REGULARITY AND EXACT CONTROLLABILITY FOR A BEAM WITH PIEZOELECTRIC ACTUATOR

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Abstract. We consider an initial and boundary value problem modelling the vibrations of a Bernoulli-Euler beam with an attached piezoelectric actuator. We show that the Sobolev regularity of the solution is by $\frac{1}{2}+\epsilon$ higher than that one obtains by simply using the Sobolev regularity of the control term. The main results concern the dependence of the space of exactly controllable initial data on the location of the actuator. Our approach is based on the Hilbert Uniqueness Method introduced by Lions (cf.[13]) combined with some results from the theory of diophantine approximation.

Key words. Exact controllability, Hilbert Uniqueness Method, piezoelectric actuator, diophantine approximation

AMS subject classifications. 93C20, 35B75, 35B60

1. Introduction . In recent years a lot of papers were devoted to the study of elastic structures with piezoelectric actuators (e.g.[2], [3], [5], [6]). The main topics covered in the papers quoted above are modelling ([2], [5], [6]), LQR and identification problems ([3]). Concerning the controllability problems, as far as we know, all published works consider finite dimensional approximations of the initial distributed control problems (see [6] and references therein). The main purpose of the present paper is to study the exact controllabillity of a beam with piezoelectric actuator by the use of the theory of infinite dimensional control systems as developed in [13]. More precisely we consider the initial and boundary value problem modelling the vibrations of a Bernoulli-Euler beam which is subject to the action of an attached piezoelectric actuator. If we suppose that the beam is hinged at both ends and that the actuator is excited in a manner so as to produce pure bending moments, the model for the controlled beam can be written as (cf. [5])

$$w''(x,t)+\frac{\partial^4 w}{\partial x^4}(x,t)=u(t)\frac{d}{dx}[\delta_\eta(x)-\delta_\xi(x)],\ 0<x<\pi,\ t>0, \tag{1.1}$$

$$w(0,t)=w(\pi,t)=\frac{\partial^2 w}{\partial x^2}(0,t)=\frac{\partial^2 w}{\partial x^2}(\pi,t)=0,\ t\geq 0, \tag{1.2}$$

$$w(x,0)=w^0(x),\ w'(x,0)=w^1(x),\ 0<x<\pi. \tag{1.3}$$

In the equations above w represents the transverse deflection of the beam, $\xi,\eta\in(0,\pi)$ stand for the ends of the actuator and δ_y is the Dirac mass at the point y. Moreover by w', w'' we denoted the time derivatives of w. The control is the fonction $u:[0,T]\to\mathbf{R}$ represents the time variation of the voltage applied to the actuator. Our main purpose is to determine the initial data which can be steered to rest by means of the control function u. More precisely we can give the following definition

DEFINITION 1.1. *We say that the initial data w^0, w^1 are "exactly L^2-controllable in ξ, η at time T" if there exists $u\in L^2(0,T)$ such that the solution w of (1.1)-(1.3)*

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satisfies the condition

$$w(x,T) = w'(x,T) = 0,\ 0 < x < \pi. \tag{1.4}$$

The plan of this paper is as follows. The second section contains some notations and preliminaries. In the third section we prove an existence and regularity result for (1.1)-(1.3). The exact controllability is studied in the fourth section. Our approach was inspired by ideas and methods used in [9] and [10] for point control problems.

2. Notations and preliminaries. In order to study the wellposedness and controllabillity for (1.1)-(1.3) we introduce the function spaces $(Y_\alpha)_{\alpha\in\mathbf{R}}$ defined as follows: if $\alpha > 0$ then Y_α is the closure in $H^\alpha(\Omega)$ of the set of all functions $y \in C^\infty(\bar{\Omega})$ satisfying the conditions $\frac{d^{2n}y}{dx^{2n}}(0) = \frac{d^{2n}y}{dx^{2n}}(\pi) = 0$, for all $n \geq 0$; for negative α we define Y_α as the dual space of $Y_{-\alpha}$ constructed by means of the inner product of $Y_0 = L^2(\Omega)$.

