Diffraction at a Curved Grating. TM and TE cases, Homogenization *

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Abstract

A hierarchy of effective boundary conditions are derived for Helmholtz equation with Dirichlet or Neumann boundary conditions. These generalized impedance boundary conditions approximate the effect of thin dielectric layers with perfectly conducting inclusions covering an object. Well-posedness of the problems with these new boundary conditions are proved and optimal error estimates are given.

1 Introduction

In this paper, we present a first attempt to study the diffraction of an electro-magnetic wave by a *curved* and *finite* grating. The grating we consider is cylindrical, lit up by a TM or TE polarized incident plane wave, which leads us to the study of 2D Helmholtz equation with Dirichlet or Neumann boundary conditions on the boundary of a perfect conductor, covered by a periodic grating image of an infinite plane grating.

The purpose of this paper is to describe approximate equalities between the curved grating and a hierarchy of generalized impedance conditions. This approximation is valid for the ratio of the period of the grating to the wave length tending to 0. We prove that we can replace the grating by an equivalent boundary condition on the boundary of the conductor and we exhibit the boundary layer correctors. If we are not interested in the boundary layer, we can write the generalized impedance conditions, each of them generates a well-posed boundary value problem for Helmholtz equation and can be used for accurate numerical calculations of the electro-magnetic field in the exterior of the coated body.

The outline of this paper is as follows. We first set the problem we are dealing with. In section 3, devoted to the TM case, we prove the convergence to the limiting problem and we obtain the first error estimate proved by contradiction using compactness and uniqueness arguments. Constructing boundary layer corrector by solving problems set in the elementary cell of the plane infinite grating, we use them to derive generalized impedance conditions. The curvature appears for the first time from the second order approximation. In section 4 we derive the first equivalent boundary condition in the TE case by correcting analogous quantities to the order 2 .

Our method has a great generality in the sense that it allows very general diffraction structures. We deal with only one thin dielectric layer for the sake of simplicity, the generalization to several superposed layers does not present new major difficulties.

We finally mention that the problem studied in this paper belongs to the subject which is commonly referred to as diffractive optics in grating material. Interesting engineering and

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numerical aspects can be found in the collection of articles edited by R. Petit [15]. Related work was done by T. Abboud and H. Ammari [1] and [2], Y. Achdou [4], Y. Achdou, O. Pironneau and A. Zebic [5], T.S. Angel, R.E. Kleinman and F. Hettlich [6], M. Artola and M. Cessenat [7], A. Bendali and K. Lemrabet [8], B. Engquist and J.C. Nédélec [10], R.F. Harrington and J.R. Mautz [11], S.N. Karp and F.C. Karal [12] and T.B.A. Senior and J.L. Volakis [14].

2 Setting of the problem

The curved grating is constructed in the following way : let $\Omega_0^c \subset \mathbb{R}^2$ be a bounded smooth open set, $\Gamma = \partial\Omega_0^c$. For $\mathbf{x}$ in $\mathbb{R}^2 \setminus \overline{\Omega}_0^c$, sufficiently close to Γ, we denote by $\mathbf{x}_\Gamma$ the orthogonal projection of $\mathbf{x}$ on Γ, s a curvilinear abscissa of $\mathbf{x}_\Gamma$,

$$n \quad = \quad \mid \mathbf{x} - \mathbf{x}_\Gamma \mid .$$

(s, n) is a parameterization of the neighborhood of Γ. Consider now a bi-dimensional infinite plane periodic grating of period unity in $\mathbb{R} \times \mathbb{R}_+^*$, made up of a dielectric layer with perfectly conducting inclusions, which is in contact with a thick perfect conductor. Consider N a large integer and let us set $\delta = l/N$. The mapping

$$\phi_\delta : (y_1, y_2) \in \mathbb{R} \times \mathbb{R}_+^* \mapsto \phi_\delta(y_1, y_2) = \mathbf{x}(s, n) \in \mathbb{R}^2 \setminus \overline{\Omega}_0^c,$$

where

$$s = \delta y_1 \bmod (l) \quad \text{and} \quad n = \delta y_2,$$

sends the plane grating onto the curved one. Let Ω_N^c denote the conducting inclusions,

$$\Omega \quad = \quad \mathbb{R}^2 \setminus \overline{\Omega}_0^c, \Omega_\delta^c \quad = \quad \Omega_0^c \cup \Omega_N^c,$$

Ω_δ^d the dielectric medium, Ω_δ^e the exterior domain,

$$\Omega^\delta \quad = \quad \mathbb{R}^2 \setminus \overline{\Omega_\delta^c}$$

and Γ^δ the interface between the dielectric material and the exterior domain. As usual in the periodic homogenization theory, we first define a so called unit cell, which, upon rescaling to size δ, becomes the period of a periodic structure. We denote by O the unit cell of the plane grating, O^c the scatterer, O^d the dielectric material :

$$O \quad = \quad]0, 1[\times \mathbb{R}_+^* \setminus \overline{O}^c,$$

$$G \quad = \quad]0, 1[\times \{0\},$$

G^c the boundary of O^c and G^d the interface between the dielectric material and the exterior domain,

$$G_h = \Big\{ \mathbf{y} = (y_1, y_2) \in O, y_2 = h \Big\}, \ O_h \ = \ O \cap \{y_2 < h\}, \ O^e \ = \ O \setminus \overline{O^d},$$

and

$$O_h^e \quad = \quad O^e \cap \{y_2 < h\} \ .$$

These open sets are supposed to be smooth.

We shall need some differential operators on the curve Γ that we will introduce. The normal vector on Γ is the opposite of the gradient of the function n. We use this property to define

$$\mathbf{n}(\mathbf{x}) = -\nabla n(\mathbf{x}) = \mathbf{n}(\mathbf{x}_\Gamma).$$

The curvature is defined as

$$c(s) = \mathrm{div}\mathbf{n}.$$

We also introduce the family of parallel curves Γ_n :

$$\Gamma_n = \Big\{ \mathbf{x} \in \mathbb{R}^2 \setminus \Omega_0^c \; ; \mathbf{x} = \mathbf{x}_\Gamma(s) + n\mathbf{n}(s) \; , \mathbf{x}_\Gamma(s) \in \Gamma \Big\}.$$

The vector field $\mathbf{n}$ is the field of the normal vectors to the curves Γ_n. To any function u defined on the curve Γ, we associate the function $\tilde{u}$ defined on the interior domain by

$$\tilde{u}(\mathbf{x}) = u(\mathbf{x}_\Gamma).$$

The tangential gradient of the function u is :

$$\nabla_\Gamma u = \nabla \tilde{u} \,|_\Gamma \, .$$

This vector belongs to the tangent to the curve Γ. We denote by Δ_Γ the vectorial operator of Beltrami-Laplace on Γ. The length element ds_n on the curve Γ_n at the point

$$\mathbf{x} = \mathbf{x}_\Gamma(s) + n\mathbf{n}(s),$$

is related to the length element of the curve Γ at the point $\mathbf{x}_\Gamma$ by :

$$ds_n = (1 + c(s)n) \, ds.$$

Let u be a smooth function defined in Ω, we have :

$$\Delta u = \frac{\partial^2 u}{\partial s_n^2} + \frac{\partial^2 u}{\partial n^2} + c(s,n)\frac{\partial u}{\partial n} \quad \text{and} \quad \nabla u = \nabla_{\Gamma_n} u + \frac{\partial u}{\partial n}\, \mathbf{n}. \tag{1}$$

where $c(s,n)$ denotes the curvature of Γ_n.

In this section, we also describe the functional setting needed to state our boundary value problems in a suitable mathematical form. We shall use standard notations for the usual functional spaces : $L^2(O)$ is the space of complex square integrable functions defined on O, $\mathcal{C}_0^\infty(O)$ is the space of regular functions defined on O with compact support in O. $H^s(\Omega_R)$ is the usual Sobolev space of order $s \in \mathbb{R}$, $H^1_\Delta(\Omega_R)$ is the set of functions u in $H^1(\Omega_R)$, with Δu in $L^2(\Omega_R)$. $H^{s,loc}_\Delta(\overline{O})$ is the set of functions in $H^s_\Delta(\overline{O}_R)$ for every positive R.

We need to define Sobolev spaces of periodic functions in y_1 of period 1. Let $\mathcal{C}^\infty_{per}(\mathbb{R}^2)$ denote the set of regular periodic functions in y_1 of period 1. $\mathcal{C}^\infty_{per}(O)$ will be the space of restrictions to O of functions $\mathcal{C}^\infty_{per}(\mathbb{R}^2)$. This enables us to consider the periodic Sobolev space $H^1_{per}(O)$, closure of $\mathcal{C}^\infty_{per}(O)$ in $H^1(O)$.

