Finite Gratings at High Frequencies. Part II : The Time Harmonic Maxwell Equations

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Abstract

We construct geometric-optics solution to a problem of scattering of the Maxwell equations by a finite periodic dielectric grating placed on the interface between semi-infinite dielectric media. The first order approximation is the solution of the infinite periodic grating.

1 Introduction and main results

Let (r_1, r_2, r_3) be an orthonormal basis of IR^3 related to a Cartesian system coordinate (x_1, x_2, x_3). Let δ, l_1, l_2, d_1 and d_2 be positive constants such that

$$\frac{l_1}{\delta d_1} \in IN \; and \; \frac{l_2}{\delta d_2} \in IN.$$

Define $L = (l_1, l_2) \in IR^2$ and $d = (d_1, d_2) \in IR^2$. Consider a surface S^δ :

$$x_3 \; = \; \delta f(\frac{x_1}{\delta}, \frac{x_2}{\delta}) H(l_1 - |x_1|) H(l_2 - |x_2|),$$

where f is doubly periodic, *i.e.*

$$f(y_1 + m d_1, y_2 + m d_2) \; = \; f(y_1, y_2)$$

for all integers m, n and $y = (y_1, y_2) \in IR^2$. The Heaviside function $H(t) = 0$ or 1 for $t < 0$ or $t \geq 0$, respectively.

Above S^δ the dielectric coefficient is a constant ε_1 and the magnetic permeability is a constant μ_1 and below it the dielectric coefficient is assumed to be another constant $\varepsilon_2, \varepsilon_1 \neq \varepsilon_2$ and the magnetic permeability is assumed to be another constant $\mu_2, \mu_1 \neq \mu_2$.

Let

$$G^\delta \; = \Big\{ x = (x_1, x_2, x_3) \in IR^3 \; such \; that \; x_3 = \delta f(\frac{x_1}{\delta}, \frac{x_2}{\delta}) \Big\}.$$

G^δ defines an infinite plane grating placed on the interface between semi-infinite dielectric media.

We are interested in constructing an asymptotic solution to the scattering problem of Maxwell's equations for a linear electromagnetic plane wave E^{in} coming down from $x_3 = +\infty$:

$$E^{in} \; = \; p \, e^{ikq \cdot x}.$$

Here $p, q \in IR^3$ are constants representing the polarization and incidence, respectively. We assume that the wave number $k \gg 1$ and $k\delta = \nu$ where ν is a positive constant.

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Our main result is the derivation of geometric-optics solution satisfying the diffraction problem by the finite grating S^δ to arbitrarily order in δ. The first order approximation of the scattered field by the finite grating S^δ is the unique solution of the scattering problem by the infinite plane grating G^δ.
The approach of this paper, like the one in [2], is based on geometric-optics techniques. However it differs from [2] in a number of important technical details.

2 The time harmonic Maxwell equations

The surface S^δ divides IR^3 into two open connected components. Let Ω_1^δ denote the component containing some element x such that $x_3 > \sup|f^\delta(x_1, x_2)|$, and let $\Omega_2^\delta = (S^\delta \cup \Omega_1^\delta)^c$.
Suppose that Ω_1^δ and Ω_2^δ are filled with materials in such a way that the magnetic permeability μ satisfies

$$\mu(x) = \begin{cases} \mu_1 & in\ \Omega_1^\delta, \\ \mu_2 & in\ \Omega_2^\delta, \end{cases}$$

and the dielectric coefficient $\varepsilon(x)$ satisfies

$$\varepsilon(x) = \begin{cases} \varepsilon_1 & in\ \Omega_1^\delta, \\ \varepsilon_2 & in\ \Omega_2^\delta, \end{cases}$$

where $\mu_1, \mu_2, \varepsilon_1$ and ε_2 are positive constants such that $\mu_1 \neq \mu_2$ and $\varepsilon_1 \neq \varepsilon_2$.
Setting

$$E_j^\delta = E^\delta|_{\Omega_j^\delta} \quad (j = 1, 2),$$

the Maxwell equations in each Ω_j^δ are

$$\vec{curl}\, \frac{1}{\mu_j}\, \vec{curl}\, E_j^\delta - k^2\, \varepsilon_j\, E_j^\delta = 0 \quad in\ \Omega_j^\delta. \tag{1}$$