Let us consider now the homogenous initial and boundary value problem

$$\phi"(x,t) + \frac{\partial^4\phi}{\partial x^4}(x,t) = 0,\ 0 < x < \pi,\ t \in (0,T) \tag{2.1}$$

$$\phi(0,t) = \phi(\pi,t) = \frac{\partial^2\phi}{\partial x^2}(0,t) = \frac{\partial^2\phi}{\partial x^2}(\pi,t) = 0,\ t \in (0,T), \tag{2.2}$$

$$\phi(x,0) = \phi^0(x),\ \phi'(x,0) = \phi^1(x),\ 0 < x < \pi. \tag{2.3}$$

It is by now wellknown that the initial and boundary value problem (2.1)-(2.3) is well posed in $Y_{\alpha+2} \times Y_\alpha$, for all $\alpha \geq -2$. Moreover, as a consequence of the Hilbert Uniqueness Method (HUM), introduced in [13], the following result holds

PROPOSITION 2.1. *All initial data in $Y_{\alpha+2} \times Y_\alpha$ are "exactly L^2-controllable in ξ, η at time T" if and only if there exists a constant $c > 0$ such that*

$$\int_0^T [\frac{\partial\phi}{\partial x}(\xi,t) - \frac{\partial\phi}{\partial x}(\eta,t)]^2 dt \geq c(\|\phi^0\|^2_{H^{-\alpha}(\Omega)} + \|\phi^1\|^2_{H^{-\alpha-2}(\Omega)}], \tag{2.4}$$

for all

$$(\phi^0, \phi^1) \in Y_3 \times Y_1.$$

We shall also need some results from the theory of diophantine approximation. For a real number ρ, we denote by $|||\rho|||$ the diference, taken positively, between ρ and the nearest integer, i.e.

$$|||\rho||| = \min_{n\in\mathbf{Z}} |\rho - n|.$$

Let us also denote by A the set of all irrationals $\rho \in]0,1[$ such that if $[0, a_1, \ldots, a_n, \ldots]$ is the expansion of ρ as a continued fraction, then (a_n) is bounded. The set A plays a very important role in our control problem. Let us notice that A is is obviously uncountable and, by classical results on diophantine approximation (cf. [4, p.120]), its

Lebesgue measure is equal to zero. In particular, by Euler-Lagrange theorem(cf.[11, p.57]) A contains the irrational quadratic numbers (i.e. satisfying a second degree equation with rational coefficients). We shall esentially use the fact that the element of A are badly approximable by rational numbers. More precisely, the following result holds true(cf.[11, p.24])

PROPOSITION 2.2. *A number $\rho \in (0,1)$ is in A if and only if there exists a constant $C > 0$ such that*

$$|||q\rho||| \geq \frac{C}{q}, \tag{2.5}$$

for all stictly positive integer q.

We shall also use the following result on simultanous approximation (cf. [4, p.14])

PROPOSITION 2.3. *Let $\rho_1, \ldots, \rho_k$ be k irrationals in $(0,1)$. Then there exists a strictly increasing sequence of natural numbers q_n such that*

$$q_n^{\frac{1}{k}} \max_{i=1,k}(|||q_n\rho_1|||, \ldots, |||q_n\rho_i|||, \ldots, |||q_n\rho_k|||) \leq \frac{k}{k+1}, \ \forall n \geq 1.$$

The next proposition, which is proved in [4, p.120], shows that an inequality slightly weaker than (2.5) holds for almost all points in $(0,1)$.

PROPOSITION 2.4. *There exists a set $B \subset (0,\pi)$ having the Lebesgue measure equal to π and a constant $C > 0$, such that for any $\rho \in B$*

$$|||q\rho||| \geq \frac{C}{q^{1+\epsilon}}, \tag{2.6}$$

for any strictly positive integer q.

Let us notice that by Roth's theorem (cf. [4, p.104]) B contains all algebraic irrationals numbers in $(0,\pi)$.

