ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR THE WAVE EQUATION WITH DAMPING CONCENTRATED IN INTERIOR CURVES

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Abstract. We consider an initial and boundary value problem for the two dimensional wave equation with nonlinear damping concentrated on an interior curve. By the use of the theory of nonlinear semigroups we prove a wellposedness result. The main result asserts that generically (i.e. for almost all interior curves) the solutions decay to zero in the energy space. We also show, that whatever the interior curve is, the decay is not uniform.

Key words. maximal monotone operator, invariance principle, unique continuation, strong stabilization.

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1. Introduction . Let $\Omega \subset \mathbf{R}^2$, be an open bounded set and denote by Γ the boundary of Ω. We also consider $\gamma \subset \Omega$ a closed simple curve. The main goal of the present paper is to study the asymptotic behavior of solutions for the folllowing initial and boundary value problem:

$$u" - \Delta u + g(u')\delta_\gamma = 0, \text{ in } \Omega \times (0,\infty) \tag{1.1}$$

$$u = 0, \text{ on } \Gamma \times (0,\infty) \tag{1.2}$$

$$u(x,0) = u^0(x), \ u'(x,0) = u^1(x), \text{ in } \Omega. \tag{1.3}$$

where $T > 0$, δ_γ is the Dirac mass concentrated on γ and by u', u'' we denoted the time derivatives of u. The function $g : \mathbf{R} \to \mathbf{R}$ is supposed to be continuous and strictly monotone, with $g(0) = 0$. Equations (1.1)-(1.3) are dissipative since

$$E'(t) = -\int_\gamma g(u'(x,t))u'(x,t)d\gamma, \tag{1.4}$$

where $E = E(t)$ is the energy

$$E(t) = \frac{1}{2}\int_\Omega [|u'(x,t)|^2 + |\nabla u(x,t)|^2]dx.$$

The system (1.1)-(1.3) was suggested to us by various models of vibrating elastic structures provided with piezoelectric actuators (see [2], [8], [11]). The main novelty in this paper is that the damping is not acting at the boundary or in an interior open set as in other related papers (see [16], [17], [23]). In the case we study the damping term is concentrated on an interior curve. So a first difficulty arises as the existence and regularity of solutions of (1.1)-(1.3) has not been studied in the literature. On

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the other hand it is clear that there exist curves γ (the "nodal curves") such that some eigenfunction ϕ, corresponding to the eigenvalue $-\lambda$ of the Laplace operator, vanishes on γ, so $e^{i\sqrt{\lambda}t}\phi(x)$ is a constant energy solution of (1.1)-(1.3). However we shall prove that "generically" with respect to γ the energy of every solution goes to zero as $t \to \infty$, in other words strong stabilization holds. In the one dimensional case γ is reduced to a point and the corresponding pointwise stabilization problem was studied in [7], [14]. The main results obtained in these papers assert that we have strong stabilization for a dense set of control points, called "strategic". In the same one dimensional case, by the use of Russell's "stabilizability-controllability" arguement (see [21]), we can see that the energy decay is not exponential (see [12], [14]). We shall generalize the results obtained in the one dimensional case by proving the existence of a dense set of "strategic curves" such that every solution of (1.1)-(1.3) decays to zero when t tends to infinity and that ,whatever the interior curve is, the decay rate is not uniform.

The plan of this paper is as follows: in the second section we introduce some notations and preliminaries; the third section contains the existence and uniqueness results for (1.1)-(1.3); in the fourth section we study the asymptotic behavior of solutions for (1.1)-(1.3); in the fifth section we prove that the decay is never uniform; we end up with a section devoted to further comments and other related questions.

2. Notations and preliminaries. We begin this section by introducing the set of curves which we shall admit as damping regions. More precisely we shall denote by $\mathcal{K}$ the set of functions $\gamma : [0,1] \to \Omega \subset \mathbf{R}^2$ satisfying to the conditions:

(C1) γ is of class C^1,

(C2) $\gamma(t_1) \neq \gamma(t_2)$, for all $0 \leq t_1 < t_2 < 1$.

(C3) $\gamma(0) = \gamma(1)$.

(C4) $\gamma(t) \notin \Gamma$,for all $t \in [0,1]$.

