Diffraction at a Curved Grating. Approximation by an Infinite Plane Grating *

Toufic Abboud † Habib Ammari †

Abstract

We study the influence of the curvature of a circular cylindrical grating on the diffraction of a TE or TM polarized electro-magnetic plane wave. For every ratio of the wave length λ to the period d, we prove the local approximation in the near field by a plane infinite grating, when the radius R of the circular grating goes to infinity. We show the existence of regions, optically different, where the error is of order $O(d/R)$, separated by transition regions where the error is like $O(d/(\lambda^{2/3}R^{1/3}))$. In each region, we calculate the first corrector, solution of an infinite plane grating diffraction problem.

1 Introduction and main results

In [1], [2] and [3], we have presented a first attempt to study the diffraction of an electro-magnetic wave by a curved and finite grating. We have studied the influence of a thin layer of dielectric material with conducting inclusions, covering a cylindrical body, on the diffraction of an electro-magnetic incident wave. In these studies, the ratio η of the period d of the grating to the wave length λ is assumed to be small.

In this paper, we study the influence of the curvature of a circular cylindrical grating with a radius R, on the diffraction of a TE or a TM polarized electro-magnetic plane wave, for every η. Our main result is that, locally in the near field, and for every ratio η, we can replace the curved structure by an infinite plane grating. The approximation is less satisfying in the neighborhood of some angles of incidence. The number of these transition regions is approximately 4η and the size of each is of order $\lambda^{2/3}R^{1/3}$ except the usual regions separating the region of direct illumination from the region of shadow which is of order $\lambda^{1/3}R^{2/3}$. The error of approximation is of order $O(d/R)$ outside the transition regions where it is of order $O(d/(\lambda^{2/3}R^{1/3})$. In each region, we calculate the first corrector, solution of an infinite plane grating diffraction problem.

The outline of this paper is as follows. We first set the problem we are dealing with. Section 3 is devoted to reduce the problem to diffraction problems set in an elementary truncated cell. In section 4, we prove our approximation result. Section 5 is devoted to construct in each region the first corrector.

2 Setting of the problem

We consider a circular cylindrical grating, lit up by a TE or TM polarized incident *plane* wave, which leads us to the study of a $2D$ Helmholtz equation (P). The curved grating is

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†Centre de Mathématiques Appliquées, CNRS URA 756, École Polytechnique, 91128 Palaiseau Cedex, France.

the image of an infinite plane one by the mapping :

$$\phi : (y_1, y_2) \in \mathbb{R} \times \mathbb{R}_+^* \mapsto \phi(y_1, y_2) = \mathbf{x}(r, \theta),$$

where

$$R_0\theta = y_1 \ mod(2\pi R_0) \text{ and } r = R_0 + y_2,$$

in cylindrical coordinates. R_0 is the radius of the cylinder, d is the period of the plane grating. They are such that $N = 2\pi R_0/d$, the number of periods, is a great integer. Let Ω^c denote the conducting medium, Ω^d the dielectric material, Ω^e the exterior domain and $\Omega = \mathbb{R}^2 \setminus \overline{\Omega^c}$. We denote by O the elementary cell of the plane infinite grating, $O =]0, d[\times \mathbb{R}_+^* \setminus \overline{O}^c$, O^c the elementary conducting scatterer, O^d the dielectric one, G^c the boundary of O^c. We introduce Ω_{θ_l} and $\Omega^c_{\theta_l}$, the intersection between Ω and Ω^c and the sector

$$\Big\{ \theta_l < \theta < \theta_l + 2\pi/N \Big\}.$$

S_R is the circle of center O and radius R and

$$S_{R,\theta_l} \quad = \quad \Big\{ r = R, \theta_l < \theta < \theta_l + 2\pi/N \Big\}.$$

Let

$$O_h \quad = \quad \Big\{ (y_1, y_2) \in O : \ y_2 < h \Big\},$$

where $h = R - R_0 =$ is a constant and

$$G_h \quad = \quad \Big\{ (y_1, y_2) \in O : \ y_2 = h \Big\}.$$

In the case of a TM or a TE polarized incident plane wave, Maxwell's equations become :

$$(P) \begin{cases} \operatorname{div} \dfrac{1}{\mu} \nabla u + k^2 \varepsilon\, u = 0 & \text{in } \Omega, \\ u = 0 \quad \text{or} \quad \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ u - u^{in} \quad \text{satisfying an outgoing radiation condition.} \end{cases}$$

$u^{in}(\mathbf{x}) = e^{i\mathbf{k}.\mathbf{x}}$, $k = |\mathbf{k}|$, $\varepsilon \in L^\infty$, $\mu \in L^\infty$, $\mu = \mu' + i\mu''$ and $\varepsilon = \varepsilon' + i\varepsilon''$ where

$$\mu'(r, \theta) \geq \mu_0 > 0, \ \mu''(r, \theta) \geq 0, \ \mu(r, \theta + 2\pi/N) = \mu(r, \theta),$$

$$\varepsilon'(r, \theta) \geq \varepsilon_0 > 0, \ \varepsilon''(r, \theta) \geq 0 \text{ and } \varepsilon(r, \theta + 2\pi/N) = \varepsilon(r, \theta).$$

We assume that $\mu = 1$ and $\varepsilon = 1$ in the exterior domain Ω^e.

