Inverse Boundary Value Problems for Helmholtz Equation in an Infinite Periodic Structure*

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Abstract

In this paper, we begin by proving that the Helmhotz equation in an infinite grating of dielectric material with a $C^{1,1}$ boundary has a unique solution provided the frequency is not in a discrete set and then we show that the tangential variations of the electric permittivity and the magnetic permeability of the dielectric material can be determined by steady measurements at the boundary.

1 Introduction and main results

Let $(\mathbf{x}', x_n) \in \mathbb{R}^n \times \mathbb{R}$. Let $q(\mathbf{x})$ be a periodic real-valued function, in $L^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^{n+1}$ is an infinite periodic domain[1] with a $C^{1,1}$ boundary and such that it is bounded in the direction x_n. This means that there exists a constant $\rho_0 > 0$ such that if $\mathbf{x} \in \Omega$ then $|x_n| < \rho_0$. We recall that an open set $D \subset \mathbb{R}^{n+1}$ is called a $C^{m,1}$ domain with m nonnegative integer if for each $\mathbf{p} \in \partial D$ there exist a neighborhood $U(\mathbf{p})$ of $\mathbf{p}$ in $\mathbb{R}^{n+1}$, and a new orthogonal coordinates $(\mathbf{x}', x_n) \in \mathbb{R}^n \times \mathbb{R}$, such that :

(i) $U(\mathbf{p})$ is an hypercube in the new coordinates $U(\mathbf{p}) = \Big\{ (\mathbf{x}', x_n),\ -a_j < x_j < a_j \Big\}$;

(ii) there exists a function φ of class $C^{m,1}$, defined in $U'(\mathbf{p}) = \Big\{ \mathbf{x}',\ -\frac{a_j}{2} < x_j < \frac{a_j}{2} \Big\}$, and such that

$$|\varphi(\mathbf{x}')| \leq \frac{a_n}{2} \quad \text{for every } \mathbf{x}' \in U'(\mathbf{p}), \quad U \cap D = \Big\{ (\mathbf{x}', x_n) : x_n < \varphi(\mathbf{x}') \Big\},$$

and $U \cap \partial D = \Big\{ (\mathbf{x}', x_n) : x_n = \varphi(\mathbf{x}') \Big\}$.

We assume that $q(\mathbf{x}' + \mathbf{d}_J, x_n) = q(\mathbf{x}', x_n)$, where $\mathbf{d}_J = (2\pi j_1 d_1, \ldots, 2\pi j_n d_n)$[2], and $q = 0$ outside Ω. Consider the following Helmholtz equation :

$$(1) \qquad \begin{cases} \Big(\Delta + k^2 + q(\mathbf{x})\Big) u = 0 \quad \text{in } \mathbb{R}^{n+1}, \\ u - u^{in} \quad \text{satisfying an outgoing radiation condition at infinity,} \end{cases}$$

with u^{in} is a $\theta-$ quasi-periodic incident plane wave. It satisfies :

$$(2) \qquad u^{in}(\mathbf{x} + \mathbf{x}_J) = e^{i\theta\cdot\mathbf{d}_J}\, u^{in}(\mathbf{x}), \quad \forall\, \mathbf{x} \in \mathbb{R}^{n+1}, \forall\, J \in \mathbb{Z}^n,$$

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[1] If $\mathbf{x} \in \Omega$ then $\mathbf{x} + \mathbf{x}_J \in \Omega, \quad \forall J \in \mathbb{Z}^n$.

[2] For $1 \leq i \leq n$, d_i is assumed to be strictly positive.

where $\theta \in \mathbb{R}^n$ and $\mathbf{x}_J = (\mathbf{d}_J, 0)$. Throughout, functions satisfying (2) are called $\theta-$quasi-periodic functions.

Let d be defined as $d = (2\pi)^n \prod_{i=1}^{n} d_i$. Let us introduce $\alpha_J = (j_1/d_1, \ldots, j_n/d_n) + \theta$ and

$$\mu_J = \begin{cases} \sqrt{k^2 - |\alpha_J|^2} & \text{if } k \geq |\alpha_J|, \\ i\sqrt{|\alpha_J|^2 - k^2} & \text{else.} \end{cases}$$

We also introduce ω as

$$\omega = \Big\{ \mathbf{x} = (\mathbf{x}', x_n) \in \Omega, \mathbf{x}' \in \prod_{i=1}^{n} \,]\, 0, 2\pi d_i \, [\Big\}, \tag{3}$$

and

$$\mathcal{V}_\theta = \Big\{ \mathbf{w} = \sum_{J \in \mathbb{Z}^n} \mathbf{w}_J \, e^{i\mu_J(\rho_0 - x_n) + i\alpha_J \,.\, \mathbf{x}'}, \quad x_n \leq \rho_0 \Big\},$$

where $\mathbf{w}_J$ are the Fourier coefficients of $\mathbf{w}$, *i.e.*

$$\mathbf{w} = \sum_{J \in \mathbb{Z}^n} \mathbf{w}_J \, e^{i\beta_J \,.\, \mathbf{x}'} \quad \text{with} \quad \beta_J = (2\pi j_1/d_1, \ldots, 2\pi j_n/d_n).$$

Let $\Gamma^{\pm}$ defined as

$$\partial\omega = \Gamma^{\pm} \cup \Big((\{0\} \times \mathbb{R}) \cap \Omega \Big) \cup \Big((\{\mathbf{d}\} \times \mathbb{R}) \cap \Omega \Big),$$

where $\partial\omega$ is the boundary of ω. Let ω_ρ denote the following hypercube

$$\omega_\rho = \Big\{ \mathbf{x} = (\mathbf{x}', x_n) \in \mathbb{R}^{n+1}, \mathbf{x}' \in \prod_{i=1}^{n} \,]\, 0, 2\pi d_i \, [, \, |x_n| < \rho \Big\}.$$

Finally, we define

$$\Pi^n = \mathbb{R}^n / \mathbf{d} \,.\, \mathbb{Z}^n \times \mathbb{R}, \quad \omega' = \Pi^n \setminus \overline{\omega}, \quad \omega'_\rho = \omega' \cap \Big\{ |x_n| < \rho \Big\},$$

and

$$\Gamma^{\pm}_\rho = \Big\{ \mathbf{x} = (\mathbf{x}', x_n) \in \mathbb{R}^{n+1}, \, \mathbf{x}' \in \prod_{i=1}^{n} \,]\, 0, 2\pi d_i \, [\text{ and } x_n = \pm\rho \Big\}.$$

Section 2 is devoted to prove the following uniqueness and existence theorem :

THEOREM 1.1. *Suppose that Ω is with a $C^{1,1}$ boundary, q is periodic real valued, in $L^\infty(\Omega)$. Then, except for some discrete set $\mathcal{N}$ of values of k with accumulation point at infinity, there is a unique solution $u \in H^{2,loc}_\theta(\overline{\omega'})$ to the problem* (1).

A precise definition of the quasi-periodic Sobolev space $H^{2,loc}_\theta(\overline{\omega'})$ will be given below. When k is in $\mathcal{N}$, the problem (1) will still have at least one solution but uniqueness is no longer guaranteed.

The main result in section 3 is the uniqueness result for the inverse problem. If we suppose that $q(\mathbf{x}) = q(\mathbf{x}')$ then we obtain the following theorems.

THEOREM 1.2. *Let Ω be with a $C^{1,1}$ boundary. Suppose that q is periodic real valued, in $L^\infty(\Omega)$ with a lower bound $> -k^2$. Let $u(\mathbf{x}, \mathbf{y})$ be the solution of* (1)*. Then $q(\mathbf{x}')$ is uniquely determined by the surface data $u(\mathbf{x})$ given at all $\mathbf{x} \in \Gamma^{\pm}_\rho$ and for all $u^{in} \in \mathcal{V}_\theta$.*

THEOREM 1.3. *$q(\mathbf{x}')$ is uniquely determined from the knowledge of its $\theta-$Dirichlet-to-Neumann operator.*

We will show that $q(\mathbf{x}')$ can be reconstructed by solving certain integral equations on $\Gamma^{\pm}$. It is impossible to prove similar results for $q = q(\mathbf{x})$.

The arguments leading to the inversion procedure for q (Theorem 3.1) are easily generalized to treat the following problem. Consider the scattering $\theta-$quasi-periodic solution generated by a point source at $\mathbf{y}$ oscillating harmonically with frequency k and satisfying (2). From the measure of the surface data $u(\mathbf{x},\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in \Gamma^{\pm}_{\rho}$ and a fixed $k > 0$ and $\theta \in \mathbb{R}^n$, we find $q(\mathbf{x}') \in L^{\infty}(\omega_{\rho_0})$ such that

$$\Big(\Delta + k^2 + q(\mathbf{x}')\Big) u(\mathbf{x},\mathbf{y}) = -\delta_\theta(\mathbf{x}-\mathbf{y}) \quad \text{in } \mathbb{R}^{n+1}, \tag{4}$$

with

$$\delta_\theta = \sum_{J\in\mathbb{Z}^n} \delta_J \, e^{i\theta\,.\,\mathbf{d}_J},$$

where δ_J is the Dirac measure at $\mathbf{x}_J = (\mathbf{d}_J, 0)$.

THEOREM 1.4. *Let Ω be with a $C^{1,1}$ boundary. Suppose that q is periodic real valued, in $L^{\infty}(\Omega)$ with a lower bound $> -k^2$. Let $u(\mathbf{x},\mathbf{y})$ be the solution of (4) satisfying the $\theta-$Sommerfeld radiation condition[3] for $|x_n| \to +\infty$. Then $q(\mathbf{x}')$ is uniquely determined by the surface data $u(\mathbf{x},\mathbf{y})$ given at all $\mathbf{x},\mathbf{y} \in \Gamma^{\pm}_{\rho}$.*

The proof of this Theorem follows arguments of Nachman in [14].

Now, given the surface data $u(\mathbf{x},\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in \Gamma^{\pm}_{\rho}$ and a fixed $k > 0$ and $\theta \in \mathbb{R}^n$, find $q_1(\mathbf{x}') \in C^{1,1}(\overline{\omega})$[4] and $q_2(\mathbf{x}') \in L^{\infty}(\omega)$ such that

$$\Big(\operatorname{div} q_1(\mathbf{x}')\,\nabla + k^2 + q_2(\mathbf{x}')\Big) u(\mathbf{x},\mathbf{y}) = -\delta_\theta(\mathbf{x}-\mathbf{y}) \quad \text{in } \mathbb{R}^{n+1}. \tag{5}$$

Once the unique solvability of (5) is established in Theorem 1.5, we prove Theorem 1.6 below which expresses in a precise way our inverse boundary value problem for (5).

THEOREM 1.5. *Suppose that Ω is with a $C^{1,1}$ boundary, q_1 is periodic real valued, in $C^{1,1}(\overline{\Omega})$ and q_2 is periodic real valued, in $L^{\infty}(\Omega)$. Then, except for some discrete set of values of k with accumulation point at infinity, there is a unique solution $u \in H^{2,loc}_{\theta}(\overline{\omega'})$ to the problem (6).*

$$\begin{cases} \Big(\operatorname{div} q_1\,\nabla + k^2 + q_2\Big) u \;=\; 0 & \text{in } \mathbb{R}^{n+1}, \\ u - u^{in} \quad \text{satisfying an outgoing radiation condition at infinity.} \end{cases} \tag{6}$$

We will prove the following Theorem and we will give a constructive procedure for recovering q_1 and q_2.

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[3]The radiation condition at infinity is not the usual one. The new radiation condition is called $\theta-$Sommerfeld radiation condition.

