Homogenization of The Time Harmonic Maxwell Equations in a Curved Doubly Periodic Structure *

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Abstract

Effective boundary conditions are derived for Maxwell's equations. These generalized impedance boundary conditions approximate the effect of thin dielectric layers with doubly-periodic conducting inclusions covering a cylindrical object on the diffraction of an electro-magnetic incident plane wave. We use a new formulation of Maxwell's equations to prove the convergence results and to give optimal error estimates. The curvature appears from the first order approximation.

1 Introduction

In the previous paper [2], we have studied the influence of a thin layer of dielectric material with metallic inclusions, covering a cylindrical body, on the diffraction of a TM or a TE polarized electro-magnetic wave. We were led to the study of a 2D Helmholtz equation with Dirichlet or Neumann boundary conditions. In this paper, we consider the general case of *Maxwell's equations* and *a doubly periodic curved grating* lit up by an incident plane wave. The present paper differs from [2] in a number of important technical details.

The purpose of this paper is to describe an approximate equality between the curved grating and an impedance condition. This approximation is valid for δ small enough, where δ is the ratio of the period of the grating to the wave length. We prove that we can replace the grating by an equivalent boundary condition on the boundary of the conductor and we exhibit the first boundary layer corrector.

The outline of this paper is as follows. We first set the problem we are dealing with and we introduce the framework to be considered. In section 3, our purpose is to establish a uniqueness theorem. We look for quasi-periodic solutions of Maxwell's equations. This allows us to set the problem in a truncated domain thanks to the Dirichlet to Neumann operator. The variational formulation of this problem is a compact perturbation of a saddle point problem. Therefore we prove that, up to a possible sequence of eigenfrequencies going to infinity, there exists a unique solution of Maxwell's equations satisfying an appropriate radiation condition at infinity. Remark that this new formulation permits to generalize the existence results obtained in the case of an infinite plane grating by Chen and Friedman in [6], Dobson and Friedman in [7] and Nédélec and Starling in [9]. In section 4, devoted to prove the convergence to the limiting problem, we first introduce an extension operator into the conductor inclusions, derived from an extension operator on the elementary truncated cell of the plane infinite grating. Then, we prove the convergence to the limiting problem and we obtain the first error estimate of order $\delta^{1/2}$. The next step is to seek a better

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approximation of the field by carrying out an asymptotic expansion including boundary layer terms. We first calculate the boundary layer corrector of order 0 by correcting the boundary condition on the boundary of the conductor inclusions and the jump of $[\,\varepsilon^\delta\,\mathbf{E}^\delta\,.\,\mathbf{n}\,]$ to the order 2 and the jump of $[\frac{1}{\mu^\delta}\ \mathrm{curl}\,\mathbf{E}^{(0)} \wedge \vec{n}]$ to the order 1 across the interface between the dielectric material and the exterior domain. In the TM case, the first order approximation is derived by correcting analogous quantities only to the order 1. The first boundary layer corrector is obtained by solving elementary problems set in the elementary cell of the plane infinite grating. It allows us to obtain the first order corrector. If we are not interested in the boundary layer, we can write the first impedance boundary condition on the boundary of the conductor. The curvature appears explicitly from the first order approximation. This boundary condition leads to a well-posed problem and we obtain error estimate of order $\delta^{3/2}$.

In the case of a homogeneous dielectric layer without conducting inclusions, we recover the generalized impedance boundary condition given by Engquist and Nédélec in [8]. We finally mention that the first order approximation generalizes the conditions obtained in the case of a plane infinite grating by Achdou in [4] and Artola and Cessenat in [5].

2 Setting of the problem

Let $\omega_0^c \subset \mathbb{R}^2$ be a smooth open bounded set, $\Omega_0^c = \omega_0^c \times \mathbb{R}$, $\Gamma = \partial\omega_0^c \times \mathbb{R}$, $\mathbf{n}$ the outward normal. Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be an orthonormal basis of $\mathbb{R}^3$, related to a system of coordinates (x_1, x_2, x_3) with an origin in Ω_0^c. Let (r, θ, z) denote the cylindrical coordinates and $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ the local basis. For $\mathbf{x} \in \Omega = \mathbb{R}^3 \setminus \overline{\Omega_0^c}$ sufficiently close to Γ, we denote by $\mathbf{x}_\Gamma$ the orthogonal projection of $\mathbf{x}$ on Γ, s curvilinear abscissa on $\partial\omega_0^c$, $0 \le s \le l$ =length $(\partial\omega_0^c)$, $n = \mid \mathbf{x} - \mathbf{x}_\Gamma \mid$, (s, n, z) is a parameterization of the neighborhood of Γ :

$$\mathbf{x}(s,n,z) \quad = \quad \mathbf{x}_\Gamma(s,z) + n\,\mathbf{n}(s)$$

Consider now a doubly periodic infinite plane grating of period unity in y_1 and $y_3, (y_2 > 0)$, made up of conductors and dielectric material. Consider N a large integer and set $\delta = l/N$. The mapping :

$$(y_1, y_2, y_3) \longrightarrow \phi_\delta(y_1, y_2, y_3) = \mathbf{x}(s, n, z),$$

where

$$s = \delta\, y_1 \; mod(l), \quad n = \delta\, y_2 \quad \text{and} \quad z = \delta\, y_3,$$

sends the plane grating onto the curved one. Let Ω_N^c be the metallic inclusions, $\Omega_\delta^c = \Omega_0^c \cup \Omega_N^c$, Ω_δ^d the dielectric material, $\Omega^\delta = \mathbb{R}^3 \setminus \overline{\Omega_\delta^c}$, Γ^δ the interface between the dielectric material and the exterior domain. We also define $\Omega_l^\delta = \Omega^\delta \cap \Big\{\, z \in]0, l[\, \Big\}$, $\Omega_{l,R}^\delta = \Omega^\delta \cap \Big\{\, r < R, z \in]0, l[\, \Big\}$, $\Omega_{l,R} = \Omega \cap \Big\{\, r < R, z \in]0, l[\, \Big\}$ and $\Sigma_{l,R} = \Big\{\, r = R,$ $z \in]0, l[\, \Big\}$.

As usual in the periodic homogenization theory, we define a so called unit cell, which, upon rescaling to size δ, becomes the period of the periodic structure. We denote by O the unit cell of the plane grating, O^c the conductor, G the basis of the elementary cell:

$$O \quad = \quad]0,1[\times \mathbb{R}_+^* \times]0,1[\setminus \overline{O}^c,$$

$G =]0,1[\times \{0\} \times]0,1[$, G^c the boundary of O^c, G^d the interface between the dielectric material and the exterior domain. These open sets are supposed to be smooth. We also suppose that the conductor O^c is connected.

We shall need some differential operators on the curve Γ that we will introduce. The normal vector on Γ is the opposite of the gradient of the function n. We use this property to define

$$\mathbf{n}(\mathbf{x}) = -\mathbf{grad}\, n(\mathbf{x}) = \mathbf{n}(\mathbf{x}_\Gamma).$$

The curvature is defined as $c(s) = \operatorname{div}\mathbf{n}$. We also introduce the family of parallel curves Γ_n :

$$\Gamma_n = \Big\{ \mathbf{x} \in \mathbb{R}^3 \setminus \Omega_0^c : \mathbf{x} = \mathbf{x}_\Gamma(s,z) + n\, \mathbf{n}(s)\ , \mathbf{x}_\Gamma(s,z) \in \Gamma \Big\}.$$

The vector field $\mathbf{n}$ is the field of the normal vectors to the curves Γ_n. To any function u defined on the curve Γ, we associate the function pu defined for $\mathbf{x}$ near Γ by

$$pu(\mathbf{x}) = u(\mathbf{x}_\Gamma).$$

We also define for $\mathbf{u}$ a regular vectorial function on the tangent plane of Γ, $p\mathbf{u}$ as

$$p\mathbf{u}(\mathbf{x} + s\mathbf{n}(s)) = \mathbf{u}(\mathbf{x}) - sR\mathbf{u}(\mathbf{x}),$$

where R is the tensor of curvature at the point $\mathbf{x}_\Gamma$. It is defined as $R = \mathbf{grad}\,\mathbf{n}$. The tangential gradient of the function u is :

$$\mathbf{grad}_\Gamma\, u = \mathbf{grad}\, pu\,|_\Gamma\ .$$

This vector belongs to the tangent to the curve Γ. For $\mathbf{u}$ on the tangent plane of Γ, let us define $\operatorname{div}_\Gamma \mathbf{u}$ as

$$\operatorname{div}_\Gamma \mathbf{u} = \operatorname{div} p\mathbf{u}\,|_\Gamma\ .$$

Let $\operatorname{curl}_\Gamma$ denote the tangential rotational operator on Γ. It is such that :

$$\operatorname{curl}_\Gamma u = \mathbf{grad}_\Gamma\, u\ \wedge\ \mathbf{n},$$

for any function u defined on the curve Γ. The scalar tangential rotational operator is defined as

$$\operatorname{curl}_\Gamma \mathbf{u} = \operatorname{curl} p\mathbf{u}\ .\ \mathbf{n}\,|_\Gamma,$$

where $\mathbf{u}$ is a tangential vector to Γ and $p\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}_\Gamma)$.

LEMMA 2.1. *Let* $\mathbf{u}$ *be a smooth vectorial function defined in* Ω*, we have :*

(1) $$\gamma_0(\operatorname{div}\mathbf{u}) = \operatorname{div}_\Gamma \mathbf{u}_\Gamma + c(s)\ \mathbf{u}\,.\,\mathbf{n} + \gamma_1(\mathbf{u}\,.\,\mathbf{n}),$$

and

(2) $$\mathbf{grad}_\Gamma(\mathbf{u}\,.\,\mathbf{n}) = \gamma_1(\mathbf{u}_\Gamma) + \mathbf{n}\wedge \operatorname{curl}\mathbf{u} + R(\mathbf{u}_\Gamma),$$

where γ_0 *and* γ_1 *are respectively the first and the second trace and* $\mathbf{u}_\Gamma$ *the tangential component of* $\mathbf{u}$*.*

Proof. We have

$$\operatorname{div}_\Gamma \mathbf{u}_\Gamma = \operatorname{div}\Big\{\ \mathbf{n}\ \wedge\ (\ \mathbf{u}\ \wedge\ \mathbf{n}\ )\ \Big\}.$$

This yields

$$\operatorname{div}_\Gamma \mathbf{u}_\Gamma = \mathbf{n}\,.\operatorname{curl}(\ \mathbf{u}\ \wedge\ \mathbf{n}\ ).$$

So that

$$\mathrm{div}_\Gamma \mathbf{u}_\Gamma = \mathbf{n}\,.\Big\{ \mathbf{n}\,\mathrm{div}\,\mathbf{u} - \mathbf{u}\,\mathrm{div}\,\mathbf{n} + (\,\mathbf{n}\,.\,\mathbf{grad}\,)\,\mathbf{u} - (\,\mathbf{u}\,.\,\mathbf{grad}\,)\,\mathbf{n}\,\Big\}.$$

Therefore, we obtain

$$\mathrm{div}_\Gamma \mathbf{u}_\Gamma = \mathrm{div}\,\mathbf{u} - c(s)\,\mathbf{u}\,.\,\mathbf{n} - \gamma_1(\mathbf{u}\,.\,\mathbf{n}).$$

In order to show (2), direct computation yields

$$\mathbf{grad}\,(\mathbf{u}\,.\,\mathbf{n}) = (\,\mathbf{n}\,.\,\mathbf{grad}\,)\,\mathbf{u} + (\,\mathbf{u}\,.\,\mathbf{grad}\,)\,\mathbf{n} + \mathbf{n}\wedge \mathrm{curl}\,\mathbf{u}.$$

Thus, it follows

$$\mathbf{grad}_\Gamma(\mathbf{u}\,.\,\mathbf{n}) = \gamma_1(\mathbf{u}_\Gamma) + \mathbf{n}\wedge\mathrm{curl}\,\mathbf{u} + R(\mathbf{u}_\Gamma).$$

□

We shall also need quasi-periodic Sobolev spaces. For every function $\mathbf{u}$ defined on Ω^δ, we associate its quasi-periodic extension $\mathbf{u}_{k_z}$ given by :

$$\mathbf{u}_{k_z}(\mathbf{x}', z+ml) = e^{imlk_z}\mathbf{u}(\mathbf{x}', z), \quad \forall\, m\in\mathbb{Z}, \forall\,\mathbf{x} = (\mathbf{x}', z)\in\Omega^\delta.$$

We define then $H_{k_z}(\mathrm{curl},\Omega^\delta)$ (resp. $H_{k_z}(\mathrm{div},\Omega^\delta)$) as the space of functions $\mathbf{u}$ in $H(\mathrm{curl},\Omega^\delta)$ (resp. $H(\mathrm{div},\Omega^\delta)$) such that :

$$\varphi\mathbf{u}_{k_z}\in H(\mathrm{curl},\mathbb{R}^3\setminus\overline{\Omega}_\delta^c) \quad (resp. H(\mathrm{div},\mathbb{R}^3\setminus\overline{\Omega}_\delta^c)), \quad \forall\varphi\in\mathcal{D}(\mathbb{R}^3).$$

We also introduce $H_{per}(\mathrm{curl},O_h)$ the set of periodic functions in x satisfying

$$\mathbf{u}(\mathbf{x}', z+ml) = e^{imlk_z}\mathbf{u}(\mathbf{x}', z) \quad \text{in } H(\mathrm{curl},O_h)$$

and we define $H^{loc}_{per}(\mathrm{curl},\overline{O})$ as the set of functions in $H_{per}(\mathrm{curl},O_h)$ for every positive h, where $O_h = O\cap\{y_2<h\}$.