Now, we also describe the functional setting needed to state our boundary value problems in suitable mathematical forms. We shall use standard notations for the usual functional spaces. We need to define Sobolev spaces of quasi-periodic functions. To this end, let $H^1_{j,\theta}(\Omega_{\theta_l,R})$ be the closure of the set

$$\Big\{ \psi \in \mathcal{C}^\infty(\overline{\Omega}) : \ \psi(r, \theta + 2\pi/N) = e^{ij2\pi/N} \psi(r, \theta) \Big\}$$

in $H^1(\Omega_{\theta_l,R})$, $H^1_{j,\theta}(\Delta, \Omega_{\theta_l,R})$ is the closure of the same set in $H^1(\Delta, \Omega_{\theta_l,R})$. Let $H^1_{\theta_l}(O_h)$ denote the closure of the set

$$\Big\{ \psi \in \mathcal{C}^\infty(\overline{O}_h) : \psi(y_1 + d, y_2) = e^{ikd\sin\theta_l} \psi(y_1, y_2) \Big\}$$

in $H^1(O_h)$ and

$$\mathcal{H}^1_{\theta_l}(\Omega_{\theta_l,R}) \quad = \quad \Big\{ \ \psi \circ \phi^{-1} : \ \psi \in H^1_{\theta_l}(O_h) \ \Big\}.$$

It is not difficult to show that we can extend a periodic version of the first trace on G_h and S_{R,θ_l}, well defined for functions in

$$\Big\{ \ \psi \in \mathcal{C}^\infty(\overline{O}_h) : \ \psi(y_1 + d, y_2) = e^{ikd\sin\theta_l}\psi(y_1, y_2) \ \Big\}$$

and in

$$\Big\{ \ \psi \in \mathcal{C}^\infty(\overline{\Omega}_{\theta_l,R}) : \ \psi(y_1 + d, y_2) = e^{ikd\sin\theta_l}\psi(y_1, y_2) \ \Big\},$$

denoted by $H^{1/2}_{\theta_l}(G_h)$ and $\mathcal{H}^{1/2}_{\theta_l}(S_{R,\theta_l})$.

3 Quasi-periodic problems

We start by reducing the problem (P) to N diffraction problems set in the elementary truncated cell $\Omega_{\theta_l,R}$ (for a fixed l) of the circular cylindrical grating. The idea is to use the invariance of the grating by the rotation of angle $2\pi/N$ and to introduce the family $(\mathcal{T}_j)$ of Dirichlet to Neumann operators to write transparent boundary conditions on S_{R,θ_l}. Let us set :

$$u^{in}_j(r,\theta) = \sum_{n\in\mathbb{Z}} u^{in}(r, \theta + n2\pi/N)e^{-inj2\pi/N}, 0 \le j \le N-1.$$

u^{in}_j satisfies :

$$u^{in}_j(r, \theta + 2\pi/N) = u^{in}_j(r,\theta)e^{ij2\pi/N},$$

and we have

$$u^{in} = \sum_{j=0}^{N-1} u^{in}_j.$$

Consider now $(P)_j$ the following diffraction problem corresponding to the incident wave u^{in}_j.

$$(P)_j \begin{cases} \operatorname{div}\dfrac{1}{\mu}\nabla u_j + k^2\varepsilon\, u_j = 0 & \text{in } \Omega, \\ u_j = 0 \quad \text{or} \quad \dfrac{\partial u_j}{\partial n} = 0 & \text{on } \partial\Omega, \\ u_j - u^{in}_j \quad \text{satisfying an outgoing radiation condition.} \end{cases}$$

Due to the quasi-periodicity of u_j, we are able to set an equivalent problem in Ω_{θ_l}. An application of the elliptic regularity theory will let us to expand u_j in a Fourier series for large r. Notice that the decomposition of u^{in} into u^{in}_j allows to seek each $u_j \in H^{1,loc}_{j,\theta}(\Omega_{\theta_l})$ and we may prove that there exists a unique solution $u_j \in H^{1,loc}_{j,\theta}(\Omega_{\theta_l})$ of $(P)_j$ and u is the sum of u_j in $H^{1,loc}(\Omega)$. More precisely, we prove the following lemma :

LEMMA 3.1. *The unique solution of the problem (P) is the sum of the N solutions of the following problems $(P)_{j,R}$, set in the elementary truncated cell $\Omega_{\theta_l,R}$:*

$$(P)_{j,R}\begin{cases} \text{seek } u_j \in H^1_{j,\theta}(\Delta,\Omega_{\theta_l,R}) & \\ \operatorname{div}\dfrac{1}{\mu}\nabla u_j + k^2\varepsilon\, u_j = 0 & \text{in } \Omega_{\theta_l,R} \\ u_j = 0 \quad \text{or} \quad \dfrac{\partial u_j}{\partial n} = 0 & \text{on } \partial\Omega^c_{\theta_l}, \\ \dfrac{\partial}{\partial n}(u_j - u_j^{in}) = \mathcal{T}_j(u_j - u_j^{in}) & \text{on } S_{R,\theta_l}, \end{cases}$$

where

$$\mathcal{T}_j : v = \sum_{n\in\mathbb{Z}} v_{nN+j} e^{i(nN+j)\theta} \longmapsto \mathcal{T}_j v = \sum_{n\in\mathbb{Z}} Z_{nN+j}(k,R) v_{nN+j} e^{i(nN+j)\theta}.$$

$$Z_n(k,R) \quad = \quad k\frac{H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)},$$

where $(H_n^{(1)})$ are the Hankel functions satisfying the outgoing radiation condition.
The scattering problem $(P)_{j,R}$ has an equivalent variational formulation :
find $u_j \in H^1_{j,\theta}(\Omega_{\theta_l,R})$ ($u_j = 0$ on $\partial\Omega^c_{\theta_l}$ in the TM case) such that

$$a_j(u_j,v_j) = -\langle \mathcal{T}_j(u_j^{in}) - \frac{\partial u_j^{in}}{\partial n}, v_j\rangle, \quad \forall v_j \in H^1_{j,\theta}(\Omega_{\theta_l,R}).$$