It is easy to see that, equipped with the usual H^1 norm and Hibertian scalar product, $H^1_{per}(O)$ becomes a Hilbert space. We introduce $H^1_{\Delta,per}(O) = H^1_{per}(O) \cap H^1_\Delta(O)$. We will also use the Fréchet space which is locally in $H^1_{\Delta,per}(O)$, in the same sense as above. It will be designed by $H^{1,loc}_{\Delta,per}(\overline{O})$. It is not difficult to show that we can extend a periodic version of the first trace on G_h, well defined for functions in $\mathcal{C}^\infty_{per}(O_h)$ to $H^1_{per}(O_h)$, with values in a fractionnary order Sobolev space of periodic functions over G_h denoted by $H^{1/2}_{per}(G_h)$.

In the case of a TM polarized incident electro-magnetic field :

$$\mathbf{E}^{in}(x_1,x_2,x_3) \quad = \quad u^{in}(x_1,x_2)\,\mathbf{e}_3,$$

the scattered electric field has the following form :

$$\mathbf{E}^\delta(\mathbf{x}) \quad = \quad u^\delta(x_1,x_2)\,\mathbf{e}_3,$$

and Maxwell's equations become :

$$(2) \qquad \left\{ \begin{array}{rcll} \mathrm{div}\dfrac{1}{\mu_r^\delta}\nabla u^\delta + k^2\varepsilon_r^\delta u^\delta &=& 0 & \text{in } \Omega^\delta, \\ u^\delta &=& 0 & \text{on } \partial\Omega_\delta^c, \\ u^\delta - u^{in} & \multicolumn{3}{l}{\text{satisfying an outgoing radiation condition at infinity.}} \end{array} \right.$$

With a TE polarized incident electro-magnetic field, the scattered magnetic field has the following form :

$$\mathbf{H}^\delta(\mathbf{x}) \quad = \quad v^\delta(x_1,x_2)\,\mathbf{e}_3,$$

and Maxwell's equations become :

$$(3) \qquad \left\{ \begin{array}{rcll} \mathrm{div}\dfrac{1}{\varepsilon_r^\delta}\nabla v^\delta + k^2\mu_r^\delta v^\delta &=& 0 & \text{in } \Omega^\delta, \\ \dfrac{\partial v^\delta}{\partial n} &=& 0 & \text{on } \partial\Omega_\delta^c, \\ v^\delta - v^{in} & \multicolumn{3}{l}{\text{satisfying an outgoing radiation condition at infinity.}} \end{array} \right.$$

where

$$\varepsilon_r^\delta(\mathbf{x}) \quad = \quad \varepsilon_r^0(\phi_\delta^{-1}(\mathbf{x}))$$

and

$$\mu_r^\delta(\mathbf{x}) \quad = \quad \mu_r^0(\phi_\delta^{-1}(\mathbf{x})).$$

ε_r^0 and μ_r^0 are respectively the electric and the magnetic permeability of the plane grating, they can be written as $\varepsilon_r^0 = \varepsilon_r' + i\varepsilon_r''$ and $\mu_r^0 = \mu_r' + i\mu_r''$, where $\varepsilon_r^0 \in L^\infty(\mathbb{R}^2), \mu_r^0 \in L^\infty(\mathbb{R}^2)$, $\varepsilon_r'(\mathbf{x}) \geq \varepsilon_0 > 0$, $\varepsilon_r''(\mathbf{x}) \geq 0$, $\mu_r'(\mathbf{x}) \geq \mu_0 > 0$ and $\mu_r''(\mathbf{x}) \geq 0$. $\varepsilon_r^0 = \mu_r^0 = 1$ in the exterior domain.

We introduce a ball B_R of radius $R > 0$ containing $\overline{\Omega_\delta^c \cup \Omega_\delta^d}$. S_R denotes the boundary of B_R . We can express the exterior normal derivative $\dfrac{\partial u^\delta}{\partial n}(r,\theta)$ and this gives us the Dirichlet-Neumann operator $\mathcal{T}$ on S_R, a continuous operator from $H^{1/2}(S_R)$ into $H^{-1/2}(S_R)$, this operator is such that (see Nédélec [13] for the proof) :

$$(4) \qquad -\Re e\langle \mathcal{T}u, u\rangle \geq \frac{1}{2R}\,|u|^2_{L^2(S_R)} \quad \text{and} \quad \Im m\langle \mathcal{T}u, u\rangle > 0 \quad \text{if } u \neq 0.$$

Helmholtz equations (2) and (3) have unique solutions in $H^1_{loc}(\overline{\Omega}^\delta)$ satisfying outgoing radiation condition at infinity. The proofs are based on Fredholm's alternative and on the inequalities (4).

3 Effective boundary conditions. The TM case

3.1 Limiting problem, first estimate

We want to replace the differential equation (2) in the domain Ω^δ by an approximate boundary value problem in Ω, when $\delta \to 0$. We will call p^δ the extension operator into the conductor inclusions Ω_N^c by 0. We introduce $\Omega = \mathbb{R} \setminus \overline{\Omega}_0^c$, $\Omega_R = \Omega \cap B_R$. The proofs of the error estimates are based on the following lemma :

LEMMA 3.1. *Consider X a separable Hilbert space, $(a_0^n)_{n\in\mathbb{N}}, (a_r^n)_{n\in\mathbb{N}}, a_0$ and a_r continuous sesquilinear forms on X satisfying the conditions :*

$$(i)\ \exists\, \alpha > 0, \forall\, n \in \mathbb{N}, \forall\, v \in X, \Re e\, a_0^n(v,v) \geq \alpha \ \| v \|_X^2\,;$$

$$(ii)\ \forall\, (u_n) \in X^{\mathbb{N}},\ u_n \rightharpoonup u \Rightarrow a_r^n(u_n,u_n) \to a_r(u,u)\,;$$

$$(iii)\ \forall\, (u_n) \in X^{\mathbb{N}},\ u_n \to u \Rightarrow a_0^n(u_n,u_n) \to a_0(u,u)\,;$$

$$(iv)\ a(u,u) = a_0(u,u) + a_r(u,u) = 0 \implies u = 0\,;$$

Let (u_n) be a sequence in X, then

$$a_0^n(u_n,u_n) + a_r^n(u_n,u_n) = O(n^{-s}) \parallel u_n \parallel_X \ \ (s > 0) \implies \parallel u_n \parallel_X = O(n^{-s}).$$

Proof. This lemma is proved by contradiction, using compactness and uniqueness arguments. If we suppose that lemma (3.1) were not true, then for every $C > 0, n_0 \in \mathbb{N}$, there would exist $n \geq n_0$ such that $|| u_n ||_X \geq Cn^{-s}$. So there exists a subsequence of $(n^s u_n)$, still denoted by $(n^s u_n)$ such that $|| n^s u_n ||_X$ converges to $+\infty$. We introduce

$$v_n = \frac{u_n}{|| u_n ||_X},$$

we have :

$$|| v_n ||_X = 1 \Rightarrow v_n \rightharpoonup v. \tag{5}$$

Now, we shall show that there exists a subsequence of (v_n) that converges to 0 in X. The existence of such subsequence would contradict (5).

$$a^n(u_n,u_n) = a_0^n(u_n,u_n) + a_r^n(u_n,u_n) = O(n^{-s}) \parallel u_n \parallel_X$$

implies that :

$$a^n(v_n,v_n) = a_0^n(v_n,v_n) + a_r^n(v_n,v_n) \to 0. \tag{6}$$

Furthermore, since (v_n) is bounded and a_r^n satisfies (ii), there exists a subsequence, still denoted by (v_n), such that

$$a_r^n(v_n,v_n) \to a_r(v,v).$$

Hence, using (6) and (iii), v_n converges strongly to v in X. It follows that :

$$a^n(v_n,v_n) \to a(v,v) = 0.$$

Finally, using property (iv), it follows that $v = 0$. This contradicts (5). Hence lemma (3.1) is proved. □

We have the following theorem :

THEOREM 3.1. *For any $R > 0$, the sequence of solutions $p^\delta u^\delta$ of (2) converges in $H^1(\Omega_R)$ strongly to u^0 the unique solution of the Helmholtz equation :*

$$(7) \qquad \left\{ \begin{array}{lll} \Delta u^0 + k^2 u^0 & = 0 & \text{in } \Omega, \\ u^0 & = 0 & \text{on } \Gamma, \\ u^0 - u^{in} & \multicolumn{2}{l}{\text{satisfying an outgoing radiation condition at infinity.}} \end{array} \right.$$

Moreover, there exists a constant $C(R) > 0$ independent of δ such that :

$$(8) \qquad \|p^\delta u^\delta - u^0\|_{H^1(\Omega_R)} \quad \leq \quad C(R)\, \delta^{1/2}.$$

Proof. We shall show that there exists a constant $C > 0$ such that :

$$(9) \qquad \int_{\Omega_R} |\nabla p^\delta u^\delta|^2 \; dx \; + \; \frac{1}{2R} \int_{S_R} |p^\delta u^\delta|^2 \; d\sigma \; \leq \; C \int_{\Omega_R} |p^\delta u^\delta|^2 \; dx.$$