[4]We recall that saying that φ belongs to the class $C^{m,1}(\overline{\Omega})(m \geq 1$, possibly equals to $+\infty)$ means that it is m times continuously differentiable and its derivatives of order m are Lipchitz continuous in $\overline{\Omega}$.

Theorem 1.6. *Let Ω be with a $C^{1,1}$ boundary. Suppose that q_1 is uniformly Lipschitz in $\overline{\Omega}$ with a strictly positive lower bound and $q_2 \in L^\infty(\Omega)$ with a lower bound $> -k^2$. Let $u(\mathbf{x},\mathbf{y})$ be the solution of (5) satisfying the $\theta-$Sommerfeld radiation condition for $|x_n| \to +\infty$. Then $q_1(\mathbf{x}')$ and $q_2(\mathbf{x}')$ are uniquely determined by the surface data $u(\mathbf{x},\mathbf{y})$ given at all $\mathbf{x},\mathbf{y} \in \Gamma_\rho^\pm$ for two admissible frequencies.*

2 Integral equation and uniqueness result

In this section we prove Theorem 1.1 and Theorem 1.5. We assume throughout that Ω has a $C^{1,1}$ boundary and q is a periodic real valued potential in $L^\infty(\Omega)$.

2.1 $\theta-$quasi-periodic Green kernel

We begin with some basic properties of the $\theta-$Green's function G_θ. Let G_θ be defined as

$$(7) \quad G_\theta(\mathbf{x},k) = \frac{i}{4}\Big(\frac{k}{2\pi}\Big)^{(n-1)/2} \sum_{J\in\mathbb{Z}^n} \frac{1}{\mid \mathbf{x}+\mathbf{x}_J \mid^{(n-1)/2}}\, e^{-i\theta\,.\,\mathbf{d}_J}\, H^{(1)}_{(n-1)/2}\Big(k \mid \mathbf{x}+\mathbf{x}_J \mid \Big),$$

where $H^{(1)}_{(n-1)/2}$ is the Hankel function of order $(n-1)/2$ satisfying the outgoing condition at infinity.
Let $\mathcal{K}$ be defined as

$$\mathcal{K} = \Big\{ k \in \mathbb{R}^+, k \neq 0 \text{ and } k \neq |\alpha_J|, \ \forall J \in \mathbb{Z}^n \Big\}.$$

We assume throughout that $\mu_J \neq 0, \forall J \in \mathbb{Z}^n$. We show that $G_\theta(\mathbf{x})$ is equal to

$$\frac{i}{2d} \sum_{J\in\mathbb{Z}^n} \frac{1}{\mu_J}\, e^{i\mu_J\,|x_n|}\, e^{i\alpha_J\,.\,\mathbf{x}'}.$$

Lemma 2.1. *Let $\mathbf{x}_J = (\mathbf{d}_J, 0) \neq (\mathbf{x}', x_n) \in \mathbb{R}^{n+1}$ and let $k \in \mathcal{K}$. Then,*

$$(8) \qquad G_\theta(\mathbf{x},k) \;=\; \frac{i}{2d} \sum_{J\in\mathbb{Z}^n} \frac{1}{\mu_J}\, e^{i\mu_J\,|x_n|}\, e^{i\alpha_J\,.\,\mathbf{x}'}$$

converges uniformly in compact sets.
Proof. We assume that $\Im m(k) > 0$. Then, by (7) it follows that $e^{-i\theta\,.\,\mathbf{x}'}\, G_\theta(\mathbf{x}', x_n)$ is a periodic function of $\mathbf{x}'$. Its $J-$th Fourier coefficient is then given by

$$a_J(x_n) \;=\; \frac{1}{d}\int_{\mathbb{R}^n/\mathbf{d}\,.\,\mathbb{Z}^n} e^{i\theta\,.\,\mathbf{x}'}\, G_\theta(\mathbf{x}', x_n)\, e^{i\beta_J\,.\,\mathbf{x}'}\, d\mathbf{x}'.$$

Since $H^{(1)}_{(n-1)/2}\Big(k \mid \mathbf{x}+\mathbf{x}_J \mid \Big)$ is a fundamental solution of $\Delta + k^2$, it follows that a_J satisfies

$$\frac{d^2 a_J}{dx_n^2} \;+\; \mu_J^2\, a_J \;=\; -\frac{1}{d}\,\delta(x_n).$$

Using that

$$H^{(1)}_\tau(z) \;=\; (\frac{2}{\pi z})^{1/2}\, e^{i(z-\pi\tau/2-\pi/4)}\, \Big(1 + O(\frac{1}{|z|})\Big)$$

(see Olver [18]), we may prove that

$$(9) \qquad |G_\theta(\mathbf{x}', x_n)| \longrightarrow 0 \quad \text{as} \quad |x_n| \longrightarrow +\infty,$$

uniformly for $\mathbf{x}' \in \mathbb{R}^n/\mathbf{d}\,.\,\mathbb{Z}^n$. Now, from (9), we conclude that

$$a_J(x_n) = \begin{cases} a_J^+ \, e^{i\mu_J x_n} & \text{if } x_n \geq 0, \\ a_J^- \, e^{-i\mu_J x_n} & \text{if } x_n \leq 0, \end{cases}$$

for some constants a_J^+ and a_J^-. Since we know that for $\Im m(k) > 0$ the equality (8) holds, and since G_θ is continuous in $\mathcal{K}$, it follows that the equality (8) remains valid for $\Im m(k) = 0$. □

Now, we verify that G_θ is a $\theta-$quasi-periodic fundamental solution of (1).

LEMMA 2.2. *G_θ given by* (7) *satisfies*

$$(\Delta + k^2)\, G_\theta = \delta_\theta = \sum_{J \in \mathbb{Z}^n} \delta_J \, e^{i\theta\,.\,\mathbf{d}_J} \quad \text{in } \mathcal{D}'. \tag{10}$$

Proof. Let

$$\psi^N \;=\; \frac{i}{2d} \sum_{|J| \leq N} \frac{1}{\mu_J}\, e^{i\mu_J\,|x_n|}\, e^{i\alpha_J\,.\,\mathbf{x}'}.$$

Using the fact that $\mu_J^2 = k^2 - |\alpha_J|^2$, we have

$$\Delta\,\psi^N + k^2\psi^N \;=\; \frac{1}{d} \sum_{|J| \leq N} e^{i\alpha_J\,.\,\mathbf{x}'}\, \delta(x_n).$$

Hence for $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$,

$$(\Delta + k^2)\,\psi^N(\varphi) \;=\; \frac{1}{d} \sum_{|J| \leq N} \int_{\mathbb{R}^n} e^{i\alpha_J\,.\,\mathbf{x}'}\, \varphi(\mathbf{x}', 0)\, d\mathbf{x}'.$$

$$(\Delta + k^2)\,\psi^N(\varphi) \;=\; \frac{1}{d} \sum_{|J| \leq N} \widehat{\varphi}(\alpha_J, 0),$$

where $\widehat{\varphi}$ is the Fourier transform which is with respect to $\mathbf{x}'$ and with $x_n = 0$.
Letting $N \to +\infty$ and applying the Poisson summation formula yields

$$(\Delta + k^2)\, G_\theta(\varphi) \;=\; \sum_{J \in \mathbb{Z}^n} \varphi(\mathbf{x}_J)\, e^{-i\theta\,.\,\mathbf{d}_J}.$$

The proof is then complete. □

REMARK 2.1. *The series* (8) *cannot be term twice differentiated with respect to x_n at $x_n = 0$.*

LEMMA 2.3. *$G_\theta(\mathbf{x}, k)$ is $L^{2,loc}(\mathbb{R}^{n+1})$.*

Proof. Since the sum (8) contains only a finite number of terms with non exponential decay in x_n then $G_\theta(\mathbf{x}, k)$ is easily checked to be locally integrable as a function of $\mathbf{x}$ in $\mathbb{R}^{n+1}$. □

LEMMA 2.4.

$$G_\theta(\mathbf{x}, k) \;=\; G_\theta^{(r)}(\mathbf{x}, k) + G_\theta^{(0)}(\mathbf{x}),$$

with

$$G_\theta^{(r)}(\mathbf{x}, k) = \frac{i}{2} \sum_{J \in \mathbb{Z}^n} \frac{e^{-i\theta\,.\,\mathbf{d}_J}}{|\,\mathbf{x} + \mathbf{x}_J\,|^{(n-3)/2}} \int_0^k \Big(\frac{\tau}{2\pi}\Big)^{(n-1)/2} H^{(1)}_{(n-3)/2}\Big(\tau\ |\,\mathbf{x} + \mathbf{x}_J\,|\,\Big)\, d\tau, \tag{11}$$

and

$$G_\theta^{(0)}(\mathbf{x}) = \frac{1}{\gamma_n} \sum_{J \in \mathbb{Z}^n} e^{i\theta \cdot \mathbf{d}_J} \; | \mathbf{x} + \mathbf{x}_J |^{1-n} . \tag{12}$$

where

$$\gamma_n = \frac{2\,\pi^{(n+1)/2}}{(n-1)\,\Gamma((n+1)/2)}.$$

$G_\theta^{(r)}(\mathbf{x},k)$ is C^∞ for $\mathbf{x}$ away from $\mathbf{x}_J = (\mathbf{d}_J, 0), \forall J \in \mathbb{Z}^n$ and its local singularity near $\mathbf{x}_J$ can be expressed as it is shown in the following lemma.

LEMMA 2.5. *For fixed k and $\mathbf{x}$ near $\mathbf{x}_J = (\mathbf{d}_J, 0)$ we have the following estimates :*

$$|G_\theta^{(r)}(\mathbf{x},k)| \quad \leq \quad C\,|\mathbf{x} - \mathbf{x}_J|^{3-n} \quad \text{when } n \neq 3. \tag{13}$$

$$|G_\theta^{(r)}(\mathbf{x},k)| \quad \leq \quad C\,\log|\mathbf{x} - \mathbf{x}_J| \quad \text{when } n = 3. \tag{14}$$

$$|\nabla\, G_\theta^{(r)}(\mathbf{x},k)| \quad \leq \quad C\,|\mathbf{x} - \mathbf{x}_J|^{2-n}. \tag{15}$$

$$|\frac{\partial^2 G_\theta^{(r)}}{\partial x_j \partial x_l}(\mathbf{x},k)| \quad \leq \quad C\,|\mathbf{x} - \mathbf{x}_J|^{1-n}. \tag{16}$$

We begin by showing the following lemma.