For $s\in\mathbb{R}$, we introduce the following spaces :

$$\begin{aligned}
H^s_{k_z}(\Sigma_{l,R}) &= \Big\{u\ :\ \|u\|_s^2 = \sum_{(m,n)\in\mathbb{Z}^2}(1+(n/R)^2+\beta_m^2)^s|u_n^m|^2<\infty\Big\},\\
H^s_{k_z}(\mathrm{div},\Sigma_{l,R}) &= \Big\{U\ :\ \|U\|_{s,\mathrm{div}}^2 = \sum_{(m,n)\in\mathbb{Z}^2}(1+(n/R)^2+\beta_m^2)^s(|\mathbf{U}_n^m|^2+|\mathbf{k}_n^m.\mathbf{U}_n^m|^2)<\infty\Big\},\\
H^s_{k_z}(\mathrm{curl},\Sigma_{l,R}) &= \Big\{U\ :\ \|U\|_{s,\mathrm{curl}}^2 = \sum_{(m,n)\in\mathbb{Z}^2}(1+(n/R)^2+\beta_m^2)^s(|\mathbf{U}_n^m|^2+|\mathbf{k}_n^m\wedge\mathbf{U}_n^m|^2)<\infty\Big\},
\end{aligned}$$

where

$$u = \sum_{(m,n)\in\mathbb{Z}^2} u_n^m\, e^{i(n\theta+\beta_m z)},$$

$$U(\mathbf{x}) = \sum_{(m,n)\in\mathbb{Z}^2}\mathbf{U}_n^m\, e^{i(n\theta+\beta_m z)} = \sum_{(m,n)\in\mathbb{Z}^2}\Big\{\,U^m_{n,\theta}\,\mathbf{e}_\theta + U^m_{n,z}\,\mathbf{e}_z\,\Big\}\, e^{i(n\theta+\beta_m z)},$$

and

$$\mathbf{k}_n^m = \frac{n}{R}\,\mathbf{e}_\theta + \beta_m\mathbf{e}_\mathbf{z}.$$

They are Hilbert spaces and we have the following duality results :

$$\Big(H^s_{k_z}(\Sigma_{l,R})\Big)' = H^{-s}_{k_z}(\Sigma_{l,R}) \quad \text{and} \quad \Big(H^s_{k_z}(\mathrm{div},\Sigma_{l,R})\Big)' = H^{-s-1}_{k_z}(\mathrm{curl},\Sigma_{l,R}).$$

LEMMA 2.2. *The mapping*

$$\mathbf{u} \in \mathcal{C}^\infty_{k_z}(\Omega^\delta_{l,R}) \longmapsto \mathbf{n} \wedge \mathbf{u}\,|_{\Sigma_{l,R}}$$

can be extended continuously to a linear continuous mapping from $H(\mathrm{curl},\Omega^\delta_{l,R})$ *onto* $H^{-1/2}_{k_z}(\mathrm{div},\Sigma_{l,R})$*, and there exists a constant* $C > 0$ *such that for every* $\eta > 0$ *and* $\mathbf{u} \in H(\mathrm{curl},\Omega^\delta_{l,R})$ *:*

$$(3) \qquad \|\, \mathbf{n} \wedge \mathbf{u} \,\|_{H^{-1/2}_{k_z}(\Sigma_{l,R})} \;\leq\; C\Big\{\eta \,\|\, \mathrm{curl}\,\mathbf{u}\,\|_{L^2(\Omega^\delta_{l,R})} + (1+\frac{1}{\eta})\,\|\,\mathbf{u}\,\|_{L^2(\Omega^\delta_{l,R})}\Big\}$$

Proof. Consider $\mathbf{u} \in \mathcal{C}^\infty_{k_z}(\Omega^\delta_{l,R})$ defined as

$$\mathbf{u} \;=\; \sum_{(m,n)\in\mathbb{Z}^2} \mathbf{u}^m_n \; e^{i(n\theta+\beta_m z)}.$$

$\mathrm{curl}\,\mathbf{u}$ can be written as

$$\mathrm{curl}\,\mathbf{u} \;=\; \sum_{(m,n)\in\mathbb{Z}^2} \mathbf{r}^m_n \; e^{i(n\theta+\beta_m z)}.$$

Direct computation yields

$$\mathrm{div}\,(\mathbf{n}\wedge\mathbf{u}) \;=\; \sum_{(m,n)\in\mathbb{Z}^2} \Big(\frac{-in}{r} u^m_{z,n} + i\beta_m u^m_{\theta,n}\Big)\, e^{i(n\theta+\beta_m z)}.$$

So that we have

$$\|\, \mathrm{div}\,(\mathbf{n}\wedge\mathbf{u})\,\|_{H^{-1/2}_{k_z}(\Sigma_{l,R})} \;\leq\; \|\,\mathbf{u}\,\|_{L^2(\Omega^\delta_{l,R})},$$

and it follows that the mapping

$$\mathbf{u} \in \mathcal{C}^\infty_{k_z}(\Omega^\delta_{l,R}) \longmapsto \mathbf{n} \wedge \mathbf{u}\,|_{\Sigma_{l,R}}$$

is continuous from $H(\mathrm{curl},\Omega^\delta_{l,R})$ onto $H^{-1/2}_{k_z}(\mathrm{div},\Sigma_{l,R})$.

Now, to establish (3), we first compute $\mathrm{curl}\,\mathbf{u}\wedge\mathbf{e}_r$

$$\mathrm{curl}\,\mathbf{u}\wedge\mathbf{e}_r = \sum_{(m,n)\in\mathbb{Z}^2} \Big\{(\frac{1}{r}u^m_{\theta,n} - i\frac{n}{r}u^m_{r,n} + \frac{du^m_{\theta,n}}{dr})\,\mathbf{e}_\theta + (\frac{du^m_{z,n}}{dr} - i\beta_m u^m_{r,n})\,\mathbf{e}_z\Big\}\, e^{i(n\theta+\beta_m z)}.$$

Using the relation :

$$|\,\mathbf{n}\wedge\mathbf{u}^m_n\,|^2\,(R) = \int \frac{d}{dr}\Big\{\varphi(r)(|\,u^m_{\theta,n}\,|^2 + |\,u^m_{r,n}\,|^2)\Big\}dr,$$

we find that

$$(4)\quad \sum_{(m,n)\in\mathbb{Z}^2} (1+(\frac{n}{R})^2+\beta_m^2)^{-1/2} \mid \mathbf{n}\wedge\mathbf{u}_n^m \mid^2 (R) \quad = \quad \sum_{(m,n)\in\mathbb{Z}^2} (1+(\frac{n}{R})^2+\beta_m^2)^{-1/2}\Big\{$$

$$\int \frac{d\varphi}{dr}\Big\{ \mid u_{\theta,n}^m \mid^2 + \mid u_{\theta,n}^m \mid^2\Big\} dr + 2\Re\mathrm{e} \int \varphi(r) \,\Big\{\, \mathbf{r}_n^m \,.\, \mathbf{n}\wedge\overline{\mathbf{u}}_n^m + (\mathbf{u}_n^m \,.\, \mathbf{k}_n^m(r))(\mathbf{u}_n^m \,.\, \mathbf{e}_r) \,\Big\} dr\Big\},$$

where $\mathbf{k}_n^m(r) = n/r\ \mathbf{e}_\theta + \beta_m \mathbf{e}_\mathbf{z}$.

By using Cauchy inequality :

$$\mid ab \mid \quad \leq \quad \eta\, a^2 + \frac{1}{\eta}\, b^2,\ \forall\, \eta > 0,$$

we see that

$$\|\, \mathbf{n}\wedge\mathbf{u}\, \|^2_{H_{k_z}^{-1/2}(\Sigma_{l,R})} \quad \leq \quad C\,\Big\{\eta \,\|\, \mathrm{curl}\,\mathbf{u}\, \|^2_{L^2(\Omega^\delta_{l,R})} + (1+\frac{1}{\eta}) \,\|\, \mathbf{u}\, \|^2_{L^2(\Omega^\delta_{l,R})}\,\Big\}.$$

Therefore, (3) holds and the proof of the lemma is complete. □

Now, we are in position to give the exact formulation of the boundary value problem describing the scattering of an electro-magnetic field by the curved doubly periodic structure Ω_δ^c.

$$(5)\qquad \begin{cases} \mathrm{curl}\,\dfrac{1}{\mu^\delta}\mathrm{curl}\,\mathbf{E}^\delta - k^2\varepsilon^\delta\mathbf{E}^\delta \;=\; \mathbf{0} \quad \text{in } \Omega^\delta, \\ \mathbf{n}\wedge\mathbf{E}^\delta \;=\; \mathbf{0} \quad \text{on } \partial\Omega_\delta^c, \\ \mathbf{E}^\delta - \mathbf{E}^{in} \quad \text{satisfies an outgoing radiation condition at infinity,} \end{cases}$$

where $\mathbf{n}$ is the outgoing normal to Ω_δ^c, $\varepsilon^\delta(\mathbf{x}) = \varepsilon\circ\phi_\delta^{-1}(\mathbf{x}), \mu^\delta(\mathbf{x}) = \mu\circ\phi_\delta^{-1}(\mathbf{x})$. The material coefficients ε and μ can be written as $\varepsilon = \varepsilon' + i\varepsilon''$ and $\mu = \mu' + i\mu''$, where ε', ε'', μ' and μ'' are real functions in $L^\infty(\mathbb{R}^3)$ and satisfy : $\varepsilon'(x) \geq \varepsilon'_1 > 0,\ \mu'(x) \geq \mu'_1 > 0,\ \varepsilon''(x) \geq 0,\ \mu''(x) \geq 0$. ε and μ are equal to 1 in the exterior domain.

