Finite Gratings at High Frequencies.
Part I : The Helmholtz Equation

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Abstract

We construct geometric-optics solution to a problem of scattering of the Helmholtz equation by a finite periodic dielectric grating placed on the interface between semi-infinite dielectric media. The first order approximation is the solution of the infinite periodic grating.

1 Introduction

There is a great interest in the mathematical study of radiation patterns created by gratings. The gratings are arrays of grooves having periodic structure. They are manufactured by micro-electronic techniques. They are used as anti-reflecting surfaces, as wave guide selectors, for beam splitting, and as beams mirrors in high lasers.
In this paper, the problem of scattering by a finite periodic dielectric grating placed on the interface between semi-infinite dielectric media is dressed.
This type of grating has been treated as an infinite, periodic structure with the resulting simplification that only a single cell of the periodic structure need to be considered (Alber [2], Wilcox [14], Nédélec and Starling [10], Chen and Friedman [5], Dobson and Friedman [7], Dobson [6], Bruno and Reitich [4], Bao [3], ...).
However, it is important to prove that this approximation is valid and to construct the correctors of the effect introduced by the truncation of the grating.
Recently, the finite grating problem has been addressed in the literature and major attention has been focused on numerical resolution of the diffraction problem by the finite gratings (Ko and Mittra [8], Pozar [12], Matsushima and Itakwa [9], Petropoulous and Kriegsmann [11], ...).
The justification of the approximation at high frequencies of the finite grating is considered in this paper. Our approach is based on geometric-optics techniques. A very close approach has been applied by Abboud, Ammari and Lebeau in [1] to study the high frequencies approximation of the diffraction at a curved grating by the infinite tangent plane grating outside transition regions corresponding to tangent Bragg's modes.
The outline of this paper is as follows. We first set the problem we are dealing with. Section 3 is devoted to derive a geometric-optics solution of the Helmholtz equation valid for a wave number k large enough. The results are summarized at the end of this paper.

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2 Notations and formulation of the diffraction problem

Let n be an integer ($n \geq 1$). Let $\delta, l_1, l_2, \ldots, l_n, d_1, d_2, \ldots$ and d_n be positive constants such that

$$\frac{l_1}{\delta d_1} \in IN, \frac{l_2}{\delta d_2} \in IN, \ldots \; and \; \frac{l_n}{\delta d_n} \in IN.$$

Define $L = (l_1, \ldots, l_n) \in IR^n$ and $d = (d_1, \ldots, d_n) \in IR^n$. Let f^δ be a real valued function satisfying

$$f^\delta(x') = \delta f(\frac{x'}{\delta}) H(l_1 - |x_1|) \ldots H(l_n - |x_n|),$$

where the Heaviside function $H(t) = 0$ or 1 for $t < 0$ or $t \geq 0$, respectively and f a periodic smooth function in x' of period d such that $f(d) = 0$.
Consider S^δ the surface imbedded in IR^{n+1} given as

$$x_{n+1} = f^\delta(x') \quad for \; all \; x = (x', x_{n+1}) \in IR^{n+1}.$$

$$\Gamma^\delta = \Big\{ x = (x', f^\delta(x')) \in S^\delta \; such \; that \; x' \in \;] -l_1, l_1 [\; \times \ldots \times \;] -l_n, l_n [\Big\}$$

is periodic with respect to the lattice Λ^δ

$$\Lambda^\delta = d_1 \delta Z\!\!Z / \,] -l_1, l_1 [\times \ldots \times d_n \delta Z\!\!Z / \,] -l_n, l_n [\times \{0\} \subset IR^{n+1} / \,] -l_1, l_1 [\times \ldots \times] -l_n, l_n [.$$