LEMMA 2.6.

$$\lim_{k \to 0} G_\theta(\mathbf{x},k) = G_\theta^{(0)}(\mathbf{x}). \tag{17}$$

Proof. From the explicit formula (7) for $G_\theta(\mathbf{x},k)$ and the asymptotic form of the Hankel function of small arguments

$$H_\tau^{(1)}(r) = -i\frac{\Gamma(\tau)}{\pi}\,\frac{2^\tau}{r^\tau}\,(1 + o(r)), (\tau > 0), \tag{18}$$

we see that

$$\lim_{k \to 0} G_\theta(\mathbf{x},k) = G_\theta^{(0)}(\mathbf{x}).$$

We now return to the proof of Lemma 2.5. Computation of the derivation of $G_\theta(\mathbf{x},k)$ with respect to k yields

$$\frac{\partial G_\theta}{\partial k}(\mathbf{x},k) = \frac{i}{8\pi}\Big(\frac{k}{2\pi}\Big)^{(n-3)/2} \sum_{J \in \mathbb{Z}^n} \frac{e^{-i\theta \cdot \mathbf{d}_J}}{| \mathbf{x} + \mathbf{x}_J |^{(n-1)/2}} \quad g_n(| \mathbf{x} + \mathbf{x}_J |, k),$$

where

$$g_n(| \mathbf{x} + \mathbf{x}_J |, k) = k\ | \mathbf{x} + \mathbf{x}_J |\ H_{(n-1)/2}^{(1)'}\Big(\,k\ | \mathbf{x} + \mathbf{x}_J |\,\Big) + \frac{n-1}{2} H_{(n-1)/2}^{(1)}\Big(\,k\ | \mathbf{x} + \mathbf{x}_J |\Big).$$

By using (17) and the relation

$$rH_\tau^{(1)'}(r) + \tau H_\tau^{(1)}(r) = rH_{\tau-1}^{(1)}(r), \tag{19}$$

we find

$$G_\theta^{(r)}(\mathbf{x},k) = \frac{i}{2} \sum_{J \in \mathbb{Z}^n} \frac{e^{-i\theta \cdot \mathbf{d}_J}}{| \mathbf{x} + \mathbf{x}_J |^{(n-3)/2}} \int_0^k \Big(\frac{\tau}{2\pi}\Big)^{(n-1)/2} H_{(n-3)/2}^{(1)}\Big(\,\tau\ | \mathbf{x} + \mathbf{x}_J |\,\Big)\, d\tau.$$

and (11) follows. The inequality (13) is an immediate consequence of (11). For $n = 2$ and $n = 3$, (18) should be replaced by the following formulae for $\tau = 0$ and for $\tau < 0$.

$$H_{\tau}^{(1)}(r) = e^{-i\pi\tau} H_{-\tau}^{(1)}(r), \quad \text{for } \tau < 0,$$

and

$$H_0^{(1)} = \frac{2i}{\pi} \log r + O(1).$$

Now, by differentiating (11) with respect to $\mathbf{x}$ it follows

$$\nabla G_\theta^{(r)}(\mathbf{x}, k) = \frac{i}{2} \frac{\mathbf{x}+\mathbf{x}_J}{|\mathbf{x}+\mathbf{x}_J|} \sum_{J\in\mathbb{Z}^n} \frac{e^{-i\theta\,.\,\mathbf{d}_J}}{|\mathbf{x}+\mathbf{x}_J|^{(n-3)/2}} \int_0^k \Big(\frac{\tau}{2\pi}\Big)^{(n-1)/2} H_{(n-1)/2}^{(1)}\Big(\tau\, |\mathbf{x}+\mathbf{x}_J|\Big)\, d\tau.$$

and then (15) follows. □

To complete this subsection, we define the regularized version of the Green's kernel G_θ as

$$(20)\quad G_\theta^\varepsilon(\mathbf{x}, k) = \lambda(n,k) \sum_{J\in\mathbb{Z}^n} \frac{e^{-i\theta\,.\,\mathbf{d}_J}}{(|\mathbf{x}+\mathbf{x}_J|^2 + \varepsilon^2)^{(n-1)/4}} H_{(n-1)/2}^{(1)}\Big(k\sqrt{|\mathbf{x}+\mathbf{x}_J|^2+\varepsilon^2}\Big),$$

for $\varepsilon > 0$ and with

$$\lambda(n,k) = \frac{i}{4}\Big(\frac{k}{2\pi}\Big)^{(n-1)/2}.$$

Direct computation yields

$$\lim_{\varepsilon\to 0} \Big(\Delta G_\theta^\varepsilon(\mathbf{x}-\mathbf{y}, k) + k^2 G_\theta^\varepsilon(\mathbf{x}-\mathbf{y}, k)\Big) = \delta_\theta(\mathbf{x}-\mathbf{y}).$$

$\Delta G_\theta^\varepsilon(\mathbf{x}-\mathbf{y}, k) + k^2 G_\theta^\varepsilon(\mathbf{x}-\mathbf{y}, k)$ is then an approximation of the identity.

2.2 θ−quasi-periodic Sobolev spaces

In this subsection, we describe the functional setting needed to state our boundary value problems in a suitable mathematical form. We state the only properties we need without proofs. Let ω be an open subset of $\mathbb{R}^{n+1}$ with a boundary of class $C^{m,1}$. We shall use standard notations for the usual functional spaces : $L^2(\omega)$ is the space of complex square integrable functions defined on ω, $\mathcal{C}_0^\infty(\omega)$ is the space of smooth functions defined on ω with compact support in ω. The Sobolev space of order s, $H^s(\omega)$ is defined as the space of all distributions u defined in ω, such that :

(a) $D^\alpha u \in L^2(\omega)$, for $|\alpha| \le s$, when s is a nonnegative integer,

(b) $u \in H^{s'}(\omega)$ and

$$\int_\omega\int_\omega \frac{|D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{n+1+2\sigma}}\, d\mathbf{x}\, d\mathbf{y} < \infty, \quad \text{for } |\alpha| = s',$$

when s' is a nonnegative integer, $s = s' + \sigma$ is nonnegative and is not an integer and $0 < \sigma < 1$. We define a Banach norm on $H^s(\omega)$ by

$$||u||^2_{H^s(\omega)} = \sum_{|\alpha|\le s} \int_\omega |D^\alpha u(\mathbf{x})|^2\, d\mathbf{x}$$

in the case (a), and

$$||u||^2_{H^s(\omega)} = ||u||^2_{H^{s'}(\omega)} + \sum_{|\alpha|=s'} \int_\omega \int_\omega \frac{|D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{n+1+2\sigma}}\, d\mathbf{x}\, d\mathbf{y}$$

in the case (b). In general the space $\mathcal{C}_0^\infty(\omega)$ is not dense in $H^s(\omega)$. Let us define for $s < 0$, $H^s(\omega)$ as the dual space of $\left(\overline{\mathcal{C}_0^\infty(\omega)}\right)^{H^{-s}(\omega)}$. We also define $H^{s,loc}(\overline{\omega})$ to be the set of functions in $H^s(\overline{K})$ for any $K \subset\subset \omega$. For proofs of the results above, we refer to Nečas [15].

Now, let Γ be an open subset of a $C^{m,1}$ manifold. We consider distributions whose order is less than or equal to $m+1$; those span the dual space of the space of all $C^{m,1}$ functions with compact support in Γ. Functions are identified with distributions by means of the usual injection $u \mapsto Tu$, defined by

$$\langle Tu, v\rangle = \int_\Gamma u\, v\, d\gamma,$$

where $d\gamma$ is the usual hypersurface measure on Γ (defined provided Γ is a Lipschitz hypersurface). A distribution u on Γ belongs to $H^s(\Gamma)$ with $|s| \leq m+1$ if

$$u \circ \Phi \in H^s\Big(U' \cap \Phi^{-1}(\Gamma \cap U)\Big),$$

for all U and φ fulfilling the assumptions of the definition of $C^{m,1}$ domains (given in page 1) and $\Phi(\mathbf{x}) = (\mathbf{x}', \varphi(\mathbf{x}'))$. We denote by γ_0 the operator defined by $\gamma_0(u) = u|_\Gamma$ when u is a smooth function. We also introduce ν the unit outward normal a.e. on Γ[5].

Assume that $s = l + 1/2 + \sigma$, where l is a nonnegative integer and $0 < \sigma < 1$, then the mapping

$$u \longmapsto \Big\{\, \gamma_0(u), \gamma_0(\frac{\partial u}{\partial \nu}), \dots, \gamma_0(\frac{\partial^l u}{\partial \nu^l})\,\Big\},$$

which is defined for $u \in C^{m+1}(\overline{\omega})$ has a unique continuous extension as an operator from $H^s(\Omega)$ onto $\displaystyle\prod_{j=0}^{l} H^{s-j-1/2}(\Gamma)$. This operator has a right continuous inverse.

We need to define Sobolev spaces of $\theta-$quasi-periodic functions in $\mathbf{x}'$ of period $\mathbf{d}$. Let $\mathcal{C}_\theta^\infty(\mathbb{R}^{n+1})$ denote the set of smooth $\theta-$quasi-periodic functions in $\mathbf{x}'$ of period $\mathbf{d}$. $\mathcal{C}_\theta^\infty(\omega)$ will be the space of restrictions to ω of functions $\mathcal{C}_\theta^\infty(\mathbb{R}^{n+1})$. This enables us to consider the $\theta-$quasi-periodic Sobolev space $H_\theta^s(\omega)$, (for $|s| \leq 2$) closure of $\mathcal{C}_\theta^\infty(\omega)$ in $H^s(\omega)$ defined above.

It is easy to see that, equipped with the H^s norm and Hibertian scalar product, $H_\theta^s(\omega)$ becomes a Hilbert space. We will also use the Fréchet space which is locally in $H_\theta^s(\omega)$. It will be designed by $H_\theta^{s,loc}(\overline{\omega})$. It is not difficult to show that we can extend a periodic

[5] ν is defined a.e. (for the measure $d\gamma$ on Γ) by

$$\nu(\mathbf{x}', \varphi(\mathbf{x}')) = -\Big(\frac{\partial \varphi}{\partial x_0}, \dots, \frac{\partial \varphi}{\partial x_{n-1}}, 1\Big) \,/\, \sqrt{1 + (\frac{\partial \varphi}{\partial x_0})^2 + \dots + (\frac{\partial \varphi}{\partial x_{n-1}})^2}, \quad \text{for } \mathbf{x}' \in U'.$$

This vector field is easily extended to the whole U by defining it independently of x_n. Finally, by a partition of unity, we define an L^∞ vector field in a neighborhood of $\overline{\omega}$, such that ν is the unit outward normal a.e. on Γ. The vector field ν is clearly of class $C^{m-1,1}$.

version of the first trace on $\Gamma^{\pm}_{\rho}$ ($\Gamma^{\pm}_{\rho}$ is smooth), well defined for functions in $\mathcal{C}^{\infty}_{\theta}(\omega_\rho)$ to $H^s_\theta(\omega_\rho)$, with values in a fractionnary order Sobolev space of $\theta-$quasi-periodic functions over $\Gamma^{\pm}_{\rho}$ denoted by $H^{s-1/2}_\theta(\Gamma^{\pm}_{\rho})$. Because of the invariance of the Sobolev spaces under uniform Lipschitz coordinate transformations for $|s| \le 2$, we can define the spaces $H^s_\theta(\Gamma^{\pm})$ in the usual way using local coordinate representations of uniformly Lipschitz manifold $\Gamma^{\pm}$.