The weak form of the Maxwell equations in a neighborhood of S^δ gives the following jump relations on S^δ :

$$\begin{cases} \left(\varepsilon_1\, E_1^\delta - \varepsilon_2\, E_2^\delta\right).n = 0, \\ \left(\frac{1}{\mu_1}\vec{curl}\, E_1^\delta - \frac{1}{\mu_2}\vec{curl}\, E_2^\delta\right) \wedge n = 0, \end{cases}$$

where n is normal to S^δ.
E^{in} is an incident electromagnetic plane wave :

$$E^{in}(x) = p e^{i\, k\, q\, .\, x}.$$

The constants $p, q \in IR^3$ must satisfy

$$q.q = \mu_1 \varepsilon_1 \ and\ q.p = 0,$$

in order for E^{in} to satisfy the Maxwell equations in Ω_1^δ.
Many authors (Nédélec and Starling [5], Chen and Friedman [3], Dobson and Friedman [4], Abboud [1], ...) have investigated the uniqueness and existence results for the problem of diffraction of Maxwell's equations by a periodic plane grating. It is well known that except for some discrete set of values of k with accumulation point at infinity, there exists a unique

solution to the diffraction problem of the Maxwell equations by the infinite periodic grating G^δ. It will be assumed throughout what follows :

$(H.\,1)$: $k \gg 1$ and $k\delta = \nu$, where ν is a fixed positive constant.

$(H.\,2)$: For the given values of the frequency k, the incident field E^{in} and the period δd, the diffraction problem by the infinite plane grating G^δ admits a unique solution.

3 Geometric-optics asymptotic expansion for the Maxwell equations

Let h be a positive constant such that

$$h > \sup_{y' \in I\!R^3} |f(y')|.$$

Let x_0 be a point on $]-l_1, l_1[\times]-l_2, l_2[\times \{0\}$. One shall construct an asymptotic expansion for the solution E_1^δ of Maxwell's equations, in $x_3 \geq \delta h$, in the form :

$$E_1^\delta(x) = E^{in}(x) + E^e(x) = E^{in}(x) + \sum_{J \in \mathbb{Z}^2} e^{ik\varphi_J^1(x)}\, e_J^1(x,\delta), \tag{2}$$

where each phase $\varphi_J^1(x)$ is to satisfy the eikonal equation :

$$\begin{cases} |\,\nabla\varphi_J^1(x)\,| = \varepsilon_1\,\mu_1, \\ \varphi_J^1(x', x_3 = \delta h) = \varphi^{in}(x', x_3 = \delta h) + \dfrac{J\,.\,x'}{\nu}, \\ \dfrac{\partial\varphi_J^1}{\partial x_3}(x', x_3 = \delta h) > 0 \quad if \quad \sqrt{\varepsilon_1\,\mu_1} > |\,\dfrac{\partial\varphi^{in}}{\partial x'} + \dfrac{J}{\nu}\,|, \\ \Im m\,\dfrac{\partial\varphi_J^1}{\partial x_3}(x', x_3 = \delta h) > 0 \quad else, \end{cases} \tag{3}$$

where

$$\varphi^{in}(x) = q\,.\,x,$$

and $e_J^1(x,\delta)$ taking the form :

$$e_J^1(x,\delta) = \sum_{0 \leq m \leq M} e_J^{m,1}(x)\,\delta^m,$$

are to satisfy vectorial transport equations. They are determined from tangential components on $x_3 = \delta h$. The first-order transport and conditional equations are given by :

$$\begin{cases} \Delta\,\varphi_J^1\; e_J^{0,1} + 2\,(\nabla\,\varphi_J^1\,.\,\nabla)\; e_J^{0,1} = 0, \\ \nabla\,\varphi_J^1\,.\,e_J^{0,1} = 0. \end{cases}$$

To resolve the eikonal equations (3), one must assume the following hypothesis :

$(H.\,3)$: $\quad |\,\dfrac{J}{\nu} + \dfrac{\partial\varphi^{in}}{\partial x_3}(x_0)\,| \neq \sqrt{\varepsilon_1\,\mu_1},\ \forall\, J \in \mathbb{Z}^2.$

