# Uniqueness theorems for an inverse problem in a doubly periodic structure *

Habib ammari †

February 6, 1995

**Abstract**

Consider an electromagnetic plane wave incident on a biperiodic structure in $\mathbb{R}^3$. The inverse problem is to determine the shape of the structure from the scattered field. In this paper, uniqueness theorems are proved by applying the uniqueness theorem of Cauchy-Kowalewska, by extending Isakov approach [6] and using a result on local injectivity of maps between finite-dimensional spaces.

## 1 Introduction and main results.

Consider a time-harmonic electromagnetic plane wave incident from the top on a biperiodic structure in $\mathbb{R}^3$. The biperiodic structure separates two regions. In one region, above the biperiodic structure, the dielectric coefficient is assumed to be a fixed constant. The other region contains perfectly reflecting material. Let the scattering profile be described by the curve $\Gamma = \{\ (x_1, x_2, x_3) :\ x_3 = \varphi(x_1, x_2)\ \}$, with a biperiodic function $\varphi$ of period $2\pi$ in $x_1$ and in $x_2$. The function $\varphi$ is assumed to be sufficiently smooth ( twice continuously differentiable at least ).

In this paper we will consider the inverse problem which consists of the determination of $\Gamma$ from the total field $\mathbf{E}$, quasi-periodic solution of Maxwell's equations.

This paper is organized as follows. In section 2 we formulate the direct diffraction problem in a doubly periodic structure and collect results on existence, uniqueness and dependence of the solution on the parameters. Section 3 is devoted to the inverse problem where the dielectric coefficient is with non-zero imaginary part corresponding to the energy absorption. Here we prove, with one incident plane wave, that the knowledge of the tangential component of the electric field on $\Sigma_h = \{(x_1, x_2, x_3) :\ 0 < x_1 < 2\pi,\ 0 < x_2 < 2\pi \text{ and } x_3 = h\}$ determines the profile of the structure. Our approach is similar to Borovikov [4] and Bao [3]. However, the result and many technical details differ. Where the dielectric coefficient is assumed to be real, with a single incident plane wave, it is impossible to prove a similar uniqueness theorem for the inverse problem. A simple counter-example may be constructed in the case of the flat surface with a normal incidence. Then the solution of the scattering problem can be written down explicitly. The non-uniqueness is obvious since the scattering fields will remain the same when one moves the flat surface up or down in certain multiples of the wave length. Therefore, in order to get uniqueness result for the inverse problem, additional information must be provided. However, by extending Isakov approach [6], we prove, in Section 4, that the knowledge of the tangential components of

*This paper is submitted for publication to Inverse Problems.

†Centre de Mathématiques Appliquées - CNRS URA 756, École Polytechnique, 91128 Palaiseau Cedex, France.

the electric fields on $\Sigma_h$ for various incident waves, determines the shape of the surface. We show that the scattered fields corresponding to two functions $(\varphi)_{i=1,2}$ coincide when we probe the regions $\Omega_i = \{(x_1, x_2, x_3) : \ x_3 > \varphi_i(x_1, x_2)\}$ by point sources (8). If $\varphi_1$ and $\varphi_2$ were different we could find a point $\mathbf{x}_0 \in \Omega_2 \cap \partial\Omega_1$ (or in $\Omega_1 \cap \partial\Omega_2$, interchange the roles of $\varphi_1$ and $\varphi_2$ ). The contradiction is constructed by letting the sources tend to $\mathbf{x}_0$. From the pratical point of view, this result is not quite satisfactory since it requires the knowledge of all modes of the scattered electromagnetic waves. Far away from the grating the components corresponding to surface waves cannot be measured, thus necessary information is lost. We show for small perturbations of the plane that only the slow variations of the biperiodic grating profile can be recovered. This result is proved, as in Kirsh [7], by applying a well known result on local injectivity of maps between finite-dimensional spaces.

These uniqueness theorems generalize the results obtained recently by Bao [3], Dobson [5] and Kirsh [7] for a periodic structure in $\mathbb{R}^2$ lit up by a TM polarized incident plane wave.