3. Existence and regularity of solutions. The regularity of the solutions of (1.1)-(1.3) is a problem which is similar in many respects to the regularity for the wave or Euler-Bernoulli equations with interior point control (see [14], [16]). As in the cases mentioned above one can prove that the regularity of the solutions of (1.1)-(1.3) is higher than that one obtains by simply using the Sobolev regularity of the right hand side of (1.1). More precisely if we denote by Ω the interval $(0,1)$ we notice that if $u \in L^2(0,T)$ the function $u(\cdot)\frac{d}{dx}(\delta_\xi - \delta_\eta)$ is in $L^2(0,T;H^{-\frac{3}{2}-\epsilon}(\Omega))$, for any $\epsilon > 0$. Standard regularity for the Bernoulli-Euler equation (cf. [13]) implies the existence and the uniqueness of a solution of
(1.1)-(1.3) in

$$C([0,T], Y_{\frac{1}{2}-\epsilon}) \cap C^1([0,T], Y_{-\frac{3}{2}-\epsilon}). \tag{3.1}$$

The main result of this section, given bellow, shows that the space regularity in (3.1) can be improved by $\frac{1}{2}+\epsilon$.

THEOREM 3.1. *Suppose that $w^0 \in Y_1$, $w^1 \in Y_{-1}$. Then the initial and boundary value problem (1.1)-(1.3) admits an unique solution having the regularity*

$$w \in C([0,T], Y_1) \cap C^1([0,T], Y_{-1}). \tag{3.2}$$

Consider $\tau \in [0,T]$. In order to prove Theorem 3.1, following [14], we introduce the homogenous initial and boundary value problem

$$v"(x,t) + \frac{\partial^4 v}{\partial x^4}(x,t) = 0,\ 0 < x < \pi,\ t \in (0,\tau) \tag{3.3}$$

$$v(0,t) = v(\pi,t) = \frac{\partial^2 v}{\partial x^2}(0,t) = \frac{\partial^2 v}{\partial x^2}(\pi,t) = 0,\ t \in (0,\tau), \tag{3.4}$$

$$v(x,\tau) = 0,\ v'(x,\tau) = g(x),\ 0 < x < \pi. \tag{3.5}$$

The following lemma shows that the solution of (3.3)-(3.5) has a point regularity property which is essential for the proof of Theorem 3.1.

LEMMA 3.2. *For any $g \in Y_{-1}$ the initial and boundary value problem (3.3)-(3.5) admits an unique solution having the regularity*

$$v \in C([0,T], Y_1) \cap C^1([0,T], Y_{-1}). \tag{3.6}$$

Moreover, for any $\rho \in (0,\pi)$ the function $\frac{\partial v}{\partial x}(\rho,\cdot)$ is in $L^2(0,T)$ and there exists a constant $C > 0$ such that

$$\|\frac{\partial v}{\partial x}(\rho,\cdot)\|_{L^2(0,T)} \leq C\|g\|_{Y_{-1}}. \tag{3.7}$$

Proof. . The existence and uniqueness of a solution having the regularity (3.6) is a standard result for the Bernoulli-Euler equation. In order to prove (3.7) we put

$$g(x) = \sum_{n=1}^{\infty} n^2 a_n \sin(nx).$$

By density it is enough to show that (3.7) holds for $g \in C_0^\infty(\Omega)$. Obviously the solution of (3.3)-(3.5) is given by

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin(n^2 t) \sin(nx),$$

which implies that

$$\frac{\partial v}{\partial x}(\rho,t) = \sum_{n=1}^{\infty} n a_n \sin(n^2 t) \cos(n\rho). \tag{3.8}$$

If we consider the right hand side of (3.8) as a Fourier series in t (see Theorem 4.1 from [7] for details) we obtain the existence of a constant C depending on T such that

$$\|\frac{\partial v}{\partial x}(\rho,\cdot)\|_{L^2(0,T)} \leq C \sum_{n=1}^{\infty} n^2 a_n^2 < \infty, \tag{3.9}$$

which is exactly (3.7). $\square$

Remark 3.1 *The result of Lemma 3.2 is optimal in the sense that one can easily find $g \in Y_{-1}$, $T > 0$, and $\rho \in (0, \pi)$ such that equality holds in (3.9). We also remark that another possibility to prove Lemma 3.2 is a multiplier technique, as used in [7] for the wave equation with point control.*