The surface S^δ divides IR^{n+1} into two open connected components. Let Ω_1^δ denote the component containing some element x such that $x_{n+1} > \sup |f^\delta(x')|$, and let $\Omega_2^\delta = (S^\delta \cup \Omega_1^\delta)^c$. Suppose that Ω_1^δ and Ω_2^δ are filled with materials in such a way that the magnetic permeability μ satisfies

$$\mu(x) = \begin{cases} \mu_1 & in \; \Omega_1^\delta, \\ \mu_2 & in \; \Omega_2^\delta, \end{cases}$$

and the dielectric coefficient $\varepsilon(x)$ satisfies

$$\varepsilon(x) = \begin{cases} \varepsilon_1 & in \; \Omega_1^\delta, \\ \varepsilon_2 & in \; \Omega_2^\delta, \end{cases}$$

where $\mu_1, \mu_2, \varepsilon_1$ and ε_2 are positive constants such that $\mu_1 \neq \mu_2$ and $\varepsilon_1 \neq \varepsilon_2$.
Setting

$$u_j^\delta = u^\delta|_{\Omega_j^\delta} \quad (j = 1, 2),$$

the Helmholtz equation in each Ω_j^δ is

$$div \frac{1}{\mu_j} \nabla u_j^\delta + k^2 \, \varepsilon_j \, u_j^\delta = 0 \quad in \; \Omega_j^\delta. \tag{1}$$

The weak form of the Helmholtz equation in a neighborhood of S^δ gives the following jump relations on S^δ :

$$\begin{cases} \dfrac{1}{\mu_1} \dfrac{\partial u_1^\delta}{\partial n} - \dfrac{1}{\mu_2} \dfrac{\partial u_2^\delta}{\partial n} & = 0, \\ u_1^\delta - u_2^\delta & = 0, \end{cases}$$

where n is normal to S^δ.
We are interested in the scattering problem for the Helmholtz equation (1) for a plane

incident wave coming down from $x_{n+1} = +\infty$. The plane incident wave u^{in} on S^δ from Ω_1^δ is in the form

$$u^{in}(x) = e^{ik\varphi^{in}(x)} \quad where\ \varphi^{in}(x) = q\,.\,x.$$

the constant $q = (q', q_{n+1}) \in I\!R^{n+1}$ must satisfy

$$|q\,.\,q|^2 = \mu_1\,\varepsilon_1,$$

in order for u^{in} to satisfy

$$div\frac{1}{\mu_1}\nabla u^{in} + k^2\,\varepsilon_1\,u^{in} = 0 \quad in\ \Omega_1^\delta.$$

Now, one introduces the infinite plane grating. Let Γ_J^δ denote

$$\Gamma_J^\delta = \Gamma^\delta + 2(j_1 l_1, \ldots, j_n l_n, 0) \quad for\ all\ J = (j_1, \ldots, j_n) \in Z\!\!\!Z^n,$$

and

$$G^\delta = \bigcup_{J \in Z\!\!\!Z^n} \Gamma_J^\delta = \Big\{x = (x', x_{n+1}) \in I\!R^{n+1}\ such\ that\ x_{n+1} = \delta f(\frac{x'}{\delta})\Big\}.$$

G^δ defines an infinite plane grating placed on the interface between semi-infinite dielectric media.

Many authors (Alber [2], Wilcox [14], Dobson [6], Bruno and Reitich [4], ...) have investigated the uniqueness and existence results for the problem of diffraction by a periodic plane in the TM and the TE polarizations. It is well known that except for some discrete set of values of k with accumulation point at infinity, there exists a unique solution to the diffraction problem of the Helmholtz equation by the infinite periodic grating G^δ. It will be assumed throughout what follows that $(H.\,1)$ and $(H.\,2)$ are satisfied.

$(H.\,1)$: $k \gg 1$ and $k\delta = \nu$, where ν is a fixed positive constant.

$(H.\,2)$: For the given values of the frequency k, the incident wave u^{in} and the period δd, the diffraction problem by the infinite plane grating G^δ admits a unique solution.