In (5), we assume that the electric incident field $\mathbf{E}^{in}$ is an incident plane wave :

$$\mathbf{E}^{in}(\mathbf{x}', z) \quad = \quad \mathbf{E}_0^{in}\ e^{i(\mathbf{K}.\mathbf{x}'+k_z z)}$$

where $\mathbf{k} = (\mathbf{K}, k_z)$ is the wave vector, $\mathbf{x}' = (x_1, x_2)$ and $\mathbf{E}_0^{in}$ is a constant vector verifying $\mathbf{E}_0^{in}\,.\,\mathbf{k} = 0$, in order to satisfy $\mathrm{div}\,\mathbf{E}^{in} = 0$. The meaning of the radiation condition is not the usual one (Sommerfeld, Silver-Muller). The periodicity of the grating, together with the form of the incident wave, imply that the physical solution $\mathbf{E}^\delta$ must to be quasi-periodic, *i.e.*

$$(6)\quad \mathbf{E}^\delta(r,\theta+2\pi n, z+ml) = e^{imlk_z}\mathbf{E}^\delta(r,\theta,z),\ \forall\,(m,n)\in\mathbb{Z}^2,\ \forall\,(r,\theta,z)\in\mathbb{R}^3\setminus\overline{\Omega}^c.$$

This property is somehow necessary if we want uniqueness of the solution. Now, we are able to give an exact meaning to the radiation condition. $e^{-ik_z z}\mathbf{E}^\delta$ is periodic in the variable (θ, z). Hence, by standard arguments of elliptic regularity theory, we can expand $\mathbf{E}^\delta(r,\theta,z)$ in Fourier series for r large enough :

$$(7)\qquad \mathbf{E}^\delta(r,\theta,z) \quad = \quad \sum_{(m,n)\in\mathbb{Z}^2} \mathbf{E}_n^m(r)\ e^{i(n\theta+\beta_m z)} + \mathbf{E}^{in}(r,\theta,z),$$

where

$$\beta_m = k_z + \frac{2\pi m}{l}.$$

In the following, we assume that

$$k^2 - \beta_m^2 \neq 0.$$

From (5) and (7), we find that $\mathbf{E}_n^m$ satisfies :

$$\frac{d^2}{dr^2}\mathbf{E}_n^m(r) + \frac{1}{r}\frac{d}{dr}\mathbf{E}_n^m(r) + (k^2 - \beta_m^2 - \frac{n^2}{r^2})\mathbf{E}_n^m(r) = \mathbf{0}. \tag{8}$$

Explicitly, if $\mathbf{E}_n^m(r)$ solves equation (8), then

$$\mathbf{E}_n^m(r) = \frac{H_n^{(1)}((k^2-\beta_m^2)^{1/2}r)}{H_n^{(1)}((k^2-\beta_m^2)^{1/2}R)}\,\mathbf{E}_n^m.$$

Here, $\mathbf{E}_n^m$ is a constant vector and $H_n^{(1)}((k^2-\beta_m^2)^{1/2}r)$ the Hankel functions satisfying the outgoing condition at infinity.

3 The variational formulation and the Dirichlet to Neumann operator

In this section, we present a uniqueness result. We shall show that the equations (5) admit a unique solution satisfying the radiation condition provided that the imaginary part of one of the dielectric constants is positive or δ is small enough. we shall set an equivalent problem in the truncated domain $\Omega_{l,R}^\delta$. For this, we need the following two lemmas (see also [3]), the first of which results from a simple calculation.

LEMMA 3.1. *There exists an analytic pseudo-differential operator $\mathcal{R}$ defined on $\Sigma_{l,R}$ of order* 1 *such that :*

$$\Pi(\operatorname{curl}\mathbf{E}^\delta) = \mathcal{R}(\mathbf{E}^\delta \wedge \mathbf{e}_r) + \Pi(\operatorname{curl}\mathbf{E}^{in}) - \mathcal{R}(\mathbf{E}^{in}\wedge\mathbf{e}_r), \tag{9}$$

where $\mathbf{E}^\delta$ is a solution of (5) *satisfying the outgoing condition at infinity and*
$\Pi = -\mathbf{e}_r \wedge (\mathbf{e}_r \wedge .)$.

The pseudo-differential operator $\mathcal{R}$ is defined as :

$$\mathcal{R}(U) = \sum_{J=(m,n)} \Bigg\{ \Big(\frac{n\beta_m}{RZ_J}\,U_{n,\theta}^m + \frac{\beta_m^2 - k^2}{Z_J}\,U_{n,z}^m\Big)\,\mathbf{e}_\theta$$

$$+ \Big(\frac{n\beta_m}{RZ_J}\,U_{n,z}^m - \Big\{ \frac{\beta_m^2}{Z_J(\beta_m^2-k^2)}\big(Z_J^2 - \frac{n^2}{R}\big) + Z_J \Big\} U_{n,z}^m \Big)\,\mathbf{e}_z \Bigg\}\; e^{i(n\theta+\beta_m z)},$$

where

$$Z_J = (k^2-\beta_m^2)^{1/2}\frac{H_n^{(1)\prime}((k^2-\beta_m^2)^{1/2}R)}{H_n^{(1)}((k^2-\beta_m^2)^{1/2}R)},$$

$$U(\mathbf{x}) = \sum_{(m,n)\in\mathbb{Z}} U_{n,\theta}^m(R)\,e^{i(n\theta+\beta_m z)}\,\mathbf{e}_\theta + \sum_{(m,n)\in\mathbb{Z}} U_{n,z}^m(R)\,e^{i(n\theta+\beta_m z)}\,\mathbf{e}_z, \quad \forall\mathbf{x}\in\Sigma_{l,R},$$

and $(H_n^{(1)})$ are the Hankel functions satisfying the outgoing condition at infinity.

$\mathcal{R}$ explains the outgoing condition at infinity. We may prove that it is a continuous mapping from $TH^{-1/2}(\mathrm{div}, \Sigma_{l,R})$ into its dual $TH^{-1/2}(\mathrm{curl}, \Sigma_{l,R})$ and we list some of its basic properties which will be needed in the sequel.

LEMMA 3.2. *The pseudo-differential operator $\mathcal{R}$ satisfies*

$$- \Re e \, \langle \mathcal{R}(\mathbf{v}^\delta), \overline{\mathbf{v}}^\delta \rangle \quad \geq \quad C \, ||\mathbf{v}^\delta||^2_{H^{-1/2}_{k_z}(\mathrm{div}, \Sigma_{l,R})} - C' \, ||\mathbf{v}^\delta||^2_{H^{-1/2}_{k_z}(\Sigma_{l,R})}, \tag{10}$$

and

$$- \Im m \, \langle \mathcal{R}(\mathbf{v}^\delta), \overline{\mathbf{v}}^\delta \rangle \quad \leq \quad 0, \quad \forall \, \mathbf{v}^\delta \in H^{-1/2}_{k_z}(\mathrm{div}, \Sigma_{l,R}), \tag{11}$$

where C and C' are two positive constants.

Proof. From the explicit formula of the pseudo-differential operator $\mathcal{R}$ and the asymptotic form of the Hankel functions

$$Z_J \quad = \quad -\sqrt{\beta_m^2 + n^2/R^2 - k^2} + O(1/R),$$

we see that (10) and (11) hold. $\square$

Now, we can set the problem (5) in $\Omega^\delta_{l,R}$ by replacing the radiation condition at infinity by the boundary condition :

$$\Pi(\mathrm{curl}\, \mathbf{E}^\delta) = \mathcal{R}(\mathbf{E}^\delta \wedge \mathbf{e}_r) + \Pi(\mathrm{curl}\, \mathbf{E}^{in}) - \mathcal{R}(\mathbf{E}^{in} \wedge \mathbf{e}_r) \quad \text{on } \Sigma_{l,R}.$$

Therefore, (5) becomes :

$$\begin{cases} \mathrm{curl}\, \dfrac{1}{\mu^\delta}\, \mathrm{curl}\, \mathbf{E}^\delta - k^2 \, \varepsilon^\delta \, \mathbf{E}^\delta = \mathbf{0} & \text{in } \Omega^\delta_{l,R}, \\ \mathbf{E}^\delta \wedge \mathbf{e}_r = \mathbf{0} & \text{on } \partial\Omega^\delta_c, \\ \Pi(\mathrm{curl}\, (\mathbf{E}^\delta - \mathbf{E}^{in})) = \mathcal{R}(\mathbf{e}_r \wedge (\mathbf{E}^\delta - \mathbf{E}^{in})) & \text{on } \Sigma_{l,R}. \end{cases} \tag{12}$$

We will give a saddle point formulation of the following problem which is equivalent to (5). If we seek for the electric field in $H(\mathrm{curl}, \Omega^\delta_{l,R})$. we decompose it to a smooth field and a gradient, then we write a saddle point formulation from which it is seen that Fredholm's alternative holds, and then uniqueness implies existence of the solution.

If $\mathbf{E}^\delta$ is written as

$$\mathbf{E}^\delta \quad = \quad \mathbf{u}^\delta + \mathbf{grad}\, p^\delta,$$

with

$$\mathrm{div}\, \varepsilon^\delta \, \mathbf{u}^\delta \quad = \quad 0,$$

then, p^δ appears as a Lagrange multiplier and the problem takes the form of a saddle point.

Let us define the Hilbert spaces :

$$X^\delta \quad = \quad \left\{ \mathbf{u} \in H_{k_z}(\mathrm{curl}, \Omega^\delta_{l,R}) \; : \; \mathbf{n} \wedge \mathbf{u} = \mathbf{0} \; \text{ on } \partial\Omega^c_{\delta,l} \right\}$$

and

$$M^\delta \quad = \quad \left\{ \psi \in H^1_{k_z}(\Omega^\delta_{l,R}) \; : \; \psi = 0 \; \text{ on } \partial\Omega^c_{\delta,l} \right\}.$$

Considering $a^\delta(.,.) : X^\delta \times X^\delta \to \mathbb{C}$ and $b^\delta(.,.) : X^\delta \times M^\delta \to \mathbb{C}$ defined as :

(13) $a^\delta(\mathbf{u}^\delta, \mathbf{v}) = \int_{\Omega^\delta_{l,R}} \frac{1}{\mu^\delta} \operatorname{curl} \mathbf{u}^\delta . \operatorname{curl} \overline{\mathbf{v}} \, \mathrm{dx} \; - k^2 \int_{\Omega^\delta_{l,R}} \varepsilon^\delta \, \mathbf{u}^\delta . \overline{\mathbf{v}} \, \mathrm{dx} \; - \langle \mathcal{R}(\mathbf{e}_r \wedge \mathbf{u}^\delta), \mathbf{e}_r \wedge \overline{\mathbf{v}} \rangle$

and

$$b^\delta(\mathbf{v}, \psi) \quad = \quad \overline{a^\delta(\mathbf{grad}\psi, \mathbf{v})}, \tag{14}$$

it is clear that the following variational problem :

$$\left\{ \begin{array}{rcll} \multicolumn{4}{l}{\text{find } (\mathbf{u}^\delta, \varphi^\delta) \in X^\delta \times M^\delta \text{ such that :}} \\ a^\delta(\mathbf{u}^\delta, \mathbf{v}) + \overline{b^\delta(\mathbf{v}, \varphi^\delta)} & = & \langle \mathbf{g}^{in}, \mathbf{e}_r \wedge \overline{\mathbf{v}} \rangle & \forall\, \mathbf{v} \in X^\delta, \\ b^\delta(\mathbf{u}^\delta, \psi) & = & \langle\, \overline{\mathbf{g}^{in}}, \mathbf{e}_r \wedge \mathbf{grad}\, \psi \,\rangle & \forall\, \psi \in M^\delta, \end{array} \right. \tag{15}$$

where

$$\mathbf{g}^{in} \quad = \quad \mathcal{R}(\mathbf{e}_r \wedge \mathbf{E}^{in}) - \Pi(\operatorname{curl} \mathbf{E}^{in}),$$

is equivalent to (5). The form b^δ satisfies the inf-sup condition : for a given $\varphi \in M^\delta$ we take $\mathbf{v}^\delta = \mathbf{grad}\, \varphi^\delta$ and we use Poincaré's inequality. On the other hand its kernel V^δ is given by :

$$V^\delta \quad = \quad \Big\{ \mathbf{v} \in X^\delta, \varepsilon^\delta \mathbf{v} \in H_{k_z}(\mathrm{div}, \Omega^\delta_{l,R}), \mathrm{div}\, \varepsilon^\delta \mathbf{v} = 0 \text{ and } \mathbf{v}.\mathbf{e}_r = \mathcal{D}(\mathbf{e}_r \;\wedge\; \mathbf{v}) \text{ on } \Sigma_{l,R} \Big\},$$

where $\mathcal{D}$ is a continuous operator from $H^s_{k_z}(\mathrm{div}, \Sigma_{l,R})$ into $H^s_{k_z}(\Sigma_{l,R})$, defined by :

$$\mathcal{D}(U) \quad = \quad -i \sum_{J=(m,n)} \frac{1}{Z_J} \left\{ \, \frac{n}{R} \, U^m_{n,z} + \frac{\beta_m}{\beta_m^2 - k^2} \, (\, Z_J^2 - \frac{n^2}{R^2} \,) \, U^m_{n,\theta} \, \right\} e^{i(n\theta + \beta_m z)}.$$