Proof of Theorem 3.1. Due to the linearity of (1.1) and to well known properties of the Bernoulli-Euler equation it is enough to consider the case $w^0 = w^1 = 0$. Suppose again $g \in C_0^\infty(\Omega)$ and let v be the solution of (3.3)-(3.5). A simple calculation shows that

$$\int_0^\pi w(x,\tau)g(x)dx = -\int_0^T u(t)[\frac{\partial v}{\partial x}(\xi,t) - \frac{\partial v}{\partial x}(\eta,t)]dt. \tag{3.10}$$

Lemma 3.2 implies that

$$\int_0^T u(t)[\frac{\partial v}{\partial x}(\xi,t) - \frac{\partial v}{\partial x}(\eta,t)]dt \leq C\|u\|_{L^2(0,T)}\|g\|_{Y_{-1}},$$

so, by (3.10), we obtain that $w(\cdot,\tau) \in Y_1$, for all $\tau \in [0,T]$. By replacing τ by $\tau + h$ in (3.10) we easily get that

$$w \in C([0,T], Y_1), \tag{3.11}$$

which implies that

$$\frac{\partial^4 w}{\partial x^4} \in C([0,T], Y_{-3}). \tag{3.12}$$

As w satisfies satisfies (1.1) from (3.12) we obtain that

$$w\text{''} \in L^2((0,T), Y_{-3}),. \tag{3.13}$$

From (3.11) and (3.13), by applying the intermediate derivative theorem (cf.[15, p.19]) it follows that

$$w' \in L^2((0,T), Y_{-1}). \tag{3.14}$$

The conclusion (3.2) is now a consequence of (3.11)- (3.14) and of the general lifting result from [12].

Remark 3.2 *As by (3.10) the interior regularity of w is equivalent to a point regularity property, Remark 3.1 implies that the result in Theorem 3.1 is sharp.*

4. The exact controllability results. In this section we shall study the space of initial data which are exactly L^2-controllable in ξ, η at time T. Let us put

$$\phi^0(x) = \sum_{n=1}^\infty a_n \sin(nx), \ \phi^1(x) = \sum_{n=1}^\infty n^2 b_n \sin(nx). \tag{4.1}$$

A simple calculation shows that the solution of (2.1)-(2.3) is given by

$$\phi(x,t) = \sum_{n\geq 1}[a_n \cos(n^2 t)\sin(nx) + b_n \sin(n^2 t)\sin(nx)],$$

which implies that

$$\int_0^T [\frac{\partial \phi}{\partial x}(\xi,t) - \frac{\partial \phi}{\partial x}(\eta,t)]^2 dt = \tag{4.2}$$

$$= 4\int_0^T \{\sum_{n\geq 1}[na_n\cos(n^2t)+nb_n\sin(n^2t)]^2\sin^2\left[\frac{n(\eta+\xi)}{2}\right]\sin^2\left[\frac{n(\eta-\xi)}{2}\right]dt.$$

Relation (4.2) implies that (2.4) is false for any α if $\frac{\xi+\eta}{2}$ or $\frac{\eta-\xi}{2}$ is rational and by Proposition 2.1 it follows that the condition

$$\frac{\xi+\eta}{2\pi},\ \frac{\eta-\xi}{2\pi}\in \mathbf{R}-\mathbf{Q}, \tag{4.3}$$

is necessary in order to have exact controllability of all initial states in $Y_{\alpha+2}\times Y_\alpha$. The following result shows that the condition above is not sufficient, in the sense that there are $\xi,\ \eta$ satisfying (4.3) which doesn't allow the control of arbitrary regular initial data. More precisely we have:

Proposition 4.1. *For any $\alpha>-2$ there exist $\xi,\ \eta\in(0,\pi)$ satisfying (4.3) such that the space $Y_{\alpha+2}\times Y_\alpha$ contains initial data which are not exactly L^2-controllable in $\xi,\ \eta$ at time T, for any $T>0$.*