The solution φ_J^1 to the eikonal equation (3) is holomorphic in a complex neighborhood of x_0 independent on δ and J. For J large enough, it is easily seen that φ_J^1 satisfies the following estimate :

$$\Im m\, \varphi_J^1 \;\geq\; C\,|J|\,(x_3 - \delta h),$$

where C is a positive constant independent of δ and J.
The convergence of the series (2) is obtained from the following lemma :

LEMMA 3.1. *One has the following estimate :*

$$\sup_{0\leq m\leq M} |\, e_J^{m,1}(x)\,| \leq\; C_1\; e^{-\eta_1\,|J|}. \tag{4}$$

C_1 and η_1 are two strictly positive constants.
The proof of this lemma will be given at the end of this paper.

One constructs an asymptotic expansion for E_2^δ in $x_3 \leq -\delta h$, in the form :

$$E_2^\delta(x) \;=\; \sum_{J\in\mathbb{Z}^2} e^{ik\varphi_J^2(x)}\; e_J^2(x,\delta), \tag{5}$$

where each phase $\varphi_J^2(x)$ is to satisfy the eikonal equation :

$$\begin{cases} |\,\nabla\varphi_J^2(x)\,| = \varepsilon_2\,\mu_2, \\ \varphi_J^2(x', x_3 = -\delta h) \;=\; \varphi^{in}(x', x_3 = -\delta h) + \dfrac{J\,.\,x'}{\nu}, \\ \Im m\, \dfrac{\partial\varphi_J^2}{\partial x_3} \;<\; 0 \quad if \quad \sqrt{\varepsilon_2\,\mu_2} > |\, \dfrac{\partial\varphi^{in}}{\partial x'} + \dfrac{J}{\nu}\,|, \\ \Im m\, \dfrac{\partial\varphi_J^2}{\partial x_3}(x', x_3 = -\delta h) \;<\; 0 \quad else, \end{cases}$$

and $e_J^2(x,\delta)$ taking the form :

$$e_J^2(x,\delta) \;=\; \sum_{0\leq m\leq M} e_J^{m,2}(x)\;\delta^m,$$

are to satisfy vectorial transport equations. They are determined from tangential components on $x_3 = -\delta h$. To resolve the eikonal equations (??), one must assume the following hypothesis :

$$(H.\,4)\;:\qquad |\,\frac{J}{\nu} + \frac{\partial\varphi^{in}}{\partial x_3}(x_0)\,| \neq\; \sqrt{\varepsilon_2\,\mu_2},\;\forall\; J\in\mathbb{Z}^2.$$

The convergence of the series (5) is obtained from the following lemma :

LEMMA 3.2. *One has the following estimate :*

$$\sup_{0\leq m\leq M} |\, e_J^{m,2}(x)\,| \leq\; C_2\; e^{-\eta_2\,|J|}. \tag{6}$$

C_2 and η_2 are two strictly positive constants.

The operators of trace on $x_3 = \delta h$ are given by :

$$E^e(x', \delta h) \wedge r_3 = e^{ik\varphi^{in}(x', \delta h)} \sum_{J \in \mathbb{Z}^2} e^{iJ \cdot y'} f_J^1(x', \delta),$$

where

$$f_J^1(x', \delta) = \sum_{0 \le m \le M} e_J^{1,m}(x', x_3 = \delta h) \wedge r_3 \ \delta^m,$$

and

$$\Pi(\vec{curl}\, E^e)(x', \delta h) = e^{ik\varphi^{in}(x', \delta h)} \sum_{J \in \mathbb{Z}^2} e^{iJ \cdot y'} \Big(- ik\, \Pi(\nabla \varphi_J^1 \wedge e_J^1)(x', \delta h; \delta) + \Pi(\vec{curl} e_J^1)(x', \delta h; \delta) \Big),$$

where $\Pi = -r_3(\, \wedge r_3 \wedge .)$.