The imbedding $V^\delta \to (L^2(\Omega^\delta_{l,R}))^3$ is compact. The form a^δ satisfies the following lemma :

LEMMA 3.3. *There exist two constants $C > 0$ and $C' > 0$ such that :*

$$\Re e\, a^\delta(\mathbf{u}^\delta, \mathbf{u}^\delta) \quad \geq \quad C\, ||\operatorname{curl} \mathbf{u}^\delta||^2_{L^2(\Omega^\delta_{l,R})} - C'\, ||\mathbf{u}^\delta||^2_{L^2(\Omega^\delta_{l,R})}. \tag{16}$$

Proof. (13) yields

$$\Re e\, a^\delta(\mathbf{u}^\delta, \mathbf{u}^\delta) = \Re e \int_{\Omega^\delta_{l,R}} \frac{1}{\mu^\delta} \mid \operatorname{curl} \mathbf{u}^\delta \mid^2 \, \mathrm{dx} - k^2 \, \Re e \int_{\Omega^\delta_{l,R}} \varepsilon^\delta \mid \mathbf{u}^\delta \mid^2 \, \mathrm{dx} - \Re e\, \langle \mathcal{R}(\mathbf{e}_r \wedge \mathbf{u}^\delta), \mathbf{e}_r \wedge \overline{\mathbf{u}^\delta} \rangle.$$

Using the inequality (10), we find that there exist two positive constants C and C' such that

$$\Re e\, a^\delta(\mathbf{u}^\delta, \mathbf{u}^\delta) \geq C \left\{ \, ||\operatorname{curl} \mathbf{u}^\delta||^2_{L^2(\Omega^\delta_{l,R})} - ||\mathbf{u}^\delta||^2_{L^2(\Omega^\delta_{l,R})} + ||\mathbf{u}^\delta||^2_{H^{-1/2}_{k_z}(\mathrm{div}, \Sigma_{l,R})} \right\} - C'\, ||\mathbf{u}^\delta||^2_{H^{-1/2}_{k_z}(\Sigma_{l,R})}. \tag{17}$$

We recall that there exists a positive constant C'' such that, $\forall \eta > 0$,

$$||\, \mathbf{u}^\delta \,||_{H^{-1/2}_{k_z}(\Sigma_{l,R})} \quad \leq \quad C'' \left\{ \eta \, ||\, \operatorname{curl} \mathbf{u}^\delta \,||_{L^2(\Omega^\delta_{l,R})} + (1 + \frac{1}{\eta}) \, ||\, \mathbf{u}^\delta \,||_{L^2(\Omega^\delta_{l,R})} \right\}. \tag{18}$$

Making use of (18) above, we obtain that there exist two positive constants C and C' such that

$$\Re e\, a^\delta(\mathbf{u}^\delta, \mathbf{u}^\delta) \quad \geq \quad C\, ||\operatorname{curl} \mathbf{u}^\delta||^2_{L^2(\Omega^\delta_{l,R})} - C'\, ||\mathbf{u}^\delta||^2_{L^2(\Omega^\delta_{l,R})}.$$

Using Fredholm's alternative, existence of solutions of (5) follows then from uniqueness. It suffices then to show the uniqueness of the solution of (5). If ε^δ or μ^δ has a non zero imaginary part, it is easily seen that the Maxwell equations (5) admit at most a solution. If $\mathbf{E}^\delta$ is an element in the kernel, then we have :

$$(19)\ \int_{\Omega^\delta_{l,R}} \frac{1}{\mu^\delta} \mid \operatorname{curl} \mathbf{E}^\delta \mid^2 \ \mathrm{dx} \ - \ k^2 \int_{\Omega^\delta_{l,R}} \varepsilon^\delta \ \mid \mathbf{E}^\delta \mid^2 \ \mathrm{dx} \quad = \quad \langle\, \mathcal{R}(\mathbf{e}_r \wedge \mathbf{E}^\delta), \mathbf{e}_r \wedge \overline{\mathbf{E}^\delta}\,\rangle.$$

By (11) and taking the imaginary part in the expression (19) above, we see that

$$\int_{\Omega^\delta_{l,R}} \frac{\Im m\ \mu^\delta}{\mid \mu^\delta \mid^2} \mid \operatorname{curl} \mathbf{E}^\delta \mid^2 \ \mathrm{dx} \ + \ k^2 \int_{\Omega^\delta_{l,R}} \Im m\ \varepsilon^\delta \ \mid \mathbf{E}^\delta \mid^2 \ \mathrm{dx} \quad = \quad 0.$$

Applying the uniqueness theorem of Cauchy-Kowaleska, we conclude that (5) admits at most a solution. In the following, we suppose that ε^δ and μ^δ are real valued. We define

$$J_k^+ \quad = \quad \Big\{\ J = (m,n) \in \mathbb{Z}^2 \text{ such that } \beta_m^2 - k^2 \geq 0\ \Big\},$$

and

$$J_k^- \quad = \quad \Big\{J = (m,n) \in \mathbb{Z}^2 \text{ such that } \beta_m^2 - k^2 \leq 0\ \Big\}.$$

If $\mathbf{E}^\delta$ is an element in the kernel then, by taking the imaginary part in the expression (19) above, we find that :

$$\mathbf{E}_n^m \quad = \quad \mathbf{0}, \quad \forall\, J = (m,n) \in J_k^-.$$

There is no propagation in the direction $\mathbf{e}_r$. So, we introduce the following operator :

$$\mathcal{R}'U(\mathbf{x}') \quad = \quad \sum_{J \in J_k^+} \Big\{ \Big(\frac{n\beta_m}{RZ_J}\, U^m_{n,\theta} + \frac{\beta_m^2 - k^2}{Z_J}\, U^m_{n,z} \Big)\, \mathbf{e}_\theta$$

$$+ \Big(\frac{n\beta_m}{RZ_J}\, U^m_{n,z} - \Big\{ \frac{\beta_m^2}{Z_J(\beta_m^2 - k^2)} \big(Z_J^2 - \frac{n^2}{R} \big) + Z_J \Big\} U^m_{n,z} \Big)\, \mathbf{e}_z \Big\}\ e^{i(n\theta + \beta_m z)}.$$

Therefore, $\mathbf{E}^\delta$ satisfies the boundary condition on $\Sigma_{l,R}$:

$$\Pi(\operatorname{curl} \mathbf{E}^\delta) \quad = \quad \mathcal{R}'(\mathbf{E}^\delta \wedge \mathbf{e}_r)$$

We also introduce the continuous sesquilinear forms :

$$a^\delta_\omega(\mathbf{u}^\delta, \mathbf{v}^\delta) \quad = \quad \int_{\Omega^\delta_{l,R}} \frac{1}{\mu^\delta} \operatorname{curl} \mathbf{u}^\delta \,.\, \operatorname{curl} \overline{\mathbf{v}^\delta} \ + \ \omega^2 \int_{\Omega^\delta_{l,R}} \varepsilon^\delta\ \mathbf{u}^\delta \,.\, \overline{\mathbf{v}^\delta} - \langle \mathcal{R}'(\mathbf{e}_r \wedge \mathbf{u}^\delta), (\mathbf{e}_r \wedge \overline{\mathbf{v}^\delta}) \rangle,$$

and

$$a^\delta_0(\mathbf{u}^\delta, \mathbf{v}^\delta) \quad = \quad \int_{\Omega^\delta_{l,R}} \varepsilon^\delta\ \mathbf{u}^\delta \,.\, \overline{\mathbf{v}^\delta}.$$

As shown above, a^δ_ω is X^δ-coercive for $\omega^2 > 0$ sufficiently large. Let b^δ_ω be the following continuous form :

$$b^\delta_\omega(\mathbf{u}^\delta, p^\delta) \quad = \quad \overline{a^\delta_\omega(\mathbf{grad}\, p^\delta, \mathbf{u}^\delta)}.$$

Consider the following eigenvalue problem :

$$\left\{\begin{array}{rcll} \multicolumn{4}{l}{\text{find } \lambda \in \mathbb{R}, (\mathbf{u}^\delta, p^\delta) \in X^\delta \times M^\delta \text{ such that :}} \\ a^\delta_\omega(\mathbf{u}^\delta,\mathbf{v}^\delta) + \overline{b^\delta_\omega(\mathbf{v}^\delta,p^\delta)} &=& \lambda\, a^\delta_0(\mathbf{u}^\delta,\mathbf{v}^\delta) & \forall\, \mathbf{v}^\delta \in X^\delta, \\ b^\delta_\omega(\mathbf{u}^\delta,q^\delta) &=& 0 & \forall\, q^\delta \in M^\delta, \end{array}\right.$$

$\mathbf{E}^\delta$ is then solution to this problem with

$$\lambda \quad = \quad \omega^2 - k^2.$$

We may prove that the modified truncated Maxwell's operator A :

$$D(A) = \Big\{ \ \mathbf{u} \in H_{k_z}(\mathrm{curl}, \Omega^\delta_{l,R}, \varepsilon^\delta dx), \mathrm{curl}\,\mathbf{u} \in H_{k_z}(\mathrm{curl}, \Omega^\delta_{l,R}, \varepsilon^\delta dx),$$

$$\mathbf{u}\wedge\mathbf{n} = \mathbf{0} \text{ on } \Gamma, \mathrm{div}\,\varepsilon^\delta\,\mathbf{u} = 0 \text{ and } \Pi(\mathrm{curl}\,\mathbf{u}) = \mathcal{R}'(\mathbf{u}\wedge\mathbf{e}_r) \ \Big\},$$

where

$$A \quad = \quad -\mathrm{curl}\,\frac{1}{\mu^\delta}\mathrm{curl},$$

is a self adjoint, densely defined operator with compact resolvent. Therefore, we can apply to A standard results on self adjoint operators, bounded from below with compact resolvent. Thus, we can state that A has an orthonormal basis of eigenvectors depending on k via $\mathcal{R}'$. Finally, we summarize the results of this section in the following theorem. The conclusions below remain valid with much weaker regularity of the boundary of Ω^c_δ. The following theorem holds when the open set Ω^c_δ satisfies the restricted cone property by applying the result of Weber [10] to prove that the imbedding $V^\delta \longmapsto L^2(\Omega^\delta_{l,R})$ is compact.

THEOREM 3.1. *Up to a possible sequence of eigenfrequencies k going to infinity, there exists a unique solution of* (5) *satisfying the appropriate radiation condition at infinity.*

4 Convergence to the limiting problem, the first estimate

We now focus on the main problem of this paper. We want to replace the equations (5) in the domain Ω^δ by an approximate boundary value problem in Ω. Remark that for δ small enough, Maxwell's equations (5) admit a unique solution even ε^δ and μ^δ are real valued. We also notice that : $\mathbf{E}^\delta = \mathbf{u}^\delta(\varphi^\delta = 0)$, this follows from $\mathbf{grad}\,M^\delta \subset X^\delta$.