3 Geometric-optics asymptotic expansion for the Helmholtz equation

Let M be an integer $(M \geq 1)$. Let h be a positive constant such that

$$h > \sup_{y' \in I\!R^n} |f(y')|.$$

Let x_0 be a point on $]-l_1, l_1[\ \times \ldots \times\ ]-l_n, l_n[\ \times \{0\}$. One shall construct an asymptotic expansion for u_1^δ in $x_{n+1} \geq \delta h$, in the form :

$$u_1^\delta(x) = u^{in}(x) + u^e(x) = u^{in}(x) + \sum_{J \in Z\!\!\!Z^n} e^{ik\varphi_J^1(x)}\,a_J^1(x, \delta) \tag{2}$$

where each phase $\varphi^1_J(x)$ is to satisfy the eikonal equation :

$$(3) \qquad \left\{ \begin{array}{l} \mid \nabla \varphi^1_J(x) \mid^2 = \mu_1 \varepsilon_1, \\ \varphi^1_J(x', x_, = \delta h) \; = \; \varphi^{in}(x', x_{n+1} = \delta h) + \dfrac{J\,.\,x'}{\nu}, \\ \dfrac{\partial \varphi^1_J}{\partial x_{n+1}}(x', x_{n+1} = \delta h) \; > \; 0 \quad if \quad \sqrt{\varepsilon_1\,\mu_1} > \mid \dfrac{J}{\nu} + \dfrac{\partial \varphi^{in}}{\partial x'}(x_0) \mid, \\ \Im m \; \dfrac{\partial \varphi^1_J}{\partial x_{n+1}}(x', x_{n+1} = \delta h) \; > \; 0 \quad else, \end{array} \right.$$

and $a^1_J(x, \delta)$ taking the form :

$$a^1_J(x, \delta) \; = \; \sum_{0 \leq m \leq M} a^{m,1}_J(x) \; \delta^m,$$

are to satisfy transport equations :

$$\left\{ \begin{array}{l} 2\nabla \, \varphi^1_J \, . \, \nabla \, a^{0,1}_J + \Delta \, \varphi^1_J \, a^{0,1}_J \; = \; 0, \\ 2\nabla \, \varphi^1_J \, . \, \nabla \, a^{1,1}_J + \Delta \, \varphi^1_J \, a^{1,1}_J \; = \; -\Delta \, a^{0,1}_J, \\ \vdots \end{array} \right.$$

$a^{m,1}_J$ are determined from their traces on $x_{n+1} = \delta h$. To resolve the eikonal equations (3), one must assume the following hypothesis :

$$(H.\,3) \; : \qquad \mid \frac{J}{\nu} + \frac{\partial \varphi^{in}}{\partial x_{n+1}}(x_0) \mid \neq \sqrt{\varepsilon_1\,\mu_1}, \; \forall \, J \in \mathbb{Z}^n.$$

The solution φ^1_J to the eikonal equation (3) is holomorphic in a complex neighborhood of x_0 independent on δ and J. For J large enough, it is easily seen that φ^1_J satisfies the following estimate :

$$\Im m \, \varphi^1_J \; \geq \; C \, |J| \, (x_{n+1} - \delta h).$$

The convergence of the series (2) above is obtained from the following lemma :

LEMMA 3.1. *One has the following estimate :*

$$(4) \qquad \sup_{0 \leq m \leq M,\, x \sim x_0} \mid a^{m,1}_J(x) \mid \leq C_1 e^{-\eta_1 \, |J|},$$

where C_1 and η_1 are two strictly positive constants.
The proof of this lemma will be given at the end of this paper.

Under the hypothesis $(H.\,3)$, the solution u^δ admits the following asymptotic expansion in $x_{n+1} \leq -\delta h$:

$$(5) \qquad u^\delta_2(x) \; = \; \sum_{J \in \mathbb{Z}^n} e^{ik\varphi^2_J(x)} \; a^2_J(x, \delta)$$

where each phase $\varphi_J^2(x)$ is to satisfy the eikonal equation :