($v_j = 0$ on $\partial\Omega^c_{\theta_l}$ in the TM case). Here the sesquilinear form is defined by

$$a_j(u_j,v_j) = \int_{\Omega_{\theta_l,R}} \frac{1}{\mu}\nabla u_j \nabla\overline{v_j} - k^2\int_{\Omega_{\theta_l,R}} \varepsilon\, u_j\overline{v_j} - \langle \mathcal{T}_j u_j, v_j\rangle,$$

and $\langle .,.\rangle$ represents the dual pairing of $H^{1/2}_{j,\theta}(S_{R,\theta_l})$ with $H^{-1/2}_{j,\theta}(S_{R,\theta_l})$. The operator

$$\mathcal{T}_j \; : H^{1/2}_{j,\theta}(S_{R,\theta_l}) \longmapsto H^{-1/2}_{j,\theta}(S_{R,\theta_l})$$

is continuous and such that (see Nédélec [6] for the proof) :

$$-\Re e\langle \mathcal{T}_j v_j, v_j\rangle \quad \geq \quad \frac{1}{2R}\,||v_j||^2_{L^2(S_R)}, \tag{1}$$

and

$$\Im m\langle \mathcal{T}_j v_j, v_j\rangle \quad > \quad 0 \quad \text{if } v_j \neq 0. \tag{2}$$

The proof of the lemma 3.1 is based on Fredholm's alternative and on the inequalities (1) and (2). From (1), we show that

$$\Re e\; a_j(u_j,u_j) \quad \geq \quad C_1\,||\nabla u_j||^2_{L^2(S_R)} - C_2\,||u_j||^2_{L^2(S_R)},$$

where C_1 and C_2 are two strictly positive constants. Now, since the imbedding $H^1(\Omega_{\theta_l,R})$ in $L^2(\Omega_{\theta_l,R})$ is compact, existence of solutions follows from uniqueness. Inequality (2) may show uniqueness and therefore, the proof of lemma 3.1 is complete.

4 Convergence to the plane infinite grating

We suppose that the wave number k and the period d are fixed. We want to replace the curved grating by an infinite plane grating and to estimate the error due to this approximation. Notice that the solution $u^0_{\theta_l}$ of the plane infinite grating is quasi-periodic, since it satisfies :

$$u^0_{\theta_l}(y_1+d,y_2) \quad = \quad e^{ikd\,sin\theta_l}u^0_{\theta_l}(y_1,y_2).$$

Our program is therefore the following : first, prove that the solution u of the problem (P) is locally almost quasi-periodic; Second, using the last result, replace u by $u^a_{\theta_l} \in \mathcal{H}^1_{\theta_l}(\Omega_{\theta_l,R})$ solution of the diffraction problem in Ω_{θ_l}; Third, compare the two variational formulations of the curved and the plane grating problems. The goal of the next subsection is to prove that u is almost quasi-periodic in the truncated cell $\Omega_{\theta_l,R}$.

4.1 Approximation of the quasi-periodicity

We have the following lemma :

LEMMA 4.1. *There exists a constant $C>0$ such that :*

$$\| \, u(r,\theta+2\frac{\pi}{N}) - e^{ikd\,sin\theta_l}u(r,\theta)\,\|_{H^1(\Omega_{\theta_l,R})} \leq C\,\frac{d}{R}\sup_{1\leq j\leq N}\| \, u \, \|_{H^1(\Omega_{\theta_j,R})} \tag{3}$$

Proof. Let us consider the function

$$w(r,\theta) \quad = \quad u(r,\theta+2\frac{\pi}{N}) - e^{ikdsin\theta}\,u(r,\theta).$$

It satisfies

$$\begin{cases} \mathrm{div}\dfrac{1}{\mu}\nabla w + k^2\varepsilon\,w = f & \text{in } \Omega, \\ w=-g^{in} \quad \text{or} \quad \dfrac{\partial w}{\partial n} = -\dfrac{\partial g^{in}}{\partial n} & \text{on } \partial\Omega, \\ w \quad \text{satisfying an outgoing radiation condition.} \end{cases}$$

where

$$f \quad = \quad \frac{kd}{r^2}\,\{\,2cos\theta\,\frac{\partial u}{\partial\theta} + \eta cos^2\theta\,u - sin\theta\,u\,\}\,e^{ikdsin\theta},$$

and

$$g^{in} \quad = \quad u^{in}(r,\theta+2\frac{\pi}{N}) - e^{ikdsin\theta}u^{in}(r,\theta).$$

Recall that the incident plane wave is given by

$$u^{in}(r,\theta) \quad = \quad e^{ikd\sin\theta}.$$

We will look for w under the sum of quasi-periodic w_p. We write f as $\sum f_p$ and $g^{in}=\sum g^{in}_p$. Each w_p satisfies the following equations :

$$\begin{cases} \mathrm{div}\dfrac{1}{\mu}\nabla w_p + k^2\varepsilon\,w_p = f_p & \text{in } \Omega, \\ w_p=-g^{in}_p \quad \text{or} \quad \dfrac{\partial w_p}{\partial n} = -\dfrac{\partial g^{in}_p}{\partial n} & \text{on } \partial\Omega, \\ w_p \quad \text{satisfying an outgoing radiation condition.} \end{cases}$$

Therefore, there exists a constant $C > 0$ independent of p, N and R such that

$$\| w_p \|^2_{H^1(\Omega_{\theta_l,R})} \leq C\Big\{ \| f_p \|^2_{L^2(\Omega_{\theta_l,R})} + \| g_p^{in} \|^2_{H^1(\Omega_{\theta_l,R})}\Big\}.$$