The operators of trace on $x_3 = -\delta h$ are given by :

$$E_2^\delta(x', -\delta h) \wedge r_3 = e^{ik\varphi^{in}(x', -\delta h)} \sum_{J \in \mathbb{Z}^2} e^{iJ \cdot y'} f_J^2(x', \delta),$$

where

$$f_J^2(x', \delta) = \sum_{0 \le m \le M} e_J^{2,m}(x', x_3 = -\delta h) \wedge r_3 \ \delta^m,$$

and

$$\Pi(\vec{curl}\, E_2^\delta)(x', -\delta h) = e^{ik\varphi^{in}(x', -\delta h)} \sum_{J \in \mathbb{Z}^2} e^{iJ \cdot y'} \Big(- ik\, \Pi(\nabla \varphi_J^2 \wedge e_J^2)(x', -\delta h; \delta) + \Pi(\vec{curl} e_J^2)(x', -\delta h; \delta) \Big).$$

Let M be an integer ($M \ge 1$). In $-\delta h \le x_3 \le \delta h$, one seeks for E^δ in the form :

$$\begin{aligned} E^\delta(x) &= e^{ik\varphi^{in}(x', 0)}\, e^\delta(x', x'/\delta, x_3/\delta) \\ &= e^{ik\varphi^{in}(x', 0)} \sum_{0 \le m \le M} e_m(x', x'/\delta, x_3/\delta)\, \delta^m. \end{aligned} \tag{7}$$

In order to determine a_m one first verifies that

$$\delta^2 \Big(\vec{curl}_x \frac{1}{\mu} \vec{curl}_x + k^2\, \varepsilon \Big)\, E^\delta = P(\sum e_m\, \delta^m),$$

where

$$P = P_0 + \delta P_1 + \delta^2 P_2,$$

with

$$P_0(x', \nu, y, \partial y) = (\partial_{y'} + i\nu \frac{\partial \varphi^{in}}{\partial x'}(x', 0)) \wedge (\frac{1}{\mu})(\partial_{y'} + i\nu \frac{\partial \varphi^{in}}{\partial x'}(x', 0)) \wedge + \partial_{y_3} \wedge \frac{1}{\mu} \partial_{y_3} \wedge + \nu^2\, \varepsilon,$$

$$P_1(x', \nu, y, \partial y) = \partial_{x'} \wedge (\frac{1}{\mu}) \Big(\partial_{y'} + i\nu \frac{\partial \varphi^{in}}{\partial x'} \Big) \wedge + \Big(\partial_{y'} + i\nu \frac{\partial \varphi^{in}}{\partial x'} \Big) \wedge (\frac{1}{\mu})\, \partial_{x'} \wedge,$$

and

$$P_2(x',\nu,y,\partial y) \;=\; \partial_{x'} \wedge (\frac{1}{\mu})\,\partial_{x'} \wedge$$

For the sake of simplicity, $\partial_{x'}$ denotes $\partial_{(x',0)}$, $\partial_{y'}$ denotes $\partial_{(y',0)}$ and ∂_{y_3} denotes $\partial_{(0,y_3)}$.

Equating coefficients of equal powers of δ yields the following sequence of problems :

$$(8)\qquad \begin{cases} P_0\, e_0 = 0 & \text{in } \; I\!R^2/d.\mathbb{Z}^2 \;\times\;]\,-h,h\,[\,, \\ i\nu\,\varepsilon(y_3)\,q'\,.\,e_0 + div\,\varepsilon\, e_0 = 0, & \text{in } I\!R^2/d.\mathbb{Z}^2 \;\times\;]\,-h,h\,[\,, \\ e_0 \wedge r_3 = v_0^1(x',\delta) & \text{on } y_3 = h, \\ e_0 \wedge r_3 = v_0^2(x',\delta) & \text{on } y_3 = -h, \end{cases}$$

$$(9)\qquad \begin{cases} P_0\, e_1 = -P_1 e_0 & \text{in } \; I\!R^2/d.\mathbb{Z}^2 \;\times\;]\,-h,h\,[\,, \\ i\nu\,\varepsilon(y_3)\,q'\,.\,e_1 + div\,\varepsilon\, e_1 = -\partial_{x'}\,(\varepsilon(y_3)\,e_0), & \text{in } I\!R^2/d.\mathbb{Z}^2 \;\times\;]\,-h,h\,[\,, \\ e_1 \wedge r_3 = v_1^1(x',\delta) & \text{on } y_3 = h, \\ e_1 \wedge r_3 = v_1^2(x',\delta) & \text{on } y_3 = -h, \end{cases}$$