## 2 Direct problem.

We consider a doubly periodic conductor in $\mathbb{R}^3$. Let $\Omega^c$ denote the conductor medium and $\Omega = \mathbb{R}^3 \setminus \Omega^c$ the exterior domain. we assume that $\partial\Omega$ is described by the curve $x_3 = \varphi(x_1, x_2), (x_1, x_2) \in \mathbb{R}^2$, with biperiodic function $\varphi \in \mathcal{C}^2(\mathbb{R}^2)$. By a simple change of variables we can assume the period in $x_1$ and $x_2$ to be $2\pi$. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal basis of $\mathbb{R}^3$ such that the open sets $\Omega^c$ and $\Omega$ are invariant by the transformations :

$$\mathbf{x} \longmapsto \mathbf{x}_J = \mathbf{x} + 2\pi j_1 \mathbf{e}_1 + 2\pi j_2 \mathbf{e}_2, \quad \forall\, J = (j_1, j_2) \in \mathbb{Z}^2.$$

We introduce $\Sigma_h = \{(x_1, x_2, x_3) : \ 0 < x_1 < 2\pi, \ 0 < x_2 < 2\pi \text{ and } x_3 = h\}$. We are interested by the scattering of an electromagnetic field by this structure. Assuming a time variation of the form $e^{-i\omega t}$, Maxwell's equations become :

$$(1) \qquad \begin{cases} \vec{\mathrm{curl}}\,\vec{\mathrm{curl}}\,\mathbf{E} - k^2\,\mathbf{E} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{E} \wedge \mathbf{n} = \mathbf{0} & \text{on } \Gamma = \partial\Omega, \\ \mathbf{E} - \mathbf{E}^{in} \text{ satisfies an outgoing condition at infinity.} \end{cases}$$

Where $\mathbf{n}$ is the outgoing normal to $\Omega$ and $\mathbf{E}^{in}$ is the electromagnetic incident field. We assume that it is an incident plane wave :

$$\mathbf{E}^{in}(\mathbf{x}', x_3) \quad = \quad \mathbf{E}_0^{in}\ e^{i(\mathbf{K}.\mathbf{x}' + k_3 x_3)}.$$

$\mathbf{k} = (\mathbf{K}, k_3)$ is the wave vector, $|\ \mathbf{k}\ |$ the wave number, $\mathbf{x}' = (x_1, x_2)$ and $\mathbf{E}_0^{in}$ a constant vector satisfying $\mathbf{E}_0^{in}.\,\mathbf{k} = 0$.

The direct problem consists of the determination of the quasi-periodic solution of (1) satisfying :

$$\mathbf{E}(\mathbf{x}_J) \quad = \quad \mathbf{E}(\mathbf{x})\ e^{i\mathbf{K}.\mathbf{x}'_J} \quad , \forall\ J \in \mathbb{Z}^2, \forall\ \mathbf{x} \in \Omega.$$

Since $\mathbf{E}(\mathbf{x})\ e^{-i\mathbf{K}.\mathbf{x}'}$ is periodic in the variable $(x_1, x_2)$, we can can expand $\mathbf{E}$ in a Fourier series :

$$\mathbf{E}(\mathbf{x}) \quad = \quad \mathbf{E}^{in} + \sum_{J \in \mathbb{Z}^2} \mathbf{E}_J e^{i\mu_J x_3} e^{i(\mathbf{K} + \mathbf{K}_J).\mathbf{x}'},$$

where $K_J = (j_1, j_2)$ and $\mu_J = e^{i\theta_J/2} \ | k^2 - | \mathbf{K} + \mathbf{K}_J |^2|^{1/2}$. $\theta_J = arg(k^2 - | \mathbf{K} + \mathbf{K}_J |^2)$. $\mathbf{E}_J$ are constant vectors satisfying :

$$(\mathbf{K} + \mathbf{K}_J) \,.\, \mathbf{E}_J + \mu_J (\mathbf{E}_J)_3 = 0.$$

We assume that $\mu_J \neq 0 \ \forall J \in \mathbb{Z}^2$. We look for quasi-periodic solutions of (1). This allows us to set the problem in