So that

$$\| w \|^2_{H^1(\Omega_{\theta_l,R})} \leq C\Big\{ \sum_{p=1}^{N} \| f_p \|^2_{L^2(\Omega_{\theta_l,R})} + \sum_{p=1}^{N} \| g_p^{in} \|^2_{H^1(\Omega_{\theta_l,R})}\Big\}$$

Notice that

$$\| f \|^2_{L^2(\Omega_R)} = N \sum_{p=1}^{N} \| f_p \|^2_{L^2(\Omega_{\theta_l,R})}$$

and

$$\| g^{in} \|^2_{H^1(\Omega_R)} = N \sum_{p=1}^{N} \| g_p^{in} \|^2_{H^1(\Omega_{\theta_l,R})}$$

As a consequence, there exists $C > 0$ independent of p, N and R such that

$$\| w \|^2_{H^1(\Omega_{\theta_l,R})} \leq C \frac{1}{N}\Big\{ \| f \|^2_{L^2(\Omega_R)} + \| g^{in} \|^2_{H^1(\Omega_R)}\Big\},$$

and therefore, by estimating $\| g^{in} \|^2_{H^1(\Omega_R)}$ and $\| f \|^2_{L^2(\Omega_R)}$, the proof of lemma 4.1 is complete.

□

We can also obtain a global estimate by the following lemma :

LEMMA 4.2. *There exists a constant $C > 0$ such that :*

$$\| u(r, \theta + 2\frac{\pi}{N}) - e^{ikdsin\theta} u(r,\theta) \|_{H^1(\Omega_R)} \leq C \frac{d}{R} \| u \|_{H^1(\Omega_R)} \tag{4}$$

These lemmas allow us to consider $u^a_{\theta_l} \in \mathcal{X}_l = \mathcal{H}^1_{\theta_l}(\Omega_{\theta_l,R})$ (null trace on $\partial\Omega^c_{\theta_l}$ in the TM case), unique solution of the diffraction quasi-periodic problem in Ω_{θ_l}. This scattering problem has an equivalent variational formulation : find $u^a_{\theta_l} \in \mathcal{X}_l$ such that :

$$b_{\theta_l}(u^a_{\theta_l}, w) = \int_{\Omega_{\theta_l,R}} \frac{1}{\mu} \nabla u^a_{\theta_l} \nabla \overline{w} - k^2 \int_{\Omega_{\theta_l,R}} \varepsilon\, u^a_{\theta_l} \overline{w} - \langle \mathcal{T}^a_{\theta_l} u^a_{\theta_l}, w\rangle = \langle \frac{\partial u^{in}}{\partial n} - \mathcal{T}^a_{\theta_l} u^{in}, w\rangle, \tag{5}$$

for any $w \in \mathcal{X}_l$. $\mathcal{T}^a_{\theta_l}$ is the Dirichlet to Neumann operator given by :

$$v = \sum_{n\in\mathbb{Z}} v_n e^{ikR(n/\eta + sin\theta_l)\theta} \longmapsto \mathcal{T}^a_{\theta_l} v = \sum_{n\in\mathbb{Z}} z_{\nu_n}(k,R) v_n e^{ikR(n/\eta+sin\theta_l)\theta},$$

where

$$\nu_n = \frac{n}{\eta} + sin\theta_l$$

and

$$z_{\nu_n}(k,R) = Z_{kR\nu_n}(k,R).$$

We start by the existence and the uniqueness of solution of (5) :

LEMMA 4.3. *There exists a unique solution $u^a_{\theta_l} \in \mathcal{X}_l$ of the variational problem* (5).

This lemma is quite similar to lemma 3.1 thanks to inequalities :

$$-\Re e\langle \mathcal{T}^a_{\theta_l} w, w\rangle \quad \geq \quad \frac{1}{2R}\,||w||^2_{L^2(S_R)},$$

and

$$\Im m\langle \mathcal{T}^a_{\theta_l} w, w\rangle \quad > \quad 0 \quad \text{if } w \neq 0,$$

and we omit the proof.

4.2 Approximation of the Dirichlet to Neumann operator

The diffraction problem at an infinite plane grating at the incidence θ_l is equivalent to the following problem set in the elementary truncated cell O_h :

$$\left\{\begin{array}{ll} \text{seek } v \in H^1_{\theta_l}(\Delta, O_h) & \\ \operatorname{div}\dfrac{1}{\mu}\nabla v + k^2\varepsilon\, v = 0 & \text{in } O_h \\ v = 0 \quad \text{or} \quad \dfrac{\partial v}{\partial n} = 0 & \text{on } G^c \\ \dfrac{\partial}{\partial n}(v - v^{in}) = T_{\theta_l}(v - v^{in}) & \text{on } G_h \end{array}\right.$$

Where $v^{in}(y_1, y_2) = e^{ik(y_1 \sin\theta_l - y_2 \cos\theta_l)}$ and T_{θ_l} is the Dirichlet to Neumann operator given by :

$$v = \sum_{n\in\mathbb{Z}} v_n e^{i(2\pi n/d + ksin\theta_l)y_1} \longmapsto T_{\theta_l} v = \sum_{n\in\mathbb{Z}} i\mu_n v_n e^{i(2\pi n/d+ksin\theta_l)y_1}$$

$$\mu_n = \left\{\begin{array}{ll} \sqrt{k^2 - (2\pi n/d + ksin\theta_l)^2} & \text{if } k > \mid 2\pi n/d + ksin\theta_l \mid, \\ i\sqrt{(2\pi n/d + ksin\theta_l)^2 - k^2} & \text{else.} \end{array}\right.$$

This scattering problem has the following variational formulation :

$$(6)\quad a_{\theta_l}(v, w) = \langle \frac{\partial v^{in}}{\partial n} - T_{\theta_l} v^{in}, w\rangle \;\; \forall\, w \in X_l = H^1_{\theta_l}(O_h)(\ w = 0 \text{ on } G^c \text{ in the TM case }).$$

a_{θ_l} is the continuous sesquilinear form on X_l, given by:

$$(7)\qquad a_{\theta_l}(v, w) = \int_{O_h} \frac{1}{\mu}\nabla v \nabla \overline{w} - k^2 \int_{O_h} \varepsilon\, v\overline{w} - \langle T_{\theta_l} v, w\rangle$$

As in [7], we prove that up to a possible sequence of wave numbers k, this variational problem has a unique solution.