$$(10)\qquad \begin{cases} P_0\, e_2 = -P_2 e_0 - P_1 e_1 & \text{in } \; I\!R^2/d.\mathbb{Z}^2 \;\times\;]\,-h,h\,[\,, \\ i\nu\,\varepsilon(y_3)\,q'\,.\,e_2 + div\,\varepsilon\, e_2 = -\partial_{x'}\,(\varepsilon(y_3)\,e_1), & \text{in } I\!R^2/d.\mathbb{Z}^2 \;\times\;]\,-h,h\,[\,, \\ e_2 \wedge r_3 = v_2^1(x',\delta) & \text{on } y_3 = h, \\ e_2 \wedge r_3 = v_2^2(x',\delta) & \text{on } y_3 = -h, \end{cases}$$

$$\vdots$$

$$(11)\qquad \begin{cases} P_0\, e_M = -P_2 e_{M-2} - P_1 e_{M-1} & \text{in } \; I\!R^2/d.\mathbb{Z}^2 \;\times\;]\,-h,h\,[\,, \\ i\nu\,\varepsilon(y_3)\,q'\,.\,e_M + div\,\varepsilon\, e_M = -\partial_{x'}\,(\varepsilon(y_3)\,e_{M-1}), & \text{in } I\!R^2/d.\mathbb{Z}^2 \;\times\;]\,-h,h\,[\,, \\ e_M \wedge r_3 = v_M^1(x',\delta) & \text{on } y_3 = h, \\ e_M \wedge r_3 = v_M^2(x',\delta) & \text{on } y_3 = -h, \end{cases}$$

where

$$v_m^1(x',\delta) \;=\; e_m(x',x'/\delta,h) \wedge r_3,$$

and

$$v_m^2(x',\delta) \;=\; e_m(x',x'/\delta,-h) \wedge r_3,$$

are the tangential components of the functions e_m on $y_3 = h$ and on $y_3 = -h$. $(e_m)_{m=1,\dots M}$ are periodic in y' with period $d = (d_1, d_2)$.

The operators $P_j = P_j(x',\nu,y,\partial_y)$ for $j = 0,1,2$ are $\mathcal{C}^\infty$ in x' and they are at most of second order.
One remarks that x' appears as a parameter in these problems and that $(P_j)_{j=1,2}$ are independent on ∂_{y_3}. Therefore we have the following important regularity lemma.

LEMMA 3.3. *For $0 \le m \le M$,*

$$e_m \in \mathcal{C}^\infty(x', H_{per}(\vec{curl}, y)) \; and \; \varepsilon(y_3) e_m \in \mathcal{C}^\infty(x', H_{per}(div, y))$$

where

$$H_{per}(\vec{curl}, y) = H(\vec{curl}, I\!R^2/d\,.\,\mathbb{Z}\!\!\mathbb{Z}^2 \times\ ]\ -h, h\ [\ )$$

and $H_{per}(div, y) = H(div, I\!R^2/d\,.\,\mathbb{Z}\!\!\mathbb{Z}^2 \times\ ]\ -h, h\ [\ )$.

Proof. It is easily seen that P_j maps $\mathcal{C}^\infty(x', H_{per}(\vec{curl}, y))$ onto $\mathcal{C}^\infty(x', H_{per}(div, y))$. It remains to verify that the solution of the variationnal problem $P_0 u = v$, where $v \in \mathcal{C}^\infty(x', H_{per}(div, y))$ is in $\mathcal{C}^\infty(x', Hper(\vec{curl}, y))$. For all $w \in H_{per}(\vec{curl}, y)$ with null tangent trace on $y_3 = -h$ and $y_3 = h$,

$$-\int_{O_h} \nu^2 \varepsilon(y_3)\, u(y)\, \overline{w}(y)\, dy + \int_{O_h} 1/\mu(y_3)\, \Big((\partial_{y'} + i\nu q')\ \wedge\ u(y)\Big)\,.\,\Big((\partial_{y'} + i\nu q')\ \wedge\ \overline{w}(y)\Big)\, dy$$