$$(6)\qquad \begin{cases} \mid \nabla\varphi_J^2(x) \mid^2 = \mu_2\varepsilon_2, \\ \varphi_J^2(x', x_{\prime} = -\delta h) \;=\; \varphi^{in}(x', x_{n+1} = -\delta h) + \dfrac{J\,.\,x'}{\nu}, \\ \dfrac{\partial\varphi_J^2}{\partial x_{n+1}}(x', x_{n+1} = -\delta h) \;<\; 0 \quad if \quad \sqrt{\varepsilon_2\,\mu_2} > \mid \dfrac{J}{\nu} + \dfrac{\partial\varphi^{in}}{\partial x'}(x_0) \mid, \\ \Im m\, \dfrac{\partial\varphi_J^2}{\partial x_{n+1}}(x', x_{n+1} = -\delta h) \;<\; 0 \quad else, \end{cases}$$

and $a_J^2(x, \delta)$ defined as :

$$a_J^2(x, \delta) \;=\; \sum_{0\le m\le M} a_J^{m,2}(x)\;\delta^m,$$

are to satisfy transport equations :

$$\begin{cases} 2\nabla\,\varphi_J^2\,.\,\nabla\,a_J^{0,2} + \Delta\,\varphi_J^2\,a_J^{0,2} \;=\; 0, \\ 2\nabla\,\varphi_J^2\,.\,\nabla\,a_J^{1,2} + \Delta\,\varphi_J^2\,a_J^{1,2} \;=\; -\Delta\,a_J^{0,2}, \\ \vdots \end{cases}$$

$a_J^{m,2}$ are determined from their traces on $x_{n+1} = -\delta h$. For resolving the eikonal equations (6), one needs the following hypothesis :

$$(H.\,4)\ :\qquad \mid \frac{J}{\nu} + \frac{\partial\varphi^{in}}{\partial x_{n+1}}(x_0) \mid \neq \sqrt{\varepsilon_2\,\mu_2},\ \forall\ J \in \mathbb{Z}^n.$$

The convergence of the series (5) above is obtained from the following estimate :

$$(7)\qquad \sup_{0\le m\le M,\, x\sim x_0} \mid a_J^{m,2}(x) \mid \le C_2 e^{-\eta_2\,|J|},$$

where C_2 and η_2 are two strictly positive constants.

The operators of trace on $x_{n+1} = \delta h$ are given by :

$$u^e(x', \delta h) \;=\; e^{ik\varphi^{in}(x',\delta h)} \sum_{J\in\mathbb{Z}^n} e^{iJ\,\cdot\,y'}\; b_J^1(x', \delta),$$

where

$$b_J^1(x', \delta) \;=\; \sum_{0\le m\le M} a_J^{m,1}(x', \delta h)\;\delta^m,$$

and, by differentiating (2) with respect to x_{n+1}, one sees that :

$$\frac{\partial u^e}{\partial x_{n+1}}(x', \delta h) \quad=\quad e^{ik\varphi^{in}(x',\delta h)} \sum_{J\in\mathbb{Z}^n} e^{iJ\,\cdot\,y'}\;\Big(ik\psi_J^1(x', \delta h) b_J^1(x', \delta) + \frac{\partial a_J^1}{\partial x_{n+1}}(x', \delta h; \delta)\Big),$$

where

$$\psi_J^1(x', x_{n+1}) \;=\; \frac{\partial\varphi_J^1}{\partial x_{n+1}}(x', x_{n+1}).$$

The operators of trace on $x_{n+1} = -\delta h$ are defined as :

$$u_2^\delta(x', -\delta h) \;=\; e^{ik\varphi^{in}(x',-\delta h)} \sum_{J\in\mathbb{Z}^n} e^{iJ\,\cdot\,y'}\; b_J^2(x', \delta),$$

where

$$b^2_J(x',\delta) \;=\; \sum_{0\le m\le M} a^{m,2}_J(x',-\delta h)\;\delta^m.$$

By differentiating (5) with respect to x_{n+1}, one obtains :

$$\frac{\partial u^\delta_2}{\partial x_{n+1}}(x',-\delta h) \;=\; e^{ik\varphi^{in}(x',-\delta h)} \sum_{J\in \mathbb{Z}^n} e^{iJ\,\cdot\, y'}\,\Big(ik\psi^2_J(x',-\delta h)b^2_J(x',\delta) + \frac{\partial a^2_J}{\partial x_{n+1}}(x',-\delta h;\delta)\Big),$$

where

$$\psi^2_J(x',x_{n+1}) \;=\; \frac{\partial \varphi^2_J}{\partial x_{n+1}}(x',x_{n+1}).$$