Let us notice that, by Jordan's theorem an admissible curve (i.e. $\gamma \in \mathcal{K}$) divides Ω in two connected components Ω_1 (interior to γ) and Ω_2. Concerning the boundary Γ of Ω we shall suppose that it is locally Lipschitz.

Remark *We notice that (1.1)-(1.3) can be understood as the coupling of two wave equations posed in Ω_1 and Ω_2 respectively. Indeed it is easy to check that (1.1) is equivalent to*

$$u" - \Delta u = 0, \ \ in \ \Omega_i \times (0,\infty), \ \ i = 1,2,$$

$$[u] = 0, \ \ \left[\frac{\partial u}{\partial \nu}\right] = g(u'), \ \ on \ \gamma \times (0,\infty),$$

where $[F]$ denotes the jump of the function F across γ.

On $\mathcal{K}$ we introduce the Hausdorff distance between two curves γ_1 and γ_2, denoted by $d(\gamma_1,\gamma_2)$ and defined in the following way (cf. [19, p.31]):

$$\delta(x,\gamma) = \inf_{t\in[0,1]} |x - \gamma(t)|, \ \ \forall x \in \mathbf{R}^2, \ \ \forall \gamma \in \mathcal{K},$$

$$\rho(\gamma_1,\gamma_2) = \sup_{t\in[0,1]} \delta(\gamma_1(t),\gamma_2),$$

$$d(\gamma_1,\gamma_2) = \max\{\rho(\gamma_1,\gamma_2), \rho(\gamma_2,\gamma_1)\}.$$

It is easy to check that $\mathcal{K}$ endowed with the distance d forms a metric space.

Let us also recall a topological notion. If X is a topological space, the subset $A \subset X$ is called residual in X if A is a countable intersection of open, dense sets in X. If A is residual then A is dense in X. Moreover, the complement of A is a set of first category in X.

3. Existence and uniqueness of solutions. The wellposedness of (1.1)-(1.3) will be studied by the use of the theory of nonlinear monotone operators. We begin by writing the system (1.1)-(1.3) in the variational form

$$< u'', \phi > + < \nabla u, \nabla \phi > + \int_\gamma g(u')\phi d\gamma = 0, \ \forall \phi \in H_0^1(\Omega), \tag{3.1}$$

where $< \cdot, \cdot >$ denotes both the scalar product in $L^2(\Omega)$ and $[L^2(\Omega)]^2$. Let A denote the Riesz isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, i.e.

$$< A\phi, \hat{\phi} >= \int_\Omega \nabla\phi \nabla\hat{\phi} dx, \ \forall \phi, \ \hat{\phi} \in H_0^1(\Omega). \tag{3.2}$$

We also introduce, in a formal way for the moment, a nonlinear operator B defined by

$$< B(\phi), \hat{\phi} >= \int_\gamma g(\phi)\hat{\phi} d\gamma, \ \forall \phi, \ \hat{\phi} \in H_0^1(\Omega). \tag{3.3}$$

LEMMA 3.1. *Let γ be an admissible curve. Moreover assume that g is continuous, increasing and that*

$$|g(\xi)| = O(|\xi|^r) \ (|\xi| \to \infty), \tag{3.4}$$

for some $r \in \mathbf{R}$. Then B is a monotone, hemicontinuous operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ and $B(0) = 0$.

Proof. If $\phi \in H_0^1(\Omega)$ then $\phi_{|\gamma} \in H^{\frac{1}{2}}(\gamma)$. Since γ is an one dimensional manifold, by Sobolev's imbedding theorem we obtain that $\phi_{|\gamma} \in L^q(\gamma)$, for all $q \in [1, \infty)$. By the continuity of g and (3.4) it follows that $g(\phi) \in L^2(\gamma)$, hence $B(\phi) \in H^{-1}(\Omega)$, for all $\phi \in H_0^1(\Omega)$. Monotonicity of B and the fact that $B(0) = 0$ follow from the corresponding properties of g while hemicontinuity of B is a consequence of the continuity of g, (3.4) and the Lebesgue dominated convergence theorem. □

By using the notations (3.2), (3.3) relation (3.1) may be written as

$$u'' + B(u') + Au = 0, \ \text{in } H^{-1}(\Omega). \tag{3.5}$$

Let us denote by X the energy space $H_0^1(\Omega) \times L^2(\Omega)$ and introduce the operators

$$\mathcal{C} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \ \mathcal{A} = \begin{pmatrix} 0 & -A \\ A & B \end{pmatrix}.$$

Then $\mathcal{C}$ is the Riesz isomorphism from X onto X^* and (3.5) becomes

$$\mathcal{C}U' + \mathcal{A}U = 0, \quad \text{in } X^* = H^{-1}(\Omega) \times L^2(\Omega), \tag{3.6}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$ and $v = u'$.