$$+\int_{O_h} 1/\mu(y_3)\, \Big(\partial_{y_3}\ \wedge\ u(y)\Big)\,.\,\Big(\partial_{y_3}\ \wedge\ \overline{w}(y)\Big)\, dy\ =\ \int_{O_h} v(y)\, \overline{w}(y)\, dy, \tag{12}$$

where

$$O_h\ =\ I\!R^n/d\,.\,\mathbb{Z}\!\!\mathbb{Z}^n \times\ ]\ -h, h\ [\ .$$

From $\varepsilon(y_3)$ and $\mu(y_3) \in \mathcal{C}^\infty(x', L^\infty(y))$, Lemma 3.3 above holds by differentiating (12) with respect to x'. □

One must choose h such that 0 is not an eigenvalue of the operator $P_0 u = 0, i\,\nu\,\varepsilon(y_3)\, q'\,.\,u + div\,\varepsilon\, u = 0,$ *in* $I\!R^2/d.\mathbb{Z}\!\!\mathbb{Z}^2\ \times\ ]\ -h, h\ [$ and $u \wedge r_3 = 0$ on $y_3 = -h$ and $y_3 = h$.

LEMMA 3.4. *The set* H *defined as*
$\Big\{h \in I\!R, h > 1$ *such that* 0 *is not an eigenvalue of the operator defined above* $\Big\}$ *is an open dense set in* $\{h > 1\}$.

The proof of this lemma is essentially the same as in [2] and then one omits it.
Each of problems (8) – (11) has then a unique solution.

One introduces the analytic pseudo-differential elliptic operator of order 1, given by :

$$\mathcal{R}^- : v = (v^1, v^2) \in \Big(TH^{-1/2}(div, I\!R^2/d.\mathbb{Z}\!\!\mathbb{Z}^2)\Big)^2$$

$$\longmapsto \mathcal{R}^- v\ = \Big(\Pi(\vec{curl}\, u)\ |_{y_3=h}, \Pi(\vec{curl}\, u)\ |_{y_3=-h}\Big) \in \Big(TH^{-1/2}(curl, I\!R^2/d.\mathbb{Z}\!\!\mathbb{Z}^2)\Big)^2,$$

where u is defined as the unique solution of the following problem :

$$\begin{cases} P_0\, u = 0 & \text{in } I\!R^2/d.\mathbb{Z}\!\!\mathbb{Z}^2\ \times\ ]\ -h, h\ [\ , \\ i\nu\varepsilon(y_3) q'\,.\,u + div\,\varepsilon\, u = 0, & \text{in } I\!R^2/d.\mathbb{Z}\!\!\mathbb{Z}^2\ \times\ ]\ -h, h\ [\ , \\ u \wedge r_3 = v^1 & \text{on } y_3 = h, \\ u \wedge r_3 = v^2 & \text{on } y_3 = -h. \end{cases}$$

Now, one links up the two asymptotic expansions (2) and (7) on $x_3 = \delta h$ and (5) and (7) on $x_3 = -\delta h$.

Direct computation yields :

$$q \wedge r_3\ +\ \sum_{J \in \mathbb{Z}\!\!\mathbb{Z}^2} e^{iJ.y'}\ f_J^1(x', \delta)\ =\ e^{ih\nu q_3}\Big\{v_0^1(x', \delta) + \delta v_1^1(x', \delta) + \ldots\Big\},$$

and

$$\sum_{J\in\mathbb{Z}^2} e^{iJ.y'}\Big(-i\nu\,\Pi(\nabla\,\varphi_J^1\wedge e_J^1)+\delta\,\Pi(\vec{curl}\,e_J^1)\Big)+i\nu\,\Pi(q)\,\wedge p))$$
$$= e^{ih\nu q_3}\Big\{\Pi(\vec{curl}\,e_0)+\delta\,\Pi(\vec{curl}\,e_1)+\ldots\Big\},$$