Proof. By Proposition (2.1) it is enough to show that (2.4) is false for any $\alpha,\ c>0$. Let us fixe $\alpha>0$ and $\nu>\frac{3\alpha+2}{2}$. We choose then

$$\frac{\xi+\eta}{2\pi}=\sum_{n=1}^{\infty}\frac{a_n}{10^{n!}}, \tag{4.4}$$

where $a_n\in\{0,1,\dots,9\}$, for all $n\geq 1$ and a_n is not identicaly zero for great n. According to [17] the right hand side of (4.4) is a Liouville number, i.e. it is transcendental and there exists a strictly increasing sequence of integers q_n such that

$$|||q_n\frac{\xi+\eta}{2\pi}|||\leq\frac{1}{q_n^\nu},\forall n\geq 1. \tag{4.5}$$

We notice that

$$|\sin(q_n\frac{\xi+\eta}{2})|=|\sin[\pi(q_n\frac{\xi+\eta}{2\pi}-p)]|\leq\pi|q_n\frac{\xi+\eta}{2\pi}-p|,$$

for any integer p. The relation above and (4.5) imply that

$$|\sin(q_n\frac{\xi+\eta}{2})|\leq\frac{\pi}{q_n^\nu},\ \forall n\geq 1. \tag{4.6}$$

Consider now the sequence of initial data

$$\phi_n^0(x)=q_n^{\frac{3\alpha}{2}}\sin(q_nx),\ \phi_n^1(x)=0,\forall x\in(0,\pi). \tag{4.7}$$

A simple calculation shows that $\{\phi_n^0,\phi_n^1\}\in Y_{\alpha+2}\times Y_\alpha$, and

$$||\phi_n^0||^2_{Y_{-\alpha}}+||\phi_n^1||^2_{Y_{-\alpha-2}}\to\infty,\ \forall\alpha>0. \tag{4.8}$$

Moreover the solution of (1.1)-(1.3) with the initial data given by (4.7) is

$$\phi_n(x,t)=q_n^{\frac{3\alpha}{2}}\cos(q_n^2t)\sin(q_nx),$$

so, by (4.6) we have

$$\int_0^T [\frac{\partial \phi_n}{\partial x}(\xi,t) - \frac{\partial \phi_n}{\partial x}(\eta,t)]^2 dt = \tag{4.9}$$

$$= 4q_n^{3\alpha+2} \sin^2 \left[\frac{q_n(\eta+\xi)}{2}\right] \sin^2 \left[\frac{q_n(\eta-\xi)}{2}\right] \int_0^T cos^2(q_n^2 t)dt \leq$$

$$\leq 4Tq_n^{3\alpha+2} \sin^2 \left[\frac{q_n(\eta+\xi)}{2}\right] \to 0, \text{ when } n \to \infty.$$

Relations (4.8) and (4.9) imply that (2.4) is false for any $\alpha > -2$ and $c > 0$. □

Remark 4.1. *The estimates in Proposition 4.1 are not sharp. In fact it seems likely that the methods developed in [10] can be used to prove the existence of ξ, η satisfying (4.3) such that $\cap_{\alpha\geq 0} Y_{\alpha+2} \times Y_\alpha$ contains initial data which are not exactly L^2-controllable in ξ, η at time T, for any $T > 0$. We also remark that, by standard duality arguements, condition (4.3) implies approximate controllability in $Y_{\alpha+2} \times Y_\alpha$.*

Proposition 4.1 shows that, in order to obtain exact controllability, we need assumptions stronger than the irrationality of $\frac{\xi+\eta}{2}$ and $\frac{\eta-\xi}{2}$. At this point we shall use the number theoretic preliminaries stated in the second section. A first result in this direction is

THEOREM 4.2. *Suppose that that $\frac{\xi+\eta}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to the set A (introduced in the second section). Than all initial data in $Y_3 \times Y_1$ are exactly L^2-controllable in ξ, η at time T, for any $T > 0$.*