If we define

$$\mathcal{D}(\mathcal{A}) = \{U \in X | Au + Bv \in L^2(\Omega)\},$$

and we suppose that $U(t) \in \mathcal{D}(\mathcal{A})$ equation (3.6) becomes

$$U' + \mathcal{C}^{-1}\mathcal{A}U = 0, \quad \text{in } X. \tag{3.7}$$

If to (3.7) we adjoin the initial condition

$$U(0) = U^0, \ U^0 = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in X, \tag{3.8}$$

we notice that (3.7), (3.8) represent the abstract form of (1.1)-(1.3). The wellposedness theorem for (1.1)-(1.3) will be a simple consequence of the following result:

LEMMA 3.2. *Under the assumptions of Lemma 3.1 $\mathcal{D}(\mathcal{A})$ is dense in X and $-\mathcal{C}^{-1}\mathcal{A}$ is the generator of a continous semigroup of nonlinear contractions in X.*

Proof. The density of $\mathcal{D}(\mathcal{A})$ in X is a consequence of the fact that $\mathcal{D}(\mathcal{A})$ contains V, where

$$V = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in (H^2(\Omega_1) \cap H^2(\Omega_2) \cap H_0^1(\Omega)) \times H_0^1(\Omega) | \ g(v)_{|\gamma} = [\frac{\partial u}{\partial \nu}]_\gamma \right\}, \tag{3.9}$$

where by $[\frac{\partial u}{\partial \nu}]_\gamma$ we denoted the jump on γ of the normal derivative of u.

If we suppose that X is endowed with the scalar product

$$\left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_X = \int_\Omega (\nabla u_1 \nabla u_2 + v_1 v_2) dx,$$

a simple calculation shows that

$$\left(\mathcal{C}^{-1} \left(\mathcal{A} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \mathcal{A} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right), \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_X =$$

$$= \int_\gamma (g(v_1) - g(v_2))(v_1 - v_2) d\gamma \geq 0, \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(\mathcal{A}),$$

so $\mathcal{C}^{-1}\mathcal{A}$ is monotone. In order to prove that $-\mathcal{C}^{-1}\mathcal{A}$ generates a semigroup of nonlinear contractions on X we have only to show that $(\mathcal{I} + \mathcal{C}^{-1}\mathcal{A})(\mathcal{D}(\mathcal{A})) = X$, where $\mathcal{I}$ is the identity map on X.

Let $\begin{pmatrix} f \\ h \end{pmatrix} \in X$ and consider the equation

$$\mathcal{C} \begin{pmatrix} u \\ v \end{pmatrix} + \mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{C} \begin{pmatrix} f \\ h \end{pmatrix},$$

which can be written as

$$Au - Av = Af, , \tag{3.10}$$

$$v + Au + B(v) = h. \tag{3.11}$$

From (3.10) it follows that $u = v + f$, so (3.11) becomes

$$v + Av + B(v) = h - Af \in H^{-1}(\Omega). \tag{3.12}$$

The operator $I + A + B$ (where I is the injection of $H_0^1(\Omega)$ into $H^{-1}(\Omega)$) is monotone, hemicontinous and strictly coercive on $H_0^1(\Omega)$. By theorem 1.1 in [6] it follows that $I + A + B$ is onto from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, so (3.12) has a unique solution $v \in H_0^1(\Omega)$. Going back to (3.10)-(3.11) we obtain that $u = v + f \in H_0^1(\Omega)$ and $Au + B(v) = h - v \in L^2(\Omega)$ so $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(\mathcal{A})$. □

As a consequence of Lemma 3.2 and of Theorem 3.1 from [3, p.152] we obtain the following existence and uniqueness result

THEOREM 3.3. *If* $U^0 = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$ *then the initial and boundary value problem (1.1)-(1.3) admits a unique strong solution* u *(i.e.* $\begin{pmatrix} u \\ u' \end{pmatrix} \in C([0,\infty), \mathcal{D}(\mathcal{A}))$*). Moreover, if* $U^0 = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in X$ *then the initial and boundary value problem (1.1)-(1.3) admits a unique mild solution (i.e.* $\begin{pmatrix} u \\ u' \end{pmatrix} \in C([0,\infty), X)$*), which is defined as limit of strong solutions.*

In order to study the asymptotic behavior of (1.1)- (1.3) we shall need the following compactness result.