In $-\delta h \le x_{n+1} \le \delta h$, one seeks for u^δ in the form :

$$(8)\qquad \begin{aligned} u^\delta(x) \;&=\; e^{ik\varphi^{in}(x',0)}\; a^\delta(x',x'/\delta,x_{n+1}/\delta)\\ &=\; e^{ik\varphi^{in}(x',0)} \sum_{0\le m\le M} a_m(x',x'/\delta,x_{n+1}/\delta)\;\delta^m. \end{aligned}$$

In order to determine a_m one first verifies that

$$\delta^2\,\Big(div_x\frac{1}{\mu}\nabla_x + k^2\,\varepsilon\Big)\,u^\delta \;=\; P(\sum a_j\;\delta^j),$$

where

$$P \;=\; P_0 + \delta P_1 + \delta^2 P_2,$$

with

$$P_0(x',\nu,y,\partial y) \;=\; (\partial_{y'} + i\nu\frac{\partial\varphi^{in}}{\partial x'}(x',0))(\frac{1}{\mu})(\partial_{y'} + i\nu\frac{\partial\varphi^{in}}{\partial x'}(x',0)) + \partial_{y_{n+1}}\frac{1}{\mu}\partial_{y_{n+1}} + \nu^2\,\varepsilon,$$

$$P_1(x',\nu,y,\partial y) \;=\; \partial_{x'}(\frac{1}{\mu})\Big(\,\partial_{y'} + i\nu\frac{\partial\varphi^{in}}{\partial x'}\,\Big) + \Big(\,\partial_{y'} + i\nu\frac{\partial\varphi^{in}}{\partial x'}\,\Big)\,(\frac{1}{\mu})\,\partial_{x'},$$

and

$$P_2(x',\nu,y,\partial y) \;=\; \partial_{x'}(\frac{1}{\mu})\partial_{x'}.$$

The operators $P_j = P_j(x',\nu,y,\partial_y)$ for $j = 0,1,2$ are $\mathcal{C}^\infty$ in x' and they are at most of second order. Equating coefficients of equal powers of δ yields the following sequence of problems :

$$(9)\qquad \begin{cases} P_0 a_0 = 0 & \text{in } I\!R^n/d\,.\,\mathbb{Z}^n \times\;]-h,h[\,,\\ a_0 = v^1_0(x',y') & \text{on } y_{n+1} = h,\\ a_0 = v^2_0(x',y') & \text{on } y_{n+1} = -h, \end{cases}$$

$$(10)\qquad \begin{cases} P_0 a_1 = -P_1 a_0 & \text{in } I\!R^n/d\,.\,\mathbb{Z}^n \times\;]-h,h[\,,\\ a_1 = v^1_1(x',y') & \text{on } y_{n+1} = h,\\ a_1 = v^2_1(x',y') & \text{on } y_{n+1} = -h, \end{cases}$$

$$(11)\qquad \begin{cases} P_0 a_m = -P_2 a_{m-2} - P_1 a_{m-1} & \text{in } I\!R^n/d\,.\,\mathbb{Z}^n \times\;]-h,h[\,,\\ a_m = v^1_m(x',y') & \text{on } y_{n+1} = h,\\ a_m = v^2_m(x',y') & \text{on } y_{n+1} = -h, \end{cases}$$

$\vdots$

$$(12)\qquad \begin{cases} P_0 a_M = -P_2 a_{M-2} - P_1 a_{M-1} & \text{in } I\!R^n/d\,.\,Z\!\!Z^n \times \;]-h,h\,[\,, \\ a_M = v^1_M(x',y') & \text{on } y_{n+1}=h, \\ a_M = v^2_M(x',y') & \text{on } y_{n+1}=-h, \end{cases}$$

where

$$v^1_m(x',y') \;=\; a_m(x',x'/\delta,h)$$
$$v^2_m(x',y') \;=\; a_m(x',x'/\delta,-h)$$

are the traces of the functions a_m on $y_{n+1}=h$ and $y_{n+1}=-h$. One remarks that x' appears as a parameter in these problems and that $(P_j)_{j=1,2}$ are independent on $\partial_{y_{n+1}}$. Therefore we have the following important regularity lemma.