Finally, to complete our preliminaries, we recall that Green's formula is valid in any bounded Lipschitz domain, as is shown in Nečas [15]. For u and v in $H^2_\theta(\omega)$, we have

$$\int_\omega \Delta u\, \overline{v}\, d\mathbf{x} - \int_\omega u\, \Delta \overline{v}\, d\mathbf{x} \;=\; \int_{\Gamma^{\pm}} \Big(\frac{\partial u}{\partial \nu}\, \overline{v} - u\, \frac{\partial \overline{v}}{\partial \nu} \Big)\, d\gamma(\mathbf{x}).$$

2.3 $\theta-$quasi-periodic layers potentials

Using the $\theta-$Green's function $G_\theta(\mathbf{x}, k)$ defined above we now define analogues of the classical single and double layer potentials. Let

$$S_\theta f(\mathbf{x}) \;=\; \int_{\Gamma^{\pm}} G_\theta(\mathbf{x},\mathbf{y})\; f(\mathbf{y})\; d\gamma(\mathbf{y}) \tag{21}$$

be the $\theta-$acoustic simple layer operator and

$$D_\theta f(\mathbf{x}) \;=\; \int_{\Gamma^{\pm}} \frac{\partial G_\theta}{\partial \nu(\mathbf{y})}(\mathbf{x},\mathbf{y})\; f(\mathbf{y})\; d\gamma(\mathbf{y}) \quad \text{for } \mathbf{x} \notin \Gamma^{\pm}, \tag{22}$$

the $\theta-$double layer operator, where $G_\theta(\mathbf{x},\mathbf{y}) = G_\theta(\mathbf{x}-\mathbf{y})$, f is θ quasi-periodic-periodic on $\Gamma^{\pm}$ and $\mathbf{x} \in \mathbb{R}^{n+1} \setminus \Gamma^{\pm}$. Since the singularity of G_θ for $\mathbf{x}$ near $\mathbf{y}$ is the same as that of $G^{(0)}_\theta$, it is locally integrable on $\Gamma^{\pm}$ and the trace $\theta-$simple layer potential is given by (21). Let us define the trace $\theta-$double boundary potential as

$$B_\theta f(\mathbf{x}) \;=\; p.v.\; \int_{\Gamma^{\pm}} \frac{\partial G_\theta}{\partial \nu(\mathbf{y})}(\mathbf{x},\mathbf{y})\; f(\mathbf{y})\; d\gamma(\mathbf{y}) \quad \text{for } \mathbf{x} \in \Gamma^{\pm}. \tag{23}$$

The following lemmas list some of the basic properties of the $\theta-$acoustic and the $\theta-$generalized layer potentials which will be needed in the sequel. All these properties are well-known for the classical layer potentials when Ω is a bounded smooth domain[6]. Some of these properties have been shown to hold in the case of a bounded domain even when the boundary is Lipschitz [7](see Verghota in [20], Costabel in [9] and Nachman [14] for the $C^{1,1}$ domains).

LEMMA 2.7. *Suppose that ω is with a $C^{1,1}$ boundary $\Gamma^{\pm}$. Then for $0 \le s \le 1$, there exists a positive constant $C(k,s)$ such that*

$$||S_\theta f||_{H^{s+1}_\theta(\Gamma^{\pm})} \quad \le \quad C(k,s)\; ||f||_{H^s_\theta(\Gamma^{\pm})}. \tag{24}$$

[6]These properties were shown, in the case of bounded C^1 domains by Fabes, Jodeit and Riviére in [11].

[7]An open set $D \subset \mathbb{R}^{n+1}$ is called a Lipschitz domain if for each $\mathbf{p} \in \partial D$ there exist a rectangular coordinate system $(\mathbf{x}', x_n) \in \mathbb{R}^n \times \mathbb{R}$, a neighborhood, $U(\mathbf{p}) \in \mathbb{R}^{n+1}$ containing $\mathbf{p}$, and a function $\Phi_\mathbf{p} : \mathbb{R}^n \mapsto \mathbb{R}$ such that :

(i) $|\Phi_\mathbf{p}(\mathbf{x}') - \Phi_\mathbf{p}(\mathbf{y}')| \le C(\mathbf{p})\,|\mathbf{x}' - \mathbf{y}'|$ for all $\mathbf{x}', \mathbf{y}' \in \mathbb{R}^n, C(\mathbf{p}) < \infty$;

(ii) $U \cap D = \Big\{ (\mathbf{x}', x_n) : x_n > \Phi_\mathbf{p}(\mathbf{x}') \Big\} \cap U.$

The coordinate systems, $(\mathbf{x}', x_n)$, may always be taken to be a rotation and translation of standard rectangular coordinates of $\mathbb{R}^{n+1}$.

Proof. We sketch a proof. We recall since $\Gamma^\pm$ is of class $C^{1,1}$ for every point $\mathbf{x} \in \Gamma^\pm$ there exist a neighborhood U of $\mathbf{x}$ in $\mathbb{R}^{n+1}$, and a new orthogonal coordinates $(\mathbf{y}', y_n) \in \mathbb{R}^n \times \mathbb{R}$, such that :

(i) U is an hypercube in the new coordinates $U = \Big\{ (\mathbf{y}', y_n),\ -a_j < y_j < a_j \Big\}$;

(ii) there exists a function φ of class $C^{1,1}$, defined in $U' = \Big\{ \mathbf{y}',\ -\frac{a_j}{2} < y_j < \frac{a_j}{2} \Big\}$, and such that

$$|\varphi(\mathbf{y}')| \leq \frac{a_n}{2} \quad \text{for every } \mathbf{y}' \in U', \quad U \cap \omega = \Big\{ (\mathbf{y}', y_n) : \, y_n < \varphi(\mathbf{y}') \Big\},$$

and $U \cap \Gamma^\pm = \Big\{ (\mathbf{y}', y_n) : \, y_n = \varphi(\mathbf{y}') \Big\}$.

In these local coordinates, for f supported in $\Gamma^\pm \cap U$, $S_\theta f$ is equal to

$$\begin{aligned} S_\theta f &= \int_{U'} G_\theta^{(r)}(\mathbf{x}' - \mathbf{y}' + \mathbf{x}_J, \varphi(\mathbf{x}') - \varphi(\mathbf{y}'))\, f(\mathbf{y}', \varphi(\mathbf{y}'))(1 + |\nabla\varphi(\mathbf{y}')|^2)^{1/2}\, d\mathbf{y} \\ &+ \frac{1}{\gamma_n} \sum_{J \in \mathbb{Z}^n} e^{i\theta \,.\, \mathbf{d}_J} \int_{U'} \Phi_J^n(\mathbf{x}', \mathbf{y}')\, f(\mathbf{y}', \varphi(\mathbf{y}'))\, (1 + |\nabla\varphi(\mathbf{y}')|^2)^{1/2}\, d\mathbf{y}, \end{aligned}$$

where

$$\Phi_J^n(\mathbf{x}', \mathbf{y}') = \Big(|\mathbf{x}' - \mathbf{y}' + \mathbf{x}_J|^2 + (\varphi(\mathbf{x}') - \varphi(\mathbf{y}'))^2 \Big)^{-(1+n)/2}.$$

Let

$$\tilde{f}(\mathbf{y}') = f(\mathbf{y}', \varphi(\mathbf{y}')).$$

For $0 \leq s \leq 2$, $\tilde{f} \in H_\theta^s(U')$ if and only if $f \in H_\theta^s(\Gamma^\pm)$. A partition of unity argument reduces (24) to proving that the operator

$$\tilde{f} \longmapsto S_\theta \tilde{f}$$

is bounded from $H_\theta^s(U')$ to $H_\theta^{s+1}(U')$. Since $\varphi \in C^{1,1}(U')$, multiplying by

$$(1 + |\nabla\varphi(\mathbf{y}')|^2)^{1/2} \in C^{0,1}(U')$$

is bounded on $H^s(U')$ for $0 \leq s \leq 1$.

We shall consider the operator K_θ on U' defined as

$$K_\theta f = \sum_{J \in \mathbb{Z}^n} e^{i\theta \,.\, \mathbf{d}_J} \int_{U'} \Phi_J^n(\mathbf{x}', \mathbf{y}')\, f(\mathbf{y}', \varphi(\mathbf{y}'))\, (1 + |\nabla\varphi(\mathbf{y}')|^2)^{1/2}\, d\mathbf{y}.$$

$\nabla_{\mathbf{x}} K_\theta$ is the principal value operator with kernel

$$\nabla_{\mathbf{x}} K_\theta(\mathbf{x}', \mathbf{y}') = \sum_{J \in \mathbb{Z}^n} e^{i\theta \,.\, \mathbf{d}_J}\, \Phi_J^n(\mathbf{x}', \mathbf{y}') \Big(\mathbf{x}' - \mathbf{y}' + \mathbf{x}_J + (\varphi(\mathbf{x}') - \varphi(\mathbf{y}'))\, \nabla\varphi \Big).$$

From a simple generalization of the Theorem of Coifman, McIntosh and Meyer and the boundeness of $\nabla\varphi$ we conclude that

$$||K_\theta f||_{H^1(\mathbb{R}^{n+1})} \quad \leq \quad C\, ||f||_{L^2(\mathbb{R}^{n+1})}.$$

We recall the celebrate Theorem of Coifman, McIntosh and Meyer.

Theorem 2.1. *(R.R. Coifman, A. McIntosh, Y. Meyer) Let A, φ be Lipschitz functions on $\mathbb{R}^m$. The following singular operator with kernel*

$$\frac{A(\mathbf{x})-A(\mathbf{y})}{\Big(|\mathbf{x}-\mathbf{y}|^2+(\varphi(\mathbf{x})-\varphi(\mathbf{y}))^2\Big)^{m/2}}$$

is bounded on $L^2(\mathbb{R}^m)$.

For the H^1 to H^2 estimate, we prove that $\dfrac{\partial^2 K_\theta}{\partial x_j \partial x_l}$ is L^2 bounded. Thus

$$||K_\theta f||_{H^2(\mathbb{R}^{n+1})} \quad \leq \quad C\,||f||_{H^1(\mathbb{R}^{n+1})}.$$

The inequality (24) follows easily by interpolation. $\square$

Lemma 2.8. *For $0 \leq s \leq 3/2$, B_θ is bounded on $H^s_\theta(\Gamma^\pm)$.*

Proof. We begin by establishing the identity

$$\Big(\, S_\theta T^\pm_\theta(0) - B_\theta - \frac{1}{2}\Big) f \;=\; k^2\gamma_0 \int_\omega G_\theta(\mathbf{x}-\mathbf{y})\mathcal{P}_0 f(\mathbf{y})\, d\mathbf{y},$$

where γ_0 is the trace operator on $\Gamma^\pm$, $\mathcal{P}_0$ given by $\mathcal{P}_0 f = u \in H^2_\theta(\omega)$ for $f \in H^{3/2}_\theta(\Gamma^\pm)$ with u the unique solution in $H^2_\theta(\omega)$ to

$$\left\{\begin{array}{rcll} (\Delta + k^2)\, u & = & 0 & \text{in } \omega, \\ u & = & f & \textit{on } \Gamma^\pm, \end{array}\right. \tag{25}$$

and $T^\pm_\theta(0)$ the $\theta-$Dirichlet-to-Neumann operator defined as

$$T^\pm_\theta(0): \quad f \in H^{3/2}_\theta(\Gamma^\pm) \longmapsto \frac{\partial(\mathcal{P}_0 f)}{\partial \nu} \in H^{1/2}_\theta(\Gamma^\pm).$$

The boundeness of the operator B_θ on $H^s_\theta(\Gamma^\pm)$ for $0 \leq s \leq 3/2$ can be deduced from the inequality (24). $\square$

Lemma 2.9. *Suppose that ω is with a $C^{1,1}$ boundary $\Gamma^\pm$. Then if $f \in H^{1/2}_\theta(\Gamma^\pm)$, the function $u = S_\theta f$ satisfies the following properties :*

(a) $\quad (\Delta + k^2)\, u = 0 \quad \text{in } \mathbb{R}^{n+1} \setminus \Gamma^\pm.$

(b) $\quad$ *u is in $H^2_\theta(\omega)$ and in $H^{2,loc}_\theta(\omega')$.*

(c) $$\lim_{\rho\to\infty} \int_{\Gamma^\pm_\rho} \Big\{ \frac{\partial G_\theta}{\partial \nu(\mathbf{y})}(\mathbf{x},\mathbf{y})\; u(\mathbf{y}) - G_\theta(\mathbf{x},\mathbf{y})\; \frac{\partial u}{\partial \nu}(\mathbf{y}) \Big\}\; d\gamma(\mathbf{y}) = 0,$$

for almost every $\mathbf{x}$ with $|\,\mathbf{x}\,| < \rho$.