Since $p^\delta u^\delta$ satisfies :

$$\int_{\Omega_R} \frac{1}{\mu_r^\delta} |\nabla p^\delta u^\delta|^2 \; dx - k^2 \int_{\Omega_R} \varepsilon_r^\delta \, |p^\delta u^\delta|^2 \; dx - \langle \mathcal{T} p^\delta u^\delta, p^\delta u^\delta \rangle = \langle \mathcal{T} u^{in} - \frac{\partial u^{in}}{\partial n}, p^\delta u^\delta \rangle.$$

(4) implies that :

$$\int_{\Omega_R} |\nabla p^\delta u^\delta|^2 \; dx \; + \; \frac{1}{2R} \int_{S_R} |p^\delta u^\delta|^2 \; d\sigma \leq C \int_{\Omega_R} |p^\delta u^\delta|^2 \; dx + \; |\langle \mathcal{T} u^{in} - \frac{\partial u^{in}}{\partial n}, p^\delta u^\delta \rangle| \; .$$

Furthermore, there exists $C > 0$ such that :

$$|\langle \mathcal{T} u^{in} - \frac{\partial u^{in}}{\partial n}, p^\delta u^\delta \rangle| \; \leq C \, || \, \mathcal{T} u^{in} - \frac{\partial u^{in}}{\partial n} \, ||_{L^2(S_R)} \; || \, p^\delta u^\delta \, ||_{L^2(S_R)} \; .$$

Using :

$$\forall \eta > 0, \; \exists C(\eta), \; \forall \varphi \in H^1(\Omega_R), \;\; || \, \varphi \, ||_{L^2(S_R)} \leq \eta \, || \, \nabla \varphi \, ||_{L^2(\Omega_R)} + C(\eta) \, || \, \varphi \, ||_{L^2(\Omega_R)},$$

we then deduce (9).

We are now ready to establish the convergence to the limiting problem. Let us first prove by contradiction that $(p^\delta u^\delta)$ is bounded in $H^1(\Omega_R)$. Suppose that we have a subsequence, still denoted by $(p^\delta u^\delta)$, such that $|| \, p^\delta u^\delta \, ||_{L^2(\Omega_R)}$ converges to $+\infty$, we introduce

$$w^\delta \quad = \quad \frac{p^\delta u^\delta}{|| \, p^\delta u^\delta \, ||_{L^2(\Omega_R)}}.$$

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ equal to 1 in a sufficiently large neighborhood of 0, we define :

$$\varphi^\delta(\mathbf{x}(s,n)) \quad = \quad \varphi(\frac{n}{\delta}).$$

By using $(1 - \varphi^\delta)\ \psi$ where $\psi \in H^1(\Omega_R)$ such that $\psi = 0$ on Γ as a test function on the variational formulation of (2), we obtain :

$$(10) \qquad \int_{\Omega_R} \frac{1}{\mu_r^\delta} \nabla w^\delta \nabla \overline{\psi} \, dx \; - k^2 \int_{\Omega_R} \varepsilon_r^\delta w^\delta \overline{\psi} \, dx \; - \langle \mathcal{T} w^\delta, \psi \rangle \quad \longrightarrow \quad 0.$$

Thus we deduce that w^δ converges to 0, $H^1(\Omega_R)$ weakly, but

$$|| \, w^\delta \, ||_{L^2(\Omega_R)} \quad = \quad 1,$$

which leads to a contradiction. The strong convergence is deduced from the compact imbedding of $H^1(\Omega_R)$ in $L^2(\Omega_R)$.

To prove the error estimate (8), we define

$$\begin{aligned} X &= \Big\{ \, v \in H^1(\Omega_R) \; : \; v = 0 \; \text{ on } \; \Gamma \, \Big\}, \\ a^\delta(u,v) &= \int_{\Omega_R} \frac{1}{\mu_r^\delta} \nabla u . \nabla v \, dx - k^2 \int_{\Omega_R} \varepsilon_r^\delta \, u \, v \, dx - \langle \mathcal{T} u, v \rangle . \end{aligned}$$

(a^δ) is a family of sesquilinear forms on X satisfying the conditions of the lemma 3.1. We can prove that there exists a constant $C > 0$ independent of δ such that :

$$|a^\delta(p^\delta u^\delta - u^0, p^\delta u^\delta - u^0)| \quad \leq \quad \|\varphi^\delta u^0\|_{H^1(\Omega_R)} \|p^\delta u^\delta - u^0\|_{H^1(\Omega_R)} + C \|\varphi^\delta u^0\|^2_{H^1(\Omega_R)} .$$

Regularity of u^0 shows that

$$\|\varphi^\delta u^0\|_{H^1(\Omega_R)} \quad = \quad O(\delta^{1/2})$$

and Lemma (3.1) allows us to conclude. □

3.2 First correctors and generalized impedance conditions

We exhibit the boundary layer correctors u^0_{BL} and u^1_{BL} of order 0 and 1, the correctors u^1 and u^2, and we prove error estimates.

The boundary layer corrector u^0_{BL} corrects the boundary condition on $\partial\Omega^c_N$ and the jump of $[\, \dfrac{1}{\mu_r^\delta} \dfrac{\partial u^0}{\partial n} \,]$ across Γ^δ. For this we introduce $(\alpha^{(0)}, a^{(0)}) \in H^{1,loc}_{\Delta,per}(\overline{O}) \times \mathbb{C}$ unique solution of the following problem :

$$(11) \qquad \left\{ \begin{array}{l} \operatorname{div} \dfrac{1}{\mu_r^0} \nabla \alpha^{(0)} = 0 \quad \text{in } O^e \cup O^d, \\ \alpha^{(0)} = -y_2 \quad \text{on } G^c, \;\; \alpha^{(0)} = 0 \quad \text{on } G, \\ [\, \dfrac{1}{\mu_r^0} \dfrac{\partial \alpha^{(0)}}{\partial n} \,] = [\, - \dfrac{1}{\mu_r^0} \,] \quad \text{on } G^d \text{ and } \alpha^{(0)} - a^{(0)} \in \mathcal{S}. \end{array} \right.$$

$\mathcal{S}$ denotes the space of functions $\mathcal{C}^\infty$, periodic in y_1, with exponential decay in $y_2 (y_2 \to +\infty)$ and $[.]$ is the jump across G^d.

The linearity of the equations and the smoothness of the data and boundaries insure that, if $\alpha^{(0)}$ exists, it is $\mathcal{C}^\infty$, and its trace on every section $y_2 = h$, for h large enough, admits a Fourier expansion :

$$\forall \, y_2 \geq h, \;\; \alpha^{(0)} = \sum_{j \in \mathbb{Z}} \alpha_j^{(0)}(h) \, e^{-2\pi |j| (y_2 - h)} \, e^{i 2\pi j y_1} .$$

So,

$$\frac{\partial \alpha^{(0)}}{\partial y_2} = -2\pi \sum_{j \in \mathbb{Z}} \mid j \mid \alpha_j^{(0)}(h) \, e^{-2\pi |j| (y_2 - h)} \, e^{i 2\pi j y_1} .$$

Hence,

$$\frac{\partial \alpha^{(0)}}{\partial y_2} \mid_{y_2=h} = T_{per}\alpha^{(0)} \mid_{y_2=h} . \tag{12}$$

T_{per} can be extended into a continuous operator from $H^{1/2}_{per}(G_h)$ into its dual $H^{-1/2}_{per}(G_h)$. The problem (11) is equivalent to the truncated problem :

$$\begin{cases} \mathrm{div}\dfrac{1}{\mu_r^0}\nabla\alpha^{(0)} = 0 \quad \text{in } O^e \cup O^d, \\ \alpha^{(0)} = -y_2 \quad \text{on } G^c, \ \ \alpha^{(0)} = 0 \quad \text{on } G, \\ [\ \dfrac{1}{\mu_r^0}\dfrac{\partial\alpha^{(0)}}{\partial n}\ ] = [\ -\dfrac{1}{\mu_r^0}\ ] \quad \text{on } G^d \text{ and } \dfrac{\partial \alpha^{(0)}}{\partial y_2} = T_{per}\alpha^{(0)} \quad \text{on } G_h. \end{cases} \tag{13}$$

and $a^{(0)} = lim_{y_2 \to +\infty}\alpha_0^{(0)}$. Once the existence and the uniqueness have been shown for (13), then, plugging the Fourier expansion of the trace of $\alpha^{(0)}$ into the extension formula (12) leads to the solution of (11). A variational formulation for (13) is :

$$\int_{O_h} \frac{1}{\mu_r^0}\nabla\alpha^{(0)}\nabla\beta - \langle T_{per}\alpha^{(0)}, \beta\rangle = \int_{G^d} [\ \frac{1}{\mu_r^0}\ ]\beta \ \ \forall \beta \in H^1(O_h), \ \beta = 0 \text{ on } G \cup G^c. \tag{14}$$

The fact that :

$$-\langle T_{per}\alpha, \alpha\rangle \ \geq \ 0 \quad \forall \ \ \alpha \in H^{1/2}_{per}(G_h), \tag{15}$$

shows that the sesquilinear form is coercive. (13) has a unique solution in $H^1_{\Delta,per}(O_h)$. Hence, problem (11) has a unique solution in $H^{1,loc}_{\Delta,per}(\overline{O})$.