LEMMA 4.4. *Up to a possible sequence of wave numbers k, there exists a unique solution of* (6).

Proof. We may consider the following operator

$$A_k = -\operatorname{div}\frac{1}{\mu}\nabla \quad \text{in } D(A_k) = \Big\{\ v \in H^1_\theta(O_h, \varepsilon dx) \text{ and } \frac{\partial v}{\partial n} = T^+_{\theta_l} \text{ on } G_h \ \Big\},$$

where the pseudo-differential operator $T^+_{\theta_l}$ is given by :

$$v = \sum_{n\in\mathbb{Z}} v_n e^{i(2\pi n/d+ksin\theta_l)y_1} \longmapsto T^+_{\theta_l} v = \sum_{|2\pi n/d + ksin\theta_l| > k} i\mu_n v_n e^{i(2\pi n/d+ksin\theta_l)y_1}.$$

A_k is a self adjoint, densely defined operator with compact resolvent and bounded from below. Consequently, there is a complete orthonormal basis of $L^2(O_h, \varepsilon dx)$ $\{\phi_n\}$ in $D(A_k)$ such that

$$A_k\phi_n = \lambda_n\phi_n \quad \text{with } \lambda_1 \leq \lambda_2 \leq \lambda_n \ldots \to +\infty$$

and

$$\lambda_n = \lambda_n(A_k)$$

is given by the Min-Max principle. It is a decreasing continuous function of k. Hence, when the wave number k does not verify :

$$\lambda_n(k) = k^2,$$

there exists a unique solution and the proof is over.

□

We assume that $u^0_{\theta_l}$ is unique . We introduce the pseudo-differential operator $T^a_{\theta_l}$ related to $\mathcal{T}^a_{\theta_l}$ by :

$$\mathcal{T}^a_{\theta_l}(w) = T^a_{\theta_l}(w \circ \phi^{-1}).$$

We also assume that $kR \gg 1$. The following asymptotic expansions for Bessel functions lead us to make the next definition :

DEFINITION 4.1. *$\Omega_{\theta_l,R}$ is in a transition region if there exists $n \in \mathbb{Z}$, such that :*

$$|\frac{n}{\eta} + sin\theta_l| = 1 + \xi(kR)^{-2/3} \tag{8}$$

where $|\ \xi\ | < c$ and c is a constant.

Out side the transition regions, the asymptotic expansions of Debye for the Bessel functions $(H^{(1)}_\nu)_{\nu \in \mathbb{R}^+}$ hold (see, for instance, F.W.J. Olver [8]).

It follows that for each fixed $n \in \mathbb{Z}$:

$$z_{\nu_n}(k,R) = i\mu_n + O(\frac{1}{R}). \tag{9}$$

According to [8], we have

$$J_{\nu_n}(kR) = (\frac{2}{\pi kR(1-\nu_n^2)^{1/2}})^{1/2}\Big\{\cos\psi_n + \frac{3}{24kR(1-\nu_n^2)^{1/2}}\sin\psi_n\Big\} + O(\frac{1}{(kR)^{5/3}}), \tag{10}$$

and

$$Y_{\nu_n}(kR) = (\frac{2}{\pi kR(1-\nu_n^2)^{1/2}})^{1/2}\Big\{\sin\psi_n - \frac{3}{24kR(1-\nu_n^2)^{1/2}}\cos\psi_n\Big\} + O(\frac{1}{(kR)^{5/3}}), \tag{11}$$

where

$$\psi_n = -kR\nu_n \operatorname{arcos}\nu_n - kR(1-\nu_n^2)^{1/2} - \frac{\pi}{4}.$$

The asymptotic expansions of $J'_{\nu_n}(kR)$ and $Y'_{\nu_n}(kR)$ are obtained in a manner analogous to (10) and (11) :

$$J'_{\nu_n}(kR) = (\frac{2}{\pi kR(1-\nu_n^2)^{1/2}})^{1/2}\Big\{-\sin\psi_n - \frac{9}{24kR(1-\nu_n^2)^{1/2}}\cos\psi_n\Big\} + O(\frac{1}{(kR)^{5/3}}). \tag{12}$$

(13) $Y'_{\nu_n}(kR) = (\frac{2}{\pi kR(1-\nu_n^2)^{1/2}})^{1/2}\Big\{\cos\psi_n - \frac{9}{24kR(1-\nu_n^2)^{1/2}}\sin\psi_n\Big\} + O(\frac{1}{(kR)^{5/3}})$.