LEMMA 3.4. *Suppose that* $\gamma \in \mathcal{K}$*. Then* $\mathcal{D}(\mathcal{A})$ *is compactly imbedded in* X *and* $(\mathcal{I} + \mathcal{C}^{-1}\mathcal{A})^{-1}$ *is compact from* X *into* X*.*

Proof. It is enough to notice that $(\mathcal{I} + \mathcal{C}^{-1}\mathcal{A})^{-1}$ maps continuously X onto $\mathcal{D}(\mathcal{A})$, $L^2(\Omega) \times H^{-1}(\Omega)$ onto X and that X is compactly imbedded in $L^2(\Omega) \times H^{-1}(\Omega)$. □

Remark. *If we assume that the curves* γ *and* Γ *are of class* C^2 *we can apply Nirenberg's translations method (cf. [18]) to show that* $\mathcal{D}(\mathcal{A})$ *is equal to* V*, where* V *is defined by (3.9).*

4. Asymptotic behavior of solutions. In this section we shall study the energy decay of solutions of (1.1)- (1.3) when $t \to \infty$. It is by now wellknown that, by the invariance principle of LaSalle the energy decay property is reduced to a unique continuation result. However, for the sake of completeness, we shall sketch the proof of the following result

LEMMA 4.1. *Suppose that the only function satisfying the conditions*

$$w" - \Delta w = 0, \ \text{in } \Omega \times (0,\infty) \tag{4.1}$$

$$w = 0, \ \text{on } \Gamma \times (0,\infty), \tag{4.2}$$

$$w' = 0, \ \text{on } \gamma \times (0,\infty), \tag{4.3}$$

is $w \equiv 0$*. Then, for any* $u^0 \in H_0^1(\Omega)$, $u^1 \in L^2(\Omega)$ *the solution of (1.1)-(1.3) satisfies the energy estimate*

$$\lim_{t\to\infty} (\|u(t)\|_{H^1(\Omega)} + \|u'(t)\|_{L^2(\Omega)}) = 0. \tag{4.4}$$

Proof. We follow the outline of [9]. Let us denote by $S(t)$ the semigroup of contractions generated by the operator $\mathcal{C}^{-1}\mathcal{A}$ introduced in section 3. By density it is enough to prove that $S(t)\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \to 0$ for all $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$. By Lemma 3.4 $(\mathcal{I}+\mathcal{C}^{-1}\mathcal{A})^{-1} : X \to X$ is compact, so, by using Theorem 3 from [10], we obtain that the orbit $orb\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} = \bigcup_{t\geq 0} S(t)\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}$ is precompact in X for all $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$. It follows that the ω-limit set $\omega\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} = \{y \in X | y = \lim_{n\to\infty} S(t_n)\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}, \text{ with } t_n \to \infty\}$ is nonvoid for all $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$.

On the other hand, by LaSalle's invariance principle (see [9],[10]), if $\begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \in \omega\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}$ then

$$S(t)\begin{pmatrix} w^0 \\ w^1 \end{pmatrix} = \begin{pmatrix} w(t) \\ w'(t) \end{pmatrix} \in \omega\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}, \ \forall t \geq 0,$$

$$\left\| \begin{pmatrix} w(t) \\ w'(t) \end{pmatrix} \right\|_X = \left\| \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \right\|_X, \ \forall t \geq 0.$$

Relations above and (1.4) imply that w satisfies (4.1)-(4.3) so $w \equiv 0$ which implies that $\omega\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ and the conclusion of the present lemma.□

For particular geometries of Ω it is easy to construct examples of curves γ such that the assumption of Lemma 4.1 doesn't hold true (the so called "nodal curves", where some eigenfunctions of the laplacian are vanishing). However we shall prove that generically the assumption of Lemma 4.1 holds true. Our main result, generalizing the well known assertions concerning the genericity of strategic points in one space dimension, is