We define u^0_{BL} by :

$$u^0_{BL}(\mathbf{x}) \quad = \quad \delta\, \frac{\partial u^0}{\partial n}(\mathbf{x}_\Gamma) \left\{ \ \alpha^{(0)}(\frac{s}{\delta}, \frac{n}{\delta}) - a^{(0)} \ \right\}. \tag{16}$$

The mapping ϕ_δ modifies the operator $\mathrm{div}\dfrac{1}{\mu_r}\nabla$, in fact, using (1), we obtain :

$$\begin{aligned} \Delta_{\mathbf{x}} u^0_{BL}(\mathbf{x}) \quad = \quad & \frac{\partial u^0}{\partial n}(\mathbf{x}_\Gamma)\underbrace{\frac{1}{\delta}\Delta_{\mathbf{y}}\alpha^{(0)}(\frac{s}{\delta}, \frac{n}{\delta})}_{=0} + 2\frac{\partial^2 u^0}{\partial s \partial n}(\mathbf{x}_\Gamma)\frac{\partial \alpha^{(0)}}{\partial y_1}(\frac{s}{\delta}, \frac{n}{\delta}) \\ & + c(s)\frac{\partial u^0}{\partial n}(\mathbf{x}_\Gamma)\left\{ \frac{\partial \alpha^{(0)}}{\partial y_2}(\frac{s}{\delta}, \frac{n}{\delta}) - 2\frac{n}{\delta}\frac{\partial^2\alpha^{(0)}}{\partial y_1^2}(\frac{s}{\delta}, \frac{n}{\delta}) \right\} + O(\delta). \end{aligned}$$

These terms will be corrected by u^1_{BL}. That is where the curvature appears explicitly for the first time. We define u^1 which corrects the boundary condition on Γ as the unique solution of :

$$\begin{cases} \Delta u^1 + k^2 u^1 \quad = \quad 0 \qquad\qquad \text{in } \Omega, \\ \qquad\qquad u^1 \quad = \quad a^{(0)}\dfrac{\partial u^0}{\partial n} \qquad \text{on } \Gamma, \\ u^1 \quad \text{satisfying an outgoing radiation condition at infinity.} \end{cases} \tag{17}$$

The constant $a^{(0)}$ may be obtained explicitly (see the last paragraph in the next section).

To exhibit u^1_{BL}, which corrects the boundary condition on $\partial\Omega^c_N$, the jump of $[\ \frac{1}{\mu^\delta_r}\frac{\partial u^0}{\partial n}\ ]$ across Γ^δ to the order 2, and the PDE in the boundary layer. In fact, we have :

$$\operatorname{div}\frac{1}{\mu^\delta_r}\nabla(u^0+u^0_{BL})+k^2\varepsilon^\delta_r(u^0+u^0_{BL}) \quad = \quad O(1),$$

in the thin dielectric layer. Then we are led to introduce $(\alpha^{(1)}_1, a^{(1)}_1)\in H^{1,loc}_{\Delta,per}(\overline{O})\times\mathbb{C}$ and $(\alpha^{(1)}_2, a^{(1)}_2)\in H^{1,loc}_{\Delta,per}(\overline{O})\times\mathbb{C}$ unique solutions of the following problems :

$$\tag{18}\left\{\begin{array}{l}\operatorname{div}\frac{1}{\mu^0_r}\nabla\alpha^{(1)}_1=\frac{\partial\alpha^{(0)}}{\partial y_1}\quad\text{in } O^e\cup O^d,\\ \alpha^{(1)}_1=-y_2\quad\text{on } G^c,\ \ \alpha^{(1)}_1=0\quad\text{on } G,\\ [\ \frac{1}{\mu^0_r}\frac{\partial\alpha^{(1)}_1}{\partial n}\ ]=[\ -\frac{1}{\mu^0_r}\ ]\quad\text{on } G^d \text{ and } \alpha^{(1)}_1-a^{(1)}_1\in\mathcal{S}.\end{array}\right.$$

$$\tag{19}\left\{\begin{array}{l}\operatorname{div}\frac{1}{\mu^0_r}\nabla\alpha^{(1)}_2=\frac{\partial\alpha^{(0)}}{\partial y_2}-2y_2\frac{\partial^2\alpha^{(0)}}{\partial y_1^2}\quad\text{in } O^e\cup O^d,\\ \alpha^{(1)}_2=-\frac{y_2^2}{2}\quad\text{on } G^c,\ \ \alpha^{(1)}_2=0\quad\text{on } G,\\ [\ \frac{1}{\mu^0_r}\frac{\partial\alpha^{(1)}_2}{\partial n}\ ]=[\ -\frac{1}{\mu^0_r}\ ]\quad\text{on } G^d \text{ and } \alpha^{(1)}_2-a^{(1)}_2\in\mathcal{S}.\end{array}\right.$$

As in the proof of existence and uniqueness of $\alpha^{(0)}$, a truncated problem is introduced. Here, another difficulty comes from the fact that $\alpha^{(0)}$ does not have a compact support in O, but decays exponentially in y_2. However, since $\alpha^{(0)}$ satisfies (11), we can obtain a transparent boundary condition analogous to (12) :

$$\tag{20}\frac{\partial\alpha^{(1)}_1}{\partial y_2}\,|_{y_2=h} = T_{per}\alpha^{(1)}_1\,|_{y_2=h}+c(\alpha^{(0)}).$$

Where T_{per} is the operator defined by (12) and c reads :

$$c(\alpha^{(0)}) \quad = \quad -i\pi\sum_{j\in\mathbb{Z}^*}\frac{j}{|\,j\,|}\ \alpha^{(0)}_j(h)\,e^{-2\pi|j|h}\,e^{i2\pi jy_1}.$$

The truncated problem reads : seek $\alpha^{(1)}_1\in H^{1,loc}_{\Delta,per}(\overline{O})$, such that :

$$\tag{21}\left\{\begin{array}{l}\operatorname{div}\frac{1}{\mu^0_r}\nabla\alpha^{(1)}_1=\frac{\partial\alpha^{(0)}}{\partial y_1}\quad\text{in } O^e\cup O^d,\\ \alpha^{(1)}_1=-y_2\quad\text{on } G^c,\ \ \alpha^{(1)}_1=0\quad\text{on } G,\\ [\ \frac{1}{\mu^0_r}\frac{\partial\alpha^{(1)}_1}{\partial n}\ ]=[\ -\frac{1}{\mu^0_r}\ ]\quad\text{on } G^d \text{ and } \frac{\partial\alpha^{(1)}_1}{\partial y_2}=T_{per}(\alpha^{(1)}_1)+c(\alpha^{(0)})\quad\text{on } G^h.\end{array}\right.$$

The variational formulation for problem (21) is :

$$\tag{22}\int_{O_h}\frac{1}{\mu^0_r}\nabla\alpha^{(1)}_1\,\nabla\beta\ -\ \langle T_{per}\alpha^{(1)}_1,\beta\rangle \quad = \quad \int_{G^d}[\ \frac{1}{\mu^0_r}\ ]\beta\ +\ \langle c(\alpha^{(0)}),\beta\rangle$$

$\forall\, \beta \in H^1(O_h),\ \beta = 0$ on $G \cup G^c$.
This sesquilinear form is coercive. Hence, (18) has a unique solution in $H^{1,loc}_{\Delta,per}(\overline{O})$.