$$\omega_h \quad = \quad \{ \, (x_1, x_2, x_3), \varphi(x_1, x_2) < x_3 < h, 0 < x_1 < 2\pi \text{ and } 0 < x_1 < 2\pi \, \},$$

thanks to a Dirichlet-Neumann operator. Then we seek for the electric field in $H(\text{curl}, \Omega)$. We decompose it to a smooth field and a gradient, then we write a saddle point formulation. By using the Dirichlet-Neumann operator to write a transparent boundary condition on the artificial boundary $\Sigma_h$, we show that Fredholm's alternative holds, and then uniqueness implies existence of the solution. For details we refer to Abboud [1]. It is easy to prove uniqueness of the solution if imaginary part of the dielectric coefficient is non-zero. If *Im* $k = 0$ then we do not have uniqueness of the solution of Maxwell's equations (1). Here, since the boundary of the doubly periodic conductor is graph of a regular function, then we may generalize the results of Alber [2] and Wilcox [11], to prove the uniqueness of the direct problem. We formulate the existence result in the following theorem.

THEOREM 2.1. *There exists a unique quasi-periodic solution* $\mathbf{E} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *of Maxwell's equations* (1).

The solution of Maxwell's equations (1) depends continuously on $H^{-1/2}(\text{div}, \partial\Omega)$-boundary data. Furthermore, we have the following result concerning lipschtz-continuous dependence of the solution on $\varphi$.

THEOREM 2.2. *Let* $\varphi_1 \in C^2(\mathbb{R}^2)$ *be* $2\pi-$*periodic in* $(x_1, x_2)$ *and* $\mathbf{E}_1 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *be the unique solution of Maxwell's equations* (1) . *Let* $\omega \subset \Omega$ *be a compact. Then there exists* $\varepsilon > 0$ *and a constant* $C(\omega)$ *such that for all* $2\pi-$*periodic* $\varphi_2 \in C^2(\mathbb{R}^2)$ *with* $|| \varphi_1 - \varphi_2 ||_{1,\infty} \leq \varepsilon$ *the unique solution* $\mathbf{E}_2$ *of Maxwell's equations* (1) *corresponding to* $\varphi_2$ *satisfies :*

$$|| \mathbf{E}_1 - \mathbf{E}_2 ||_{H(\text{curl}, \omega)} \quad \leq \quad C(\omega) \ || \varphi_1 - \varphi_2 ||_{1,\infty} \,.$$

*Here,* $|| \varphi ||_{1,\infty} = || \varphi ||_\infty + || \nabla\varphi ||_\infty$ *denotes the norm in* $C^1((0, 2\pi) \times (0, 2\pi))$.

We recall that the quasi-periodic Grenn's kernel is given by :

$$G(\mathbf{x}, \mathbf{y}) \quad = \frac{1}{8\pi^2} \quad \sum_{J \in \mathbb{Z}^2} \frac{1}{i\mu_J} \ e^{i\mu_J |x_3 - y_3|} \ e^{i(\mathbf{K}+\mathbf{K}_J).(\mathbf{x}' - \mathbf{y}')},$$

and the ansatz for $\mathbf{E}$ as

$$\text{(2)} \qquad \mathbf{E} \quad = \quad -\nabla \int_\Gamma G(\mathbf{x}, \mathbf{y}) \ \text{div}_\Gamma \mathbf{p}(\mathbf{y}) \ d\gamma(\mathbf{y}) + \int_\Gamma G(\mathbf{x}, \mathbf{y}) \ \mathbf{p}(\mathbf{y}) \ d\gamma(\mathbf{y}) \qquad \text{for } \mathbf{x} \notin \Gamma.$$

with quasi-periodic and continuous tangential density

$$\mathbf{p} \quad = [ \ \vec{\text{curl}}\, \mathbf{E} \ \wedge \ \mathbf{n} \ ]$$

reduces the boundary value problem (1) to an integral equation which can be shown uniquely solvable ( see Nédélec and Starling [10] for the proof).