THEOREM 4.2. *There exists a set $\mathcal{N}$, residual in $\mathcal{K}$, such that if $\gamma \in \mathcal{N}$, then $w \equiv 0$ is the only solution of (4.1)-(4.3).*

As a consequence of Lemma 4.1 and Theorem 4.2 we easily obtain

THEOREM 4.3. *For any $\gamma \in \mathcal{N}$ the solution u of (1.1)-(1.3) satisfies*

$$\lim_{t\to\infty} (||u(t)||_{H^1(\Omega)} + ||u'(t)||_{L^2(\Omega)}) = 0. \tag{4.5}$$

Let us denote by $-\lambda_i, i \geq 1$ the distinct eigenvalues of the Laplace operator, with $0 < \lambda_1 < \lambda_2 < \ldots \lambda_n < \ldots$.

Definition 4.1 *We say that a curve $\gamma \in \mathcal{K}$ is a decay curve for (1.1)-(1.3) and we write $\gamma \in \mathcal{N}$ if for all $\phi \in \cup_{i=1}^{\infty} ker(\Delta + \lambda_i I)$, $\phi \neq 0$ we have $\phi_{|\gamma} \not\equiv 0$.*

Remark *According to the definition above we see that, if $(u_i)_{i\geq 1}$ is a basis of $L^2(\Omega)$ formed by eigenfunctions of the laplacian, the condition $\gamma \in \mathcal{N}$ is stronger than the condition $u_{i|\gamma} \neq 0, \forall i \geq 1$. In fact, by Definition 4.1, if $\gamma \in \mathcal{N}$ then all linear combinations formed by elements of (u_i) corresponding to the same eigenvalue are not vanishing on γ.*

The rest of this section is devoted to the proof of Theorem 4.2. We begin by stating a lemma which will easily imply Theorem 4.2 (the proof of the lemma will be given at the end of this section).

LEMMA 4.4. *The set $\mathcal{N}$ is residual in $\mathcal{K}$.*

Proof of Theorem 4.2. Consider $\gamma \in \mathcal{N}$ and

$$w \in C([0,\infty), H_0^1(\Omega)) \cap C^1([0,\infty), L^2(\Omega)),$$

satisfying (4.1)-(4.3). If we denote w' by ψ we see that ψ satisfies the conditions

$$\psi'' - \Delta\psi = 0, \text{ in } \Omega \times (0,\infty) \tag{4.6}$$

$$\psi = 0, \text{ on } \Gamma \times (0,\infty), \tag{4.7}$$

$$\psi = 0, \text{ on } \gamma \times (0,\infty). \tag{4.8}$$

Suppose that $\psi(0)$ (repectively $\psi'(0)$) admits the Fourier expansion

$$\psi(x,0) = \sum_{n\geq 1} a_n w_n(x) (\text{ respectively } \psi'(x,0) = \sum_{n\geq 1} b_n w_n(x)),$$

where $w_n \in ker(\Delta + \lambda_n \mathcal{I}),\ ||w_n|| = 1$. A simple calculation shows that $\psi(x,t)$ will be given by

$$\psi(x,t) = \sum_{n\geq 1} [a_n w_n(x)\cos(\sqrt{\lambda_n}t) + \frac{b_n}{\sqrt{\lambda_n}} w_n(x)\sin(\sqrt{\lambda_n}t)],\ \forall t \geq 0. \tag{4.9}$$

As the functions $(e^{i\sqrt{\lambda_n}t})_{n\geq 1}$ are orthonormal in the space of almost periodic functions AP($\mathbf{R}$) (cf.[15]), from (4.9) it follows that $\psi_{|\gamma} \equiv 0$ for all $t \geq 0$ if and only if

$$a_n w_n(x) = b_n w_n(x) = 0,\ \forall x \in \gamma,\ \forall n \geq 1. \tag{4.10}$$

As $\gamma \in \mathcal{N}$ from (4.10) we obtain that $a_n = b_n = 0,\ \forall n \geq 1$, so $\psi \equiv 0$ in $\Omega \times (0,T)$. By using the relation $\psi = w'$ and (4.6) it follows that $\Delta w = 0$, which combined with (4.7) gives $w \equiv 0$ in $\Omega \times (0,T)$.