Let us first introduce p^δ an extension operator into the conductor inclusions Ω^c_N satisfying :

$$p^\delta\mathbf{E}^\delta \in W = \Big\{ \ \mathbf{w} \in H_{k_z}(\mathrm{curl}, \Omega_{l,R}), \varepsilon^\delta\,\mathbf{w} \in H_{k_z}(\mathrm{div}, \Omega_{l,R}), \mathbf{v}\wedge\mathbf{n} = \mathbf{0} \ \text{ on } \ \Gamma \ \Big\}$$

and the following estimate :

$$|| \, p^\delta \mathbf{v}^\delta \, ||_{H_{k_z}(\mathrm{curl},\Omega_{l,R})} \quad \leq \quad C \quad || \, \mathbf{v}^\delta \, ||_{H_{k_z}(\mathrm{curl},\Omega^\delta_{l,R})}, \tag{20}$$

where C is a positive constant independent of δ. p^δ can be easily derived from p an extension operator on the elementary truncated cell $O_h = O \cap \{y_2 < h\}(h > 1)$, from

$$\Big\{\mathbf{w} \in H_{per}(\mathrm{curl}, O_h), \mathrm{div}\,\varepsilon\,\mathbf{w} = 0, \mathbf{n}\wedge\mathbf{w} = \mathbf{0} \text{ on } G^c\Big\}$$

into

$$\left\{ \mathbf{w} \in H_{per}(\mathrm{curl}, O_h \cup \overline{O}^c), \varepsilon\, \mathbf{w} \in H_{per}(\mathrm{div}, O_h \cup \overline{O}^c) \right\},$$

satisfying

$$\| p\mathbf{v} \|_{H_{per}(\mathrm{curl}, O_h \cup \overline{O^c})} \quad \leq \quad C \quad \| \mathbf{v} \|_{H_{per}(\mathrm{curl}, O_h)} .$$

p^δ is defined as

$$p^\delta \mathbf{v}^\delta \quad = \quad p(\mathbf{v}^\delta \circ \phi_\delta) \circ \phi_\delta^{-1}.$$

It is now possible to show the following theorem :

THEOREM 4.1. *For any $R > 0$, the sequence of solutions $p^\delta \mathbf{E}^\delta$ of (5) converges strongly in $H_{k_z}(\mathrm{curl}, \Omega_{l,R})$, to $\mathbf{E}^{(0)}$ unique solution of :*

$$\left\{ \begin{array}{rcll} \mathrm{curl}\,\mathrm{curl}\, \mathbf{E}^{(0)} - k^2\, \mathbf{E}^{(0)} & = & \mathbf{0} & \text{in } \Omega_l, \\ \mathbf{n} \wedge \mathbf{E}^{(0)} & = & \mathbf{0} & \text{on } \partial\Omega_l, \\ \mathbf{E}^{(0)} - \mathbf{E}^{in} & \multicolumn{3}{l}{\text{satisfies an outgoing radiation condition at infinity.}} \end{array} \right. \tag{21}$$

Moreover, there exists a constant $C(R) > 0$ such that :

$$\| p^\delta \mathbf{E}^\delta - \mathbf{E}^{(0)} \|_{H_{k_z}(\mathrm{curl}, \Omega_{l,R})} \quad \leq \quad C(R)\, \delta^{1/2}. \tag{22}$$

Proof. The general outline of this proof follows the same lines as the proof of the convergence result for the Helmholtz equation given in [2]. We first show that there exists a positive constant C independent of δ such that :

$$\| p^\delta \mathbf{E}^\delta \|_{H_{k_z}(\mathrm{curl}, \Omega_{l,R})} \quad \leq \quad C \, \| p^\delta \mathbf{E}^\delta \|_{L^2(\Omega_{l,R})} . \tag{23}$$

Since $\mathbf{E}^\delta$ satisfies

$$\Re e\, a^\delta(\mathbf{E}^\delta, \mathbf{E}^\delta) \quad \leq \quad \|\mathbf{g}^{in}\|_{H^{1/2}_{k_z}(\Sigma_{l,R})} \, \|\mathbf{e}_r \wedge \mathbf{E}^\delta\|^2_{H^{-1/2}_{k_z}(\Sigma_{l,R})},$$

then

$$\Re e\, a^\delta(\mathbf{E}^\delta, \mathbf{E}^\delta) \quad \leq \quad C \left\{ \eta \, \| \,\mathrm{curl}\, \mathbf{E}^\delta \, \|_{L^2(\Omega^\delta_{l,R})} + (1 + \frac{1}{\eta}) \, \| \, \mathbf{E}^\delta \, \|_{L^2(\Omega^\delta_{l,R})} \right\}.$$

Hence, from (16) and (20), we deduce that there exists a positive constant C' independent of δ such that

$$\| \, \mathbf{E}^\delta \, \|_{H_{k_z}(\mathrm{curl}, \Omega_{l,R})} \quad \leq \quad C' \, \| \, p^\delta \mathbf{E}^\delta \, \|_{L^2(\Omega_{l,R})},$$

and then (23) holds from (20).

Now, let us prove by contradiction that $p^\delta \mathbf{E}^\delta$ is bounded in $H_{k_z}(\mathrm{curl}, \Omega_{l,R})$. Suppose that we have a subsequence still denoted by $p^\delta \mathbf{E}^\delta$ such that $\| \, p^\delta \mathbf{E}^\delta \, \|_{L^2(\Omega_{l,R})}$ converges to $+\infty$, we introduce

$$\mathbf{u}^\delta \quad = \quad \frac{p^\delta \mathbf{E}^\delta}{\| \, p^\delta \mathbf{E}^\delta \, \|_{L^2(\Omega_R)}}.$$

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ equal to 1 in a sufficiently large neighborhood of 0, we define :

$$\varphi^\delta(\mathbf{x}(s, n, z)) \quad = \quad \varphi(\frac{n}{\delta}).$$

By using $(1-\varphi^\delta)\,\mathbf{v}$ where $\mathbf{v}\in H_{k_z}(\mathrm{curl},\Omega_{l,R})$ such that $\mathbf{v}\wedge\mathbf{n}=\mathbf{0}$ on Γ as a test-function on the variational formulation of (5), we obtain :

$$(24)\quad \int_{\Omega_{l,R}} \frac{1}{\mu^\delta}\mathrm{curl}\,\mathbf{u}^\delta \,.\, \mathrm{curl}\,\mathbf{v}\ \mathrm{dx}\ - k^2\int_{\Omega_{l,R}} \varepsilon^\delta \mathbf{u}^\delta \,.\, \mathbf{v}\ \mathrm{dx}\ - \langle \mathcal{R}(\mathbf{u}^\delta\wedge\mathbf{e}_r),\mathbf{v}\wedge\mathbf{e}_r\rangle \quad\longrightarrow\quad 0.$$

Thus we deduce that $\mathbf{u}^\delta$ converges to 0, $H_{k_z}(\mathrm{curl},\Omega_{l,R})$ weakly, since

$$\|\,\mathbf{u}^\delta\,\|_{L^2(\Omega_{l,R})} \quad=\quad 1,$$

this leads to a contradiction. The strong convergence is deduced from the compact imbedding of W in $L^2(\Omega_{l,R})$.

To prove the error estimate (22), let us define

$$a^\delta(\mathbf{u},\mathbf{v}) \quad=\quad \int_{\Omega_{l,R}} \frac{1}{\mu^\delta}\,\mathrm{curl}\,\mathbf{u}\,.\,\mathrm{curl}\,\mathbf{v}\ \mathrm{dx} - k^2\int_{\Omega_{l,R}} \varepsilon^\delta\,\mathbf{u}\,.\,\mathbf{v}\ \mathrm{dx} - \langle\mathcal{R}(\mathbf{u}\wedge\mathbf{e}_r,\mathbf{v}\wedge\mathbf{e}_r\rangle.$$

(a^δ) is a family of sesquilinear forms on W satisfying the conditions of the following lemma (see [2] for the proof) where

$$a_0(\mathbf{u},\mathbf{v}) \quad=\quad \int_{\Omega_{l,R}} \mathrm{curl}\,\mathbf{u}\,.\,\mathrm{curl}\,\mathbf{v}\ \mathrm{dx} + \int_{\Omega_{l,R}} \mathbf{u}\,.\,\mathbf{v}\ \mathrm{dx}\ -\ \langle\mathcal{R}(\mathbf{u}\wedge\mathbf{e}_r,\mathbf{v}\wedge\mathbf{e}_r\rangle,$$

and

$$a_r(\mathbf{u},\mathbf{v}) \quad=\quad -(1+k^2)\int_{\Omega_{l,R}} \mathbf{u}\,.\,\mathbf{v}\ \mathrm{dx}.$$

Lemma 4.1. *Consider $(a_0^n)_{n\in\mathbb{N}},(a_r^n)_{n\in\mathbb{N}},a_0$ and a_r continuous sesquilinear forms on W satisfying the conditions :*

(i) $\exists\,\alpha>0,\,\forall\,n\in\mathbb{N},\,\forall\,v\in W,\,\Re e\,a_0^n(v,v)\geq\alpha\ \|\,v\,\|_W^2\,;$

(ii) $\forall\,(u_n)\in W^{\mathbb{N}},\ u_n\rightharpoonup u\Rightarrow a_r^n(u_n,u_n)\to a_r(u,u)\,;$

(iii) $\forall\,(u_n)\in W^{\mathbb{N}},\ u_n\to u\Rightarrow a_0^n(u_n,u_n)\to a_0(u,u)\,;$

(iv) $a(u,u)=a_0(u,u)+a_r(u,u)=0\Longrightarrow u=0\,;$

Let (u_n) be a sequence in W, then

$$a_0^n(u_n,u_n)+a_r^n(u_n,u_n)=O(n^{-s})\,\|\,u_n\,\|_W\quad(s>0)\ \Longrightarrow\|\,u_n\,\|_W=O(n^{-s}).$$

Regularity of $\mathbf{E}^0$ shows that there exists a constant $C>0$ independent of δ such that :

$$(25)\qquad |\,a^\delta(\mathbf{E}^\delta-\mathbf{E}^{(0)},\mathbf{E}^\delta-\mathbf{E}^{(0)})\,| \quad\leq\quad C\,\delta^{1/2}\ \|\,p^\delta\mathbf{E}^\delta-\mathbf{E}^{(0)}\,\|_{H_{k_z}(\mathrm{curl},\Omega_{l,R})},$$

and Lemma 4.1 allows us to conclude. $\square$

5 The first equivalent boundary condition

We start by exhibiting the boundary layer corrector $\mathbf{E}_{BL}^{(0)}$ of order 0. This permits us to derive the first impedance condition. $\mathbf{E}_{BL}^{(0)}$ corrects the boundary conditions on Ω_N^c, the jumps of $[\frac{1}{\mu^\delta}\,\mathrm{curl}\,\mathbf{E}^{(0)} \wedge \mathbf{n}]$ and $[\,\varepsilon^\delta\,\mathbf{E}^{(0)}\,.\,\mathbf{n}\,]$ across Γ^δ. To construct $\mathbf{E}_{BL}^{(0)}$, we need first the following lemma.

LEMMA 5.1. *There exists a unique* $\alpha_0^{(0)} \in H^{loc}_{per}(\mathrm{curl}, \overline{O})$, *such that :*

$$\left\{\begin{array}{rcll} \mathrm{curl}\,\alpha_0^{(0)} &=& \mathbf{0} & \text{in } O, \\ \mathrm{div}\,\varepsilon\,\alpha_0^{(0)} &=& 0 & \text{in } O, \\ \alpha_0^{(0)} \wedge \mathbf{n} &=& -\mathbf{e}_2 \wedge \mathbf{n} & \text{on } G^c, \\ [\,\varepsilon\,\alpha_0^{(0)}.\mathbf{n}\,] &=& -[\,\varepsilon\,] & \text{on } G^d, \\ \alpha_0^{(0)} \wedge \mathbf{n} &=& \mathbf{0} & \text{on } G, \\ \multicolumn{4}{l}{\displaystyle\int_G \varepsilon\,\alpha_0^{(0)}\,.\,\mathbf{e}_2\,dy_1dy_3 = -[\,\varepsilon\,].} \end{array}\right. \tag{26}$$

The first step in the proof of this lemma is to show the existence of $\alpha_0^{(0)}$ by solving a scalar problem. In the second step, to show the uniqueness of $\alpha_0^{(0)}$, the following lemma is used.