In order to get the asymptotic expansion of $z_{\nu_n}(k,R)$, we combine (10) to (13) to obtain :

$$z_{\nu_n}(k,R) \quad = \quad i\mu_n - \frac{1}{2R} + O(\frac{1}{k^{2/3}R^{5/3}}).$$

In the transition regions we use the same asymptotic expansions up to the integer satisfying (8) where the asymptotic expansion of $z_{\nu_n}(k,R)$ may be obtained by use of uniform asymptotic expressions of $J_{kR+\xi(kR)^{1/3}}(kR)$, $Y_{kR+\xi(kR)^{1/3}}(kR)$, $J'_{kR+\xi(kR)^{1/3}}(kR)$ and $Y'_{kR+\xi(kR)^{1/3}}(kR)$ in terms of Airy functions. We obtain :

$$z_{kR+\xi(kR)^{1/3}}(k,R) \quad = \quad c(\xi)\frac{k^{2/3}}{R^{1/3}} + 0(\frac{1}{R}), \tag{14}$$

where $c(\xi)$ is expressible in terms of the Airy functions.
According to [8], we have

$$J_{kR+\xi(kR)^{1/3}}(kR) \quad = \quad \frac{2^{1/3}}{(kR)^{1/3}}A_i(2^{1/3}\xi) + O(\frac{1}{kR}),$$

and

$$Y_{kR+\xi(kR)^{1/3}}(kR) \quad = \quad -\frac{2^{1/3}}{(kR)^{1/3}}B_i(2^{1/3}\xi) + O(\frac{1}{kR}).$$

The asymptotic expansions of $J'_{kR+\xi(kR)^{1/3}}(kR)$ and $Y'_{kR+\xi(kR)^{1/3}}(kR)$ are given by

$$J'_{kR+\xi(kR)^{1/3}}(kR) \quad = \quad -\frac{2^{1/3}}{(kR)^{1/3}}A'_i(2^{1/3}\xi) + O(\frac{1}{(kR)^{4/3}}),$$

and

$$Y'_{kR+\xi(kR)^{1/3}}(kR) \quad = \quad \frac{2^{1/3}}{(kR)^{1/3}}B'_i(2^{1/3}\xi) + O(\frac{1}{(kR)^{4/3}}).$$

Thus we obtain

$$z_{kR+\xi(kR)^{1/3}}(k,R) \quad = \quad c(\xi)\frac{k^{2/3}}{R^{1/3}} + 0(\frac{1}{R}),$$

where

$$c(\xi) \quad = \quad 2^{1/3}\frac{iB'_i(2^{1/3}\xi) - A'_i(2^{1/3}\xi)}{A_i(2^{1/3}\xi) - B_i(2^{1/3}\xi)}. \tag{15}$$

We remark that these transition regions are a generalization of the usual regions separating the region of direct illumination from the region of shadow. The size of each region is of order $\lambda^{2/3}R^{1/3}$ except the two regions corresponding to $n=0$ which are of order $\lambda^{1/3}R^{2/3}$. Then we have the following lemma :

LEMMA 4.5. *There exists a constant $C>0$, such that, for any f and $g \in H^{1/2}_{\theta_l}(G_h)$, one has :*

$$|\ \langle (T_{\theta_l} - T^a_{\theta_l})f, g\rangle\ | \quad \leq \quad C\ k^{\alpha_l}\ R^{-\beta_l}\ ||\ f\ ||_{H^{-1/2}(G_h)}\ ||\ g\ ||_{H^{1/2}(G_h)} \tag{16}$$

$(\alpha_l,\beta_l) = (2/3,1/3)$ if $\Omega_{\theta_l,R}$ is in a transition region, $(0,1)$ if not.

The approximation is less satisfying in the transition regions which is of order $k^{2/3}/R^{1/3}$. We prove the following theorem :

THEOREM 4.1. *We assume that $u^0_{\theta_l}$ is unique, then there exists a constant $C > 0$ such that :*

$$|| u^0_{\theta_l} \circ \phi^{-1} - u ||_{H^1_{\theta_l}(\Omega_{\theta_l,R})} \leq C\, k^{\alpha_l}\, R^{-\beta_l}\, || u ||_{H^1(\Omega_{\theta_l,R})} \tag{17}$$

$(\alpha_l, \beta_l) = (2/3, 1/3)$ if $\Omega_{\theta_l,R}$ is in a transition region, $(0, 1)$ if not.

The proof of the error estimate (17) is based on the following lemma :

LEMMA 4.6. *Consider X a separable Hilbert space, a_0 and a_r two continuous sesquilinear forms on X satisfying the conditions :*

$$\begin{array}{l} (i)\ \exists\, \alpha > 0, \forall\, v \in X, \Re e\, a_0(v,v) \geq \alpha\ \| v \|^2_X\,; \\ (ii)\ \forall\, (u_n) \in X^{\mathbb{N}},\ u_n \rightharpoonup u \Rightarrow a_r(u_n,u_n) \rightarrow a_r(u,u)\,; \\ (iii)\ a(u,u) = a_0(u,u) + a_r(u,u) = 0 \Longrightarrow u = 0\,; \end{array}$$

Let (u_n) be a sequence in X, then

$$a_0(u_n,u_n) + a_r(u_n,u_n) = O(n^{-s})\, \| u_n \|_X \quad (s > 0) \Longrightarrow \| u_n \|_X = O(n^{-s})$$

Proof. This lemma is proved by contradiction, using compactness and uniqueness arguments. If we suppose that lemma (4.6) were not true, then for every $C > 0, n_0 \in \mathbb{N}$, there would exist $n \geq n_0$ such that

$$|| u_n ||_X \quad \geq \quad C n^{-s}.$$

So there exists a subsequence of $(n^s u_n)$, still denoted by $(n^s u_n)$ such that $|| \, n^s u_n \, ||_X$ converges to $+\infty$. We introduce

$$v_n \quad = \quad \frac{u_n}{|| u_n ||_X},$$

we have :

$$|| v_n ||_X = 1 \Longrightarrow v_n \rightharpoonup v \tag{18}$$

Now, we shall show that there exists a subsequence of (v_n) that converges to 0 in X. The existence of such subsequence would contradict (18).