LEMMA 3.2. *For $0\le m\le M, a_m \in \mathcal{C}^\infty(x', H^1_{per}(y))$ where $H^1_{per}(y) = H^1(I\!R^n/d\,.\,Z\!\!Z^n \times \;]-h,h\,[\,)$.*

Proof. From the explicit formulae of the operators $(P_j)_{j\ge 0}$, it is clear that P_j maps $\mathcal{C}^\infty(x', H^1_{per}(y))$ onto $\mathcal{C}^\infty(x', H^{-1}_{per}(y))$. It remains to verify that the solution of the variationnal problem $P_0\phi=\psi$, where $\psi \in \mathcal{C}^\infty(x', H^{-1}_{per}(y))$ is in $\mathcal{C}^\infty(x', H^1_{per}(y))$. We have, for $\xi \in H^1_{per}(y)$ with null trace on $y_{n+1}=-h$ and $y_{n+1}=h$,

$$(13)\quad -\int_{O_h} \nu^2 \varepsilon(y_{n+1})\,\phi(y)\,\overline{\xi}(y)\,dy + \int_{O_h} 1/\mu(y_{n+1})\;\nabla\,\phi(y)\,\nabla\,\overline{\xi}(y)\,dy \;=\; \int_{O_h} \psi(y)\,\overline{\xi}(y)\,dy,$$

where

$$O_h \;=\; I\!R^n/d\,.\,Z\!\!Z^n \times \;]-h,h\,[\,.$$

Since $\varepsilon(y_{n+1})$ and $\mu(y_{n+1})$ are in $\mathcal{C}^\infty(x', L^\infty(y))$, then by differentiating (13) with respect to x', Lemma 3.2 above holds and the proof is over. $\square$

One must choose h such that 0 is not an eigenvalue of the operator P_0 with Dirichlet boundary conditions on $y_{n+1}=-h$ and $y_{n+1}=h$.

LEMMA 3.3. *The set H defined as $\Big\{h\in I\!R, h>1$ such that 0 is not an eigenvalue of the operator P_0 with Dirichlet conditions on $y_{n+1}=-h$ and $y_{n+1}=h\Big\}$ is an open dense set in $\{h>1\}$.*

Proof. By the change of variable

$$y_{n+1} \in [\,-h,h\,] \longmapsto \tilde{y}_{n+1} \in [\,-h_0,h_0\,]\,,$$

one gets a family of elliptic operators P^h on the fixed domain $I\!R^n/d\,.\,Z\!\!Z^n \times \,]-h_0,h_0\,[\,$. Let N_h be the kernel of P^h. N_h is a finite dimensional space and the mapping

$$h \mapsto dim N_h$$

is semi-continuous in h. For $\eta>0$, one defines q as

$$inf\Big\{\; dim\; N_h, \mid h-h_0 \mid \le \eta \Big\}.$$

One may prove by contradiction that there exists $\eta>0$ such that $q=0$. Let one assume that $q>0$. There exists then h_1 such that $\mid h_1-h_0\mid \le \eta$ and dim $N_{h_1}=d$. As consequently, dim $N_h = q$ for $h - h_1$ small enough. From the $\mathcal{C}^\infty$ regularity of P^h in a neighborhood of $x_{n+1} = \pm h$ and by applying the classical theory of elliptic operators, one deduces that there exists a basis $(\tilde{e}_j(y,h))_{1\leq j\leq q}$ of N_h where each of these vectors is $\mathcal{C}^\infty$ in a neighborhood $\tilde{y}_{n+1} = \pm h_0$. Now, one gets back to the initial coordinates. One has e_j satisfying :