LEMMA 5.2. *The functions* $\mathbf{u} \in H^{loc}_{per}(\mathrm{curl}, \overline{O})$ *which satisfy :*

$$\left\{\begin{array}{rcll} \mathrm{curl}\,\mathbf{u} &=& \mathbf{0} & \text{in } O, \\ \mathrm{div}\,\varepsilon\,\mathbf{u} &=& 0 & \text{in } O, \\ \mathbf{u} \wedge \mathbf{n} &=& \mathbf{0} & \text{on } G^c, \\ \mathbf{u} \wedge \mathbf{n} &=& \mathbf{0} & \text{on } G. \end{array}\right. \tag{27}$$

form, provided O^c is connected, a two-dimensional vectorial space N, given by

$$N = \left\{ a\mathbf{e}_2 + \mathbf{grad}\,\varphi_{a,b}, (a,b) \in \mathbb{C}^2 \right\},$$

where $\varphi_{a,b}$ is defined as

$$\left\{\begin{array}{rcll} \mathrm{div}\,\varepsilon\,\mathbf{grad}\,\varphi_{a,b} &=& 0 & \text{in } O, \\ \varphi_{a,b} &=& -ay_2 + b & \text{on } G^c, \\ \varphi_{a,b} &=& 0 & \text{on } G. \end{array}\right. \tag{28}$$

Proof. Once the existence of $\varphi_{a,b}$ for $(a,b) \in \mathbb{C}^2$ is shown, the next step is to find (a,b) such that $\mathbf{grad}\,\varphi_{a,b}$ is a solution of (27). For the proof of the uniqueness and the existence of $\varphi_{a,b}$ see lemma 3.1 in [2]. We are now able to sketch a proof of lemma 5.1. Let us define the mapping $m(a,b)$ as

$$m(a,b) = \int_G \varepsilon\,\mathbf{grad}\,\varphi_{a,b}\,.\,\mathbf{e}_2\,dy_1dy_3.$$

Let (a_1, b_1) and (a_2, b_2) two pairs of complex, such that $m(a_1, b_1) \neq 0$, $m(a_2, b_2) \neq 0$ and $m(a_1, b_1) \neq m(a_2, b_2)$. There exists a unique λ, such that

$$\lambda\, m(a_1, b_1) + (1 - \lambda)\, m(a_2, b_2) = -[\, \varepsilon\,].$$

Then, if

$$(a, b) \quad = \quad \lambda\, (a_1, b_1) + (1 - \lambda)\, (a_2, b_2),$$

then, $\varphi_{a,b}$ satisfies (28). Now, in order to show the uniqueness of $\alpha_0^{(0)}$, let us consider φ a solution to the following problem :

$$\left\{ \begin{array}{rcll} \operatorname{div} \varepsilon\, \mathbf{grad}\, \varphi & = & 0 & \text{in } O, \\ \varphi & = & a & \text{on } G^c, \\ \varphi & = & 0 & \text{on } G, \\ \multicolumn{4}{l}{\displaystyle\int_G \varepsilon\, \mathbf{grad}\, \varphi \,.\, \mathbf{e}_2\, dy_1 dy_3 = 0.} \end{array} \right. \tag{29}$$

As in lemma 3.1, we may prove that there exists an analytic pseudo-differential operator of order 1 such that

$$\frac{\partial \varphi}{\partial y_2}\, |_{y_2 = h} = \; T_{per} \varphi\, |_{y_2 = h}\, . \tag{30}$$

T_{per} can be extended into a continuous operator from $H^{1/2}_{per}(G_h)$ into its dual $H^{-1/2}_{per}(G_h)$. Using T_{per}, the problem (29) is equivalent to the truncated problem :

$$\left\{ \begin{array}{rcll} \operatorname{div} \varepsilon\, \mathbf{grad}\, \varphi & = & 0 & \text{in } O, \\ \varphi & = & a & \text{on } G^c, \\ \varphi & = & 0 & \text{on } G, \\ [\, \varepsilon\, \dfrac{\partial \varphi}{\partial n}\,] & = & 0 & \text{on } G^d, \\ \dfrac{\partial \varphi}{\partial y_2} & = & T_{per} \varphi & \text{on } G_h, \\ \displaystyle\int_G \varepsilon\, \frac{\partial \varphi}{\partial y_2}\, dy_1 dy_3 & = & 0. & \end{array} \right. \tag{31}$$

The variational formulation for (31) is given by :

$$\int_{O_h} \varepsilon\, \mathbf{grad}\, \varphi \,.\, \mathbf{grad}\, \psi \text{ dy } - \langle\, T_{per} \varphi, \psi\, \rangle = a \int_{G^c} \varepsilon\, \frac{\partial \psi}{\partial n}\, d\sigma,\ \forall \psi \in H^1(O_h), \psi = 0 \text{ on } G. \tag{32}$$

To verify that $\varphi = 0$, first let $\varphi = \psi$ in the above variational formulation. (32) yields

$$\int_{O_h} \varepsilon\, \mid \mathbf{grad}\, \varphi \mid^2 \text{ dy } - \langle\, T_{per} \varphi, \varphi\, \rangle \quad = \quad -a \int_{G^c} \varepsilon\, \frac{\partial \varphi}{\partial n} d\sigma.$$

Next, multiply $\operatorname{div} \varepsilon\, \mathbf{grad}\, \varphi$ by 1 and integrate by part. Therefore, from

$$\int_G \varepsilon\, \frac{\partial \varphi}{\partial y_2}\, dy_1 dy_3 \quad = \quad 0,$$

and $\operatorname{div} \varepsilon\, \mathbf{grad}\, \varphi = 0$, we obtain that

$$\int_{G^c} \varepsilon \, \frac{\partial \varphi}{\partial n} d\sigma \quad = \quad 0.$$

So that

(33)
$$\int_{O_h} \varepsilon \;|\, \mathbf{grad}\, \varphi \,|^2 \;\; \mathrm{dy} \;\; - \langle\, T_{per}\varphi, \varphi \,\rangle \quad = \quad 0.$$

The fact that :

(34)
$$- \langle T_{per}\alpha, \alpha \rangle \;\geq\; 0, \quad \forall \;\alpha \in H^{1/2}_{per}(G_h),$$

(see [2] for the proof) shows that the sesquilinear form is coercive and then $\varphi = 0$ follows from (33). This ends the proof of lemma 5.1. □

Let us define $b_0^{(0)}$ as $b_0^{(0)} = -[\, \frac{1}{\mu}\, \alpha_0^{(0)} \,.\, \mathbf{n}\,]$ and $\beta_0^{(0)}$ such that

$$\phi_\delta \,\Big(\, \alpha_0^{(0)}(\mathbf{x}/\delta) \,\Big) \quad = \quad \delta\, \beta_0^{(0)}(\mathbf{x}/\delta).$$

From (26), we obtain that the quantity $E^{(0)}(\mathbf{x}_\Gamma)\; \beta_0^{(0)}(\mathbf{x}/\delta)$ corrects the boundary conditions and the jump of $[\, \varepsilon^\delta\, \mathbf{E}^{(0)} \,.\, \mathbf{n}\,]$ across Γ^δ.

Now, for the sake of simplicity, we suppose that ε and μ are two constants in the layer of dielectric material. The following lemma is needed to find the correctors for the jump relation :

$$[\, \frac{1}{\mu^\delta} \operatorname{curl} \Big\{ \mathbf{E}^{(0)} + E^{(0)}(\mathbf{x}_\Gamma)\; \beta_0^{(0)}(\mathbf{x}/\delta) \;\Big\} \;\wedge \mathbf{n}\,].$$

LEMMA 5.3. $\forall i \in \{1,3\}$, *there exists a unique* $(\alpha_i^{(0)}, \mathbf{a}_i^{(0)}) \in (H^{loc}_{per}(\mathrm{curl}, \overline{O}))^3 \times \mathbb{C}^{\,3}$, *such that :*

(35)
$$\left\{ \begin{array}{rcll} \operatorname{curl} \dfrac{1}{\mu} \operatorname{curl} \alpha_i^{(0)} & = & \mathbf{0} & \text{in } O, \\ \operatorname{div} \varepsilon\, \alpha_i^{(0)} & = & 0 & \text{in } O, \\ \mathbf{n} \wedge \alpha_i^{(0)} & = & \mathbf{0} & \text{on } G, \\ \mathbf{n} \wedge \alpha_i^{(0)} & = & \mathbf{0} & \text{on } G^c, \\ [\, \dfrac{1}{\mu} \operatorname{curl} \alpha_i^{(0)} \wedge \mathbf{n}\,] & = & -\mathbf{e}_i \wedge \mathbf{n} & \text{on } G^d, \\ \multicolumn{4}{l}{\alpha_i^{(0)} - \mathbf{a}_i^{(0)} \to \mathbf{0}, \quad \mathbf{a}_i^{(0)} \,.\, \mathbf{e}_2 = 0.} \end{array} \right.$$

Proof. We consider the following more general problem :

(36)
$$\left\{ \begin{array}{rcll} \operatorname{curl} \dfrac{1}{\mu} \operatorname{curl} \alpha & = & \mathbf{f} & \text{in } O, \\ \operatorname{div} \varepsilon\, \alpha & = & 0 & \text{in } O, \\ \mathbf{n} \wedge \alpha & = & \mathbf{0} & \text{on } G, \\ \mathbf{n} \wedge \alpha & = & \mathbf{0} & \text{on } G^c, \\ [\, \dfrac{1}{\mu} \operatorname{curl} \alpha_i^{(0)} \wedge \mathbf{n}\,] & = & \mathbf{g} & \text{on } G^d, \\ [\, \varepsilon\, \alpha \,.\, \mathbf{n}\,] & = & h & \text{on } G^d, \\ \multicolumn{4}{l}{\alpha - \mathbf{a} \to \mathbf{0}, \quad \mathbf{a} \,.\, \mathbf{e}_2 = 0,} \end{array} \right.$$

where, $\mathbf{f}$ is periodic with exponential decay in y_2, $\mathbf{g} \in H_{per}^{-1/2}(\text{curl}, G^d)$ and $h \in H_{per}^{-1/2}(G^d)$. Consider

$$N_h = \left\{ a\mathbf{e}_2 + \mathbf{grad}\ \varphi_{a,b}, (a,b) \in \mathbb{C}^2 \right\},$$

where $\varphi_{a,b}$ is defined as

$$\left\{ \begin{array}{rcll} \operatorname{div} \varepsilon\, \mathbf{grad}\ \varphi_{a,b} & = & 0 & \text{in } O, \\ \varphi_{a,b} & = & -a y_2 + b & \text{on } G^c, \\ \varphi_{a,b} & = & 0 & \text{on } G, \\ [\, \varepsilon \dfrac{\partial \varphi_{a,b}}{\partial n} \,] & = & h & \text{on } G^d. \end{array} \right. \tag{37}$$

N_h is a two dimensional vectorial space. As in the proof of existence and uniqueness of $\alpha_0^{(0)}$, we introduce a truncated problem. Here, another difficulty comes from the fact that $\mathbf{f}$ does not have a compact support in O, but decays exponentially in y_2. However, since $\mathbf{f}$ is periodic, we can obtain a transparent boundary condition analogous to (10) :

$$\Pi(\alpha) = \mathcal{R}_{per}(\alpha \wedge \mathbf{e}_2) + \mathcal{N}_{per}(\mathbf{f}), \tag{38}$$

where $\mathcal{R}_{per}$ is the operator defined as

$$\mathcal{R}_{per} \ : \mathbf{u} = \sum_{J=(m,n)\in \mathbb{Z}^2} \mathbf{u}_J\, e^{i2\pi J\,.\,\mathbf{y}} \longmapsto \sum_{J=(m,n)\in \mathbb{Z}^2} \Big(J\,.\,\mathbf{u}_J \Big) \frac{J}{|\, J\,|}\, e^{i2\pi J\,.\,\mathbf{y}},$$

and $\mathcal{N}_{per}$ is a pseudo-differential operator. It can be calculated explicitly. The truncated problem reads : seek $\alpha \in H_{per}^{loc}(\text{curl}, \overline{O})$, such that :

$$\left\{ \begin{array}{rcll} \operatorname{curl} \dfrac{1}{\mu} \operatorname{curl} \alpha & = & \mathbf{f} & \text{in } O, \\ \operatorname{div} \varepsilon\, \alpha & = & 0 & \text{in } O, \\ \mathbf{n} \wedge \alpha & = & \mathbf{0} & \text{on } G, \\ \mathbf{n} \wedge \alpha & = & \mathbf{0} & \text{on } G^c, \\ [\, \dfrac{1}{\mu} \operatorname{curl} \alpha_i^{(0)} \wedge \mathbf{n} \,] & = & \mathbf{g} & \text{on } G^d, \\ [\, \varepsilon\, \alpha\,.\,\mathbf{n} \,] & = & h & \text{on } G^d, \\ \Pi(\alpha) & = & \mathcal{R}_{per}(\alpha \wedge \mathbf{e}_2) + \mathcal{N}_{per}(\mathbf{f}) & \text{on } G^d. \end{array} \right. \tag{39}$$