In section 4, we need a denseness result which we prove it using the integral formulation of Maxwell's equations (2). We formulate this lemma in the following way :

LEMMA 2.1. *Let $\varphi \in \mathcal{C}^2(\mathbb{R}^2)$ be $2\pi$ periodic in $\mathbf{x}' = (x_1, x_2)$ and*

$$V = \text{span} \left\{ e^{-i\mu_J x_3} e^{i(\mathbf{K}+\mathbf{K}_J).\mathbf{x}'}, J \in \mathbb{Z}^2 \right\}^2. \tag{3}$$

*Then $V\,|_\Gamma$ is dense in $(L^2(\Gamma))^2$. Where*

$$\Gamma = \{ (x_1, x_2, x_3), x_3 = \varphi(x_1, x_2), 0 < x_1 < 2\pi \text{ and } 0 < x_1 < 2\pi \}.$$

*Proof.* Let $\psi \in (L^2(\Gamma))^2$ with

$$\int_\Gamma \overline{\psi}(\mathbf{x})\, e^{-i\mu_J x_3} e^{i(\mathbf{K}+\mathbf{K}_J).\mathbf{x}'} d\gamma(\mathbf{x}) = 0, \quad \forall\, J \in \mathbb{Z}^2. \tag{4}$$

Let $\mathbf{E}$ be

$$\mathbf{E} = -\nabla \int_\Gamma G(\mathbf{x}, \mathbf{y})\, \text{div}_\Gamma \psi(\mathbf{y})\, d\gamma(\mathbf{y}) + \int_\Gamma G(\mathbf{x}, \mathbf{y})\, \psi(\mathbf{y})\, d\gamma(\mathbf{y}) \quad \text{for } \mathbf{x} \notin \Gamma.$$

By (4) and (2), $\mathbf{E}$ is a quasi-periodic solution of homogeneous Maxwell's equations. The uniqueness result ( theorem 2.1) yields that $\mathbf{E}$ vanishes in $\mathbb{R}^3$. Finally, from the jump condition

$$\psi = [\ \vec{\text{curl}}\mathbf{E} \wedge \mathbf{n}\ ],$$

we conclude that $\psi = \mathbf{0}$. This ends the proof. □

## 3 Uniqueness in inverse problem for lossy dielectric.

Let $\varphi_i$ denote two smooth functions periodic in $x_1$ and in $x_2$ with period $2\pi$. We introduce

$$\Omega_i = \{(x_1, x_2, x_3) :\ x_3 > \varphi_i(x_1, x_2)\}.$$

Define for $h > \sup_i \sup_{(x_1,x_2)} \varphi_i(x_1, x_2)$, the curve $\Sigma_h$ :

$$\Sigma_h = \{(x_1, x_2, x_3) :\ 0 < x_1 < 2\pi,\ 0 < x_2 < 2\pi \text{ and } x_3 = h\}.$$

Let $\mathbf{E}_i$ be the unique solution of Maxwell's equations :

$$\left\{ \begin{array}{ll} \vec{\text{curl}}\,\vec{\text{curl}}\,\mathbf{E}_i - k^2\,\mathbf{E}_i = \mathbf{0} & \text{in } \Omega_i, \\ \mathbf{E}_i \wedge \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_i. \\ \mathbf{E}_i - \mathbf{E}^{in} \text{ satisfies an outgoing condition at infinity,} & \end{array} \right. \tag{5}$$

Then we can state the main result of this section :

THEOREM 3.1. *Assume that $\mathbf{E}_1 \wedge \mathbf{n}\,|_{\Sigma_h} = \mathbf{E}_2 \wedge \mathbf{n}\,|_{\Sigma_h}$ then :*

$$\varphi_1(x_1, x_2) = \varphi_2(x_1, x_2), \quad \forall\, (x_1, x_2) \in \mathbb{R}^2.$$

The following lemma plays an important role in our proof of the uniqueness theorem 3.1 :

LEMMA 3.1. *There is a pseudodifferential operator $\mathcal{R}$ defined on $\Sigma_h$ of order* 1 *such that :*

$$\Pi(\vec{\text{curl}}\,\mathbf{E}_i) = \mathcal{R}(\mathbf{E}_i \wedge \mathbf{n}) + \Pi(\vec{\text{curl}}\,\mathbf{E}^{in}) - \mathcal{R}(\mathbf{E}^{in} \wedge \mathbf{n}). \tag{6}$$

*Where $\mathbf{E}_i$ is the unique solution of* (5) *satisfying the outgoing condition at infinity and* $\Pi = -\mathbf{n} \wedge (\mathbf{n}\ \wedge .)$.