$$\begin{cases} P_0(e_j) &= 0 \qquad \textit{in } -h < y_{n+1} < h, \\ e_j &= 0 \qquad \textit{on } y_{n+1} = h, \\ e_j &= 0 \qquad \textit{on } y_{n+1} = -h. \end{cases}$$

e_j is $\mathcal{C}^\infty$ in a neighborhood of $y_{n+1} = \pm h$. Let

$$f_j = \frac{\partial e_j}{\partial h}.$$

Then, f_j satisfies that :

$$\begin{cases} P_0(f_j) &= 0 \qquad \textit{in } 0 < y_{n+1} < h, \\ f_j + \dfrac{\partial e_j}{\partial y_{n+1}} &= 0 \qquad \textit{on } y_{n+1} = h, \\ f_j + \dfrac{\partial e_j}{\partial y_{n+1}} &= 0 \qquad \textit{on } y_{n+1} = -h. \end{cases}$$

From $e_j = 0$ *on* $y_{n+1} = \pm h$, integration by part yields :

$$\int_{-h_1}^{h_1}\int_{y'} P_0(f_j\,|_{h=h_1})\; e_j\,|_{h=h_1} = -\int_{y'} f_j\,|_{y_{n+1}=\pm h_1, h=h_1} \frac{\partial}{\partial y_{n+1}} e_j\,|_{y_{n+1}=\pm h_1, h=h_1} = 0.$$

Thus, one obtains

$$\frac{\partial e_j}{\partial y_{n+1}} = 0 \quad \textit{on } y_{n+1} = \pm h_1 \textit{ and } h = h_1.$$

One conclucles then that $e_j = 0$. This leads to a contradiction and ends the proof of the lemma. $\square$

Therefore, one can assume that h is such that each of problems (9), ..., (12) has a unique periodic solution.

Let $\mathcal{N}^-$ be the analytic pseudo-differential elliptic operator of order 1 defined as :

$$\mathcal{N}^- : \quad v \in H^{1/2}(I\!R^n/d\,.\,\mathbb{Z}^n) \times H^{1/2}(I\!R^n/d\,.\,\mathbb{Z}^n)$$

$$\longmapsto \mathcal{N}^- v = \Big(\frac{\partial u}{\partial y_{n+1}}\,|_{y_{n+1}=h}, \frac{\partial u}{\partial y_{n+1}}\,|_{y_{n+1}=-h}\Big) \in H^{-1/2}(I\!R^n/d\,.\,\mathbb{Z}^n) \times H^{-1/2}(I\!R^n/d\,.\,\mathbb{Z}^n),$$

where u is defined as the unique solution of the following problem :

$$\begin{cases} P_0\, u = 0 \qquad \text{in } I\!R^n/d\,.\,\mathbb{Z}^n \times\,]-h,h[\,, \\ u = v^1(x',y') \qquad \text{on } y_{n+1} = h, \\ u = v^2(x',y') \qquad \text{on } y_{n+1} = -h. \end{cases}$$

Now, one links up the two asymptotic expansions (2) and (8) on $x_{n+1} = \delta h$ and (5) and (8) on $x_{n+1} = -\delta h$.

Direct computation yields :

$$\sum_{J\in\mathbb{Z}^n} e^{iJ\,.\,y'}\, b_J^1(x',\delta)+1 \;=\; e^{i\nu h q_{n+1}}\Big\{v_0^1(x',y') \;+\; \delta\, v_1^1(x',y') \;+\; \ldots\Big\}, \tag{14}$$

$$\sum_{J\in\mathbb{Z}^n} e^{iJ\,.\,y'}\Big\{i\nu\psi_J^1 b_J^1+\delta\frac{\partial a_J^1}{\partial x_{n+1}}\Big\}+i\nu\frac{\partial\varphi^{in}}{\partial x_{n+1}}$$