Proof. (a) is clear. To verify (b), we observe S_θ is bounded as an operator from $L^2(\Gamma^\pm)$ to $L^2(\omega)$. Then we write

$$-\,\Delta u \;=\; k^2 u \quad \text{in } \omega \text{ (or in } \omega'_\rho\text{)}, \tag{26}$$

and use the interior regularity for (26). From Lemma 2.10 below, this argument yields

$$||S_\theta f||_{H^2_\theta(\omega)} \quad \leq \quad C(k)\, ||f||_{H^{3/2}_\theta(\Gamma^\pm)}.$$

To verify (c) we use G_θ^ε the regularized version of G_θ given by (20). By applying the Green's formula to this smooth function we have

$$\int_{\Gamma_\rho^\pm} \Big\{ \frac{\partial G_\theta}{\partial \nu(\mathbf{y})}(\mathbf{y},\mathbf{z})\, G_\theta^\varepsilon(\mathbf{x},\mathbf{y}) - G_\theta(\mathbf{y},\mathbf{z})\, \frac{\partial G_\theta^\varepsilon}{\partial \nu}(\mathbf{x},\mathbf{y}) \Big\}\, d\gamma(\mathbf{y}) \;=\; G_\theta^\varepsilon(\mathbf{x},\mathbf{z})$$

$$- \int_{|y_n|<\rho} G_\theta(\mathbf{y},\mathbf{z})\, \Delta_{\mathbf{y}}\, G_\theta^\varepsilon(\mathbf{x},\mathbf{y})\, d\mathbf{y}.$$

Now, multiplying by $f(\mathbf{z})$ and integrating over $\Gamma_\rho^\pm$ give

$$\int_{\Gamma_\rho^\pm} \Big\{ \frac{\partial u}{\partial \nu}(\mathbf{y})\, G_\theta^\varepsilon(\mathbf{x},\mathbf{y}) - u(\mathbf{y})\, \frac{\partial G_\theta^\varepsilon}{\partial \nu}(\mathbf{x},\mathbf{y}) \Big\}\, d\gamma(\mathbf{y}) \;=\; - \int_{|y_n|<\rho} u(\mathbf{y})\, \Delta_{\mathbf{y}}\, G_\theta^\varepsilon(\mathbf{x},\mathbf{y})\, d\mathbf{y}$$

$$+ \int_{\Gamma_\rho^\pm} G_\theta^\varepsilon(\mathbf{x},\mathbf{z})\, f(\mathbf{z})\, d\gamma(\mathbf{z}).$$

Letting ε tend to 0 gives

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho^\pm} \Big\{ \frac{\partial G_\theta}{\partial \nu(\mathbf{y})}(\mathbf{x},\mathbf{y})\, u(\mathbf{y}) - G_\theta(\mathbf{x},\mathbf{y})\, \frac{\partial u}{\partial \nu}(\mathbf{y}) \Big\}\, d\gamma(\mathbf{y}) = 0,$$

and then the proof of the Lemma is over. $\square$

Now, we verify the interior regularity result used in the proof above.

LEMMA 2.10. *Suppose that k^2 is not a $\theta-$Dirichlet eigenvalue*[8] *of the following interior problem* (27). *If $f \in H_\theta^{3/2}(\Gamma^\pm)$ then there exists a unique solution $u \in H_\theta^2(\omega)$ to*

$$\left\{ \begin{array}{rcll} (\Delta + k^2)\, u &=& 0 & \text{in } \omega, \\ u &=& f & \textit{on } \Gamma^\pm. \end{array} \right. \tag{27}$$

Proof. The regularity of the interior solution under these limited smoothness assumptions is classical. See, for instance, Grisvard [12]. $\square$

LEMMA 2.11. *If $f \in H_\theta^{3/2}(\Gamma^\pm)$, the function $v = D_\theta f$ satisfies the properties $(a), (b)$ and (c) given in lemma 2.3 above. The non-tangential limits $v^+ (v^-)$ of v as we approach the boundary from outside (respectively inside) Ω exist and satisfy*

$$v^\pm(\mathbf{x}) \;=\; \pm\frac{1}{2}\, f(\mathbf{x}) + B_\theta f(\mathbf{x}) \quad \text{for almost every } \mathbf{x} \in \Gamma^\pm.$$

Proof. The proof is based on a straightforward generalization of the result of Fabes, Jodeit and Riviére [11]. We may prove that the nontangential maximal function[9]

$$D_\theta f^*(\mathbf{x}) = \sup\Big\{ |D_\theta f(\mathbf{y})|, \mathbf{y} \in \omega, |\mathbf{x}-\mathbf{y}| < \delta(\alpha), (\mathbf{x}-\mathbf{y})\,.\,\nu(\mathbf{x}) > \alpha|\mathbf{x}-\mathbf{y}| \Big\}, \quad 0 < \alpha < 1$$

[8] We say that k^2 is a $\theta-$Dirchlet eigenvalue if there exists nontrivial solution $u \in H_\theta^2(\omega)$ to the following Dirichlet problem :

$$\left\{ \begin{array}{rcll} (\Delta + k^2)\, u &=& 0 & \text{in } \omega, \\ u &=& 0 & \textit{on } \Gamma^\pm. \end{array} \right.$$

[9] Let $C(\mathbf{x})$ be an open, circular truncated cone with two non-empty, convex components with vertex at $\mathbf{x}$ and one component in ω and the other in ω'. The nontangential maximal function is defined as

$$f^*(\mathbf{x}) \;=\; \sup_{\mathbf{y}\in C(\mathbf{x})} |f(\mathbf{y})|.$$

satisfies

$$||(D_\theta f)^*||_{L^2(\Gamma^\pm)} \le ||f||_{L^2(\Gamma^\pm)},$$

and, for almost $\mathbf{x} \in \Gamma^\pm$,

$$D_\theta f(\mathbf{y}) \longrightarrow -\frac{1}{2} f(\mathbf{x}) + B_\theta f(\mathbf{x})$$

as $\mathbf{y}$ approaches $\mathbf{x}$ approaches nontangentially from inside ω. □

Let $T_\theta^\pm(q)$ be the $\theta-$Dirichlet-to-Neumann operator defined as

$$T_\theta^\pm(q): \quad f \in H_\theta^s(\Gamma^\pm) \longmapsto \frac{\partial u}{\partial \nu} \in H_\theta^{s-1}(\Gamma^\pm),$$

where $u \in H_\theta^{s+1/2}(\omega)$ is the unique solution to the following problem

$$\begin{cases} (\Delta + k^2 + q)\, u &= 0 \qquad \text{in } \omega, \\ \qquad\qquad u &= f \qquad \textit{on } \Gamma^\pm. \end{cases} \tag{28}$$

$T_\theta^\pm(q)$ is well defined if and only if k^2 is not a $\theta-$Dirichlet eigenvalue of $-\Delta - q$ in ω.

We shall introduce as the same way $T_{\rho,\theta}^\pm(q)$

$$T_{\rho,\theta}^\pm(q): \quad f \in H_\theta^s(\Gamma_\rho^\pm) \longmapsto \frac{\partial u}{\partial \nu} \in H_\theta^{s-1}(\Gamma_\rho^\pm),$$

where $u \in H_\theta^{s+1/2}(\omega_\rho)$ is the unique solution to the following problem

$$\begin{cases} (\Delta + k^2 + q)\, u &= 0 \qquad \text{in } \omega_\rho, \\ \qquad\qquad u &= f \qquad \textit{on } \Gamma_\rho^\pm. \end{cases} \tag{29}$$

$T_{\rho,\theta}^\pm(q)$ is well defined if and only if k^2 is not a $\theta-$Dirichlet eigenvalue of $-\Delta - q$ in ω_ρ. Note that $T_{\rho,\theta}^\pm(q)$ gives a precise meaning of the outgoing radiation condition at infinity. u satisfies the $\theta-$Sommerfeld radiation condition if and only if

$$\frac{\partial u}{\partial \nu} = T_{\rho,\theta}^\pm(q) u \quad \text{on } \Gamma_\rho^\pm.$$

We are now able to show that equation (1) admits at most one $\theta-$quasi-periodic solution.

Proposition 2.1. (**Uniqueness**) *There exists a discrete set $\mathcal{N}$ of values of k with accumulation point at infinity such that, the problem*

$$\begin{cases} \Big(\Delta + k^2 + q(\mathbf{x})\Big) u \;=\; 0 \qquad \text{in } \mathbb{R}^{n+1}, \\ u \quad \text{satisfying } \theta-\text{quasi-periodic radiation condition at infinity}, \end{cases} \tag{30}$$

can have non trivial solutions in $L^{2,loc}(\mathbb{R}^{n+1})$ if and only if k is in $\mathcal{N}$.

Proof. The argument used in this proof generalizes the one given by Nédélec and Starling in [16]. We sketch it quickly. u appears as an eigenvalue of the operator $A = \Delta + q$ with a domain defined as

$$D(A) = \Big\{ u \in L^2(\omega_\rho), \Delta\, u \in L^2(\omega_\rho), \frac{\partial u}{\partial \nu} = T_{\rho,\theta}^\pm(q) u \text{ on } \Gamma_\rho^\pm \Big\},$$

$\mathcal{T}^{\pm}_{\rho,\theta}(q)$ is defined from $T^{\pm}_{\rho,\theta}(q)$ by

$$\mathcal{T}^{\pm}_{\rho,\theta}(q)u \;=\; T^{\pm}_{\rho,\theta}(q)\,\tilde{u},$$

where

$$\tilde{u} \;=\; \sum_{|\alpha_J|>k} u_J\, e^{i\beta_J\,.\mathbf{x}'},$$

with u_J are the Fourier coefficients of u. It is not difficult to verify that A is a self adjoint, densely defined operator, with compact resolvent. Therefore, there is a complete orthonormal basis $\{\,\phi_m\,\}$ of $L^2(\omega_\rho)$ in $D(A)$ such that

$$A\,\phi_m \;=\; \lambda_m\,\phi_m \quad \text{with } \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \to +\infty,$$

and λ_m is given by the Min-Max principle. We conclude that if $k^2 \neq \lambda_m,\,\forall\, m \in \mathbb{N}$ then (30) has at most one solution. $\square$

Now, consider the following integral equation :

$$f(\mathbf{x}) \;=\; u^{in}(\mathbf{x}) - \Big(\, S_\theta T^{\pm}_\theta(q) - B_\theta - \frac{1}{2}I\,\Big)\, f(\mathbf{x}) \quad \text{on } \Gamma^{\pm}. \tag{31}$$

PROPOSITION 2.2. *Suppose that k^2 is not a $\theta-$Dirichlet eigenvalue of $-\Delta + q$ in ω. If u solves*

$$\left\{\begin{array}{rcl} (\Delta + k^2 + q)\,u & = & 0 \quad \text{in } \omega', \\ u & \in & H^{2,loc}_\theta(\omega'), \\ \dfrac{\partial u}{\partial \nu} & = & T^{\pm}_\theta(q)u \quad \textit{on } \Gamma^{\pm}, \\ u - u^{in} & & \text{satisfying the } \theta\text{−radiation condition at infinity,} \end{array}\right. \tag{32}$$

then its trace on the boundary $\Gamma^{\pm}$ solves the integral equation (31). *Conversely, if $f \in H^{3/2}_\theta(\Gamma^{\pm})$ is a solution of* (31) *then u defined for $\mathbf{x} \in \omega'$ by*

$$u(\mathbf{x}) \;=\; u^{in}(\mathbf{x}) - \Big(\, S_\theta T^{\pm}_\theta(q) - D_\theta\,\Big) f(\mathbf{x}) \tag{33}$$

solves (32).