$$a(u_n,u_n) = a_0(u_n,u_n) + a_r(u_n,u_n) = O(n^{-s})\, \| u_n \|_X$$

implies that :

$$a(v_n,v_n) = a_0(v_n,v_n) + a_r(v_n,v_n) \rightarrow 0 \tag{19}$$

Furthermore, since (v_n) is bounded and a_r satisfies (ii), there exists a subsequence, still denoted by (v_n), such that $a_r(v_n,v_n) \rightarrow a_r(v,v)$. Hence, using (19), v_n converges strongly to v in X. It follows that : $a(v_n,v_n) \rightarrow a(v,v) = 0$. Finally, using property (iii), it follows that $v = 0$. This contradicts (18). Hence lemma (4.6) is proved. □

We now recall that $u^0_{\theta_l}$ solves

$$a_{\theta_l}(u^0_{\theta_l}, w) \quad = \quad \langle \frac{\partial v^{in}}{\partial n} - T_{\theta_l} v^{in}, w \rangle, \forall\, w \in \mathcal{X}_l,$$

and $u^a_{\theta_l}$ solves

$$b_{\theta_l}(u^a_{\theta_l}, w) \quad = \quad \langle \frac{\partial u^{in}}{\partial n} - \mathcal{T}^a_{\theta_l} u^{in}, w\rangle, \forall\, w \in X_l.$$

We see that,

$$\begin{aligned} a_{\theta_l}(u^0_{\theta_l} - u^a_{\theta_l}\circ\phi, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi) \quad &= \quad \langle \frac{\partial v^{in}}{\partial n} - T_{\theta_l} v^{in}, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi\rangle \\ - a_{\theta_l}(u^a_{\theta_l}\circ\phi, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi) \end{aligned}$$

Hence, we have

$$a_{\theta_l}(u^0_{\theta_l} - u^a_{\theta_l}\circ\phi, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi) = \langle \frac{\partial v^{in}}{\partial n} - T_{\theta_l} v^{in}, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi\rangle - b_{\theta_l}(u^a_{\theta_l}, u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l})$$
$$+ b_{\theta_l}(u^a_{\theta_l}, u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l}) - a_{\theta_l}(u^a_{\theta_l}\circ\phi, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi)$$

By using (5), we obtain

$$\begin{aligned} a_{\theta_l}(u^0_{\theta_l} - u^a_{\theta_l}\circ\phi, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi) \quad &= \quad \langle \frac{\partial v^{in}}{\partial n} - T_{\theta_l} v^{in}, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi\rangle \\ -\langle \frac{\partial u^{in}}{\partial n} - \mathcal{T}^a_{\theta_l} u^{in}, u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l}\rangle + b_{\theta_l}(u^a_{\theta_l}, u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l}) - a_{\theta_l}(u^a_{\theta_l}\circ\phi, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi) \end{aligned}$$

and

$$\begin{aligned} & b_{\theta_l}(u^a_{\theta_l}, u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l}) - a_{\theta_l}(u^a_{\theta_l}\circ\phi, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi) = \int_{\Omega_{\theta_l,R}} \frac{1}{\mu}\nabla u^a_{\theta_l} \nabla\overline{(u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l})} \\ & - k^2 \int_{\Omega_{\theta_l,R}} \varepsilon\, u^a_{\theta_l}\, \overline{(u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l})} - \langle \mathcal{T}^a_{\theta_l} u^a_{\theta_l}, u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l}\rangle - \int_{O_h} \frac{1}{\mu}\nabla u^a_{\theta_l}\circ\phi\ \nabla\overline{(u^0_{\theta_l} - u^a_{\theta_l}\circ\phi)} \\ & + k^2 \int_{O_h} \varepsilon\, u^a_{\theta_l}\circ\phi\ \overline{(u^0_{\theta_l} - u^a_{\theta_l}\circ\phi)} + \langle T_{\theta_l} u^a_{\theta_l}\circ\phi, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi\rangle. \end{aligned}$$

We introduce

$$I_1 \quad = \quad |\int_{\Omega_{\theta_l,R}} \frac{1}{\mu}\nabla u^a_{\theta_l}\ \nabla\overline{(u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l})} - \int_{O_h} \frac{1}{\mu}\nabla u^a_{\theta_l}\circ\phi\ \nabla\overline{(u^0_{\theta_l} - u^a_{\theta_l}\circ\phi)}\ |,$$

$$I_2 \quad = \quad |\int_{\Omega_{\theta_l,R}} \varepsilon\, u^a_{\theta_l}\ \overline{(u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l})} - \int_{O_h} \varepsilon\, u^a_{\theta_l}\circ\phi\ \overline{(u^0_{\theta_l} - u^a_{\theta_l}\circ\phi)}\ |,$$

$$I_3 \quad = \quad |\ \langle \mathcal{T}^a_{\theta_l} u^a_{\theta_l}, u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l}\rangle - \langle T_{\theta_l} u^a_{\theta_l}\circ\phi, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi\rangle\ |,$$

and

$$I_4 \quad = \quad |\ \langle \frac{\partial v^{in}}{\partial n} - T_{\theta_l} v^{in}, u^0_{\theta_l} - u^a_{\theta_l}\circ\phi\rangle - \langle \frac{\partial u^{in}}{\partial n} - \mathcal{T}^a_{\theta_l} u^{in}, u^0_{\theta_l}\circ\phi^{-1} - u^a_{\theta_l}\rangle\ |\ .$$