$$= e^{i\nu h q_{n+1}}\Big\{\frac{\partial a_0}{\partial y_{n+1}}|_{y_{n+1}=h} \;+\; \delta\frac{\partial a_1}{\partial y_{n+1}}|_{y_{n+1}=h} \;+\; \ldots\Big\}, \tag{15}$$

$$\sum_{J\in\mathbb{Z}^n} e^{iJy'}\, b_J^2(x',\delta) \;=\; e^{-i\nu h q_{n+1}}\Big\{v_0^2(x',y') \;+\; \delta\, v_1^2(x',y') \;+\; \ldots\Big\}, \tag{16}$$

and

$$\sum_{J\in\mathbb{Z}^n} e^{iJ\,.\,y'}\Big\{i\nu\psi_J^2 b_J^2+\delta\frac{\partial a_J^2}{\partial x_{n+1}}\Big\} \;=\; e^{-i\nu h q_{n+1}}\Big\{\frac{\partial a_0}{\partial y_{n+1}}|_{y_{n+1}=-h} \;+\; \ldots\Big\} \tag{17}$$

One introduces $\mathcal{N}^+$, the analytic pseudo-differential elliptic operator of order 1 defined as

$$c(x',y',\delta)=\sum_{J\in\mathbb{Z}^n} e^{iJ\,.\,y'}\Big(\, b_J^1(x',\delta), b_J^2(x',\delta)\,\Big)$$

$$=\sum_{0\le m\le M} c_m\,\delta^m \longmapsto \mathcal{N}^+(c)=i\,\nu\sum_{J\in\mathbb{Z}^n} e^{iJ\,.\,y'}\,\Big(\psi_J^1\, b_J^1(x,\delta),\psi_J^2\, b_J^2(x,\delta)\Big),$$

where

$$c_m \;=\; c_m(x',y',\delta) \;=\; \sum_{J\in\mathbb{Z}^n} e^{iJ\,.\,y'}\;\Big(a_J^{1,m}(x',\delta h), a_J^{2,m}(x',-\delta h)\Big).$$

Let v_m denote $\Big(v_m^1(x',y'),v_m^1(x',y')\Big)$.
In equations (14) – (17), the unknowns are $(c_m)_{0\le m\le M}$ and $(v_m)_{0\le m\le M}$. These equations lead to a triangular linear system of equations which has a unique solution since the analytic pseudo-differential operator $\mathcal{N}^+-\mathcal{N}^-$ is invertible.
One verifies that c_0 is solution of the following equation :

$$(\mathcal{N}^+-\mathcal{N}^-)(c_0) \;=\; \mathcal{N}^-\Big(1,0\Big)-\Big(i\nu q',0\Big), \tag{18}$$

and v_0 is given by

$$v_0 \;=\; c_0+\Big(1,0\Big) \tag{19}$$

It is easily seen that the two equations (18) and (19) give the solution of the diffraction problem by the infinite grating G^δ. Having principal real valued symbol, the operator $\mathcal{N}^+-\mathcal{N}^-$ is a Fredholm operator with index 0 (see [13]) and it is invertible if the diffraction problem by the infinite plane grating G^δ has a unique solution. That is where the hypothesis $(H.\,2)$ comes into play. Under this hypothesis, the system obtained from (14) – (17) is solved by recurrence and the c_m are analytic in y' because the right hand member is analytic on y'. As a consequence, one obtains the estimates (4) and (7) and the geometric-optics asymptotic expansions (2), (5) and (8) hold. The proof of lemma 3.1 is then complete.
The main results of this paper are summarized in the following theorem :

THEOREM 3.1. *One assumes that the hypothesis* $(H.\,1)-(H.\,4)$ *are satisfied. Then, for each point* $x\in\;]-l_1,l_1[\;\times\ldots\times\;]-l_n,l_n[\;\times\{0\}$*, there exists a geometric-optics asymptotic* *expansion given by* (2), (5) *and* (8) *satisfying the Helmholtz equation to arbitrarily order in* δ. *The first order in this asymptotic expansion is the unique solution of the infinite plane grating* G^δ.

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