Proof. By Proposition 2.1 the conclusion of the theorem is equivalent to the existence of a constant $c > 0$ such that

$$\int_0^T [\frac{\partial \phi}{\partial x}(\xi,t) - \frac{\partial \phi}{\partial x}(\eta,t)]^2 dt \geq c(\|\phi^0\|^2_{Y_{-1}} + \|\phi^1\|^2_{Y_{-3}}], \tag{4.10}$$

for all

$$(\phi^0, \phi^1) \in Y_3 \times Y_1.$$

By applying the Ball-Slemrod generalization of Ingham's inequality (cf.[1], [8]) from (4.2) we obtain that there exists a constant $C > 0$ such that

$$\int_0^T [\frac{\partial \phi}{\partial x}(\xi,t) - \frac{\partial \phi}{\partial x}(\eta,t)]^2 dt \geq \tag{4.11}$$

$$\geq C \sum_{n\geq 1} (n^2 a_n^2 + n^2 b_n^2) \sin^2 \left[\frac{n(\eta+\xi)}{2}\right] \sin^2 \left[\frac{n(\eta-\xi)}{2}\right] dt$$

As $\frac{\xi+\eta}{2\pi}$ and $\frac{\xi-\eta}{2\pi}$ are in A from (2.5) we see that there exists a constant $C > 0$ such that, for n large enough we have

$$|\sin\left[\frac{n(\eta\pm\xi)}{2}\right]| = |\sin\left\{\pi\left[\frac{n(\eta\pm\xi)}{2\pi} - p\right]\right\}| \geq |sin\left(\frac{\pi C}{n}\right)| \geq \frac{C}{n}, \ \forall n \geq 1. \tag{4.12}$$

Inequalities (4.11) and (4.12) imply that

$$\int_0^T [\frac{\partial \phi}{\partial x}(\xi,t) - \frac{\partial \phi}{\partial x}(\eta,t)]^2 dt \geq c \sum_{n\geq 1}(n^{-2}a_n^2 + n^{-2}b_n^2),$$

which is exactly (4.10). □

The following result shows that, for almost all choices of the ends of the actuator, we have exact controllability in Sobolev spaces more regular than $Y_3 \times Y_1$.

THEOREM 4.3. *Suupose that $\epsilon > 0$ is arbitrary and consider the set B introduced in Proposition 2.4. Then, for any ξ, $\eta \in (0,\pi)$ such that $\frac{\xi+\eta}{2\pi}$, $\frac{\xi-\eta}{2\pi} \in B$, all initial data in $Y_{3+\epsilon} \times Y_{1+\epsilon}$ are exactly L^2-controllable in ξ, η at time T, for any $T > 0$.*

Proof. As $\frac{\xi+\eta}{2\pi}$, $\frac{\eta-\xi}{2\pi} \in B$, from (2.6) it follows that

$$|\sin\left[\frac{n(\eta+\xi)}{2}\right]| \geq \frac{C}{n^{1+\epsilon}},\ |\sin\left[\frac{n(\eta-\xi)}{2}\right]| \geq \frac{C}{n^{1+\epsilon}},\ \forall n \geq 1. \tag{4.13}$$

Consider again ϕ the solution (2.1)-(2.3) with the initial data given by (4.1). By applying (4.2), (4.13) we obtain

$$\int_0^T [\frac{\partial \phi}{\partial x}(\xi,t) - \frac{\partial \phi}{\partial x}(\eta,t)]^2 dt \geq c(\|\phi^0\|^2_{Y_{-1-\epsilon}} + \|\phi^1\|^2_{Y_{-3-\epsilon}}),$$

for all

$$(\phi^0, \phi^1) \in Y_3 \times Y_1.$$

By Proposition 2.1 it follows that all initial data in $Y_{3+\epsilon} \times Y_{1+\epsilon}$ are exactly L^2-controllable in ξ, η at time T, for any $T > 0$. □