Now, we are able to say that

$$I_1 \quad \le \quad \frac{1}{R}\ ||\ u^a_{\theta_l}\ ||_{H^1(\Omega_{\theta_l,R})}||\ u^0_{\theta_l} - u^a_{\theta_l}\circ\phi\ ||_{H^1(O_h)},$$

$$I_2 \quad \le \quad \frac{1}{R}\ ||\ u^a_{\theta_l}\ ||_{H^1(\Omega_{\theta_l,R})}||\ u^0_{\theta_l} - u^a_{\theta_l}\circ\phi\ ||_{H^1(O_h)}\ .$$

By applying lemma 4.4, we may prove that

$$I_3 \quad \leq \quad \frac{k^{\alpha_l}}{R^{\beta_l}} \mid\mid u^a_{\theta_l} \mid\mid_{H^1(\Omega_{\theta_l,R})} \mid\mid u^0_{\theta_l} - u^a_{\theta_l} \circ \phi \mid\mid_{H^1(O_h)},$$

and

$$I_4 \quad \leq \quad C\, \frac{k^{\alpha_l}}{R^{\beta_l}} \mid\mid u^0_{\theta_l} - u^a_{\theta_l} \circ \phi \mid\mid_{H^1(O_h)} .$$

Finally, we get

$$\begin{aligned} \mid a_{\theta_l}(u^0_{\theta_l} - u^a_{\theta_l} \circ \phi, u^0_{\theta_l} - u^a_{\theta_l} \circ \phi) \mid \quad \leq \quad & C \Big\{ (\frac{1}{R} + \frac{k^{\alpha_l}}{R^{\beta_l}}) \mid\mid u^a_{\theta_l} \mid\mid_{H^1(\Omega_{\theta_l,R})} \mid\mid u^0_{\theta_l} - u^a_{\theta_l} \circ \phi \mid\mid_{H^1(O_h)} \\ & + \frac{k^{\alpha_l}}{R^{\beta_l}} \mid\mid u^0_{\theta_l} - u^a_{\theta_l} \circ \phi \mid\mid_{H^1(O_h)} \Big\} \end{aligned} \tag{20}$$

and lemma 4.5 allows us to conclude.

5 The first corrector

The first corrector in the elementary truncated cell $\Omega_{\theta_l,R}$ should correct the Dirichlet to Neumann operator and the error introduced by the transformation ϕ. For any $w \in \mathcal{H}^1_{\theta_l}(\Omega_{\theta_l})$ ($w = 0$ on $\partial\Omega^c_{\theta_l}$ in the TE case), we have :

$$a_{\theta_l}(u^0_{\theta_l} - u^a_{\theta_l} \circ \phi, w \circ \phi) = b_{\theta_l}(u^a_{\theta_l}, w) - a_{\theta_l}(u^a_{\theta_l} \circ \phi, w \circ \phi) + \langle \frac{\partial v^{in}}{\partial n} - T_{\theta_l} v^{in}, w \circ \phi \rangle - \langle \frac{\partial u^{in}}{\partial n} - T^a_{\theta_l} u^{in}, w \rangle$$

If $\Omega_{\theta_l,R}$ is outside the transition regions then the first corrector $u^1_{\theta_l}$ is the unique solution of the following variational problem : $\forall\, w \in H^1_{\theta_l}(O_h)$,

$$a_{\theta_l}(u^1_{\theta_l}, w) = - \int_{O_h} \frac{1}{\mu} \nabla u^0_{\theta_l} \nabla \overline{w}\; y_2 \; dy + k^2 \int_{O_h} \varepsilon\, u^0_{\theta_l} \overline{w}\; y_2 \; dy - 1/2 \langle u^0_{\theta_l} + v^{in}, w \rangle. \tag{21}$$

In this case, the solution u has the following asymptotic expansion :

$$u(r,\theta) \sim u^0_{\theta_l} \circ \phi^{-1}(r,\theta) + \frac{1}{R_0}\, u^1_{\theta_l} \circ \phi^{-1}(r,\theta) \tag{22}$$

If $\Omega_{\theta_l,R}$ is in a transition region, the first corrector is obtained as a solution of :

$$a_{\theta_l}(u^1_{\theta_l}, w) \quad = \quad \langle T^c_l u^0_{\theta_l}, w \rangle, \quad \forall\, w \in H^1_{\theta_l}(O_h), \tag{23}$$

($w = 0$ on $\partial\Omega^c_{\theta_l}$ in the TE case), where T^c_l is the pseudo-differential operator given by :

$$T^c_l : \sum_{n \in \mathbb{Z}} v_n e^{i(2\pi n/d + k sin\theta_l) y_1} \longmapsto \sum_{n_0} d(n_0) v_{n_0} e^{i(2\pi n_0/d + k sin\theta_l) y_1},$$

n_0 are the integers satisfying (8) and

$$d(\xi) \quad = \quad c(\xi) - \sqrt{2 \mid \xi \mid}.$$

We recall that $c(\xi)$ is the function defined by (15).
Finally, we derive in the transition regions the following asymptotic expansion for u :

$$u(r,\theta) \sim u^0_{\theta_l} \circ \phi^{-1}(r,\theta) + \frac{k^{2/3}}{R^{1/3}}\, u^1_{\theta_l} \circ \phi^{-1}(r,\theta). \tag{24}$$

6 Concluding remarks

We conclude with some remarks on some extensions and generalizations. First we remark that the asymptotic expansion of the scattered wave may be extended to a strictly convex grating. In [5], we derive a geometric-optics solution outside the transition regions, satisfying Helmholtz equation to arbitrarily order and we show that the first order approximation is solution of the infinite plane tangent grating diffraction problem. Finally, we remark that we can apply the same procedure in the case of Maxwell's equations and a curved doubly periodic non homogeneous structure to generalize the result of approximation by an infinite plane grating announced in [4].

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