We suppose that

$$\int_{O_h} \mathbf{f}\,.\,\mathbf{v}\ \text{dy} + \int_{G^d} \mathbf{g}\,.\,\mathbf{v}\ d\sigma + \langle \mathcal{N}_{per}(\mathbf{f}), \mathbf{v} \wedge \mathbf{e}_2 \rangle = 0, \quad \forall\, \mathbf{v} \in N_h. \tag{40}$$

Remark that

$$\lim_{y_2 \to +\infty} \mathbf{v}\,.\,\mathbf{e}_2 = 0, \quad \forall\, \mathbf{v} \in N_h. \tag{41}$$

By essentialy the same arguments as in the proof of theorem 3.1, we can write a point saddle formulation for the problem (39) with :

$$a(\mathbf{u}, \mathbf{v}) = \int_{O_h} \frac{1}{\mu} \operatorname{curl} \mathbf{u}\,.\operatorname{curl} \mathbf{v}\ \text{dy} - \langle \mathcal{R}_{per}(\mathbf{u} \wedge \mathbf{e}_2), \mathbf{v} \wedge \mathbf{e}_2 \rangle \tag{42}$$

$$= \int_{O_h} \mathbf{f} \,.\, \mathbf{v} \; \mathrm{dy} \; + \int_{G^d} \mathbf{g} \,.\, \mathbf{v} \; d\sigma \; + \; \langle \mathcal{N}_{per}(\mathbf{f}), \mathbf{v} \wedge \mathbf{e}_2 \rangle,$$

$\forall\ \mathbf{v} \in H_{per}(\mathrm{curl}, O_h)$, $\mathbf{v} \wedge \mathbf{e}_2 = \mathbf{0}$ on G, and $b(\mathbf{v}, \psi) = \overline{a(\mathbf{grad}\psi, \mathbf{v})}$, where $\psi \in H^1_{per}(O_h), \psi = 0$ on G.

Hence, from (40) the Fredholm alternative can be applied. By (41), (39) admits a unique solution in $H^{loc}_{per}(\mathrm{curl}, \overline{O})$ such that

$$\lim_{y_2 \to +\infty} \alpha \,.\, \mathbf{e}_2 \quad = \quad 0,$$

and then lemma 5.3 follows. $\square$
Let us recall that curl_Γ and $\mathbf{grad}_\Gamma$ are the tangential rotational and the tangential gradient. Let $\mathbf{E}^{(0)}_\Gamma$ denote the tangential component of $\mathbf{E}^{(0)}$.

Now, we shall correct the error in the PDE. Direct computation yields :

$$\begin{aligned}\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,E^{(0)}(\mathbf{x}_\Gamma)\,\beta^{(0)}_0 \quad &= \quad -\frac{1}{\delta}\Big\{\,\mathbf{grad}_\Gamma\,E^{(0)}(\mathbf{x}_\Gamma)\,.\,\mathbf{grad}\;\;\beta^{(0)}_0\,\Big\}\,(\mathbf{x}/\delta)\\ -\frac{2}{\mu}\,c(s)\,E^{(0)}(\mathbf{x}_\Gamma)\,\frac{n}{\delta^2}\,\Big\{\,\frac{\partial^2\alpha^{(0)}_{0,3}}{\partial y_3 \partial y_1}(\mathbf{x}/\delta)\,\mathbf{t} &- \frac{\partial^2\alpha^{(0)}_{0,3}}{\partial y_1^2}(\mathbf{x}/\delta)\,\mathbf{e}_z\,\Big\} + 0(1),\end{aligned}$$

where $\alpha^{(0)}_{0,j}$ is the j^{th} component of $\alpha^{(0)}_0$ and $c(s)$ is the curvature of Γ at the point $\mathbf{x}_\Gamma$. The following two lemmas are needed to find the correctors for the PDE terms.

LEMMA 5.4. *There exists a unique* $(\alpha^{(0)}_{0,1}, \mathbf{a}^{(0)}_{0,1}) \in (H^{loc}_{per}(\mathrm{curl}, \overline{O}))^3 \times \mathbb{C}^3$*, such that :*

$$(43)\qquad \left\{\begin{array}{rcll} \mathrm{curl}\,\dfrac{1}{\mu}\mathrm{curl}\,\alpha^{(0)}_{0,1} &=& y_2\,\Big\{\,\dfrac{\partial^2\alpha^{(0)}_{0,3}}{\partial y_3 \partial y_1}\mathbf{e}_1 - \dfrac{\partial^2\alpha^{(0)}_{0,3}}{\partial y_1^2}\mathbf{e}_3\,\Big\} & \text{in } O,\\ \mathrm{div}\varepsilon\alpha^{(0)}_{0,1} &=& 0 & \text{in } O,\\ \mathbf{n}\wedge\alpha^{(0)}_{0,1} &=& \mathbf{0} & \text{on } G,\\ \mathbf{n}\wedge\alpha^{(0)}_{0,1} &=& \mathbf{0} & \text{on } G^c,\\ \multicolumn{4}{l}{\alpha^{(0)}_{0,1} - \mathbf{a}^{(0)}_{0,1} \to \mathbf{0}, \quad \mathbf{a}^{(0)}_{0,1}\,.\,\mathbf{e}_2 = 0.} \end{array}\right.$$

It is easily checked that

$$\mathrm{div}\,y_2\,\Big\{\,\frac{\partial^2\alpha^{(0)}_{0,3}}{\partial y_3 \partial y_1}\mathbf{e}_1 - \frac{\partial^2\alpha^{(0)}_{0,3}}{\partial y_1^2}\mathbf{e}_3\,\Big\} \quad = \quad 0.$$

Therefore, lemma 5.4 holds from (36).

LEMMA 5.5. $\forall j \in \{1,3\}$, *there exists a unique* $(\alpha_{1,j}^{(0)}, \mathbf{a}_{1,j}^{(0)}) \in (H_{per}^{loc}(\mathrm{curl}, \overline{O}))^3 \times \mathbb{C}^3$, *such that :*

$$
(44) \qquad \left\{
\begin{array}{rcll}
\mathrm{curl}\, \dfrac{1}{\mu} \mathrm{curl}\, \alpha_{1,j}^{(0)} & = & \dfrac{\partial (\varepsilon\, \alpha_0^{(0)})}{\partial y_j} & \text{in } O, \\
\mathrm{div}\, \varepsilon\, \alpha_{1,j}^{(0)} & = & 0 & \text{in } O, \\
\mathbf{n} \wedge \alpha_{1,j}^{(0)} & = & \mathbf{0} & \text{on } G, \\
\mathbf{n} \wedge \alpha_{1,j}^{(0)} & = & \mathbf{0} & \text{on } G^c, \\
\alpha_{1,j}^{(0)} - \mathbf{a}_{1,j}^{(0)} \to \mathbf{0}, & & \mathbf{a}_{1,j}^{(0)} \,.\, \mathbf{e}_2 = 0. &
\end{array}
\right.
$$

The proof of this lemma follows from the one of lemma 5.3 and then we omit it.

The boundary layer corrector of order 0 is obtained by correcting the boundary conditions and the jump of $[\, \varepsilon^\delta\, \mathbf{E}^{(0)} \,.\, \mathbf{n} \,]$ across Γ^δ to the order 2. We recall that, in the TM case (see [2]), the first boundary layer corrector is derived by correcting analogous quantities only to the order 1.

LEMMA 5.6. $\forall j \in \{1,2,3\}$, *there exists a unique* $(\alpha_{2,j}^{(0)}, \mathbf{a}_{2,j}^{(0)}) \in (H_{per}^{loc}(\mathrm{curl}, \overline{O}))^3 \times \mathbb{C}^3$, *such that :*

$$
(45) \qquad \left\{
\begin{array}{rcll}
\mathrm{curl}\, \dfrac{1}{\mu} \mathrm{curl}\, \alpha_{2,j}^{(0)} & = & 0 & \text{in } O, \\
\mathrm{div}\, \varepsilon\, \alpha_{2,j}^{(0)} & = & 0 & \text{in } O, \\
\mathbf{n} \wedge \alpha_{2,j}^{(0)} & = & \mathbf{0} & \text{on } G, \\
\mathbf{n} \wedge \alpha_{2,j}^{(0)} & = & y_2\, \mathbf{e}_j \wedge \mathbf{n} & \text{on } G^c, \\
[\, \varepsilon\, \alpha_{2,3}^{(0)} \,.\, \mathbf{n} \,] & = & -[\, \varepsilon \,] & \text{on } G^d, \\
\alpha_{2,j}^{(0)} - \mathbf{a}_{2,j}^{(0)} \to \mathbf{0}, & & \mathbf{a}_{2,j}^{(0)} \,.\, \mathbf{e}_2 = 0. &
\end{array}
\right.
$$

Let $(\beta_i^{(0)}, \mathbf{b}_i^{(0)})$ and $(\beta_{i,j}^{(0)}, \mathbf{b}_{i,j}^{(0)})$ be the images of $(\alpha_i^{(0)}, \mathbf{a}_i^{(0)})$ and $(\alpha_{i,j}^{(0)}, \mathbf{a}_{i,j}^{(0)})$ by $1/\delta\ \phi_\delta$. For brevity in notation we define $\mathbf{e}^1 = \mathbf{t}$, $\mathbf{e}^2 = \mathbf{n}$ and $\mathbf{e}^3 = \mathbf{e}_z$.

The boundary layer corrector $\mathbf{E}_{BL}^{(0)}$ of order 0 is then given by :

$$
\begin{aligned}
\mathbf{E}_{BL}^{(0)}(\mathbf{x}_\Gamma, n) \quad = \quad & E^{(0)}(\mathbf{x}_\Gamma)\ \beta_0^{(0)}(\mathbf{x}/\delta) + \frac{2}{\mu}\ c(s)\ E^{(0)}(\mathbf{x}_\Gamma) \left\{\ \beta_{0,1}^{(0)}(\mathbf{x}/\delta) - \mathbf{b}_{0,1}^{(0)}\ \right\} \\
& + \delta \left\{ \frac{1}{\varepsilon} \sum_{j=1,3} \mathbf{grad}_\Gamma\, E^{(0)}(\mathbf{x}_\Gamma) \,.\, \mathbf{e}^j \left\{\ \beta_{1,j}^{(0)}(\mathbf{x}/\delta) - \mathbf{b}_{1,j}^{(0)}\ \right\} + \sum_{j=1}^{3} \frac{\partial \mathbf{E}^{(0)}}{\partial n}(\mathbf{x}_\Gamma) \,.\, \mathbf{e}^j \left\{\ \beta_{2,j}^{(0)}(\mathbf{x}/\delta) - \mathbf{b}_{2,j}^{(0)}\ \right\} \right. \\
& \left. + \sum_{j=1,3} \left\{\, [\, \frac{1}{\mu}\,]\ \mathrm{curl}\, \mathbf{E}^{(0)}(\mathbf{x}_\Gamma) \wedge \mathbf{n} + b_0^{(0)} \mathbf{grad}_\Gamma E^{(0)}(\mathbf{x}_\Gamma)\, \right\} . \mathbf{e}^j \left\{\ \beta_j^{(0)}(\mathbf{x}/\delta) - \mathbf{b}_j^{(0)}\ \right\} \wedge \mathbf{n} \right\}.
\end{aligned}
$$

We recall that

$$
E^{(0)}(\mathbf{x}_\Gamma) \quad = \quad \mathbf{E}^{(0)}(\mathbf{x}_\Gamma) \,.\, \mathbf{n}.
$$