$$\sum_{J\in\mathbb{Z}^2} e^{iJ.y'}\,f_J^2(x',\delta) = e^{-ih\nu q_3}\Big\{v_0^2(x',\delta)+\delta v_1^2(x',\delta)+\ldots\Big\},$$

and

$$\sum_{J\in\mathbb{Z}^2} e^{iJ.y'}\Big(-i\nu\,\Pi(\nabla\,\varphi_J^2\wedge e_J^2)+\delta\,\Pi(\vec{curl}\,e_J^2)\Big)$$
$$= e^{-ih\nu q_3}\Big\{\Pi(\vec{curl}\,e_0)+\delta\,\Pi(\vec{curl}\,e_1)+\ldots\Big\}.$$

One introduces $\mathcal{R}^+$, the analytic pseudo-differential elliptic operator of order 1, defined, for

$$w(x',y',\delta) = \sum_{J\in\mathbb{Z}^2} e^{iJ\,.\,y'}\Big(f_J^1(x',\delta),f_J^2(x',\delta)\Big) = \sum_{0\le m\le M} w_m\,\delta^m$$

by

$$\mathcal{R}^+(w) = \sum_{J\in\mathbb{Z}^2} e^{iJ\,.\,y'}\Big\{-i\,\nu\,\Big(\psi_J^1\,f_J^1,\psi_J^2\,f_J^2\Big)+\Big(\frac{\alpha_J}{\psi_J^1}\,.\,f_J^1\alpha_J,\frac{\alpha_J}{\psi_J^2}\,.\,f_J^2\alpha_J\Big)\Big\},$$

where

$$\begin{cases}\psi_J^1 = \dfrac{\partial\varphi_J^1}{\partial x_3}(x',\delta h),\\ \psi_J^2 = \dfrac{\partial\varphi_J^2}{\partial x_3}(x',-\delta h),\end{cases}$$

and

$$\alpha_J = \Big(q'+\frac{J}{\nu},0\Big)\,\wedge\,r_3.$$

The unknowns are $(w_m)_{0\le m\le M}$ and $(v_m)_{0\le m\le M}$. As in the case of the Helmholtz equation [2], these equations lead to a triangular linear system of equations which has a unique solution since the pseudo-differential operator $\mathcal{R}^+-\mathcal{R}^-$ is invertible. Having principal real valued symbol, this operator is a Fredholm operator with index 0 and it is invertible if the scattering problem for Maxwell's equations by the infinite plane grating G^δ has a unique solution. That is where the hypothesis $H.\,2$ comes into play. Under this hypothesis, the system is solved by recurrence and the w_m are analytic in y' because the hand member is not dependent on y'.
As a consequence, one obtains the estimates (4) and (6) and a geometric-optics asymptotic expansion for Maxwell's equations. This leads to the main result of this paper.

THEOREM 3.1. *We assume that the hypothesis* $(H.\,1)-(H.\,4)$ *are satisfied. Then, for each point on* $]-l_1,l_1[\,\times\times\,]-l_2,l_2[\,\times\{0\}$, *there exists a geometric-optics asymptotic expansion given by* (2), (5) *and* (7) *satisfying Maxwell's equations to arbitrarily order in* δ. *The first order in this asymptotic expansion is the unique solution of the infinite periodic plane grating* G^δ.

References

[1] T. Abboud, *Formulation variationnelle des équations de Maxwell dans un réseau bipériodique de IR^3*. C. R. Acad. Sci. Paris, t.317, Série I, p.245-248, 1993.
[2] H. Ammari, *Finite gratings at high frequencies. Part I : The Helmholtz equation*, Rapport interne C.M.A.P, Ecole Polytechnique (1995).
[3] X. Chen and A. Friedman, *Maxwell's equations in a periodic structure,* Trans. Amer. Math. Soc, 323, 465-507 (1992).
[4] D. Dobson and A. Friedman, *The time-harmonic Maxwell equations in a doubly periodic structure*, J. Math. Anal. Appl., 166, 507-528 (1992).
[5] J.C. Nédélec and F. Starling, *Integral equation methods in a quasi-periodic diffraction problem for the time-harmonic Maxwell's equations,* SIAM, J. Math. Anal., 22 (1991), pp. 1679-1701.

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