We define u^1_{BL} by :

$$u^1_{BL}(\mathbf{x}) \;=\; \delta\,(\frac{\partial u^1}{\partial n}(\mathbf{x}_\Gamma) + 2\frac{\partial^2 u^0}{\partial s \partial n}(\mathbf{x}_\Gamma))\Big\{\ \alpha^{(0)}(\frac{s}{\delta},\frac{n}{\delta}) - a^{(0)}\ \Big\} - \delta c(s)\frac{\partial u^0}{\partial n}(\mathbf{x}_\Gamma)\Big\{\ \alpha_2^{(1)}(\frac{s}{\delta},\frac{n}{\delta}\} - a_2^{(1)}\ \Big\}$$

$$-2\delta\frac{\partial^2 u^0}{\partial s \partial n}(\mathbf{x}_\Gamma)\ \Big\{\ \alpha_1^{(1)}(\frac{s}{\delta},\frac{n}{\delta}\} - a_1^{(1)}\ \Big\}. \tag{23}$$

The corrector of order 2 is u^2 unique solution of :

$$\left\{\begin{array}{l} \Delta u^2 + k^2 u^2 = 0 \quad \text{in } \Omega, \\ u^2 = a^{(0)}\left\{\dfrac{\partial u^1}{\partial n}(\mathbf{x}_\Gamma) + 2\dfrac{\partial^2 u^0}{\partial s \partial n}(\mathbf{x}_\Gamma)\right\} - c(s)a_2^{(1)}\dfrac{\partial u^0}{\partial n} - 2a_1^{(1)}\dfrac{\partial^2 u^0}{\partial s\partial n}(x_0) \quad \text{on } \Gamma, \\ u^2 \quad \text{satisfying an outgoing radiation condition at infinity.} \end{array}\right. \tag{24}$$

We have the following theorem :

THEOREM 3.2. *For any $R > 0$, there exists a constant $C(R) > 0$ such that :*

$$\|p^\delta u^\delta - (u^0 + u^0_{BL}) - \delta(u^1 + u^1_{BL})\|_{H^1(\Omega_R)} \le C(R)\ \delta^{3/2}. \tag{25}$$

Proof. We introduce $w^\delta = u^0 + u^0_{BL} + \delta(u^1 + u^1_{BL})$, w^δ satisfies : $\Delta w^\delta + k^2 w^\delta = 0(\delta)$, in the dielectric layer, $w^\delta = 0(\delta^2)$ on $\Omega^c_N \cup \Gamma$, and $[\ \dfrac{\partial w^\delta}{\partial n}\ ] = 0$ across Γ^d. We have :

$$a^\delta(w^\delta, w^\delta - p^\delta u^\delta) - a^\delta(p^\delta u^\delta, (1-\varphi^\delta)w^\delta - p^\delta u^\delta) = 0(\delta^{3/2})\ \|p^\delta u^\delta - w^\delta\|_{H^1(\Omega_R)}.$$

$\|\varphi^\delta w^\delta\|_{H^1(\Omega_R)} = 0(\delta^{3/2})$ implies that :

$$a^\delta(w^\delta - p^\delta u^\delta, w^\delta - p^\delta u^\delta) = 0(\delta^{3/2})\ \|p^\delta u^\delta - w^\delta\|_{H^1(\Omega_R)}.$$

Hence, lemma (3.1) allows us to prove error estimate (25). □

Starting from the boundary conditions satisfied by u^0, u^1 and u^2 on Γ, we obtain generalized impedance boundary conditions :

$$\text{order 0 :} \qquad u = 0 \qquad \text{on } \Gamma, \tag{26}$$

$$\text{order 1 :} \qquad u - \delta a^{(0)}\frac{\partial u}{\partial n} = 0 \qquad \text{on } \Gamma, \tag{27}$$

$$\text{order 2 :} \quad u - \delta a^{(0)}\frac{\partial u}{\partial n} + \delta^2\left\{a_2^{(1)}c(s)\frac{\partial u}{\partial n} + 2(a_1^{(1)} - a^{(0)})\frac{\partial^2 u}{\partial s\partial n}\right\} = 0 \quad \text{on } \Gamma. \tag{28}$$

To obtain the first order approximation (error estimate of order $\delta^{3/2}$) we replace the problem (2) by the following exterior problem with the first new boundary condition :

$$
(29) \qquad \begin{cases} \Delta u + k^2 u &= 0 \quad \text{in } \Omega, \\ u - \delta a^{(0)} \dfrac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma, \\ u - u^{in} & \text{satisfying an outgoing radiation condition at infinity.} \end{cases}
$$

and for the second order approximation (error estimate of order $\delta^{5/2}$), we use the second new boundary condition :

$$
(30) \qquad \begin{cases} \Delta u + k^2 u &= 0 \quad \text{in } \Omega, \\ u - \delta(a^{(0)} - \delta c(s) a_2^{(1)}) \dfrac{\partial u}{\partial n} - 2\delta^2 (a_1^{(1)} - a^{(0)}) \dfrac{\partial^2 u}{\partial s \partial n} &= 0 \quad \text{on } \Gamma, \\ u - u^{in} & \text{satisfying an outgoing radiation condition at infinity.} \end{cases}
$$

(29) is an exterior Robin problem. The boundary condition (27) is to be interpreted in the sense : u has boundary values $u \in H^{1/2}(\Gamma)$ and normal derivative $\dfrac{\partial u}{\partial n} \in H^{1/2}(\Gamma)$.

We shall show that $\Im m\, a^{(0)} \geq 0$, from this we may prove that problem (29) has a unique solution by using Fredholm's alternative and the inequalities (4).
Let us use $\alpha^{(0)} + y_2$ as a test function on the variational formulation (14).
Condition radiation shows that :

$$
\int_{O_h} \frac{1}{\mu_r^0} \mid \nabla \alpha^{(0)} \mid^2 + \int_{O_h} \frac{1}{\mu_r^0} \frac{\partial \alpha^{(0)}}{\partial y_2} - [\ \frac{1}{\mu_r^0}\ ] \int_{G^d} \alpha^{(0)} + y_2 \longrightarrow 0,
$$

and

$$
\int_{O_h} \frac{1}{\mu_r^0} \frac{\partial \alpha^{(0)}}{\partial y_2} \longrightarrow \int_{G^c} \frac{1}{\mu_r^0} y_2 + a^{(0)} + [\ \frac{1}{\mu_r^0}\ ] a^{(0)}.
$$

From this we may deduce that :

$$
\int_{O_h} \frac{1}{\mu_r^0} \mid \nabla \alpha^{(0)} \mid^2 + \int_{G^c} \frac{1}{\mu_r^0} y_2 \longrightarrow [\ \frac{1}{\mu_r^0}\ ](\overline{a}^{(0)} + 1) - [\ \frac{1}{\mu_r^0}\ ](a^{(0)} + 1).
$$

All the terms on the left are easily seen to have negative complex part, and it follows then that

$$
(31) \qquad \Im m\, a^{(0)} \quad \geq \quad 0.
$$

The boundary condition (28) is to be interpreted in the sense :
$u \in H^{1/2}(\Gamma)$ and $\dfrac{\partial u}{\partial n} \in H^{1/2}(\Gamma)$. From (28) we see that the solution u of (30) and its normal derivate $q = \dfrac{\partial u}{\partial n} \in H^{1/2}(\Gamma)$ satisfy :

$$
\begin{cases} \displaystyle\int_{\Omega_R} \nabla u\, \nabla \overline{w}\, dx - k^2 \int_{\Omega_R} u\, \overline{w}\, dx - \int_\Gamma q\, \overline{w}\, ds - \langle \mathcal{T} u, w \rangle &= \langle \mathcal{T} u^{in} - \dfrac{\partial u^{in}}{\partial n}, w \rangle \quad \forall\, w \in H^1(\Omega_R), \\ \displaystyle\int_\Gamma u\, \overline{p}\, ds \ - \ a_1 \int_\Gamma q\, \overline{p}\, ds - a_2 \int_\Gamma \frac{\partial q}{\partial s}\, \overline{p}\, ds &= 0 \quad \forall\, p \in H^{1/2}(\Gamma), \end{cases}
$$

where

$$
a_1 \quad = \quad \delta \left(a^{(0)} - \delta c(s) a_2^{(1)} \right)
$$

and

$$a_2 \quad = \quad 2\delta^2 \left(a_1^{(1)} - a^{(0)} \right).$$

This problem has a unique solution for δ small enough. To prove uniqueness, we use u as a test function on the previous variational formulation. Therefore,

$$\Im m \left(\overline{a_1} \parallel q \parallel^2_{L^2(\Gamma)} - \langle \mathcal{T} u, u \rangle - i\overline{a_2} \parallel q \parallel^2_{H^{1/2}(\Gamma)} \right) \quad = \quad 0.$$

Uniqueness follows then from the compact imbedding of $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$, (31) and (4).

In the case of an homogeneous dielectric layer without conducting inclusions, we found

$$a^{(0)} = \mu_r - 1, a_2^{(1)} = \frac{a^{(0)}}{2} \text{ and } a_1^{(1)} = a^{(0)}.$$

So we obtain the approximate boundary conditions given by Engquist-Nédélec [10] and Bendali-Lemrabet [8] (written on Γ instead of Γ^δ) .

If we are not interested in the boundary layer, we can write the generalized impedance conditions (27) and (28), and we obtain the following error estimates :

$$\|p^\delta u^\delta - (u^0 + \delta u^1)\|_{H^1(\omega)} \leq C(\omega)\, \delta^{3/2}, \tag{32}$$

$$\|p^\delta u^\delta - (u^0 + \delta u^1 + \delta^2 u^2)\|_{H^1(\omega)} \leq C(\omega)\, \delta^{5/2}, \tag{33}$$

for each open set $\omega \subset\subset \Omega$.