The last part of this section is devoted to the proof of Lemma 4.4. Similar ideas were used in [1] for proving genericity of simple eigenvalues for the laplacian.

We shall need the following result

LEMMA 4.5. *Consider an orthonormal basis of $L^2(\Omega)$ formed by eigenvectors of the Laplace operator and denoted by $(u_i)_{i=1,\infty}$. Then the set Z of points $x \in \Omega$ such that $u_i(x) \neq 0,\ \forall i \geq 1$ is dense in Ω.*

Proof. According to [5] the functions u_i are continous, so the set of points $x \in \Omega$ such that $u_i(x) \neq 0$, is open in Ω for all $i \geq 1$. On the other hand, by Holmgren's uniqueness theorem the interior of the set of points $x \in \Omega$ such that $u_i(x) = 0$, is void, for all $i \geq 1$, so by applying Baire's lemma we obtain that Z is dense.□

Proof of Lemma 4.4. Let us define the sets

$$A_n = \{\gamma \in \mathcal{K} \text{ s.t. } w_{|\gamma} \not\equiv 0,\ \forall w \in \cup_{i=1}^n ker(\Delta + \lambda_i \mathcal{I}),\ \text{with } ||w|| = 1\}.$$

The conclusion of Lemma 4.4 can be obtained as a consequence of the following two assertions:

(1) The set A_n is open in $\mathcal{K}, \forall n \geq 0$.

(2) The set A_{n+1} is dense in $A_n,\ \forall n \geq 0$.

In order to prove assertion (1) we shall use induction after n. Let us suppose that A_n is open for $n = 1, \dots, k$ Let us first notice that

$$A_{k+1} = A_k \cap \{\gamma \in \mathcal{K} \text{ s.t. } w_{|\gamma} \not\equiv 0,\ \forall w \in ker(\Delta + \lambda_{k+1}\mathcal{I}),\ \text{with } ||w|| = 1\}.$$

On the other hand it is simple to prove that

$$\{\gamma \in \mathcal{K} \text{ s.t. } w_{|\gamma} \not\equiv 0,\ \forall w \in ker(\Delta + \lambda_{k+1}\mathcal{I}),\ \text{with } ||w|| = 1\},$$

is open in $\mathcal{K}$ so A_{k+1} is also open.

Let us now check property (2). Consider again the orthonormal basis of $L^2(\Omega)$ formed by eigenvectors of the Laplace operator and denoted by $(u_i)_{i=1,\infty}$. For $\gamma \in A_n - A_{n+1}$ and an arbitrary $\epsilon > 0$ we shall construct a curve $\gamma'' \in \mathbf{K}$ such that

$$\gamma'' \in A_{n+1},\ d(\gamma, \gamma'') < \epsilon \tag{4.11}$$

Let us fix $x_0 \in \gamma$. By Lemma 4.5 we can deform γ into a curve γ' such that $d(\gamma, \gamma') < \frac{\epsilon}{2}$ and $u_i(x_0') \neq 0,\ \forall i \geq 1$ (here $x_0' \in \gamma'$). If λ_{n+1} is simple the proof is concluded. For the sake of simplicity we shall first suppose that the multiplicity of the eigenvalue λ_{n+1} is two and let $\{u_{i_0}, u_{i_0+1}\}$ be a basis of $ker(\lambda_{n+1}\mathcal{I} + \Delta)$. If $\gamma' \in A_{n+1}$ there is nothing to prove. If $\gamma' \notin A_{n+1}$ then there exists a constant $\alpha \neq 0$ such that

$$u_{i_0}(x) = \alpha u_{i_0+1}(x),\ \forall x \in \gamma' \tag{4.12}$$

As α in (4.12) is the same for all points $x \in \gamma'$ we obtain:

$$\alpha = \frac{u_{i_0}(x_0')}{u_{i_0+1}(x_0')}. \tag{4.13}$$

Let us consider now a function $\widetilde{\gamma} : [0,1] \times [0,1] \to \mathbf{R}^2$ having the properties:

(A) $\widetilde{\gamma}(t, \cdot) \in \mathcal{K},\ \forall t \in [0,1]$,
(B) $\widetilde{\gamma}(0, \cdot) = \gamma'$,
(C) $\widetilde{\gamma}(t, 0) = x_0',\ \forall t \in [0,1]$,
(D) $d(\widetilde{\gamma}(t, \cdot), \gamma') < \epsilon/2,\ \forall t \in [0,1]$,
(E) The interior of the set $\widetilde{\gamma}([0,1] \times [0,1])$ is non void.