The operator $\mathcal{R}$ is defined for

$$U(\mathbf{x}') = \sum_{J\in\mathbb{Z}^2} U_J\, e^{i(K+K_J).\mathbf{x}'},$$

by

$$\mathcal{R}U(\mathbf{x}') = -\sum_{J\in\mathbb{Z}^2} \frac{1}{i\mu_J}\{k^2 U_J - ((K+K_J).U_J)(K+K_J)\}e^{i(K+K_J).\mathbf{x}'}.$$

The operator $\mathcal{R}$ explains the outgoing condition at infinity. It is easy to verify that $\mathbf{E}^{in}$ does not satisfy this condition. In addition, we need a unique continuation lemma which may be proved by an application of Cauchy-Kowalewska uniqueness theorem.

LEMMA 3.2. *Let $\Omega$ be an open set in $\mathbb{R}^3$. Suppose that $\mathbf{E} \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$\vec{\text{curl}}\,\vec{\text{curl}}\,\mathbf{E} - k^2\,\mathbf{E} = \mathbf{0} \quad \text{in } \Omega.$$

*Let $\Gamma$ be an analytic curve in $\Omega$, such that :*

$$\mathbf{E} \wedge \mathbf{n} = \mathbf{0} \quad \text{and} \quad \Pi(\vec{\text{curl}}\,\mathbf{E}) = \mathbf{0} \quad \text{on } \Gamma,$$

*then* $\mathbf{E} = \mathbf{0}$ in $\overline{\Omega}$.

We are ready to prove the main result of this section.

*Proof.* Theorem 3.1 may be proved by contradiction. Let us assume that $\varphi_1$ and $\varphi_2$ are two different functions. Let $\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2$ and

$$\Omega = \{(x_1, x_2, x_3) :\ x_3 > \varphi(x_1, x_2)\},$$

with $\varphi(x_1, x_2) = \max\{\varphi_1(x_1, x_2), \varphi_2(x_1, x_2)\}$.
Then :

$$\mathbf{E} \wedge \mathbf{n}\,|_{\partial\Omega} = \begin{cases} \mathbf{0} & \text{if} \quad \varphi_1(x_1,x_2) = \varphi_2(x_1,x_2). \\ -\mathbf{E}_2(x_1,x_2,\varphi_1(x_1,x_2)) \wedge \mathbf{n} & \text{if} \quad \varphi_1(x_1,x_2) > \varphi_2(x_1,x_2). \\ \mathbf{E}_1(x_1,x_2,\varphi_2(x_1,x_2)) \wedge \mathbf{n} & \text{if} \quad \varphi_1(x_1,x_2) < \varphi_2(x_1,x_2). \end{cases}$$

From $\mathbf{E}_1 \wedge \mathbf{n}\,|_{\Sigma_h} = \mathbf{E}_2 \wedge \mathbf{n}\,|_{\Sigma_h}$, we deduce that : $\Pi(\vec{\text{curl}}\,\mathbf{E}) = \mathbf{0}$. It follows from (3.2) that $\mathbf{E} = \mathbf{0}$ in $\overline{\Omega}$. Hence : $\mathbf{E}_2(x_1, x_2, \varphi_1(x_1, x_2)) \wedge \mathbf{n} = \mathbf{0}$ and $\mathbf{E}_1(x_1, x_2, \varphi_2(x_1, x_2)) \wedge \mathbf{n} = \mathbf{0}$. Without loss of generality, assume that $\varphi_1 < \varphi_2$ for some $(x_1, x_2)$. Let $D_1 = \Omega_1 \setminus \Omega$. $D_1$ is a bounded domain in $\mathbb{R}^3$.
$\mathbf{E}_1$ satisfies :

$$\begin{cases} \vec{\text{curl}}\,\vec{\text{curl}}\,\mathbf{E}_1 - k^2\,\mathbf{E}_1 = \mathbf{0} & \text{in } D_1, \\ \mathbf{E}_1 \wedge \mathbf{n} = \mathbf{0} & \text{on } \partial D_1. \end{cases} \tag{7}$$

Since *Im* $k > 0$ then (7) shows that $\mathbf{E}_1 = \mathbf{0}$ in $D_1$. Hence $\Pi(\vec{\text{curl}}\mathbf{E}_1) = \mathbf{0}$. This contradicts the equality (6). $\square$

The ideas used in this uniqueness proof can be immediately extended to more general biperiodic structures. In particular, uniqueness for a biperiodic structure corresponding to arrays of antennas can be proved by means of the technique described in this section.