4 Effective boundary conditions. The TE case

We want to replace the differential equation (3) in the domain Ω^δ by an approximate boundary value problem in Ω, when $\delta \to 0$. We will need an extension operator into Ω^c_N, $P^\delta \in \mathcal{L}(H^1(\Omega^\delta_R), H^1(\Omega_R))$ satisfying :

$$\|\nabla P^\delta v^\delta\|_{(L^2(\Omega_R))^2} \quad \leq \quad C \; \|\nabla v^\delta\|_{(L^2(\Omega^\delta_R))^2}, \quad \forall\, v^\delta \in H^1(\Omega^\delta_R). \tag{34}$$

We first prove the existence of such operator.

Lemma 4.1. *There exists an operator $P^\delta \in \mathcal{L}(H^1(\Omega^\delta_R), H^1(\Omega_R))$ satisfying* (34)

Proof. To construct P^δ, we use an extension technique similar to the one used in [9]. We first construct an extension operator P on the elementary cell $Q = O_h \cup \overline{O}^c$, verifying :

$$\|\nabla P v\|_{(L^2(Q))^2} \quad \leq \quad C \,\|\nabla v\|_{(L^2(O_h))^2}, \; \forall\, v \in H^1(O_h), \tag{35}$$

and then we derive an extension operator P^δ on Ω_R. Let $v \in H^1(O_h)$, $m(v)$ denotes the average of v in O_h. Let $S \in \mathcal{L}(H^1(O_h), H^1(Q))$ be any extension operator (such an operator exists since the boundaries of the conductor inclusions are smooth enough). Then :

$$\|S(v - m(v))\|_{H^1(Q)} \quad \leq \quad C \,\|v - m(v)\|_{H^1(O_h)}.$$

Since the average of $v - m(v)$ is zero, we have

$$\|v - m(v)\|_{H^1(O_h)} \quad \leq \quad C \,\|\nabla v\|_{(L^2(O_h))^2}. \tag{36}$$

Set $Pv = S(v - m(v)) + m(v)$, by (36), this extension operator has the required property (35). Using the operator P, we shall show the previous lemma.
We consider P^δ defined as :

$$P^\delta v^\delta \quad = \quad P(v^\delta \circ \phi_\delta) \circ \phi_\delta^{-1}. \tag{37}$$

$P^\delta v^\delta$ satisfies the inequality (34). In fact, we have :

$$\delta^2 \int_{\Omega_R} |\nabla P^\delta v^\delta|^2(\mathbf{x})\, dx \quad \leq \quad C \int_{\Omega_R} |\nabla P(v^\delta \circ \phi_\delta)|^2(\phi_\delta^{-1}(\mathbf{x}))\, dx.$$

Thus,

$$\delta \int_{\Omega_R} |\nabla P^\delta v^\delta|^2(x) dx \quad \leq \quad C \int_Q |\nabla P(v^\delta \circ \phi_\delta)|^2(y)\, dy.$$

Using (35), we obtain that :

$$\delta \int_{\Omega_R} |\nabla P^\delta v^\delta|^2(x) dx \quad \leq \quad C \int_{O_h} |\nabla (v^\delta \circ \phi_\delta)|^2(y)\, dy.$$

It also follows that : $\displaystyle\int_{\Omega_R} |\nabla P^\delta v^\delta|^2(x)\, dx \quad \leq \quad C\, \delta^3 \int_{\delta O_h} |\nabla (v^\delta \circ \phi_\delta)|^2(y/\delta)\, dy.$
Hence,

$$\|\nabla P^\delta v^\delta\|_{(L^2(\Omega_R))^2} \quad \leq \quad C\, \|\nabla v^\delta\|_{(L^2(\Omega_R^\delta))^2}.$$

□
The behavior of $P^\delta v^\delta$ as $\delta \to 0$ will be now studied.

THEOREM 4.1. *For any $R > 0$, the sequence of solutions $P^\delta v^\delta$ of (3) converges in $H^1(\Omega_R)$ strongly to v^0 the unique solution of the Helmholtz equation :*

$$\begin{cases} \Delta v^0 + k^2 v^0 & = \quad 0 \quad \text{in } \Omega, \\ \dfrac{\partial v^0}{\partial n} & = \quad 0 \quad \text{on } \Gamma, \\ v^0 - v^{in} & \text{satisfying an outgoing radiation condition at infinity.} \end{cases} \tag{38}$$

Moreover, there exists a constant $C(R) > 0$ such that :

$$\|P^\delta v^\delta - v^0\|_{H^1(\Omega_R)} \quad \leq \quad C(R)\, \delta^{1/2}. \tag{39}$$

Proof. We shall show that there exists a constant $C > 0$ such that :

$$\int_{\Omega_R} |\, \nabla P^\delta v^\delta \,|^2\ dx \quad \leq \quad C \int_{\Omega_R} |\, P^\delta v^\delta \,|^2\ dx. \tag{40}$$

Since v^δ satisfies :

$$\int_{\Omega_R^\delta} \frac{1}{\varepsilon_r^\delta} |\, \nabla v^\delta \,|^2\ dx - k^2 \int_{\Omega_R^\delta} \mu_r^\delta \,|\, v^\delta \,|^2\ dx - \langle \mathcal{T} v^\delta, v^\delta \rangle \quad = \quad \langle \mathcal{T} v^{in} - \frac{\partial v^{in}}{\partial n}, v^\delta \rangle. \tag{41}$$

(4) implies that :

$$\int_{\Omega_R^\delta} |\, \nabla v^\delta \,|^2\ dx \quad \leq \quad C \int_{\Omega_R} |\, P^\delta v^\delta \,|^2\ dx + \,|\, \langle \mathcal{T} v^{in} - \frac{\partial v^{in}}{\partial n}, P^\delta v^\delta \rangle \,|\ .$$

From the inequality (34), we deduce (40).
Now, let us prove by contradiction that $(P^\delta v^\delta)$ is bounded in $H^1(\Omega_R)$. Suppose that we have a subsequence, still denoted by $(P^\delta v^\delta)$, such that $\| P^\delta v^\delta \|_{L^2(\Omega_R)}$ converges to $+\infty$, we introduce :

$$w^\delta = \frac{P^\delta v^\delta}{\| P^\delta v^\delta \|_{L^2(\Omega_R)}},$$

for any $\psi \in H^1(\Omega_R)$, we have :

$$\int_{\Omega_R} \frac{1}{\varepsilon_r^\delta} \nabla w^\delta \nabla \overline{\psi}\, dx - k^2 \int_{\Omega_R} \mu_r^\delta w^\delta \overline{\psi}\, dx - \langle \mathcal{T} w^\delta, \psi \rangle = \frac{1}{\| P^\delta v^\delta \|_{L^2(\Omega_R)}} \langle \mathcal{T} v^{in} - \frac{\partial v^{in}}{\partial n}, \psi \rangle.$$

Thus we deduce that w^δ converges to 0, $H^1(\Omega_R)$ weakly, but

$$\| w^\delta \|_{L^2(\Omega_R)} = 1,$$

which leads to contradiction. The strong convergence is deduced from the compact imbedding of $H^1(\Omega_R)$ in $L^2(\Omega_R)$ and the equality (41).
To prove the error estimate (39), we observe that we have :

$$\begin{aligned}
&\int_{\Omega_R^\delta} \frac{1}{\varepsilon_r^\delta} \mid \nabla (v^0 - v^\delta) \mid^2 \ dx - k^2 \int_{\Omega_R^\delta} \mu_r^\delta \mid (v^0 - v^\delta) \mid^2 \ dx - \langle \mathcal{T} v^0 - v^\delta, v^0 - v^\delta \rangle \\
&= \int_{\Omega_R} (1 - \frac{1}{\varepsilon_r^\delta}) \nabla v^0 \nabla (P^\delta v^\delta - v^0)\, dx - k^2 \int_{\Omega_R} (1 - \mu_r^\delta) v^0 (P^\delta v^\delta - v^0)\, dx \\
&\quad + \int_{\Omega_R \setminus \Omega_R^\delta} \frac{1}{\varepsilon_r^\delta} \nabla v^0 \nabla (P^\delta v^\delta - v^0)\, dx - k^2 \int_{\Omega_R \setminus \Omega_R^\delta} \mu_r^\delta v^0 (P^\delta v^\delta - v^0)\, dx.
\end{aligned} \tag{42}$$