If there exists $t_0 \in [0,1]$ such that the restrictions of u_{i_0} and u_{i_0+1} to $\widetilde{\gamma}(t_0, \cdot)$ are linearly independent the curve $\gamma'' = \gamma(t_0, \cdot)$ satisfies (4.11). Let us now assume that the restrictions of u_{i_0} and u_{i_0+1} to $\widetilde{\gamma}(t_0, \cdot)$ are linearly dependent for all $t \in [0,1]$. By assumptions (C) and (E) it follows that

$$u_{i_0}(x) = \alpha u_{i_0+1}(x),\ \forall x \in \mathcal{D}, \tag{4.14}$$

where D is an open subset of Ω and α is given by (4.13). On the other hand, the function $v(x) = u_{i_0} - \alpha u_{i_0+1}(x)$ satisfies the equation

$$\Delta v(x) + \lambda_{n+1} v(x) = 0,\ \forall x \in \Omega. \tag{4.15}$$

From (4.14) and (4.15), by Holmgren's uniqueness theorem, we obtain that $v \equiv 0$ in Ω, so $u_{i_0}(x) = \alpha u_{i_0+1}(x),\ \forall x \in \Omega$, which is obviously false. If the multiplicity of λ_{n+1} is equal to $\mu_{n+1} \geq 3$ the proof can be acomplished in a similar way by using deformations with $\mu_{n+1} - 1$ fixed points.

5. The lack of exponential decay rate. In this section we shall prove that if the damping term is linear and the curve γ satisfies conditions (C1)-(C4) the decay rate for solutions of (1.1)-(1.3) is not uniform with respect to the initial data. Our method is inspired by ideas from [4] where a similar result is obtained for a boundary stabilization problem.

Let us first notice that, by taking $g(v) = v$ in (1.1), we obtain the following initial and boundary value problem

$$u" - \Delta u + u'\delta_\gamma = 0, \text{ in } \Omega \times (0,\infty) \tag{5.1}$$

$$u = 0, \text{ on } \Gamma \times (0,\infty) \tag{5.2}$$

$$u(x,0) = u^0(x), \ u'(x,0) = u^1(x). \text{ in } \Omega. \tag{5.3}$$

It is by now well known (see [16]) that the uniform energy decay for the solutions of (5.1)-(5.3) is equivalent to the existence of two constants $C, T > 0$ such that

$$||u^0||^2 + ||u^1||^2 \leq C \int_0^T \int_\gamma |u'|^2 d\gamma dt, \ \forall u^0 \in H_0^1(\Omega), \ \forall u^1 \in L^2(\Omega). \tag{5.4}$$

In order to prove that (5.4) is false for any curve γ satisfying (C1)-(C4) we introduce the function y, solution of the following initial and boundary value problem

$$y" - \Delta y = 0, \text{ in } \Omega \times (0,\infty) \tag{5.5}$$

$$y = 0, \text{ on } \Gamma \times (0,\infty) \tag{5.6}$$

$$y(x,0) = u^0(x), \ y'(x,0) = u^1(x). \text{ in } \Omega. \tag{5.7}$$

Let us first notice that we have the following result

LEMMA 5.1. *Suppose that (5.4) holds for some $C, T > 0$ and $\gamma \in \mathcal{K}$. Then there exists a constant $C' > 0$ such that*

$$||u^0||^2 + ||u^1||^2 \leq C' \int_0^T \int_\gamma |y'|^2 d\gamma dt, \ \forall u^0 \in H_0^1(\Omega), \ \forall u^1 \in L^2(\Omega). \tag{5.8}$$