## 4 Uniqueness in inverse problem for lossless dielectric.

We assume that $Im\ k = 0$. Under this assumption, we obtain the local inverse uniqueness theorem :

Theorem 4.1. *Assume that* $\mathbf{E}_{1,n} \wedge \mathbf{n} \mid_{\Sigma_h} = \mathbf{E}_{2,n} \wedge \mathbf{n} \mid_{\Sigma_h}$ *for all incident waves* $\mathbf{E}_n^{in} \in V$, *then :*

$$\varphi_1(x_1, x_2) \quad = \quad \varphi_2(x_1, x_2), \quad \forall\, (x_1, x_2) \in \mathbb{R}^2.$$

$\mathbf{E}_{i,n}$ *is the unique quasi-periodic solution of Maxwell's equations in* $\Omega_i$ *with the boundary condition :*

$$\mathbf{E}_{i,n} \wedge \mathbf{n} \;=\; -\mathbf{E}_n^{in} \wedge \mathbf{n} \quad \text{ on } \partial\Omega_i.$$

*Proof.* We introduce:

$$\mathbf{E}^{in}(\mathbf{x}, \mathbf{x}_0) = \sum_{J \in \mathbb{Z}^2} \frac{1}{i\mu_J}\, \mathbf{E}_J\, e^{i\mu_J |x_3 - x'_{0,3}|}\, e^{i(\mathbf{K}+\mathbf{K}_J).(\mathbf{x}' - \mathbf{x}'_0)}. \tag{8}$$

$\mathbf{E}_J \wedge \mathbf{e}_3$ is independent of $J$. $\mathbf{x}_0 = (\mathbf{x}'_0, x'_{0,3})$ is the point source. $V$ is the space of incident waves given by (3).

Theorem 4.1 may be proved by contradiction. We assume that $\varphi_1 \neq \varphi_2$ and we choose $\mathbf{x}_0 = (\mathbf{x}'_0, x'_{0,3}) \in \partial\Omega_1$ such that $\varphi_1(\mathbf{x}'_0) = x'_{0,3} > \varphi_2(\mathbf{x}'_0)$. Let $\mathbf{E}^d_{i,\varepsilon}$ ( $\varepsilon > 0$) denote the electric scattered field solutions of Maxwell's equations with the following boundary conditions :

$$\mathbf{E}^d_{i,\varepsilon} \wedge \mathbf{n} \;=\; -\, \mathbf{E}^{in}(\,.\,,\, \mathbf{x}_0 + \varepsilon\, \mathbf{e}_3\,) \wedge \mathbf{n} \quad \text{ on } \partial\Omega_i.$$

By lemma 2.1, we prove that there exists a constant $C(\omega)$ such that :

$$||\; \mathbf{E}_{i,n} - \mathbf{E}^d_{i,\varepsilon}\; ||_{H(\mathrm{curl},\,\omega)} \leq C(\omega)\;\; ||\; \mathbf{E}^{in}_n - \mathbf{E}^{in}(\,.\,,\, \mathbf{x}_0 + \varepsilon\, \mathbf{e}_3\,)\; ||_{H^{-1/2}(\mathrm{div},\,\Gamma_i)}\;.$$

Where $\Gamma_i = \partial\Omega_i \cap \{0 < x_1 < 2\pi, 0 < x_2 < 2\pi\}$ and $\omega$ a small open set containing the point $\mathbf{x}_0 + \varepsilon\,\mathbf{e}_3$. The uniqueness of the solution of Maxwell's equations and an analytic continuation argument imply that

$$\mathbf{E}^d_{1,\varepsilon}(\mathbf{x}_0) \quad = \quad \mathbf{E}^d_{2,\varepsilon}(\mathbf{x}_0).$$