Proof. Assume u solves (32). Applying Green's formula to G^{ε}_θ and u in ω' and letting ε tend to 0 show that (31) holds. Conversely, the properties of S_θ and D_θ give

$$\left\{\begin{array}{rcl} (\Delta + k^2 + q)\,u & = & 0 \quad \text{in } \omega', \\ u & \in & H^{2,loc}_\theta(\omega'), \\ u - u^{in} & & \text{satisfying the } \theta\text{−radiation condition at infinity.} \end{array}\right.$$

It remains to prove that

$$\frac{\partial u}{\partial \nu} \;=\; T^{\pm}_\theta(q)u \quad \textit{on } \Gamma^{\pm}. \tag{34}$$

By the same way as in the proof of Lemma 2.9, we have that

$$u(\mathbf{x}) \;=\; u^{in}(\mathbf{x}) + \Big(\, S_\theta T^{\pm}_\theta(0) - B_\theta - \frac{1}{2}\,\Big) f\,(\mathbf{x})$$

The equation (31) shows then that $u|_{\Gamma^\pm} = f$. We also have

$$u(\mathbf{x}) \;=\; u^{in}(\mathbf{x}) + S_\theta(\frac{\partial u}{\partial \nu})(\mathbf{x}) + D_\theta f\,(\mathbf{x}).$$

Thus

$$S_\theta\Big(\frac{\partial u}{\partial \nu} - T_\theta^{\pm}(q)u \Big) \;=\; 0.$$

From the invertibility of the layer operator S_θ (see Lemma below), we conclude that (34) holds. $\square$

We prove that the solutions of the exterior problem (32) are precesily the restrictions to ω' of the $\theta-$quasi-periodic solutions of the integral equation (31).

LEMMA 2.12. *If k^2 is not a $\theta-$Dirichlet eigenvalue of $-\Delta - q$ in ω then*

$$S_\theta : \quad H_\theta^{1/2}(\Gamma^\pm) \longmapsto H_\theta^{3/2}(\Gamma^\pm)$$

is invertible.

Proof. We first show that S_θ is injective. Assume $f \in H_\theta^{1/2}(\Gamma^\pm)$ and $S_\theta f = 0$ on $\Gamma^\pm$ then

$$u \;=\; S_\theta f$$

is a $\theta-$Dirichlet eigenfunction of $-\Delta - q$ in ω. Since k^2 is not a $\theta-$Dirichlet eigenvalue of $-\Delta - q$ in ω, $u = 0$. Lemma 2.9 gives us $u = 0$ in ω'. Therefore

$$\frac{\partial u}{\partial \nu_+} \;=\; \frac{\partial u}{\partial \nu_-} \;=\; 0 \quad \text{on } \Gamma^\pm.$$

From

$$f \;=\; \frac{\partial u}{\partial \nu_-} - \frac{\partial u}{\partial \nu_+} \quad \text{on } \Gamma^\pm,$$

we conclude that $f = 0$. It remains to show that S_θ is surjective. We will exhibit the inverse of S_θ.

Integrating by parts yields

$$u(\mathbf{x}) \;=\; -\int_{\Gamma^\pm} \Big(G_\theta(\mathbf{x},\mathbf{y})\, \frac{\partial \overline{u}}{\partial \nu}(\mathbf{y}) - \frac{\partial G_\theta}{\partial \nu(\mathbf{y})}(\mathbf{x},\mathbf{y})\, \overline{u}(\mathbf{y}) \Big)\, d\gamma(\mathbf{y}) \quad \text{for a.e. } \mathbf{x} \in \omega.$$

By taking the trace on the boundary we obtain

$$S_\theta T_\theta^{\pm}(0) \;=\; -\frac{1}{2} I + B_\theta.$$

Let $\mathcal{T}_\theta^{\pm}(0)$ be defined as

$$\mathcal{T}_\theta^{\pm}(0) f \;=\; \frac{\partial u}{\partial \nu_+}.$$

By the same argument as above we find

$$S_\theta \mathcal{T}_\theta^{\pm}(0) \;=\; -\frac{1}{2} I + B_\theta.$$

Therefore

$$S_\theta \Big(T_\theta^{\pm}(0) - \mathcal{T}_\theta^{\pm}(0) \Big) \;=\; I,$$

and the proof is complete. $\square$

We will also need for the proof of Theorem 1.1 that the solutions of the exterior problem are precisely the restriction of the solutions of the integral equation (35) below.

LEMMA 2.13. *Suppose $h \in L^{2,loc}(\mathbb{R}^{n+1})$ solution to the $\theta-$Lippman-Schwinger equation :*

$$h(\mathbf{x}) = u^{in}(\mathbf{x}) - \int_\omega G_\theta(\mathbf{x}-\mathbf{y})\, q(\mathbf{y})\, h(\mathbf{y})\, d\mathbf{y}, \tag{35}$$

then the restriction of h to ω' satisfies (32). *Conversely, if h solves* (32)*, there exists a unique solution $\tilde{h} \in L^{2,loc}(\mathbb{R}^{n+1})$ of* (35) *such that $\tilde{h} = h$ in ω'.*

Proof. The proof of this Lemma is essentially the same as for smooth bounded domains and then we omit it. □

PROPOSITION 2.3. *$S_\theta T_\theta^{\pm}(q) - B_\theta - \frac{1}{2}I$ is a compact operator on $H_\theta^{3/2}(\Gamma^{\pm})$.*

Proof. Let $f \in H_\theta^{3/2}(\Gamma_\rho^{\pm})$ and let $u = \mathcal{P}_q f$ be the unique solution to the interior problem (28). By using G_θ^ε, the regularized version of G_θ, and applying Green's formula to G_θ^ε and u in ω we obtain

$$\int_\omega G_\theta(\mathbf{x}-\mathbf{y})\, \Delta\, u(\mathbf{y})\, d\mathbf{y} = \Big(S_\theta T_\theta^{\pm}(q) - D_\theta \Big) f(\mathbf{x}).$$

Now, letting $\mathbf{x}$ approach to $\Gamma^{\pm}$ (nontangentially from inside ω) gives, by the Proposition 2.1,

$$\Big(S_\theta T_\theta^{\pm}(q) - B_\theta - \frac{1}{2}I \Big) f(\mathbf{x}) = \gamma_0 \int_\omega G_\theta(\mathbf{x},\mathbf{y})\, q(\mathbf{y})\, \mathcal{P}_q f(\mathbf{y})\, d\mathbf{y},$$

with $\gamma_0 : H_\theta^2(\omega) \longmapsto H_\theta^{3/2}(\Gamma^{\pm})$ the trace operator on $\Gamma^{\pm}$ and

$$\mathcal{P}_q : f \in H_\theta^{3/2}(\Gamma^{\pm}) \longmapsto u \in H_\theta^2(\omega),$$

given by (28). We have then obtained the following factorization

$$S_\theta T_\theta^{\pm}(q) - B_\theta - \frac{1}{2}I = \gamma_0 G_\theta\, q\, \mathcal{I}\, \mathcal{P}_q,$$

where $\mathcal{I}$ is the compactness of the inclusion operator $\mathcal{I}$ from $H_\theta^2(\omega)$ in $L^2(\omega)$. Since $\mathcal{P}_q$ is a bounded operator from $H_\theta^{3/2}(\Gamma^{\pm})$ to $H_\theta^2(\omega)$, multiplication by q is a bounded operator and G_θ maps $L^2(\omega)$ to $H_\theta^2(\omega)$ (see Lemma 2.14), the result follows from the compactness of the inclusion operator from $H_\theta^2(\omega)$ in $L^2(\omega)$. □

LEMMA 2.14. *G_θ maps $L^2(\omega)$ to $H_\theta^2(\omega)$.*

Proof. Lemma above holds from a variation of the argument of Agmon in [4]. □

PROPOSITION 2.4. *Suppose that $k \notin \mathcal{N}$, then the integral equation* (31) *is uniquely solvable.*

Proof. We have from Proposition 2.2 that the integral equation (31) is uniquely solvable in $H_\theta^{3/2}(\Gamma^{\pm})$ whenever k is such that the operator $S_\theta T_\theta^{\pm}(q) - B_\theta + \frac{1}{2}I$ has a trivial null-space. Let $g \in H_\theta^{3/2}(\Gamma^{\pm})$ be in this null-space. g can be extended to h defined in ω' satisfying (32). h also extends to $\tilde{h}$ solution to the $\theta-$Lippman-Schwinger (35) in $\mathbb{R}^{n+1}$. Using the uniqueness result in Proposition 2.1, we obtain $\tilde{h} = 0$ and then $S_\theta T_\theta^{\pm}(q) - B_\theta + \frac{1}{2}I$ is invertible for any $k \notin \mathcal{N}$. Let

$$u(\mathbf{x}) = \Big(S_\theta T_\theta^{\pm}(q) - B_\theta + \frac{1}{2}I \Big)^{-1} (u^{in}).$$

□

Combining Proposition 2.2 and Proposition 2.1 we see that u extends to $\mathbb{R}^{n+1}$ is to be precesily the solution to (1). Hence we have proved Theorem 1.1.

To this end, we sketch a proof of Theorem 1.5. We begin by reducing the problem (6) to the analogue one for the operator $-\Delta + q$. The substitution $w = \sqrt{q_1}\, u$ yields

$$-\Delta w + q w \;=\; 0 \quad \text{in } \mathbb{R}^{n+1},$$

with

$$q \;=\; \frac{1}{\sqrt{q_1}} \Delta(\sqrt{q_1}) - q_2 q_1 - k^2 q_1$$

and then Theorem 1.5 holds from Theorem 1.1. We also have :

Theorem 2.2. *The operator*

$$u \longmapsto \Big(\operatorname{div} q_1 \,\nabla\, u + (k^2 + q_2)\, u,\; u|_{\Gamma^\pm}\Big)$$

is a Fredholm operator of index zero from $H^2_\theta(\omega)$ into $L^2(\omega) \times H^{3/2}_\theta(\Gamma^\pm)$.

3 Uniqueness results for the inverse problems

This section is devoted to establishing Theorems 1.2, 1.3, 1.5 and 1.6. The reader will note that many of the arguments of this section owe their conception to the results of Nachman given in [14] and Ramm in [17].