Proof. We can write $y = \xi + \eta$, where ξ and η are solutions of the problems

$$\xi" - \Delta \xi + \xi'\delta_\gamma = 0, \text{ in } \Omega \times (0,\infty) \tag{5.9}$$

$$\xi = 0, \text{ on } \Gamma \times (0,\infty) \tag{5.10}$$

$$\xi(x,0) = u^0(x), \ \xi'(x,0) = u^1(x). \text{ in } \Omega, \tag{5.11}$$

$$\eta" - \Delta \eta + \eta'\delta_\gamma = y'\delta_\gamma, \text{ in } \Omega \times (0,\infty) \tag{5.12}$$

$$\eta = 0, \text{ on } \Gamma \times (0, \infty) \tag{5.13}$$

$$\eta(x,0) = 0, \ \eta'(x,0) = 0. \text{ in } \Omega. \tag{5.14}$$

As we assumed that (5.4) holds true it follows that

$$||u^0||^2 + ||u^1||^2 \leq C \int_0^T \int_\gamma |\xi'|^2 d\gamma dt \leq 2C(\int_\gamma |y'|^2 d\gamma dt + \int_\gamma |\eta'|^2 d\gamma dt), \ \forall u^0 \in H_0^1(\Omega), \tag{5.15}$$

$$\forall u^1 \in L^2(\Omega).$$

On the other hand, by taking the inner product of (5.12) by η' we obtain

$$\int_0^T \int_\gamma |\eta'|^2 d\gamma dt \leq \int_0^T \int_\gamma |y'|^2 d\gamma dt. \tag{5.16}$$

From (5.15) and (5.16) we obviously get the conclusion (5.8). □

We can now state the main result of this section

THEOREM 5.2. *If $\gamma \in \mathcal{N}$ the decay to zero of the solutions of (1.1)-(1.3) is not uniform with respect to the initial data. More precisely, for all time T and any constant $\epsilon > 0$, there exist initial data $u^0 \in H_0^1(\Omega)$, $u^1 \in L^2(\Omega)$ such that the solution u of (1.1)-(1.3) satisfies the conditions*

$$\|\{u^0, u^1\}\|_{H_0^1(\Omega)\times L^2(\Omega)} = 1, \ \|\{u(t), u'(t)\}\|_{H_0^1(\Omega)\times L^2(\Omega)} > 1 - \epsilon, \ 0 \leq t < T.$$

Proof. Let us suppose that the (5.8) is true for a time $T > 0$ and a curve $\gamma \in \mathcal{N}$. By assumptions (C1)-(C4) there exists an open set ω such that $\gamma \subset \omega$ and $\bar{\omega} \subset \Omega$, so (5.8) implies that

$$||u^0||^2 + ||u^1||^2 \leq C' \int_0^T ||y'||^2_{H^1(\omega)} dt, \ \forall u^0 \in H_0^1(\Omega), \ \forall u^1 \in L^2(\Omega). \tag{5.17}$$

The counterexample given in [20] shows that the estimate (5.17) holds true only if any geometric ray wandering in Ω (reflected at the boundary) passes through ω in time T, condition which cannot be fullfiled if $\bar{\omega} \subset \Omega$.□

Remark. *The theorem above can also be interpreted as follows: for given $\gamma \in \mathcal{N}$ and $h : \mathbf{R}^+ \to \mathbf{R}^+$, with $h(t) \to 0$ when $t \to \infty$, there exist initial data $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ such that*

$$\|u(t)\|_{H_0^1(\Omega)} + \|u'(t)\|_{L^2(\Omega)} \geq h(t), \ \forall t \geq 0.$$

6. Comments and related questions. The results in this paper can be generalized to the case of n-space dimensions, in the following setting:

(a) $\gamma = \partial\omega$, where $\omega \subset \bar{\omega} \subset \Omega$ is an open set with regular boundary.

(b) The function g satisfies appropriate growth conditions, such that the problem (1.1)-(1.3) is well posed.

Another way to generalize the results in section 1-6 consists in weakening the assumptions on the function g. More precisely, following [13] we can replace the strict monotonicity of g by simple monotonicity and the condition $0 \notin Int[g^{-1}(0)]$.

A question related to the problem studied in this paper is the stabilization of elastic plates by the use of piezoelectric actuators (see [11] for appropriate models and [22] for the one dimensional case). In this case new difficulties arise as the control function is scalar valued, so one may hope that strong stabilization holds only in the case of simple eigenvalues.

In general, the methods used in this paper apply for a large class of equations of the form $u'' + Au + g(u')\delta_\gamma = 0$ incuding the plate equation with various boundary conditions.

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