Let $y(x,t)$ be the solution of

$$y''(x,t) + \frac{\partial^4 y}{\partial x^4}(x,t) = h(t)\frac{d}{dx}[\delta_\eta(x) - \delta_\xi(x)],\ 0 < x < \pi,\ t > 0, \tag{4.14}$$

$$y(0,t) = y(\pi,t) = \frac{\partial^2 y}{\partial x^2}(0,t) = \frac{\partial^2 y}{\partial x^2}(\pi,t) = 0,\ t \geq 0, \tag{4.15}$$

$$y(x,0) = 0,\ y'(x,0) = 0,\ 0 < x < \pi. \tag{4.16}$$

By Theorem 3.1 we have

$$y \in C([0,T], Y_1) \cap C^1([0,T], Y_{-1}).$$

By reversing the sense of the time t in (1.1)-(1.3) we easily see that Theorems 4.2 and 4.3 imply that the space regularity $y(\cdot,T)$ and $y'(\cdot,T)$ is higher than that one obtains by simply using Theorem 3.1.

COROLLARY 4.4. *Suppose that y is the solution of (4.14)-(4.16), where $h \in L^2(0,T)$ and $\frac{\xi+\eta}{2\pi}$, $\frac{\eta-\xi}{2\pi}$ belong to the set A (respectively to B). Than, for any $T > 0$, the application $h \rightarrow \{y(\cdot,T), y'(\cdot,T)\}$ maps $L^2(0,T)$ onto $Y_3 \times Y_1$ (respectively*

onto $Y_{3+\epsilon} \times Y_{1+\epsilon},\ \forall \epsilon > 0$*).*
Theorems 4.2 and 4.3 give no information on the controllability of initial data in $Y_{\alpha+2} \times Y_{\alpha}$, with $\alpha < 1$. A partial answer is given by the following result

PROPOSITION 4.5. *Suppose that* $\epsilon > 0$. *Than the set* $Y_{2-\epsilon} \times Y_{-\epsilon}$ *contains initial data which are not exactly* L^2*-controllable in* $\xi,\ \eta$ *at time* T*, for any* $T > 0$ *and* $\xi,\ \eta \in [0, \pi]$.

Proof. . By applying Proposition 2.3 we easily obtain the existence of a strictly increasing sequence of positive integers (q_n) such that

$$|\sin\left[\frac{q_n(\eta+\xi)}{2}\right]| \leq \frac{\pi}{\sqrt{q_n}},\ |\sin\left[\frac{q_n(\eta-\xi)}{2}\right]| \leq \frac{\pi}{\sqrt{q_n}},\ \forall n \geq 1. \tag{4.17}$$

Consider now the sequence of initial data

$$\phi_n^0(x) = \sin(q_n x),\ \phi_n^1(x) = 0, \forall x \in (0, \pi).$$

We notice that

$$\|\phi_n^0\|_{Y_\epsilon}^2 + \|\phi_n^1\|_{Y_{\epsilon-2}}^2 = q_n^{2\epsilon} \to \infty, \quad \text{when } n \to \infty \tag{4.18}$$

A simple calculation gives the corresponding sequence of solutions of (2.1)-(2.3) is

$$\phi_n(x,t) = \cos(q_n^2 t)\sin(q_n x),$$

so, by (4.17) we have

$$\int_0^T [\frac{\partial \phi_n}{\partial x}(\xi,t) - \frac{\partial \phi_n}{\partial x}(\eta,t)]^2 dt = \tag{4.19}$$

$$= q_n^2 \sin^2\left[\frac{q_n(\eta+\xi)}{2}\right]\sin^2\left[\frac{q_n(\eta-\xi)}{2}\right]\int_0^T cos^2(q_n^2 t)dt \leq K,\ \forall n \geq 1,$$

where K is a positive constant. In a similar manner we can show that (4.18), (4.19) hold true for the sequence of initial data

$$\phi_n^0(x) = 0,\ \phi_n^1(x) = q_n^2 \sin(q_n x), \forall x \in (0, \pi),$$

so (2.4) is false for $\alpha = -\epsilon$ and arbitrary $c > 0$. □

Remark 4.2 *The exact controllability of initial data in* $Y_{\alpha+2} \times Y_{\alpha}$*, with* $0 \leq \alpha < 1$ *seems an open question.*

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