Now we let $\varepsilon$ tend to zero. Since $\mathbf{x}_0$ is strictly above $\Gamma_2$ the scattered field $\mid \mathbf{E}^d_{2,\varepsilon}(\mathbf{x}_0) \mid$ remains bounded for $\varepsilon \to 0^+$. Since, however,

$$\mathbf{E}^d_{1,\varepsilon}(\mathbf{x}_0) \quad = \quad -\, \mathbf{E}^{in}(\, \mathbf{x}_0\,,\, \mathbf{x}_0 + \varepsilon\, \mathbf{e}_3\,),$$

we see that $\mid \mathbf{E}^d_{1,\varepsilon}(\mathbf{x}_0) \mid \to +\infty$ . This contradicts the equality $\mathbf{E}^d_{1,\varepsilon}(\mathbf{x}_0) = \mathbf{E}^d_{2,\varepsilon}(\mathbf{x}_0)$ and ends the proof. $\square$

## 5 Uniqueness theorem for a small perturbations of the plane.

In this last section we assume that $Im\ k = 0$, $Re\ k > 0$ and $k_3 \neq 0$. Far away from the doubly periodic structure the components corresponding to surface waves cannot be measured, thus necessary information is lost. The components $E_J$ such that $k \leq | \mathbf{K} + \mathbf{K}_J |$ correspond to the plane waves and may be mesured far away from the structure.
We define the far scattered field by :

$$\mathbf{E}^{\infty}(\mathbf{x}) \quad = \sum_{|\mathbf{K}+\mathbf{K}_J| \leq k} \mathbf{E}_J \, e^{i\mu_J x_3} \, e^{i(\mathbf{K}+\mathbf{K}_J).\mathbf{x}'}.$$

We note that $\mathbf{E}^{\infty}$ is an element of a finite-dimensional space. It is clear that we cannot identify the function $\varphi$ from $\mathbf{E}^{\infty}$ without further information about $\varphi$. We prove for small perturbations of the plane that only the slow variations of the doubly periodic profile can be recovered.
We consider the finite-dimensional but non-linear operator :

$$S: \ \varphi \in X_M \mapsto \mathbf{E}^{\infty} \in Y.$$

The spaces $X_M$ et $Y$ are given by :

$$X_M = \{ \sum_{|J| \leq M} \varphi_J \, e^{i\mathbf{K}_J \,.\, \mathbf{x}'}, \varphi_J \in \mathbb{C} \text{ and } \overline{\varphi}_{\mathrm{J}} = \varphi_{-\mathrm{J}} \} \text{ and } \mathrm{Y} = \{ \sum_{|\mathrm{K}+\mathrm{K_J}| \leq \mathrm{k}} \mathrm{U_J} \, \mathrm{e}^{\mathrm{i}(\mathbf{K}+\mathbf{K}_\mathrm{J})\,.\mathbf{x}'}, \mathrm{U_J} \in \mathbb{C}^3 \}.$$

Where $M = \max_{J \in \mathbb{Z}^2} \{ \ | J | \ \text{t.q} \ | K + K_J | \leq k \ \}$.

THEOREM 5.1. *There exists $\varepsilon > 0$ such that the restriction of $S$ on $X_M^{\varepsilon} = \{\varphi \in X_M :$ $\varphi$ is real − valued and $2\pi$ periodic with $|| \varphi ||_{\infty} \leq \varepsilon\}$ is one-to-one.*

