i - l_i)$ and we assume $L/2 > \delta \geq 0$. There exists $\xi_c > 0$ only depending on $\frac{b}{a}$ and $\frac{\nu c}{a^2}$ so that*

$$\rho(k) < |\rho_j(k)|\, e^{-2\delta Re(\Lambda(k))},\ \forall k \in \mathbf{R},$$

as soon as $\frac{aL}{\nu} > \xi_c$.

Proof We set $\eta = \frac{\nu k}{a}$, $\xi = \frac{aL}{\nu}$ and we have $Re(\lambda) = Re(\lambda^+ - \frac{a}{2\nu}) > \frac{a}{\nu}\frac{\sqrt{1+4\eta^2}}{2}$. By referring to the assumption $L/2 > \delta$, we get

$$\frac{|1-{\rho_j}^2| e^{-2Re(\lambda)(L-\delta)}}{1-|\rho_j|^2 e^{-2Re(\lambda)L}} \leq \frac{2e^{-\sqrt{1+4\eta^2}\frac{\xi}{2}}}{1-e^{-\sqrt{1+4\eta^2}\xi}} \leq \frac{2e^{-\frac{\xi}{2}}}{1-e^{-\xi}}.$$

For ξ larger than some ξ_0, this quantity is smaller than 1 and $\rho(k)$ is estimated by

$$\begin{aligned} \sqrt{\rho(k)} &\leq |\rho_j|^{1/2}\, e^{-\delta Re(\lambda)} \left(|\rho_j|^{1/2} \frac{1+e^{-\sqrt{1+4\eta^2}\frac{\xi}{2}}}{1-e^{-\sqrt{1+4\eta^2}\xi}} \frac{1}{1-\frac{2e^{-\sqrt{1+4\eta^2}\frac{\xi}{2}}}{1-e^{-\sqrt{1+4\eta^2}\xi}}} \right) \\ &\leq |\rho_j|^{1/2}\, e^{-\delta Re(\lambda)} \left(|\rho_j|^{1/2} \frac{1+e^{-\sqrt{1+4\eta^2}\frac{\xi}{2}}}{1-e^{-\sqrt{1+4\eta^2}\xi}-2e^{-\sqrt{1+4\eta^2}\frac{\xi}{2}}} \right). \end{aligned} \tag{23}$$

We just have to check that $f_1(\eta,\xi) = \dfrac{1-e^{-\sqrt{1+4\eta^2}\xi} - 2e^{-\sqrt{1+4\eta^2}\frac{\xi}{2}}}{1+e^{-\sqrt{1+4\eta^2}\frac{\xi}{2}}}$ is larger than $f_2(\eta) = |\rho_j(\frac{a\eta}{\nu})|^{1/2}$ or equivalently that $g(\eta,\xi) = f_1(\eta,\xi) - f_2(\eta) > 0$ for $\xi > \xi_c$. By Lemma 3.2 b),

we can find $d' > 0$ and ${\eta'}_c > 0$ depending on $\frac{b}{a}$ and $\frac{\nu c}{a^2}$ so that $f_2(\eta) \leq 1 - \frac{d'}{|\eta|}$ while $f_1(\eta, \xi_0)$ approaches 1 exponentially as $\eta \to \infty$. Hence, there is an $\eta_0 = \eta_0(\frac{\nu c}{a^2}, \frac{b}{a})$ so that $g(\eta, \xi_0) > 0$ for $|\eta| > \eta_0$. Since $f_1(\eta, \xi)$ and consequently $g(\eta, \xi)$ are increasing functions of ξ for any fixed η, we have

$$g(\eta, \xi) > 0 \text{ and } \frac{f_2(\eta)}{f_1(\eta, \xi)} < 1, \ \forall \xi, \ \eta, \ |\xi| \geq \xi_0, \ |\eta| \geq \eta_0. \tag{24}$$

For $|\eta| \leq \eta_0$, we write

$$\frac{f_2(\eta)}{f_1(\eta, \xi)} \leq \sup_{|\eta| \leq \eta_0} |\rho_j|^{1/2} \frac{1 + e^{-\xi/2}}{1 - e^{-\xi} - 2e^{-\xi/2}}$$

and it suffices to take $\xi_c \geq \xi_0$ so that the right hand side is less than 1.□

Now we can conclude the

Proof of Theorem 2.4: When $\delta > 0$, Lemma 6.1 and Theorem 2.4 give $|\rho(k)| \leq |\rho_j(k)| e^{-\frac{a\delta}{\nu}}$ and

$$\sup_{i,r \text{ or } l} |h_r^n \text{ or } {}_{l,i}(k)| \leq C|\rho_j(k)|^{An+B} e^{-\frac{a\delta}{\nu}(An+B)} \sup_{i,r \text{ or } l} |h_r^1 \text{ or } {}_{l,i}(k)|$$
$$\leq C e^{-\frac{a\delta}{\nu}(An+B)} \sup_{i,r \text{ or } l} |h_r^1 \text{ or } {}_{l,i}(k)|$$

with $A > 0$. The geometric convergence is then achieved for the three algorithms. The difficulty appears when $\delta = 0$ ($\rho(k) \leq |\rho_j(k)| < 1$). For the DSA we have

$$\sup_{i,r \text{ or } l} |h_r^n \text{ or } {}_{l,i}(k)| \leq C\rho(k)^{n-1} \sup_{i,r \text{ or } l} |h_r^1 \text{ or } {}_{l,i}(k)|$$

and Lebesgue's theorem yields the convergence in $H^{1/2-\sigma(j)}(\Gamma)$. For the ASM and the FDA we refer to Lemma 3.2 and Theorem 5.1 and we write

$$\sup_{i,r \text{ or } l} |h_r^n \text{ or } {}_{l,i}(k)| \leq C\rho(k)^{An+B} \frac{|k|}{d_{\beta\gamma}} \sup_{i,r \text{ or } l} |h_r^1 \text{ or } {}_{l,i}(k)|.$$

In these cases, the data have to be chosen so that $(h_{l,2}^1, \ldots, h_{l,N-1}^1, h_{r,1}^1, \ldots, h_{r,N-2}^1) \in H^{3/2-\sigma(j)}(\Gamma)$, that is $f \in H^1(\mathbf{R}^2)$ and $u_i^0 \in H_{st}^3(\Omega_i)$.

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