Lemma 3.1. *For any $\rho > \rho_0$, span$\Big\{\; u(u^{in}), \forall\, u^{in} \in \mathcal{V}_\theta \;\Big\}$ is $L^2(\omega_\rho)-$dense in the space of all $\theta-$ quasi-periodic solutions to the following equation :*

$$(\Delta + k^2 + q)\, u = 0 \quad \text{in } \mathbb{R}^{n+1}. \tag{36}$$

Proof. Let us prove that the closure of the linear span of the set $\Big\{\; u(u^{in}), \forall\, u^{in} \in \mathcal{V}_\theta \;\Big\}$ in $L^2(\omega_\rho)$ contains $N^\theta_{\omega_\rho}(l)$ defined as

$$N^\theta_{\omega_\rho}(l) \;= \Big\{\; v \in H^1_\theta(\omega_\rho) \text{ such that } l\, v = 0 \text{ in } \omega_\rho \;\Big\},$$

where

$$l \;=\; \Delta + k^2 + q.$$

We can assume that 0 is not an eigenvalue of the operator l in ω_ρ with Dirichlet conditions on $\Gamma^\pm_\rho$ (see the proof of Lemma 3.2 below). If this assumption holds then the set $\Big\{\; u(u^{in}), \forall\, u^{in} \in \mathcal{V}_\theta \;\Big\}$ is dense in $L^2(\omega_\rho)$ in $N^\theta_{\omega_\rho}(l)$ if and only if the set $\Big\{\; u(u^{in})|_{\Gamma^\pm_\rho}, \forall\, u^{in} \in \mathcal{V}_\theta \;\Big\}$ is dense in $H^{1/2}_\theta(\Gamma^\pm_\rho)$.
Suppose that there exists $\varphi \in H^{1/2}_\theta(\Gamma^\pm_\rho)$ such that

$$\int_{\Gamma^\pm_\rho} \varphi(\mathbf{y})\; u(u^{in})(\mathbf{y})\; d\gamma(\mathbf{y}) \;=\; 0, \quad \forall\, u^{in} \in \mathcal{V}_\theta. \tag{37}$$

Then

$$\int_{\Gamma^\pm_\rho} \varphi(\mathbf{y})\; e^{i\mu_J y_n + i\alpha_J \cdot \mathbf{y}'}\; d\gamma(\mathbf{y}) \;=\; 0, \quad \forall\, J \in \mathbb{Z}^n. \tag{38}$$

Let us prove that (38) above implies that $\varphi = 0$ on $\Gamma_\rho^\pm$.
Define v as

$$v(\mathbf{x}) = \int_{\Gamma_\rho^\pm} \varphi(\mathbf{y})\, G_\theta(\mathbf{x}-\mathbf{y})\, d\gamma(\mathbf{y}), \quad \forall \mathbf{x} \notin \Gamma_\rho^\pm.$$

It is clear that $l(v)(\mathbf{x}) = 0$ for any $\mathbf{x} \notin \Gamma_\rho^\pm$. Uniqueness of $\theta-$quasi-periodic solution implies that $v = 0$ in $\mathbb{R}^{n+1}$, thus $v = 0$ on $\Gamma_\rho^\pm$. From the jump conditions for the normal derivative $\varphi = 0$ on $\Gamma_\rho^\pm$ follows. □

LEMMA 3.2. *The set* $\Big\{ \rho \in \mathbb{R}, \rho > \rho_0$ *such that* 0 *is not an eigenvalue of the operator* l_i *in* ω_ρ *with Dirichlet conditions on* $\Gamma_\rho^\pm \Big\}$ *is an open dense set in* $]\,\rho_0, +\infty\,[$.

Proof. By the change of variable

$$x_n \in [\,0,\rho\,] \longmapsto \tilde{x}_n \in [\,0,\rho_0\,]\,,$$

one gets a family of elliptic operators l_i^ρ on the fixed domain $\mathbb{R}^n/\mathbf{d}\,.\,\mathbb{Z}^n \times]\,-\rho_0,\rho_0\,[$. Let N_ρ be the kernel of l_i^ρ. N_ρ is a finite dimensional space and the mapping

$$\rho \longmapsto dim N_\rho$$

is semi-continuous in ρ. For $\eta > 0$, one defines m as

$$\inf\Big\{ \dim N_\rho, \mid \rho - \rho_0 \mid \leq \eta \Big\}.$$

One may prove by contradiction that there exists $\eta > 0$ such that $m = 0$. Let one assume that $m > 0$. There exists then ρ_1 such that

$$\mid \rho_1 - \rho_0 \mid \quad \leq \quad \eta$$

and $\dim N_{\rho_1} = m$. As consequently, $\dim N_\rho = m$ for $\rho - \rho_1$ small enough. From the $\mathcal{C}^\infty$ regularity of l_i^ρ in a neighborhood of $x_n = \rho$ and by applying the classical theory of elliptic operators[10], one deduces that there exists a basis $(\tilde{e}_j(\mathbf{x},\rho))_{1\leq j\leq m}$ of N_ρ where each of these vectors is $\mathcal{C}^\infty$ in $x_n \leq \rho_0$. Now, one gets back to the initial coordinates. One has e_j satisfying :

$$\begin{cases} l_i^\rho(e_j) &= 0 \qquad \text{in } |x_n| < \rho, \\ e_j &= 0 \qquad \text{on } x_n = \pm\rho. \end{cases}$$

e_j is $\mathcal{C}^\infty$ in a neighborhood of $x_n = \pm\rho$.
Let

$$f_j = \frac{\partial e_j}{\partial \rho}.$$

Then, f_j satisfies that :

$$\begin{cases} l_i^\rho(f_j) &= 0 \qquad \text{in } |x_n| < \rho, \\ f_j + \dfrac{\partial e_j}{\partial x_n} &= 0 \qquad \text{on } x_n = \pm\rho. \end{cases}$$

Integration by part yields :

$$\int_{-\rho_1}^{\rho_1}\int_{\mathbf{x}'} l_i^\rho(f_j\,|_{\rho=\rho_1})\; e_j\,|_{\rho=\rho_1} = -\int_{\mathbf{x}'} f_j\,|_{x_n=\pm\rho_1,\rho=\rho_1}\;\frac{\partial}{\partial x_n} e_j\,|_{x_n=\pm\rho_1,\rho=\rho_1} = 0.$$

[10] For $\rho > \rho_0$, and in a neighborhood of $x_n = \rho$, the operator $l_i^\rho = \Delta + k^2$.

Thus, one obtains

$$\frac{\partial e_j}{\partial x_n} = 0 \quad \text{on } x_n = \pm\rho_1 \text{ and } \rho = \rho_1.$$

One concludes then that $e_j = 0$. This leads to a contradiction and ends the proof of the lemma. ☐

LEMMA 3.3. *For any $\rho > \rho_0$, span$\Big\{ u(u^{in}), \forall\, u^{in} \in \mathcal{V}_\theta \Big\}_{\forall\, 0\leq\theta\leq 2\pi}$ is $L^2(\omega_\rho)-$dense in the space of all solutions to the equation* (36) *above.*

Proof. This Lemma follows from Lemma 3.1 and the Bloch decomposition. ☐

REMARK 3.1. *Lemmas* 3.1 *holds for any bounded domain.*

Now, we give a relation between the Dirichlet to Neumann operator and the trace of the solution of (1) on $\Gamma_\rho^\pm$.

THEOREM 3.1. *$u_1 = u_2$ on $\Gamma_\rho^\pm$ for any $u^{in} \in \mathcal{V}_\theta$ if and only if $T_\theta^\pm(q_1) = T_\theta^\pm(q_2)$.*

Proof. Let $u_1 = u_2$ on $\Gamma_\rho^\pm$ for any $u^{in} \in \mathcal{V}_\theta$. $u = u_1 - u_2$ solves

$$\Big(\Delta + k^2 \Big) u = 0 \quad \text{in } \Pi^n \setminus \omega, \tag{39}$$

with

$$u = 0 \quad \text{on } \Gamma^\pm. \tag{40}$$

From the uniqueness of a solution satisfying (39) and (40) we deduce that $u = 0$ and then

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma^\pm.$$

It follows that

$$T_\theta^\pm(q_1)(u_1) = T_\theta^\pm(q_2)(u_2) \quad \text{for any } u^{in} \in \mathcal{V}_\theta. \tag{41}$$

By using Lemma 3.4, (41) yields

$$T_\theta^\pm(q_1)(\varphi) = T_\theta^\pm(q_2)(\varphi) \quad \text{for any } \varphi \in H_\theta^{1/2}(\Gamma^\pm).$$

Now, let u_i be a solution to the following Helmholtz equation :

$$\begin{cases} \Big(\Delta + k^2 + q_i(\mathbf{x}') \Big) u_i = 0 \quad \text{in } \mathbb{R}^{n+1}, \\ u_i - u^{in} \quad \text{satisfying the } \theta- \text{ quasi-periodic radiation condition.} \end{cases}$$

Since $T_\theta^\pm(q_1)u_1 = T_\theta^\pm(q_2)u_1$ we obtain that $u = u_1 - u_2$ solves

$$\Big(\Delta + k^2 \Big) u = 0 \quad \text{in } \Pi^n \setminus \omega, \tag{42}$$

with

$$\frac{\partial u}{\partial \nu} = T_\theta^\pm(q_1)(u) \quad \text{on } \Gamma^\pm. \tag{43}$$

From the uniqueness of a solution satisfying (42) and (43) we deduce that $u = 0$ and then the proof of Theorem 3.1 is complete. ☐

REMARK 3.2. *We remark that $T_{\rho,\theta}^\pm(q_1) = T_{\rho,\theta}^\pm(q_2)$ for any $\rho > \rho_0$.*

We state here a list of lemmas which are just straightforward consequences of Lemma 3.1.

LEMMA 3.4. *span$\Big\{ u(u^{in})|_{\Gamma_\rho^\pm}, \forall\, u^{in} \in \mathcal{V}_\theta \Big\}$ is dense in $H_\theta^{1/2}(\Gamma_\rho^\pm)$ for any $\rho > \rho_0$.*

LEMMA 3.5. *span* $\Big\{ u(u^{in})|_{\Gamma^\pm}, \forall\, u^{in} \in \mathcal{V}_\theta \Big\}$ *is dense in* $H^{1/2}_\theta(\Gamma^\pm)$.

LEMMA 3.6. *span* $\Big\{ u(u^{in})|_{\Gamma^\pm_\rho}, \forall\, u^{in} \in \mathcal{V}_\theta \Big\}_{\forall\, 0 \le \theta \le \pi}$ *is dense in* $H^{1/2}(\Gamma^\pm_\rho)$ *for any* $\rho > \rho_0$.

Proof. As Lemma 3.3, the Lemma 3.5 above follows from Lemma 3.4 and the Bloch decomposition and then we omit its proof. □

In order to construct a special solution to (36), we shall make use of $L^2(\Pi^n)$-based Sobolev spaces with weights at infinity. Let

$$L^2_\delta(\Pi^n) = \Big\{ f \in L^{2,loc}(\Pi^n), \int_{\Pi^n} (1+|x_n|^2)^\delta \, |f(\mathbf{x})| \, d\mathbf{x} < \infty \Big\}.$$

THEOREM 3.2. *For any* $J \in \mathbb{Z}^n$ *such that* $|J| \to +\infty$, *there exists a unique* u_J *in* $L^2_\delta(\Pi^n)$ *(for* $-1 < \delta < -1/2$ *) solving* (36) *such that*

$$u_J(\mathbf{x}) = e^{i\mu_J x_n} \, e^{i\alpha_J \,.\, \mathbf{x}'} \Big(1 + \psi_J(\mathbf{x}) \Big), \tag{44}$$

where

$$||\psi_J||_{L^2_\delta(\Pi^n)} \le \frac{C(\delta)}{|J|} \, ||q||_{L^\infty(\Omega)}. \tag{45}$$

Proof. The proof follows by using arguments given in section 2. ψ_J must satisfy

$$\Delta\, \psi_J + 2i\gamma_J \,.\, \nabla\, \psi_J + q(1+\psi_J) = 0,$$

where

$$\gamma_J = (\alpha_J, \mu_J).$$

Let G^J_θ be a fundamental solution to

$$\Delta\, G^J_\theta + 2i\gamma_J \,.\, \nabla\, G^J_\theta = \sum_{J' \in \mathbb{Z}^n} \delta_{J'} \, e^{i\theta \,.\, d_{J'}}.$$