*Proof.* We denote $\Omega = \{ \ (x_1, x_2, x_3) \text{ t.q } x_3 > \varphi(x_1, x_2) \ \}$ and we consider $\omega \subset\subset \Omega$.
By theorem 2.2, the application $T \ : \varphi \mapsto \mathbf{E}_{\varphi} \ |_{\omega}$ is lipschitz-continuous from a $C^1$ neighbourhood of $\varphi$ into $H(\text{curl}, \omega)$. It is differentiable and its Fréchet derivative is given by :

$$T'(\varphi, \psi) = \mathbf{E}(\psi) \ |_{\omega}$$

where $\mathbf{E}(\psi)$ is the unique solution of Maxwell's equations with boundary condition on $\Gamma = \partial\Omega$ :

$$\mathbf{E}(\psi) \wedge \mathbf{n} = -\psi \ \Big( \ \nabla_{\Gamma} \text{div}_{\Gamma} (\vec{\text{curl}}\, \mathbf{E}_{\varphi} \wedge \mathbf{n}) - \mathbf{n} \wedge \vec{\text{curl}}\, \mathbf{E}_{\varphi} \ \Big) \wedge \mathbf{n}.$$

We consider $X_M$ and $Y$ as finite-dimensional vector spaces over the field $\mathbb{R}$. By applying a well known result on local injectivity of maps between finite-dimensional spaces ( *cf.* Lang [9]), we have to show that $S'(0)$ is one-to-one. Let $\varphi \in X_M$ with $S'(0)\varphi = \mathbf{E}^{\infty} = \mathbf{0}$. We prove that $\mathbf{E}$ is the unique solution of Maxwell's equations with boundary condition on $\Gamma = \partial\Omega$ :

$$\mathbf{E} \wedge \mathbf{n} = -\varphi \ \Big( \ \nabla_{\Gamma} \text{div}_{\Gamma} (\vec{\text{curl}}\, \mathbf{E}^{(0)} \wedge \mathbf{n}) - \mathbf{n} \wedge \vec{\text{curl}}\, \mathbf{E}^{(0)} \ \Big) \wedge \mathbf{n},$$

with $\mathbf{E}^{(0)}$ unique solution of (1). $\mathbf{E}$ can be be determined explicity and is given by :

$$\mathbf{E}(\mathbf{x}) = 2ik_3 \Big( \ (\mathbf{E}^{in} \,.\, \mathbf{K}) \ \mathbf{K} - \mathbf{E}^{in} \ \Big) \sum_{J \in \mathbb{Z}^2} \varphi_J \ e^{i\mu_J x_3} \ e^{i(\mathbf{K}+\mathbf{K}_J).\mathbf{x}'}.$$

From this we conclude that $\varphi_J = 0$, $\forall J \in \mathbb{Z}^2$ and the proof is complete. □

## References

[1] T. Abboud, Formulation variationnelle des équations de Maxwell dans un réseau bipériodique de $\mathbb{R}^3$. *C. R. Acad. Sci. Paris*, t.317, Série I, p.245-248, 1993.

[2] H.D. Alber, A quasi-periodic boundary value problem for the Laplacian and the continuation of its resolvant, *Proc. R. Soc. Edinburgh A* 82, p.251-272, 1979.

[3] G. Bao, A uniqueness theorem for an inverse problem in periodic diffrective optics, *Inverse problems* 10, p.335-340, 1994.

[4] I.P. Borovikov, Uniqueness of solutions to one inverse diffraction problem, *Differentsial'nye Uravneniya*, p.827-831, 1992.

[5] D. Dobson, A free boundary problem from the design of periodic diffraction structures, Preprint, 1993.

[6] V. Isakov, On uniqueness in the inverse scattering problem *Commun. Par. Eq.*, 15, p.1565-1587, 1990.

[7] A. Kirsch, Uniqueness theorems in inverse scattering theory for periodic structures, *Inverse problems* 10, p.145-152, 1994.

[8] A. Kirsch and R. Kress, Uniqueness in inverse obstacle scattering, *Inverse problems* 9, p.285-299, 1993.

[9] S. Lang, Analysis I (*Reading, MA : Addison-Wesley* ) 1976.

[10] J.C. Nédélec and F. Starling, *Integral Equation Methods in a Quasi-periodic Diffraction Problem for the Time-Harmonic Maxwell's Equations*, SIAM J.Math Analysis, 22 (1991), pp. 1679-1701.

[11] C.H. Wilcox, Theory for Diffraction Gratings, *Applied Mathematical Sciences 46, Springer-Verlag,* 1984.

**Acknowledgment.** - The author is grateful to Professor J.C. Nédélec for stimulating discussions.