Regularity of v^0 and lemma (3.1) allow us to conclude. □

4.1 Boundary layer corrector and first generalized impedance condition

The boundary layer corrector v^0_{BL} corrects the boundary condition on $\partial\Omega_N^c$ (to the order 1). In fact, using (1), we have for any $\mathbf{x} \in G^c$:

$$\frac{\partial v^0}{\partial n}(\mathbf{x}) = \Big\{ (1 - c(s)n) \nabla_\Gamma v^0(\mathbf{x}) - n (\frac{\partial^2 v^0}{\partial s^2} + k^2 v^0) \mathbf{n}(\mathbf{x}_\Gamma) \Big\} . \mathbf{n}(\mathbf{x}) + 0(\delta^2). \tag{43}$$

So to exhibit v^0_{BL}, we introduce $\beta^{(0)} \in (H^{1,loc}_{\Delta,per}(\overline{O}))^2$ unique solution of the following problem :

$$\left\{ \begin{array}{l}
\mathrm{div} \frac{1}{\varepsilon_r^0} \nabla \beta^{(0)} = 0 \quad \text{in } O^e \cup O^d, \\
\frac{\partial \beta^{(0)}}{\partial n} = \mathbf{n} \quad \text{on } G^c, \ \ \frac{\partial \beta^{(0)}}{\partial n} = 0 \quad \text{on } G, \\
[\ \frac{1}{\varepsilon_r^0} \frac{\partial \beta^{(0)}}{\partial n}\ ] = 0 \quad \text{on } G^d \text{ and } \beta^{(0)} \in \mathcal{S}^2.
\end{array} \right. \tag{44}$$

We define v^0_{BL} by :

$$v^0_{BL}(\mathbf{x}) = -\delta\ \nabla_\Gamma v^0(\mathbf{x}_\Gamma) \,.\, \beta^{(0)}(\frac{s}{\delta}, \frac{n}{\delta}). \tag{45}$$

v^1_{BL} corrects

$$\frac{1}{\varepsilon_r^\delta} \Delta v^0_{BL} + k^2 \mu_r^\delta v^0_{BL} = 0(1),$$

in the boundary layer, the boundary condition (43) on $\partial\Omega_N^c$ to the order 2, the boundary conditions on Γ and the jump of

$$[\ \frac{1}{\varepsilon_r^\delta}(\frac{\partial v^0}{\partial n} + \delta\frac{\partial v^1}{\partial n})\ ]$$

across Γ^δ. The mapping ϕ_δ modifies the operator $\mathrm{div}\frac{1}{\varepsilon_r}\nabla$, in fact, using (1), we obtain :

$$\begin{aligned} \Delta_{\mathbf{x}} v_{BL}^0(\mathbf{x}) &= -c(s)\nabla_\Gamma v^0(\mathbf{x}_\Gamma)\,.\Big\{\ \frac{\partial\beta^{(0)}}{\partial y_2}(\frac{s}{\delta},\frac{n}{\delta}) - 2\frac{n}{\delta}\frac{\partial^2\beta^{(0)}}{\partial y_1^2}(\frac{s}{\delta},\frac{n}{\delta})\ \Big\} \\ &+ 2\ \Big(\ \Delta_\Gamma v^0(\mathbf{x}_\Gamma) + c(s)\frac{\partial\, v^0}{\partial s}(\mathbf{x}_\Gamma)\mathbf{n}(\mathbf{x}_\Gamma)\ \Big)\,.\,\frac{\partial\beta^{(0)}}{\partial y_1}(\frac{s}{\delta},\frac{n}{\delta}) + 0(\delta). \end{aligned}$$

These terms will be corrected by v_{BL}^1. To exhibit it, we introduce $(\beta_1^{(1)},\mathbf{b}_1^{(1)})$, $(\beta_2^{(1)},\mathbf{b}_2^{(1)})$ and $(\beta_3^{(1)},\mathbf{b}_3^{(1)}) \in (H^{1,loc}_{\Delta,per}(\overline{O}))^2 \times \mathbb{C}^2$ unique solutions of the following problems :

$$\left\{\begin{array}{l} \mathrm{div}\frac{1}{\varepsilon_r^0}\nabla\beta_1^{(1)} = -\frac{\partial\beta^{(0)}}{\partial y_1} \quad \text{in } O^e\cup O^d, \\ \frac{\partial\beta_1^{(1)}}{\partial n} = 0 \quad \text{on } G^c, \quad \frac{\partial\beta_1^{(1)}}{\partial n} = \mathbf{b}_1^{(1)} \quad \text{on } G, \\ [\ \frac{1}{\varepsilon_r^0}\frac{\partial\beta_1^{(1)}}{\partial n}\ ] = [\ \frac{1}{\varepsilon_r^0}\ ]\ \mathbf{b}_1^{(1)} \quad \text{on } G^d \text{ and } \beta_1^{(1)} \in \mathcal{S}^2. \end{array}\right. \tag{46}$$

$\mathbf{b}_1^{(1)}$ is given by : $\mathbf{b}_1^{(1)} = -\displaystyle\int_O \frac{\partial\beta^{(0)}}{\partial y_2}$.

$$\left\{\begin{array}{l} \mathrm{div}\frac{1}{\varepsilon_r^0}\nabla\beta_2^{(1)} = \frac{\partial\beta^{(0)}}{\partial y_2} - 2y_2\frac{\partial^2\beta^{(0)}}{\partial y_1^2} \quad \text{in } O^e\cup O^d, \\ \frac{\partial\beta_2^{(1)}}{\partial n} = y_2\,\mathbf{n} \quad \text{on } G^c, \quad \frac{\partial\beta_2^{(1)}}{\partial n} = \mathbf{b}_2^{(1)} \quad \text{on } G, \\ [\ \frac{1}{\varepsilon_r^0}\frac{\partial\beta_2^{(1)}}{\partial n}\ ] = [\ \frac{1}{\varepsilon_r^0}\ ]\ \mathbf{b}_2^{(1)} \quad \text{on } G^d \text{ and } \beta_2^{(1)} \in \mathcal{S}^2. \end{array}\right. \tag{47}$$

$\mathbf{b}_2^{(1)}$ is given by : $\mathbf{b}_2^{(1)} = -\displaystyle\int_{G^c} y_2 n_2 + \int_O \frac{\partial\beta^{(0)}}{\partial y_2} - 2y_2\frac{\partial^2\beta^{(0)}}{\partial y_1^2}$.

$$\left\{\begin{array}{l} \mathrm{div}\frac{1}{\varepsilon_r^0}\nabla\beta_3^{(1)} = 0 \quad \text{in } O^e\cup O^d, \\ \frac{\partial\beta_3^{(1)}}{\partial n} = y_2\,\mathbf{n} \quad \text{on } G^c, \quad \frac{\partial\beta_3^{(1)}}{\partial n} = \mathbf{b}_3^{(1)} \quad \text{on } G, \\ [\ \frac{1}{\varepsilon_r^0}\frac{\partial\beta_3^{(1)}}{\partial n}\ ] = [\ \frac{1}{\varepsilon_r^0}\ ]\ \mathbf{b}_3^{(1)} \quad \text{on } G^d \text{ and } \beta_3^{(1)} \in \mathcal{S}^2. \end{array}\right. \tag{48}$$

We have : $\mathbf{b}_3^{(1)} = -\displaystyle\int_{G^c} y_2 n_2$, and to correct

$$\mathrm{div}\frac{1}{\varepsilon_r^\delta}\nabla v^0 + k^2\mu_r^\delta v^0 = 0(1),$$

in the thin layer and the jump of

$$[\ \frac{1}{\varepsilon_r^\delta}\frac{\partial v^0}{\partial n}\ ]$$

across Γ^δ, we also introduce $(\beta_4^{(1)}, b_4^{(1)})$ and $(\beta_5^{(1)}, b_5^{(1)}) \in H^{1,loc}_{\Delta,per}(\overline{O}) \times \mathbb{C}$, unique solutions of the following problems :

$$\left\{\begin{array}{l} \operatorname{div}\frac{1}{\varepsilon_r^0}\nabla\beta_4^{(1)} = 0 \quad \text{in } O^e \cup O^d \\ \frac{\partial\beta_4^{(1)}}{\partial n} = 0 \quad \text{on } G^c, \quad \frac{\partial\beta_4^{(1)}}{\partial n} = b_4^{(1)} \quad \text{on } G, \\ [\ \frac{1}{\varepsilon_r^0}\frac{\partial\beta_4^{(1)}}{\partial n}\ ] = [\ \frac{1}{\varepsilon_r^0}\ ]\ (b_4^{(1)} - 1) \quad \text{on } G^d \text{ and } \beta_4^{(1)} \in \mathcal{S}. \end{array}\right. \tag{49}$$

$$\left\{\begin{array}{l} \operatorname{div}\frac{1}{\varepsilon_r^0}\nabla\beta_5^{(1)} = -1 \quad \text{in } O^d, \ \operatorname{div}\frac{1}{\varepsilon_r^0}\nabla\beta_5^{(1)} = 0 \quad \text{in } O^e, \\ \frac{\partial\beta_5^{(1)}}{\partial n} = 0 \quad \text{on } G^c, \quad \frac{\partial\beta_5^{(1)}}{\partial n} = b_5^{(1)} \quad \text{on } G, \\ [\ \frac{1}{\varepsilon_r^0}\frac{\partial\beta_5^{(1)}}{\partial n}\ ] = [\ \frac{1}{\varepsilon_r^0}\ ]\ b_5^{(1)} \quad \text{on } G^d \text{ and } \beta_5^{(1)} \in \mathcal{S}. \end{array}\right. \tag{50}$$