This expression of $\mathbf{E}_{BL}^{(0)}$ permits us to construct the first corrector $\mathbf{E}^{(1)}$, unique solution to the following Maxwell equations :

$$
(46) \qquad \left\{ \begin{array}{rcll} \operatorname{curl}\operatorname{curl}\mathbf{E}^{(1)} - k^2\ \mathbf{E}^{(1)} & = & \mathbf{0} & \text{in } \Omega, \\ \mathbf{n}\wedge\mathbf{E}^{(1)} & = & -\mathbf{n}\wedge\mathbf{E}_{BL}^{(0)} & \text{on } \Gamma = \partial\Omega, \\ \mathbf{E}^{(1)} & \multicolumn{3}{l}{\text{satisfies an outgoing radiation condition at infinity.}} \end{array} \right.
$$

We shall now simplify the expression of the boundary layer corrector $\mathbf{E}_{BL}^{(0)}$. From (1), we see that

$$
\gamma_0\Big(\operatorname{div}(\operatorname{curl}\mathbf{E}^{(0)}\ \wedge\ \mathbf{n})\Big) \quad = \quad \operatorname{div}_\Gamma(\operatorname{curl}\mathbf{E}^{(0)}\ \wedge\ \mathbf{n}).
$$

It follows easily by using (21) that :

$$
(47) \qquad \operatorname{div}_\Gamma(\operatorname{curl}\mathbf{E}^{(0)}\ \wedge\ \mathbf{n}) \quad = \quad -k^2\ E^{(0)}(\mathbf{x}_\Gamma).
$$

In order to compute $\dfrac{\partial\mathbf{E}^{(0)}}{\partial n}(\mathbf{x}_\Gamma)$, we recall that

$$
(48) \qquad \frac{\partial\mathbf{E}^{(0)}}{\partial n}(\mathbf{x}_\Gamma) \quad = \quad \frac{\partial\mathbf{E}_\Gamma^{(0)}}{\partial n}(\mathbf{x}_\Gamma) + \frac{\partial E^{(0)}}{\partial n}(\mathbf{x}_\Gamma)\ \mathbf{n}.
$$

From (2), we get that

$$
(49) \qquad \frac{\partial\mathbf{E}_\Gamma^{(0)}}{\partial n}(\mathbf{x}_\Gamma) \quad = \quad \mathbf{grad}_\Gamma\ E^{(0)}(\mathbf{x}_\Gamma) - \mathbf{n}\wedge\operatorname{curl}\mathbf{E}^{(0)}(\mathbf{x}_\Gamma).
$$

(1) yields

$$
(50) \qquad \frac{\partial E^{(0)}}{\partial n}(\mathbf{x}_\Gamma) \quad = \quad -2c(s)\ E^{(0)}(\mathbf{x}_\Gamma).
$$

Now, by (49) and (50), (48) yields

$$
(51) \qquad \frac{\partial\mathbf{E}^{(0)}}{\partial n}(\mathbf{x}_\Gamma) \quad = \quad \mathbf{grad}_\Gamma\ E^{(0)}(\mathbf{x}_\Gamma) - \mathbf{n}\wedge\operatorname{curl}\mathbf{E}^{(0)}(\mathbf{x}_\Gamma) - 2c(s)\ E^{(0)}(\mathbf{x}_\Gamma).
$$

Finally, by the expression of the boundary layer corrector and making use of (47) and (51) above, we obtain the following theorem.

THEOREM 5.1. *The first generalized impedance condition on Γ is given by :*

$$
(52) \qquad \begin{array}{l} \mathbf{m} = \delta \displaystyle\sum_{j=1,3} \frac{1}{k^2}\ \mathbf{grad}_\Gamma\operatorname{div}_\Gamma(\mathbf{p})\,.\,\mathbf{e}^j\ \Big\{ \frac{1}{\varepsilon}\ \mathbf{b}_{1,j}^{(0)} + \mathbf{b}_{2,j}^{(0)} + b_0^{(0)}\ \mathbf{b}_j^{(0)}\ \wedge\ \mathbf{n} \Big\} \wedge\mathbf{n} + \delta\ \dfrac{2c(s)}{k^2} \\ \operatorname{div}_\Gamma\mathbf{p}\ \Big\{ \dfrac{1}{\mu}\ \mathbf{b}_{0,1}^{(0)} - \mathbf{b}_{2,3}^{(0)} \Big\} \wedge\mathbf{n}\ +\ \delta \displaystyle\sum_{j=1,3} \mathbf{p}\,.\,\mathbf{e}^j \Big\{ \mathbf{b}_{2,j}^{(0)} + [\ \frac{1}{\mu}\ ]\ \mathbf{b}_j^{(0)}\ \wedge\ \mathbf{n} \Big\} \wedge\mathbf{n}, \end{array}
$$

where

$$
\mathbf{m} = \mathbf{E}\wedge\mathbf{n},\ \mathbf{p} = \operatorname{curl}\mathbf{E}\ \wedge\ \mathbf{n}
$$

and $\mathbf{E}$ is the unique solution to Maxwell's equations with the new boundary condition (52).

This impedance condition leads to a well-posed problem (at least for δ small enough) and finally, we establish the following error estimate :

PROPOSITION 5.1. *$\forall\, \omega \subset\subset \Omega$, there exists a positive constant $C(\omega)$ independent of δ, such that :*

$$|| \, p^\delta \mathbf{E}^\delta - \mathbf{E} \, ||_{H_{k_z}(\mathrm{curl},\omega)} \quad \leq \quad C(\omega)\, \delta^{3/2}. \tag{53}$$

Proof. Let $\mathbf{E}_{BL}^{(1)}$ be the boundary layer corrector of order 1.

$$|| \, p^\delta \mathbf{E}^\delta - \mathbf{E} \, ||_{H_{k_z}(\mathrm{curl},\omega)} \quad \leq \quad C(\omega) \;\; || \, p^\delta \mathbf{E}^\delta - (\mathbf{E}^{(0)} + \mathbf{E}_{BL}^{(0)}) - \delta(\mathbf{E}^{(1)} + \mathbf{E}_{BL}^{(1)}) \, ||_{H_{k_z}(\mathrm{curl},\Omega_{l(\omega),R(\omega)})} \, .$$

We introduce

$$\mathbf{w}^\delta \quad = \quad \mathbf{E}^{(0)} + \mathbf{E}_{BL}^{(0)} + \delta(\mathbf{E}^{(1)} + \mathbf{E}_{BL}^{(1)}),$$

$\mathbf{w}^\delta$ satisfies :

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{w}^\delta + k^2 \, \mathbf{w}^\delta \quad = \quad 0(\delta),$$

in the dielectric layer,

$$\mathbf{w}^\delta \wedge \mathbf{n} = 0(\delta^2) \text{ on } \Omega_N^c \cup \Gamma, \quad [\, \mathbf{w}^\delta \,.\, \mathbf{n} \,] = 0(\delta^2)$$

and

$$[\, \mathrm{curl}\, \mathbf{w}^\delta \, \wedge \, \mathbf{n} \,] \quad = \quad 0(\delta^2)$$

across Γ^δ. We have :

$$a^\delta(\mathbf{w}^\delta, \mathbf{w}^\delta - p^\delta \mathbf{E}^\delta) - a^\delta(p^\delta \mathbf{E}^\delta, (1-\varphi^\delta)\mathbf{w}^\delta - p^\delta \mathbf{E}^\delta) = 0(\delta^{3/2}) \; \|p^\delta \mathbf{E}^\delta - \mathbf{w}^\delta\|_{H(\mathrm{curl},\Omega_{l,R})}.$$

$\|\varphi^\delta \mathbf{w}^\delta\|_{H(\mathrm{curl},\Omega_{l,R})} = 0(\delta^{3/2})$ implies that :

$$a^\delta(\mathbf{w}^\delta - p^\delta \mathbf{E}^\delta, \mathbf{w}^\delta - p^\delta \mathbf{E}^\delta) = 0(\delta^{3/2}) \; \|p^\delta \mathbf{E}^\delta - \mathbf{w}^\delta\|_{H(\mathrm{curl},\Omega_{l,R})}.$$

Hence, lemma 4.1 allows us to prove error estimate (53). ☐

To derive the impedance condition in the case of a homogeneous dielectric layer without conducting inclusions, we compute the constants appearing in the elementary problems set in the elementary cell of the infinite plane grating. We find that :

$$\left\{ \begin{array}{lcll} \mathbf{b}_j^{(0)} \wedge \mathbf{n} & = & \mu \, \mathbf{e}^j, & \forall j \in \{1,2,3\}, \\ \mathbf{b}_{i,j}^{(0)} & = & \mathbf{0}, & \forall i,j \in \{1,2,3\}, \\ b_0^{(0)} & = & \dfrac{1}{\mu} \; [\, \dfrac{1}{\varepsilon} \,]. & \end{array} \right. \tag{54}$$

Then, the new boundary condition (52) is reduced to the following impedance condition on Γ .

$$\mathbf{m} = \delta \;\; [\, \mu \,] \; \mathbf{p} \wedge \mathbf{n} + \; \delta \, \frac{1}{k^2} \;\; [\, \frac{1}{\varepsilon} \,] \; \mathbf{grad}_\Gamma \mathrm{div}_\Gamma(\mathbf{p}) \, \wedge \, \mathbf{n}. \tag{55}$$

Here, we obtain the approximate boundary conditions given by Engquist and Nédélec in [8]. The boundary layer corrector $\mathbf{E}_{BL}^{(0)}$ of order 0 is given in this case by :

$$\mathbf{E}_{BL}^{(0)}(\mathbf{x}_\Gamma, n) \quad = \quad \frac{1}{\mu} \;\, [\, \frac{1}{\varepsilon} \,] \; E^{(0)}(\mathbf{x}_\Gamma) \; \mathbf{n} \; + \;\; [\, \mu \,] \; \mathrm{curl}\, \mathbf{E}^{(0)}(\mathbf{x}_\Gamma) \wedge \mathbf{n} + \;\; [\, \frac{1}{\varepsilon} \,] \; \mathbf{grad}_\Gamma E^{(0)}(\mathbf{x}_\Gamma).$$

We also obtain the approximate condition for an infinite plane grating. (52) yields

$$
(56) \quad \begin{aligned}
&\mathbf{m} = \ \delta \sum_{j=1,3} \frac{1}{k^2}\, \mathbf{grad}_{\,\Gamma}\mathrm{div}_\Gamma(\mathbf{p})\,.\,\mathbf{e}_j \ \left\{ \frac{1}{\varepsilon}\, \mathbf{a}_{1,j}^{(0)} + \mathbf{a}_{2,j}^{(0)} + b_0^{(0)}\ \mathbf{a}_j^{(0)} \wedge \mathbf{n} \right\} \wedge \mathbf{n} \\
&\delta \sum_{j=1,3} \mathbf{p}\,.\,\mathbf{e}_j \Big\{ \ \mathbf{a}_{2,j}^{(0)} \ + \ [\ \frac{1}{\mu}\ ]\ \mathbf{a}_j^{(0)} \ \wedge \ \mathbf{n} \ \Big\} \wedge \mathbf{n}.
\end{aligned}
$$

As can be seen from formula (56), the impedance condition for a grating without dielectric material is given by :

$$
\mathbf{m} \ = \ \delta \sum_{j=1,3} \mathbf{p}\,.\,\mathbf{e}_j\ \mathbf{a}_{2,j}^{(0)} \wedge \mathbf{n} + \ \delta \sum_{j=1,3} \frac{1}{k^2}\, \mathbf{grad}_{\,\Gamma}\mathrm{div}_\Gamma(\mathbf{p})\,.\,\mathbf{e}_j \ \left\{ \frac{1}{\varepsilon}\, \mathbf{a}_{1,j}^{(0)} + \mathbf{a}_{2,j}^{(0)} \right\} \wedge \mathbf{n}.
$$

Therefore, the equivalent boundary condition obtained by Achdou in [4] and by Artola and Cessenat in [5] is then recovered.

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