ψ_J is then given by the following integral equation

$$\psi_J(\mathbf{x}) = \int_\omega G^J_\theta(\mathbf{x}-\mathbf{y}) \, q(y) \Big(1 + \psi_J(\mathbf{y}) \Big) \, d\mathbf{y}.$$

Arguments given in section 2 insure the unique solvability of the integral equation above. □

We recall that

$$l_i = \Delta + k^2 + q_i,$$

and

$$N^\theta_{\omega_\rho}(l_i) = \Big\{ v_i \in H^1_\theta(\omega_\rho) \text{ such that } l_i v_i = 0 \text{ in } \omega_\rho \Big\}.$$

THEOREM 3.3. *If* $u_1(u^{in}) = u_2(u^{in})$ *on* $\Gamma^\pm_\rho$ *for any* $u^{in} \in \mathcal{V}_\theta$ *then*

$$\int_{\omega_\rho} (q_1 - q_2)(\mathbf{y}) \, \overline{v}_1(\mathbf{y}) \, v_2(\mathbf{y}) \, d\mathbf{y} = 0, \quad \text{for all } v_i \in N^\theta_{\omega_\rho}(l_i). \tag{46}$$

Proof. Assume that $q_i \in L^2(\omega)$. Let u_i be a solution to the following Helmholtz equation :

$$\begin{cases} \Big(\Delta + k^2 + q_i(\mathbf{x}') \Big) u_i = 0 \quad \text{in } \mathbb{R}^{n+1}, \\ u_i - u^{in} \quad \text{satisfying the } \theta\text{-quasi-periodic radiation condition.} \end{cases}$$

Since $u_1(u^{in}) = u_2(u^{in})$ on $\Gamma^{\pm}_{\rho}$ for any $u^{in} \in \mathcal{V}_\theta$ we obtain that $u = u_1 - u_2$ satisfies

$$\Big(\Delta + k^2 + q_1 \Big) u \;=\; (q_1 - q_2) u_2 \quad \text{in } \mathbb{R}^{n+1}, \tag{47}$$

and

$$\frac{\partial u}{\partial \nu} \;=\; u \;=\; 0 \quad \text{on } \Gamma^{\pm}_{\rho}. \tag{48}$$

From (47) and (48), integration by part yields

$$\int_{\omega_\rho} (q_1 - q_2)(\mathbf{y})\, \overline{v}_1(\mathbf{y})\, u_2(u^{in})(\mathbf{y})\, d\mathbf{y} = 0, \quad \text{for any } v_1 \in N^{\theta}_{\omega_\rho}(l_1) \text{ and } u^{in} \in \mathcal{V}_\theta. \tag{49}$$

Since $\Big\{ u_2(u^{in}),\ \forall u^{in} \in \mathcal{V}_\theta \Big\}$ is dense in $N^{\theta}_{\omega_\rho}(l_2)$, (46) follows then from (49). □

Our uniqueness result for the inverse problem follows by combining Theorem 3.2 and Theorem 3.3. (46) yields

$$\int_{\omega_\rho} (q_1 - q_2)(\mathbf{y})\, \overline{u^1}_{J_1}(\mathbf{y})\, u^2_{J_2}(\mathbf{y})\, d\mathbf{y} = 0, \tag{50}$$

where $u^m_{J_m}$ for $m = 1$ or 2 is the unique solution to

$$(\Delta + k^2 + q_m)\, u^m_{J_m} = 0 \quad \text{in } \mathbb{R}^{n+1},$$

such that

$$u^m_{J_m} \;=\; e^{i\mu_{J_m} x_n}\, e^{i\alpha_{J_m} \cdot \mathbf{x}'} \Big(1 + \psi_{J_m}(\mathbf{x}) \Big),$$

with

$$||\psi_{J_m}||_{L^2_\delta(\Pi^n)} \quad \leq \quad \frac{C(\delta)}{|J_m|}\, ||q_m||_{L^\infty(\Omega)}.$$

Let $J \in \mathbb{Z}^n$. There exist J_1 and J_2 such that $J = J_1 - J_2$ and $|J_1|$ and $|J_2|$ are large enough. It is clear that (50) gives

$$q_1(\mathbf{x}') \;=\; q_2(\mathbf{x}'), \quad \forall \mathbf{x}' \in \mathbb{R}^n,$$

and the proof of Theorem 1.2 is over.

Now, we show that $q(\mathbf{x}')$ can be reconstructed by solving certain integral equations on $\Gamma^{\pm}$. Let us introduce

$$Q(J, J') \;=\; \frac{1}{|\omega|} \int_\omega e^{-i\mu_{J'}|x_n| - i\alpha_{J'} \cdot \mathbf{x}'}\, e^{-i\beta_J \cdot \mathbf{x}'}\, q(\mathbf{x}')\, u_{J'}(\mathbf{x})\, d\mathbf{x}, \tag{51}$$

where $u_{J'}(\mathbf{x})$ satisfies (44).

If we expand $q(\mathbf{x}')$ in Fourier series :

$$q(\mathbf{x}') \;=\; \sum_{J \in \mathbb{Z}^n} q_J\, e^{i\beta_J \cdot \mathbf{x}'}, \tag{52}$$

then q_J may be obtained from $Q(J, J')$. More precisely, we have the following Theorem.

THEOREM 3.4.

$$q_J \;=\; \lim_{|J'| \to +\infty} Q(J, J'), \quad \forall J \in \mathbb{Z}^n.$$

Proof. From

$$Q(J,J') \;=\; \frac{1}{|\omega|}\int_\omega e^{-i\mu_{J'}\,|x_n| - i\alpha_{J'}\,.\,\mathbf{x}'}\; e^{-i\beta_J\,.\,\mathbf{x}'}\; q(\mathbf{x}')\,\Big(\,1+\psi_{J'}(\mathbf{x})\,\Big)\, d\mathbf{x},$$

and by using (45), it is clear that

$$Q(J,J') \longrightarrow \frac{1}{|\omega|}\int_\omega e^{-i\beta_J\,.\,\mathbf{x}'}\; q(\mathbf{x}')\; d\mathbf{x}.$$

This ends the proof. □

One way to recover q is then to use the following formula :

THEOREM 3.5. *Let J' satisfy*

$$\beta_J^2 + 2\beta_J\,.\,\alpha_{J'} \;=\; 0, \tag{53}$$

$$q_J = \frac{1}{|\omega|}\lim_{|J'|\to+\infty}\int_{\Gamma^\pm} e^{-i\mu_{J'}\,|x_n| - i\alpha_{J'}\,.\,\mathbf{x}'} e^{-i\beta_J\,.\,\mathbf{x}'}\Big(T^\pm(q) + i\sigma_{J'}\,.\,\nu\Big)\, u_{J'}(\mathbf{x})\; d\gamma(\mathbf{x}), \tag{54}$$

where

$$\sigma_{J'} \;=\Big(\,\alpha_{J'} + \beta_J, \mu_{J'}\,\Big).$$

Proof. (54) follows by applying Green's formula to (51). □

Starting from $T^\pm(q)$, we now know that $Q(J,J')$ can be calculated for all J and $|J'|\geq M$ satisfying (53).

Now, we sketch a proof of Theorem 1.3. Let $u(\mathbf{x},\mathbf{y})$ be solution to

$$\Big(\,\Delta + k^2 + q(\mathbf{x}')\,\Big)\, u(\mathbf{x},\mathbf{y}) = -\delta_\theta(\mathbf{x}-\mathbf{y}) \quad \text{in } \mathbb{R}^{n+1}, \tag{55}$$

satisfying the $\theta-$Sommerfeld radiation condition. $u(\mathbf{x},\mathbf{y})$ satisfies

$$k^2\, u(\mathbf{x},\mathbf{y}) \;=\; k^2\, G_\theta(\mathbf{x}-\mathbf{y}) + \int_{\omega_\rho} G_\theta(\mathbf{x}-\mathbf{z})\, q(\mathbf{z}')\, u(\mathbf{z},\mathbf{y})\; d\mathbf{z}.$$

Multiplying $\varphi(\mathbf{x})\in H^{1/2}(\Gamma_\rho^\pm)$ and integrating over $\Gamma_\rho^\pm$ yield

$$\int_{\omega_\rho}\Big(\,\int_{\Gamma_\rho^\pm} G_\theta(\mathbf{x}-\mathbf{z})\,\varphi(\mathbf{x})\, d\gamma(\mathbf{x})\,\Big)\, q(\mathbf{z}')\, u(\mathbf{z},\mathbf{y})\; d\mathbf{z}$$

From Lemma 3.1, we deduce that

$$\int_{\omega_\rho} v_1(\mathbf{z})\, q(\mathbf{z}')\, u(\mathbf{z},\mathbf{y})\; d\mathbf{z},$$

is known for all $v_1(\mathbf{z})\in N^\theta_{\omega_\rho}(\Delta+k^2)$. Multiplying $\varphi(\mathbf{y})\in H^{1/2}(\Gamma_\rho^\pm)$ and integrating over $\Gamma_\rho^\pm$ give

$$\int_{\omega_\rho} v_1(\mathbf{z})\, q(\mathbf{z}')\,\Big(\,\int_{\Gamma_\rho^\pm}\varphi(\mathbf{y})\, u(\mathbf{z},\mathbf{y})\, d\gamma(\mathbf{y})\,\Big)\, d\mathbf{z},$$

is known for all $v_1(\mathbf{z}) \in N^\theta_{\omega_\rho}(\Delta + k^2)$ and $\varphi \in H^{1/2}(\Gamma^\pm_\rho)$.
Now, given the surface data $u(\mathbf{x}, \mathbf{y})$ given at all $\mathbf{x}, \mathbf{y} \in \Gamma^\pm_\rho$, it is clear that

$$\int_{\omega_\rho} v_1(\mathbf{z})\, q(\mathbf{z}')\, v_2(\mathbf{z})\; d\mathbf{z},$$

is known for all $v_1(\mathbf{z}) \in N^\theta_{\omega_\rho}(\Delta + k^2)$ and $v_2(\mathbf{z}) \in N^\theta_{\omega_\rho}(\Delta + k^2 + q)$ and then by using the arguments given in the proof of Theorem 1.3, it is easily seen that $q(\mathbf{z}')$ is uniquely determined.

Finally, we sketch a proof of Theorem 1.6 which follows from Theorem 1.4. The substitution $w = \sqrt{q_1}\, u$ yields

$$-\Delta w + qw \;=\; 0 \quad \text{in } \mathbb{R}^{n+1},$$

with

$$q \;=\; \frac{1}{\sqrt{q_1}} \Delta(\sqrt{q_1}) - q_2 q_1 - k^2 q_1$$

Therefore, $q_1(\mathbf{x}')$ and $q_2(\mathbf{x}')$ are uniquely determined by the surface data $u(\mathbf{x}, \mathbf{y})$ given at all $\mathbf{x}, \mathbf{y} \in \Gamma^\pm_\rho$ for two frequencies.

4 Extension remarks

Finally, we remark that all the results presented in this paper remain valid for Maxwell's equations in a periodic structure with boundaru of class $C^{1,1}$. Using the saddle point formulation of Maxwell's equations given in [2], we may may prove a uniqueness result similar to Proposition 2.1. Existence holds from integral equation where the operator $T^\pm_\theta$ is replaced by the pseudo-differential operator $\mathcal{R}^\pm_\theta \;: \mathbf{E} \wedge \mathbf{n} \mapsto \vec{\mathrm{curl}}\mathbf{E} \wedge \mathbf{n}$, where $\mathbf{E}$ is the electric field solution to Maxwell's equations in the periodic structure.

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