We define v^1_{BL} by :

$$\begin{aligned} v^1_{BL} \ = \ & \delta c(s)\nabla_\Gamma v^0(\mathbf{x}_\Gamma).\beta_2^{(1)}(\frac{s}{\delta},\frac{n}{\delta}) + 2\delta\left\{\Delta_\Gamma v^0(\mathbf{x}_\Gamma) + c(s)\frac{\partial\, v^0}{\partial s}(\mathbf{x}_\Gamma)\mathbf{n}(\mathbf{x}_\Gamma)\right\}.\,\beta_1^{(1)}(\frac{s}{\delta},\frac{n}{\delta}) \\ & +\delta\left\{\ \frac{\partial^2\, v^0}{\partial s^2}(\mathbf{x}_\Gamma) + k^2 v^0(\mathbf{x}_\Gamma)\ \right\}\mathbf{n}(\mathbf{x}_\Gamma)\,.\,\beta_3^{(1)}(\frac{s}{\delta},\frac{n}{\delta}) - \delta\nabla_\Gamma v^1(\mathbf{x}_\Gamma)\,.\,\beta^{(0)}(\frac{s}{\delta},\frac{n}{\delta}) \\ & +\left\{\frac{\partial^2\, v^0}{\partial s^2}(\mathbf{x}_\Gamma) + k^2 v^0(\mathbf{x}_\Gamma)\right\}\ \beta_4^{(1)}(\frac{s}{\delta},\frac{n}{\delta}) + \delta k^2(\mu_r^\delta - \frac{1}{\varepsilon_r^\delta})v^0(\mathbf{x}_\Gamma)\beta_5^{(1)}(\frac{s}{\delta},\frac{n}{\delta}). \end{aligned} \tag{51}$$

The first corrector corrects the boundary conditions on Γ. It is defined as the unique solution of the following problem :

$$\left\{\begin{array}{lcll} \Delta v^1 + k^2 v^1 & = & 0 & \text{in } \Omega, \\ \frac{\partial v^1}{\partial n} & = & -\frac{\partial v^1_{BL}}{\partial n} & \text{on } \Gamma, \\ v^1 & \multicolumn{3}{l}{\text{satisfying an outgoing radiation condition at infinity.}} \end{array}\right. \tag{52}$$

We have the following theorem :

THEOREM 4.2. *For any $R > 0$, there exists a constant $C(R) > 0$ such that :*

$$\|P^\delta v^\delta - (v^0 + v^0_{BL}) - \delta(v^1 + v^1_{BL})\|_{H^1(\Omega_R)} \leq C(R)\ \delta^{3/2}. \tag{53}$$

The generalized impedance condition of order 0 is :

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma, \tag{54}$$

and that of order 1 is given on Γ by :

$$\begin{aligned} \frac{\partial v}{\partial n} \ = \ & -2\delta\mathbf{b}_1^{(1)}\,.\left\{\Delta_\Gamma v + c(s)\frac{\partial\, v}{\partial s}\mathbf{n}\right\} - \delta c(s)\mathbf{b}_2^{(1)}.\nabla_\Gamma v - \delta\mathbf{b}_3^{(1)}\,.\,\mathbf{n}\left\{\frac{\partial^2\, v}{\partial s^2} + k^2 v\right\} \\ & +\delta b_4^{(1)}\left\{\frac{\partial^2\, v}{\partial s^2} + k^2 v\right\} - \delta b_5^{(1)} k^2(\mu_r^\delta - \frac{1}{\varepsilon_r^\delta})v. \end{aligned} \tag{55}$$

In the case of a homogeneous dielectric layer without conducting inclusions, we found $\mathbf{b}_1^{(1)} = \mathbf{b}_2^{(1)} = \mathbf{b}_3^{(1)} = 0, b_4^{(1)} = 1 - 1/\varepsilon_r$ and $b_5^{(1)} = 1$. So we obtain the approximate boundary condition given by Engquist-Nédélec [10] and Bendali-Lemrabet [8] (written on Γ instead of Γ^δ).

4.2 On the constants appearing in the TM case

We end this section by indicating an idea to obtain explicitly, as in the TE case, the constants appearing in the TM case. For simplicity, we derive the first constant for a grating of conductors without a layer of dielectric material. We first remark that, in the TM case, the magnetic field satisfies :

$$\left\{ \begin{array}{rcll} \operatorname{curl}\operatorname{curl}\mathbf{H}^\delta - k^2\mathbf{H}^\delta &=& \mathbf{0} & \text{in } \Omega^\delta, \\ \operatorname{div}\mathbf{H}^\delta &=& 0 & \text{in } \Omega^\delta, \\ \operatorname{curl}\mathbf{H}^\delta &=& \mathbf{0} & \text{on } \partial\Omega^c_\delta, \\ \mathbf{H}^\delta - \mathbf{H}^{in} & \multicolumn{3}{l}{\text{satisfying an outgoing radiation condition at infinity.}} \end{array} \right. \tag{56}$$

$\mathbf{H}^\delta$ has the following asymptotic expansion :

$$\mathbf{H}^\delta = \mathbf{H}^{(0)} + \mathbf{H}^{(0)}_{BL} + \delta(\mathbf{H}^{(1)} + \mathbf{H}^{(1)}_{BL}) + \ldots,$$

where

$$\mathbf{H}^{(0)} = \operatorname{curl}\mathbf{E}^{(0)}$$

and $\mathbf{H}^{(0)}_{BL}$ is the boundary layer corrector of order 0. It corrects the boundary condition $\operatorname{curl}\mathbf{H}^{(0)}$ on the boundary of the conducting inclusions. $\mathbf{H}^{(0)}_{BL}$ is given by :

$$\mathbf{H}^{(0)}_{BL}(\mathbf{x}) = \delta^2 \frac{\partial u^0}{\partial n}(\mathbf{x}_\Gamma)\, \beta^{(0)}(\frac{s}{\delta}, \frac{n}{\delta}),$$

where $\beta^{(0)} \in H^{loc}_{\mathrm{curl},per}(\overline{O})$ is a solution of the following problem :

$$\left\{ \begin{array}{l} \operatorname{curl}\operatorname{curl}\beta^{(0)} = \mathbf{0}, \operatorname{div}\beta^{(0)} = 0 \quad \text{in } O \\ \operatorname{curl}\beta^{(0)} = -y_2\mathbf{e}_3 \quad \text{on } G^c, \;\; \operatorname{curl}\beta^{(0)} = \mathbf{a}^{(0)} \quad \text{on } G, \\ \mathbf{a}^{(0)}.\mathbf{e}_2 = \mathbf{0}. \end{array} \right. \tag{57}$$

Therefore, from

$$\mathbf{E}^{(0)}_{BL} = \operatorname{curl}\mathbf{H}^{(0)}_{BL},$$

we deduce that the constant $a^{(0)}$, introduced in (11), is given by :

$$a^{(0)} = \int_{G^c} y_2 n_2 d\sigma^c.$$

5 Conclusions and Extensions

In this paper we have studied the influence of a thin layer of dielectric material with conducting inclusions on the diffraction of a TM or a TE polarized electro-magnetic wave.

A closely related but more difficult problem is to derive generalized impedance conditions for a doubly periodic curved structure. The first order equivalent boundary condition is obtained in [2] for the general case of Maxwell's equations and a doubly periodic curved structure image of an infinite plane grating by an appropriate mapping. This condition generalizes the first order conditions given in this paper.

The derivation of these generalized impedance conditions assumes that the ratio η of the period d of the grating to the wave length λ is small. An interesting problem is to study the influence of the curvature of a grating on the diffraction of an electro-magnetic plane wave when η is not small. In [3], we study, for every ratio η, the influence of the curvature of a circular cylindrical grating on the diffraction of a TM or a TE polarized electro-magnetic plane wave. Our main result is that, locally in the near field, we can replace this curved structure by an infinite plane grating. The approximation is less satisfying in the neighborhood of some angles of incidence. The number of these transition regions is approximately $2 + 4\eta$ and the size of each is of order $\lambda^{2/3}R^{1/3}$ except the usual regions separating the region of direct illumination from the region of shadow which is of order $\lambda^{1/3}R^{2/3}$. The error of approximation is of order $0(d/R)$ outside the transition regions where it is of order $0(d/\lambda^{2/3}R^{1/3})$. In each region, we calculate the first corrector, solution of an infinite plane grating diffraction problem.

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